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The Bremermann-Dirichlet Problem for q-Plurisubharmonic Functions.

ZBIGNIEW SLODKOWSKI

0. – Introduction.

A smooth C^2-function in $\mathbf{C}^n$ is called q-plurisubharmonic $(0 \leqslant q \leqslant n-1)$ if its complex Hessian has at least $(n-q)$-nonnegative eigenvalues at each point. Hunt and Murray [8] gave a new definition, which is also applicable to upper semicontinuous functions, and studied systematically properties of the larger class. They proved, among other results, that, under natural assumptions on the boundary of a domain $D \subset \mathbf{C}^n$ for every continuous function $b\colon \partial D \to R$ there exists a continuous extension u of b to D which is both q-plurisubharmonic and $(n-q-1)$-plurisuperharmonic in D. Hunt and Murray [8] conjecture that there is at most one u with such properties.

The main aim of this paper is to prove this conjecture. (The proof given by Kalka [9], who considers a special case of this conjecture is—in our opinion—incorrect.)

Hunt and Murray, as well as Kalka, have observed that the above conjecture is a consequence of the following one.

(*) *If u and v are q- and r-plurisubharmonic functions respectively, then $u+v$ is $(q+r)$-plurisubharmonic.*

On the other hand the author, working on Basener's conjecture concerning higher order Shilov boundaries of tensor products of uniform algebras (cf. Basener, [2]), has reduced it to the following claim:

(**) *If u and v are q- and r-plurisubharmonic functions respectively, then $\min(u, v)$ is $(q+r+1)$-plurisubharmonic.*

The similarity between conjectures (*) and (**) suggested that a relation between them might exist. In fact, in Sect. 6 we prove that (*) implies (**). The rest of the paper is devoted to the proof of (*).

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Already Hunt and Murray [8] have noticed that (*) is simple for smooth functions and they have raised the question whether uniform approximation of continuous q-plurisubharmonic functions by smooth q-plurisubharmonic ones is possible. This seems to be still unknown. We were, however, able to achieve such an approximation by means of q-plurisubharmonic functions with second-order derivatives (in the Peano sense) existing almost everywhere (Theorem 2.9). (As a by-product, it is proved that every upper semicontinuous function from the q-plurisubharmonic class can be approximated pointwisely by continuous functions from this class.)

More specifically, we introduce the class of *functions with lower bounded Hessian*, to which our approximants belong (Sec. 2). A function of this class is q-plurisubharmonic if and only if it has almost everywhere at most q negative eigenvalues (Theorem 4.1). This and the approximation mentioned above yield the proof of Conjecture (*), and consequently the proof of uniqueness of solution to the Dirichlet problem described above (cf. Sec. 5).

Since the class of functions with lower bounded Hessian is strictly related to the class of convex functions, it is but natural that some estimates concerning convex functions play essential role in our arguments (Sec. 3).

Main results of this paper and of [13] were announced in [12]. Applications to uniform algebras and to further study of q-plurisubharmonic functions and q-pseudoconvex domains will appear in [13].

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1. – Basic properties of q-plurisubharmonic functions.

We recall, after Hunt and Murray [8], that an upper semicontinuous function $u\colon U \to [-\infty, +\infty)$, where $U \subset \mathbf{C}^n$ is open, is said to be q-plurisubharmonic if for every $(q+1)$-dimensional complex hyperplane L intersecting U, for every closed ball $\bar{B} \subset U \cap L$, and for every smooth, plurisuperharmonic function g defined in a neighbourhood of $\bar{B}$ (in L):

$$\text{if } g|\partial B > u|\partial B, \text{ then } g|\bar{B} > u|\bar{B}. \tag{1.1}$$

Actually Hunt and Murray considered the following condition

$$\text{if } g|\partial B \geqslant u|\partial B, \text{ then } g|B \geqslant u|B, \tag{1.2}$$

which is, of course, equivalent to (1.1).

REMARK. The notion of q-plurisubharmonic function is interesting only when $0 \leqslant q \leqslant n-1$. However, the definition makes sense also for $q \geqslant n$: in this case each upper semicontinuous function in $\mathbf{C}^n$ is q-plurisubharmonic.

Notation and terminology. A C^2-function u is called *strictly q-plurisubharmonic* in $U \subset \mathbf{C}^n$ if its complex Hessian has at least $(n-q)$ positive eigenvalues at each point of U.

A function u is called q-plurisuperharmonic if $-u$ is q-plurisubharmonic.

Let $E \subset \mathbf{C}^n$; usc (E) denotes the set of all upper semicontinuous functions defined in E; $C(E)$ denotes all continuous functions defined in E; $PSH_q(E)$, $q \geqslant 0$, the set of restrictions to E of all functions q-plurisubharmonic in some neighbourhood of E; if E is open, $C^2PSH_q(E)$ denotes the set of all smooth q-plurisubharmonic functions defined in E.

Let $E \subset R^N$; if u is defined in $E \subset R^N$, and $y \in R^N$ then the function $x \to u(x-y): E + y \to [-\infty, +\infty)$ is denoted by $T_y u$. The topological boundary of E is denoted by ∂E.

$B(c, r)$ denotes the open ball with center c and radius r with respect to Euclidean metric in $\mathbf{C}^n$ or R^n; $\bar{B}(c, r)$ denotes the closure of $B(c, r)$; $S(c, r)$ denotes the boundary of $B(c, r)$.

Let $f: U \to [-\infty, +\infty)$, $U \subset \mathbf{C}^n$, belong to usc (E). We say that f *has local maximum property in* U if, for every compact subset $K \subset U$

$$\max f|K \leqslant \max f|\partial K . \tag{1.3}$$

We will use the following characterization of q-plurisubharmonic functions. The methods of proof resemble those of Hunt and Murray [8].

PROPOSITION 1.1. *Let $U \subset \mathbf{C}^n$ be open and $u \in$ usc (U). Then $u \in PSH_q(U)$ if and only if one of the following two conditions holds*

(i) *for every $U' \subset U$ and $f \in C^2PSH_{n-q-1}(U')$ the function $u + f$ has local maximum property in U,*

(ii) *every point $z \in U$ has a basis $\{U_k\}$ of relatively compact neighbourhoods such that for every $f \in C^2PSH_{n-q-1}(\overline{U}_k)$ it holds*

$$(u+f)(z^*) \leqslant \max (u+f)|\partial U_k .$$

The function $u \in PSH_q(U)$ if and only if the following condition does not hold.

(iii) *there exist $z^* \in U$, $\varepsilon > 0$, $r > 0$ with $B(z^*, r) \subset U$ and a strictly $(n-q-1)$-plurisubharmonic function f, defined in $B(z^*, r)$ such that*

$$u(z^*) + f(z^*) = 0 \tag{1.4}$$

$$u(z) + f(z) \leqslant -\varepsilon|z-z^*|^2, \qquad |z-z^*| < r . \tag{1.5}$$

PROOF. It is enough to check the following implications:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \sim \text{(iii)} \Rightarrow \text{(i)} \Rightarrow (1.1) \Rightarrow \sim \text{(iii)}.$$

The first two are obvious.

$\sim$(iii) $\Rightarrow$ (i); we show that $\sim$(i) $\Rightarrow$ (iii). Suppose that there is $f \in C^2PSH_{n-q-1}(U')$, $U' \subset U$, such that $u+f$ does not have local maximum property in U', *i.e.* for some compact $K \subset U'$, $\max(f+u)|K > \max(f+u)|\partial K$. Then there is $\varepsilon > 0$ such that $M := \max(f_1+u)|K > \max(f_1+u)|\partial K$, where $f_1(z) := f(z) + 2\varepsilon|z|^2$, $z \in U$. Choose $z^* \in \operatorname{Int}(K)$ such that $(f_1+u)\cdot$ $\cdot(z^*) = M$ and set $f_2(z) = f_1(z) - M - \varepsilon|z-z^*|^2$. It is clear that $(u+f_2)\cdot$ $\cdot(z^*) = 0$, $(u+f_2)(z) \leqslant -\varepsilon|z-z^*|^2$ for $z \in K$, while $z^* \in \operatorname{Int}(K)$. Moreover $f_2(z) = f(z) + \varepsilon|z|^2 +$ linear form, and so it is strictly q-plurisubharmonic. Thus f_2 satisfies (iii) (cf. Hunt and Murray, [8], Lemma 2.7).

(i) $\Rightarrow$ (1.1). Suppose that u is not q-plurisharmonic, that is there exist a $(q+1)$-dim hyperplane L, a ball $\bar{B} \subset L \cap U$ and a 0-plurisuperharmonic function g defined in a neighbourhood of $\bar{B}$, such that $g|\partial B > u|\partial B$, but for some $z_1 \in B$, $g(z_1) \leqslant u(z_1)$. Similarly as Hunt and Murray [8, Proof of Th. 3.3], we choose new coordinates so that $L = \{z_{q+2} = \ldots = z_n = 0\}$ and $B = \{z \in L\colon |z| < R\}$, and an open convex set K such that $\bar{K} \subset U$, $K \cap L = B$ and the orthogonal projection of K onto L is exactly B. We set $g_w(z) := g(z_1, \ldots, z_{q+1}) + C(|z_{q+2}|^2 + \ldots + |z_n|^2)$. Clearly g_w is a smooth, $(n-q-1)$-plurisuperharmonic function in a neighbourhood of $\bar{K}$ for $C > 0$. If $C > 0$ is big enough then $g_w|\partial K > u|\partial K$ but $g_w(z_1) \leqslant u(z_1)$. Finally $f := -g_w \in C^2PSH_{n-q-1}(\bar{K})$ and $(u+f)(z_1) \geqslant 0 > \max(u+f)|\partial K$, which contradicts (i).

(1.1) $\Rightarrow \sim$(iii). We prove (iii) $\Rightarrow \sim$(1.1). Let f, z^*, ε be as in (iii). Since f is strictly $(n-q-1)$-plurisubharmonic, its complex Hessian at z^* has at least $(q+1)$-positive eigenvalues. Consider the $\mathbf{C}$-linear subspace spanned by corresponding eigenvectors and translate it to z^*. Then $g = -f|L$ has negative definite Hessian and so it is plurisuperharmonic near z^*, say in $B(z, 2\delta)$. Further $g(z^*) = u(z^*)$, but $u < g$ on $\partial B(z^*, \delta)$. Thus (1.1) does not hold. Q.E.D.

For easy reference we list now some properties of q-plurisubharmonic functions.

PROPOSITION 1.2. *Let $D, D_1 \subset \mathbf{C}^n$ be open, $0 \leqslant q \leqslant n-1$. Then*

(i) $u \in PSH_q(D)$, $D \subset D_1 \Rightarrow u|D_1 \in PSH_q(D_1)$

(ii) $u \in PSH_q(D)$, $y \in \mathbf{C}^n \Rightarrow T_y u \in PSH_q(D+y)$

(iii) $D \subset D_1$, $u \in PSH_q(D)$, $u_1 \in PSH_q(D_1)$ *and for every* $z \in D \cap \partial D_1$, $\limsup_{z' \to z} u_1(z') \leqslant u(z)$, *then the function*

$$v(z) = \begin{cases} u(z)\,, & z \in D \setminus D_1 \\ \max\big(u(z), u_1(z)\big)\,, & z \in D_1 \end{cases}$$

belongs to $PSH_q(D)$.

(iv) *if* $\{u_t\} \subset PSH_q(D)$ *is locally uniformly bounded from the above and* $u(z) = \sup u_t(z)$, *then* $u^* \in PSH_q(U)$, *where* u^* *denotes the upper semicontinuous regularization of* u.

(v) $PSH_q(D) + PSH_0(D) \subset PSH_q(D)$

(vi) $u \in PSH_q(D)$, $r > 0 \Rightarrow ru \in PSH_q(D)$

(vii) *the limit of a pointwise convergent and nonincreasing sequence of* q*-plurisubharmonic functions is* q*-plurisubharmonic.*

(viii) *every* $u \in PSH_q(U)$, $0 \leqslant q \leqslant n-1$, *has local maximum property in* U.

These properties, proved, for the most part, by Hunt and Murray [8] follow quickly from the criterion given above. Thus, for example, (viii) is a special case of Prop. 1.1.(i) with $f = 0$, and (iii) follows from Proposition 1.1.(ii).

2. – Regularization of q-plurisubharmonic functions.

In this and next sections we will frequently deal with functions on R^N. When we pass to $\mathbf{C}^n = R^{2n}$, the real Hessian has to be distinguished from the complex one.

DEFINITION 2.1. Let $U \subset R^N$ be open. We say that a function $u: U \to R$ has *lower bounded* (*real*) *Hessian* if there is $L > 0$ such that the function $u(x) + \frac{1}{2}L|x|^2$ is locally convex in U.

Our terminology is motivated by the following proposition. Since we will not use it, we omit its (easy) proof.

PROPOSITION 2.2. *Let* $U \subset R^N$ *be open,* $u: U \to R$ *be continuous,* $x \in R^N$ *and* $L > 0$. *Then the following conditions are equivalent:*

(i) *the function* $u(x) + \frac{1}{2}L|x - x^*|^2$ *is locally convex in* U;

(ii) *for every* $x \in U$ *there is* $\delta > 0$ *such that for every* $|h| < \delta$ $u(x+h) + u(x-h) + L|h|^2 \geqslant 2u(x)$;

(iii) *the real Hessian of* u, *in the sense of the distribution theory, is a matrix valued measure. Its singular part is a measure with values in positive semi-definite matrices and its Radon-Nikodym derivative takes a.e. values that are matrices with no eigenvalue smaller than* $-L$.

REMARK. One can check that $u \in C^{1,1}$ if and only if both u and $-u$ have lower bounded (real) Hessian.

We denote the class of all functions satisfying with constant L any of these conditions by $C^1_L(U)$.

PROPOSITION 2.3. *Let* $U_1 \subset U \subset R^N$ *be open,* $L, L_1 \geqslant 0$. *Class* $C^1_L(U)$ *has the following properties*:

(i) $u \in C^1_L(U)$, $U \subset U_1 \Rightarrow u|U_1 \in C^1_L(U_1)$;

(ii) $u \in C^1_L(U)$ *and* $y \in R^N \Rightarrow T_y u \in C^1_L(U+y)$;

(iii) C^1_L *is local in the following sense: if for every* $x \in U$ *there is* $\delta > 0$ *such that* $u|B(x, \delta) \in C^1_L(B(x, \delta))$, *then* $u \in C^1_L(U)$;

(iv) *if family* $\{u_t\} \subset C^1_L(U)$ *is pointwise bounded from the above, then* $\sup_t u_t \in C^1_L(U)$;

(v) $C^1_L(U) + C^1_{L_1}(U) \subset C^1_{L+L_1}(U)$;

(vi) $u \in C^1_L(U)$, $r > 0 \Rightarrow ru \in C^1_{rL}(U)$;

(vii) *if* $\{u_n\} \subset C^1_L(U)$ *converges pointwise to* u, *then* $u \in C^1_L(U)$;

(viii) $L_1 \leqslant L \Rightarrow C^1_{L_1}(U) \subset C^1_L(U)$;

(ix) *if* $u \in C^2(U)$ *and at each poin*$_s$ *its (real) Hessian has norm non bigger than* L, *then* $u \in C^1_L(U)$.

PROOF. These properties follow directly from the definition and corresponding properties of convex functions. We check only some of them.

(ii) $v(x) = u(x) + \frac{1}{2}L|x|^2$ is convex in U. $(T_y u)(x) + \frac{1}{2}L|x|^2 = u(x-y) + \frac{1}{2}L|x|^2 = v(x-y) - \frac{1}{2}L|x-y|^2 + \frac{1}{2}L|x|^2 = (T_y v)(x) +$ linear form, and so it is convex.

(iv) The functions $v_t(x) := u_t(x) + \frac{1}{2}L|x|^2$ are convex in U. The family $\{v_t\}$ is pointwise bounded in U and so $v(x) = \sup_t v_t(x)$ is a convex function; in particular v is continuous, cf. Rockafellar [10, Th. 10.1]. Clearly $v(x) = u(x) + \frac{1}{2}L|x|^2$ and so $u \in C^1_L(U)$.

(ix) Let $v(x) = u(x) + \frac{1}{2}L|x|^2$. Then $(\text{Hess}\, v)\,(x) = (\text{Hess}\, u)\,(x) + LI$, where I denotes the identity matrix. Clearly Hess v is positive semidefinite at each $x \in U$. By the well-known criterion v is convex and $u \in C_L^1(U)$. Q.E.D.

DEFINITION 2.4. Let u and g be non-negative bounded function (possibly discontinuous) defined on the whole R^N. The *supremum-convolution* of u and g, denoted by $u *_s g$, is defined by the formula

$$u *_s g(x) := \sup \{u(y)g(x-y) : y \in R^N\}, \qquad x \in R^N.$$

If u is defined in $U \subset R^N$ only, $u *_s g$ is understood as $\tilde{u} *_s g$, where $\tilde{u}(x) = u(x)$ for $x \in U$ and 0 otherwise.

The following results show how the operation of supremum-convolution helps us to obtain regularization of q-plurisubharmonic functions.

REMARK 2.5. If $u \geqslant 0$, $g \geqslant g_1 \geqslant 0$ are bounded functions, then $u *_s g \geqslant u *_s g_1 \geqslant 0$ (Proof obvious).

PROPOSITION 2.6. *Let $u\colon R^N \to [0, +\infty)$ and $g \in C_L^1(R^N)$, $g \geqslant 0$, $L \geqslant 0$ be bounded. Then $u *_s g \in C_{ML}^1(R^N)$, where $M = \sup u$. In particular $u *_s g$ is continuous.*

PROOF. By Def. 2.4 $u *_s g(x) = \sup\limits_{y \in R^N} \big(u(y)T_y g\big)(x)$. We use Proposition 2.3: by (ii) $T_y g \in C_L^1(R^N)$, by (vi) $u(y)T_y g \in C_{ML}^1(R^N)$, and by (iv) $u * g \in C_{ML}^1(R^N)$. Q.E.D.

LEMMA 2.7. *Let $u, g\colon \mathbf{C}^n \to [0, \infty)$ be bounded and $U \subset \mathbf{C}^n$ be open. If u is q-plurisubharmonic in U and $\operatorname{supp} g \subset B(0, r)$, $r > 0$, then the upper semi-continuous regularization of $u *_s g$ is q-plurisubharmonic in $U_r = \{x \in U : B(x, r) \subset U\}$.*

PROOF. Let us rewrite the definition of $u *_s g$:

$$\begin{aligned}(2.1)\qquad u *_s g(z) &= \sup \{u(y)g(z-y) : y \in \mathbf{C}^n = \sup \{u(w+z)g(w) : w \in \mathbf{C}^n\} \\ &= \sup_{|w| \leqslant r} \big(g(w)T_{-w}u\big)(z)\end{aligned}$$

(since $g(w) = 0$ for $|w| > r$). By Proposition 1.2.(i), (ii) and (vi) $T_{-w}u|U_r$ and $g(w)T_{-w}u|U_r$ belong to $PSH_q(U_r)$, for $|w| \leqslant r$. The lemma follows from Proposition 1.2.(iv) applied to uniformly bounded family $g(w)T_{-w}u|U_r$.

The following Corollary is a direct consequence of the last two results.

COROLLARY 2.8. *Let $U \subset \mathbf{C}^n$ be open, $u \in PSH_q(U)$, $g \in C_L^1(\mathbf{C}^n)$, $L \geqslant 0$.*

Assume that u and g are non-negative, $\sup u = M$, *and* $\operatorname{supp}(g) \subset B(0, r)$, $r > 0$. *Then* $u *_s g \in C^1_{ML}(\mathbf{C}^n) \cap PSH_q(U_r)$.

THEOREM 2.9. *Let $U \subset \mathbf{C}^n$ be open and $u \in PSH_q(U)$. Then for every compact $K \subset U$ there is a monotone, non-increasing sequence of functions $(u_k)_{k=1}$ such that:*

(i) $u_k \in C^1_{L(k)}(\mathbf{C}^n)$, $k = 1, 2, \ldots$, *for some constants* $L(k) \geqslant 0$, $k = 1, 2, \ldots$

(ii) $u_k|K \in PSH_q(K)$, $k = 1, 2, \ldots$

(iii) u_k *converge pointwise to u on K.*

If, in addition, u is continuous (in U), then the convergence is uniform on K.

REMARK 2.10. In particular, every q-plurisubharmonic function can be approximated (on a compact set K) by pointwise convergent monotonous sequence of continuous q-plurisubharmonic functions (defined in a neighbourhood of K).

PROOF OF THEOREM 2.9.

ASSERTION. Assume, in addition to the assumptions of the theorem, that $0 \leqslant u \leqslant M$ and that a continuous function $v: K \to R$ is given, such that $v > u|K$. Then there exist $L \geqslant 0$ and $w \in C^1_L(\mathbf{C}^n) \cap PSH_q(K)$, such that $v > w|K > u|K$.

PROOF OF THE ASSERTION. Choose $r > 0$ such that $K \subset U_r$ and

$$\max u|\bar{B}(z, r) < v(z) \quad \text{for every } z \in K, \tag{2.2}$$

(by the upper semicontinuity of u). Take $g \in C^\infty_0(\mathbf{C}^n)$ such that $g(0) = 1$, $0 \leqslant g \leqslant 1$ and $\operatorname{supp}(g) \subset B(0, r)$, and set $w = u *_s g$ (meant as $\tilde{u} *_s g$). Let L be the maximum of the norm of the real Hessian of g. By Proposition 2.3 (ix) $g \in C^1_L(\mathbf{C}^n)$ and by Corollary 2.8 $w \in C^1_{ML}(\mathbf{C}^n) \cap PSH_q(U_r)$. Moreover

$$w(z) = \sup\{\tilde{u}(y) g(z - y) : y \in \mathbf{C}^n\} \geqslant \tilde{u}(z) g(0) = u(z), \qquad z \in K.$$

Furthermore, for $z \in K$

$$\begin{aligned} w(z) = \tilde{u} *_s g(z) &= \sup_{|y| \leqslant r} g(y)(T_{-y}u)(z) \qquad (\text{by } (2.1)) \\ &\leqslant \sup_{|y| \leqslant r} (T_{-y}u)(z) \leqslant \max u|\bar{B}(z, r) < v(z) \qquad (\text{by } (2.2)). \end{aligned}$$

We may add a small constant to w, so that $w|K > u|K$, without changing other properties. The Assertion is established.

The general case. We may assume (shrinking U, if necessary) that u is bounded from the above. Since u is upper semicontinuous, there is a decreasing sequence $\{v_k\}$ of continuous functions on K converging pointwise to $u|K$; in particular $v_k > u|K$. To end the proof it is enough to find functions u_k and constants $L(k)$, $k = 1, 2, \ldots$, such that $L(k) \geqslant 0$, $u_k \in C^1_{L(k)}(\mathbf{C}^n) \cap \cap PSH_q(K)$, $v_k > u_k|K > u|K$ and $u_k > u_{k+1}$, $k = 1, 2, \ldots$

We proceed by induction. Assume that $u_1, \ldots, u_k$ are already constructed. (We can choose $u_1 \equiv (\sup u) + 1$). Now set $C = \max\left(-\inf u_k|K, -\inf v_{k+1}|K, 0\right)$, $u'(z) = C + \max\left(u(z), -C\right)$, $z \in U$ and $v'(z) = C + \min\left(u_k(z), v_{k+1}(z)\right)$, $z \in K$. Then u', v' fulfil the assumptions of the assertion, in particular $u' \geqslant 0$, $v' > u'|K$, $u' \in PSH_q(K)$, therefore there is $L(k+1) \geqslant 0$ and $w \in C^1_L(\mathbf{C}^n) \cap \cap PSH_q(K)$, such that $v' > w|K > u'|K$. It is easy to see that the function $u_{k+1}(z) := -C + w(z)$ has all the properties required in the induction step. Q.E.D.

3. – Convex functions: an estimate involving second order derivatives.

One may expect that the approximation theorem 2.9 will allow us—in many instances—to restrict our attention to q-plurisubharmonic functions with lower bounded Hessian, instead of considering the general class (cf. Sects. 4 and 6). Since every function in the special class is, up to quadratic polynomial $\frac{1}{2}L|x|^2$, convex, it is natural to apply convex analysis. It is well known that every convex function $u\colon U \to R$, U open in R^N, has second-order partial derivatives in the sense of distribution theory, and the real Hessian is a matrix-valued measure. It is much less known that such u has also second-order derivatives in the local (Peano) sense. Namely, for almost every $x \in U$, there are: a vector $a = (a_1, \ldots, a_N)$, and a symmetric matrix $(b_{ij})_{i,j=1}^N$ (real Hessian) such that

$$(3.1)\quad \lim_{y\to x} |y-x|^{-2}\Big(f(y)-f(x)\Big) - \sum_{i=1}^N a_i(y_i - x_i) - \frac{1}{2}\sum_{i,j=1}^N b_{ij}(y_i - x_i)(y_j - x_j)\Big) = 0\,.$$

The result is apparently due to Alexandrov [1]. For a proof the reader may refer also to Buseman [5, p. 24].

However, for our further applications in Sec. 4 we need also to handle somehow these points at which second order differential does not exist. To this purpose we introduce a quantity $K(u, x)$ which is equal to the largest eigenvalue of the real Hessian (b_{ij}) at x, provided it exists, and plays similar role at the remaining points.

DEFINITION 3.1. If grad u at x exists, $K(u, x)$ is defined by the formula

$$K(u, x) = \limsup_{\varepsilon\to 0} 2\varepsilon^{-2} \max\{u(x+\varepsilon h) - u(x) - \varepsilon(\operatorname{grad} u(x), h) : |h|=1\},$$

otherwise $K(u, x)$ is defined as $+\infty$.

We specify that the *lower density* of Lebesgue measurable set $Z \subset R^N$ at $x^* \in R^N$ is the number

$$\liminf_{\varepsilon\to 0} m_N(Z \cap B(x^*, \varepsilon))/m_N(B(x^*, \varepsilon)),$$

where m_N denotes the N-dimensional Lebesgue measure.

Our further applications of convex analysis to q-plurisubharmonic functions (cf. Theorem 4.1.) depend on the following, rather technical estimate.

THEOREM 3.2. *Let u be convex near $x^* \in R^N$. Assume that $K(u, x^*) = k^*$ is finite. Then for every $k > k^*$ the set $\{x: K(u, x) < k\}$ is Borel and its lower density at x^* is not less than $((k - k^*)/2k)^N$.*

Before starting the proof of Theorem 3.2 we give simple properties of $K(u, x)$.

We will say that a sphere $S(c, r)$ *supports the graph of u from the above at* $y = (x, u(x))$, if $y \in S(c, r)$, $B(c, r) \cap \operatorname{graph}(u) = \emptyset$ and $c_{n+1} > u(Pc)$, where P denotes the orthogonal projection of R^{N+1} onto R^N.

PROPOSITION 3.3. *Let $U \subset R^N$ be open and $u: U \to R$ be convex. Assume that u has gradient at x.*

(i) *If u has second-order Peano derivatives at x, then $K(u, x)$ is equal to the norm (i.e. the largest eigenvalue) of the (real) Hessian of u at x.*

(ii) *If $K(u, x)$ is finite, then for every $K > K(u, x)$ there is $\varepsilon > 0$ such that $u(x + h) - u(x) - (\operatorname{grad}\ u(x), h) \leqslant \frac{1}{2}K|h|^2$ for every $|h| < \varepsilon$.*

(iii) *If there is a sphere $S(c, r)$, $r > 0$ which supports the graph of u from the above at $(x, u(x))$, then*

$$K(u, x) \leqslant r^{-1}(1 + g^2)^{\frac{3}{2}},$$

where $g = |\operatorname{grad} u(x)|$.

The statements (i) and (ii) follow directly from the relevant definitions; (iii) is obtained by elementary, though lengthy calculations, which are postponed to the Appendix. The following lemma deals, in geometric terms, with the essential difficulty of Theorem 3.2.

LEMMA 3.4. *Let u be a non-negative convex function in $B(0, d) \subset R^N$, $d > 0$, such that $u(0) = 0$ and* (grad u) (0) $= 0$. *Let $R > 0$ and assume that the ball $B(c^*, R)$, $c^* = (0, \ldots, 0, R) \in R^{N+1}$, intersects the graph of u only at $0 \in R^{N+1}$. Let X_r, $0 < r < R$ denote the set of all $x \in B(0, d) \subset R^N$ such that there exist a sphere of radius r supporting the graph of u from the above at $\big(x, u(x)\big)$. Then the lower density of X_r at 0 is not less than $\big((R-r)/2r\big)^N$.*

PROOF. The number $r \in (0, R)$ will be kept fixed in the proof and we write X for X_r. Define $Z = \{(x, u(x)) \in R^{N+1} : x \in X\}$. It is clear that $Z \cap \bar{B}(0, d') \times R$ is compact for every $d' < d$. Also $X \cap \bar{B}(0, d) \times R$ is compact, since it is the image of the former set via the orthogonal projection $P: R^{N+1} \to R^N$. Thus the notion of density is applicable to both X and Z.

It is more convenient to estimate first the density of Z at 0 (with respect to N-dimensional Hausdorff measure H^N). With this in mind we will modify u outisde a small neighbourhood of 0, that is we are going to introduce new, auxiliary convex functions

$$v_\alpha : B(0, R) \to [0, \infty)\,, \qquad 0 < \alpha < \arcsin(d/R)\,.$$

Define

$$Y = \{y \in R^{N+1} : |y - c^*| = R,\ \sphericalangle\,(y - c^*, 0 - c^*) = 2\alpha\}\,.$$

Then $|y - c^*||c^*|\cos 2\alpha = (y - c^*, -c^*) = -Ry_{N+1} + R^2$, and so Y is the intersection of the sphere $S(c, R)$ with the hyperplane $y_{N+1} = R(1 - \cos 2\alpha)$. Its projection $P(Y)$ onto R^N is the sphere of radius $(R^2 - (R - y_{N+1})^2)^{\frac{1}{2}} = R \sin 2\alpha$, centered at 0. Let C_α denote the union of all closed segments $\overline{w, y}$ with one end-point w on the axis $0 \times R \subset R^{N+1}$ and tangent to the sphere $S(c, R)$ at the other end-point y about which we assume that it belongs to Y. Since $\sphericalangle\,(w - c^*, y - c^*) = 2\alpha$ and $\sphericalangle\,(w - y, y - c^*) = \pi/2$, therefore $|w - c^*| \cos 2\alpha = |y - c^*| = R$, and so $w = \big(0, \ldots, 0, R(1 - 1/\cos 2\alpha)\big)$ independently of y. We conclude that C_α is a (finite) cone, with vertex at w and base Y, tangent to $S(c^*, R)$ along Y. Define now

$$T_\alpha = \{y \in S(c^*, R) : R(1 - \cos 2\alpha) \leqslant y_{N+1} < R\}\,. \qquad (\text{Note that } C_\alpha \cap T_\alpha = Y.)$$

It is geometrically obvious that $C_\alpha \cup T_\alpha$ is the graph of some convex function $k_\alpha : B_N(0, R) \to R$. Finally we define, for $0 < \alpha < \frac{1}{2} \arcsin(d/R)$

$$v_\alpha(x) = \begin{cases} \max\big(u(x), k_\alpha(x)\big)\,, & |x| < R \sin 2\alpha\,, \\ k_\alpha(x) & R \sin 2\alpha \leqslant |x| < R\,. \end{cases}$$

In particular

$$v_\alpha(x) \geqslant u(x)\,, \quad \text{for } |x| < d\,. \tag{3.2}$$

It is clear that v_α is locally convex on the set $|x| \neq R \sin 2\alpha$. If $|x| = R\sin 2\alpha$, then $(x, u(x)) \in Y \subset S(c^*, R)$. By assumptions $S(c^*, R)$ lies above the graph of u, and so $k_\alpha|Y > u|Y$ and $v_\alpha \equiv k_\alpha$ near Y. Since u is locally convex in $B(0, R)$, it is convex.

Let v be a convex function in $B(0, R)$. Denote by $E(v)$ the convex set $\{(x, t) \in R^{N+1}: t > u(x)\}$. Define Z^v as the set of all $y = (x, v(x), |x| < R$, such that for some $c \in R^{N+1}$, $B(c, r) \subset E(v)$ and $y \in S(c, r)$. The set Z^v (like Z) is closed in $B_N(0, R) \times R$. Observe that if $y = (x, v(x)) \in Z^v$, then graph (v) has a unique supporting hyperplane at y (since any such hyperplane is tangent to $S(c, r)$), and, in turn, c is uniquely determined by y. We write $c = \gamma^v(y)$ and assert: *the map $\gamma^v: Z^v \to R^{N+1}$ is Lipschitz with constant one.* Indeed, let $y_1, y_2 \in Z^v$, and $c_i = \gamma^v(y_i)$, $i = 1, 2$. The set $E(v)$, being convex, contains $W := \operatorname{co}(B(c_1, r) \cup B(c_2, r))$. In particular $W \cap \operatorname{graph}(u) = 0$. Since $y_i \in S(c_i, r) \cap \operatorname{graph}(u)$, we get $y_i \in S(c_i, r)\setminus W$, $i = 1, 2$, and so y_i, $i = 1, 2$ do not belong to, and are separated by the open layer between two hyperplanes which are orthogonal to the segment $\overline{c_1, c_2}$ and pass through its ends. Thus $|c_1 - c_2| \leqslant |y_1 - y_2|$.

The objects Z^v and γ^v defined for $v = v_\alpha$, $\alpha \in (0, \frac{1}{2}\arcsin(d/R))$ will be denoted by Z^α and γ^α respectively.

Let us consider the set

$$U_\alpha = \operatorname{graph}(v_\alpha)\setminus(C_\alpha \cup T_\alpha)\,.$$

The following inclusions hold for $\alpha \in (0, \frac{1}{2}\arcsin(d/R))$:

$$P(U_\alpha) \subset B(0, R\sin 2\alpha)\,, \tag{3.3}$$

$$Z^\alpha \cap U_\alpha \subset Z \cap U_\alpha\,, \tag{3.4}$$

$$B_N(0, \delta) \subset P(\gamma^\alpha(Z^\alpha \cap U_\alpha))\,, \quad \text{where } \delta = (R - r)\operatorname{tg}\alpha\,. \tag{3.5}$$

The first inclusion follows directly from the definitions. Further, by definitions and by (3.2), $Z^\alpha \cap \operatorname{graph}(u) \subset Z$; since $U_\alpha \subset \operatorname{graph}(u)$, $Z^\alpha \cap U_\alpha \subset Z^\alpha \cap \operatorname{graph}(u) \subset Z$, and (3.4) follows. As for the third relation, consider $x \in R^N$, $|x| < R - r$. Then the set $\{c \in \{x\} \times R: B(c, r) \subset E(v_\alpha)\}$ is a non-empty, closed half-line. Let c be its end-point and $y \in S(c, r) \cap \operatorname{graph}(v_\alpha)$. Then $c = \gamma^\alpha(y)$ and $x = P\gamma^\alpha(y)$. Thus

$$B_N(0, R - r) \subset P\gamma^\alpha(Z)\,. \tag{3.6}$$

Since $Z^\alpha \setminus (C_\alpha \cup T_\alpha) \subset \operatorname{graph}(v_\alpha) \setminus (C_\alpha \cup T_\alpha) = U_\alpha$, therefore $Z^\alpha \setminus (C_\alpha \cup T_\alpha) \subset \subset Z^\alpha \cap U_\alpha$. Consequently $P\gamma^\alpha(Z^\alpha) \setminus P\gamma^\alpha(C_\alpha \cup T_\alpha) \subset P\gamma^\alpha(Z^\alpha \cap U_\alpha)$. This relation and (3.6) will imply (3.5), as soon as we prove that:

$$P\gamma^\alpha\big(Z^\alpha \cap (C_\alpha \cup T_\alpha)\big) \cap B_N(0, \delta) = \emptyset\,, \tag{3.7}$$

To see this, consider the family of all spheres $S(c, r)$ which support $C_\alpha \setminus Y$ from the above (at some point of $C_\alpha \setminus Y$) and are contained in the upper half space $y_{N+1} \geqslant 0$. Then the smallest value of $|P(c)|$ is attained when sphere $S(c, r)$ is tangent both to C_α and to $\{y_{N+1} = 0\}$. It is not difficult to see geometrically that in such a position $\sphericalangle\,(c - c^*, 0 - c^*) = \alpha$, and so $|P(c)| = \big(|c^*| - c_{N+1}\big) \operatorname{tg} \alpha = (R - r) \operatorname{tg} \alpha = \delta$. Thus

$$P\gamma^\alpha(Z^\alpha \cap C_\alpha) \cap B_N(0, \delta) = 0\,. \tag{3.8}$$

(The points of $Y \subset C_\alpha$ are handled as limits.) When in turn $S(c, r)$ supports $T_\alpha \setminus Y)$ from the above at some point y, the segment $\overline{c, y}$ is normal to $S(c^*, R)$ and $y_{N+1} \geqslant R(1 - \cos 2\alpha)$, therefore $\sphericalangle\,(c - c^*, 0 - c^*) \geqslant 2\alpha$ and, like above, $|P(c)| \geqslant (R - r) \operatorname{tg} 2\alpha \geqslant \delta$. Thus $P\big(\gamma^\alpha(Z^\alpha \cap T_\alpha)\big) \cap B(0, \delta) = \emptyset$, which, together with (3.8), gives (3.7).

We are now in a position to estimate the density of $X = P(Z)$. Note that the map

$$x \to \big(x, u(x)\big) \colon P(U_\alpha) \to U_\alpha \tag{3.9}$$

is Lipschitz with constant $(1 + g_\alpha^2)^{\frac{1}{2}}$, where $g_\alpha = \sup\{|\operatorname{grad} u(x)| \colon |x| < R \sin 2\alpha\}$. (In fact, by Rockafellar [10, Th. 10.4] u is Lipschitz, and by [10, Theorems 24.7, 25.5 and 25.6] g_α is a Lipschitz bound for $u|B(0, R\sin 2\alpha)$). Moreover (3.9) maps $X \cap P(U_\alpha) = P(Z \cap U_\alpha)$ onto $Z \cap U_\alpha$. Applying basic theorem about effect of Lipschitz maps on Hausdorff measures [Rogers 11, Ch. 2, Th. 29] and (3.3) we get:

$$\begin{aligned} H^N(Z \cap U_\alpha) &\leqslant (1 + g_\alpha^2)^{N/2} m_N\big(X \cap P(U_\alpha)\big) \\ &\leqslant (1 + g_\alpha^2)^{N/2} m_N\big(X \cap B(0, \varepsilon)\big)\,, \qquad \varepsilon = R \sin 2\alpha\,. \end{aligned}$$

Furthermore it holds

$$\begin{aligned} m_N\big(B(0, \delta)\big) &\leqslant m_N\big(P\gamma^\alpha(Z^\alpha \cap U_\alpha)\big) && \text{(by (3.5))} \\ &\leqslant H^N(Z^\alpha \cap U_\alpha) && \text{(since } P\gamma^\alpha \text{ is nonexpanding)} \\ &\leqslant H^N(Z \cap U_\alpha) && \text{(by (3.4))}\,. \end{aligned}$$

Combining these inequalities we obtain:

$$m_N(X \cap B(0,\varepsilon))/m_N(B(0,\varepsilon)) \geqslant (1+g_\alpha^2)^{-N/2} m_N(B(0,\delta))/m_N(B(0,\varepsilon)) = \\ = (1+g_\alpha^2)^{-N/2}\cos^{-2N}\alpha\left(\frac{R-r}{2R}\right)^N.$$

If $\varepsilon \to 0$, then $\alpha \to 0$ and $g_\alpha \to 0$ by continuity of the gradient of a convex function [Rockafellar 10, Th. 25.5]. Q.E.D.

PROOF OF THEOREM 3.2. Denote $\{x \in \operatorname{dom}(u): K(u,x) < k\}$ by $\tilde{X}_k$. It is clear that $K(u,x)$ is of first Baire class and so $\tilde{X}_k$ is Borel. We can assume without loss of generality that $x^* = 0$, $u(x^*) = 0$ and $\operatorname{grad} u(x^*) = 0$; in particular $u \geqslant 0$. Fix $k > k^* = K(u,0)$ and take K such that $k > K > k^*$. By Proposition 3.3(ii), there is $d > 0$ such that $u(h) \leqslant \frac{1}{2}K|h|^2$ for $|h| < d$. Since for $R = 1/K$, $|x| \leqslant R$ the inequality $R - (R^2 - |x|^2)^{\frac{1}{2}} \geqslant \frac{1}{2}K|x|^2$ holds, therefore the sphere $S(c^*, R)$, where $c^* = (0, \dots, 0, R) \in R^{N+1}$, supports the graph of $u|B(0,d)$ from the above at $0 \in R^{N+1}$. We can apply Lemma 3.4 to the function $u|B(0,d)$.

Take arbitrary r such that $1/k < r < R$, and let $X = X_r$ and $Z = Z_r$ have the same meaning as in Lemma 3.4 and its proof. By Proposition 3.3(iii), if $x \in X$ then $K(u,x) \leqslant r^{-1}(1+g^2)^{\frac{3}{2}}$, where $g = |\operatorname{grad} u(x)|$. Set

$$g_\varepsilon = \sup\{|\operatorname{grad} u(x)|: |x| < \varepsilon\}.$$

Then

$$K(u,x) \leqslant r^{-1}(1+g_\varepsilon^2)^{\frac{3}{2}} \quad \text{for } x \in X \cap B(0,\varepsilon).$$

By the continuity of the gradient [Rockafellar, 10, Th. 25.5] $\lim_{\varepsilon\to 0} g_\varepsilon = \operatorname{grad} u(0) = 0$. Therefore there is $0 < \varepsilon' < d$ such that $r^{-1}(1+g_\varepsilon^2)^{\frac{3}{2}} < k$ for $\varepsilon < \varepsilon'$, and so

$$B(0,\varepsilon) \cap X \subset B(0,\varepsilon) \cap \tilde{X}_k, \quad \text{for } 0 < \varepsilon < \varepsilon'.$$

It follows that

$$\liminf_{\varepsilon\to 0} m_N(\tilde{X}_k \cap B(0,\varepsilon))/m_N(B(0,\varepsilon)) \geqslant \\ \geqslant \liminf m_N(X \cap B(0,\varepsilon))/m_N(B(0,\varepsilon)) \geqslant \left(\frac{R-r}{2r}\right)^N,$$

by Lemma 3.4. Since we can choose $R = 1/K$ and $r > 1/k$ arbitrarily close to $1/k^*$ and $1/k$ respectively, we get the desired bound $((k-k^*)/2k)^N$. Q.E.D.

The following fact follows immediately from Theorem 3.2:

COROLLARY 3.5. *Let u be a locally convex function in $U \subset R^N$. Assume that $K(u, x) \geqslant M$ for almost every $x \in U$. Then $K(u, x) \geqslant M$ for all $x \in U$.*

4. – Characterization of q-plurisubharmonic functions among functions with lower bounded Hessian.

Since a function u with lower bounded real Hessian is the difference of a convex function and of quadratic polynomial $\frac{1}{2}L|x|^2$, the notion of a real Hessian makes sense at almost every point $x \in \operatorname{dom}(u)$. The formula (3.1) may be rewritten as follows:

$$u(x+h) = u(x) + \big(\operatorname{grad} u(x), h\big) + \tfrac{1}{2}B(h, h) + o\big(|h|^2\big), \tag{4.1}$$

where $B\colon R^N \times R^N \to R$, the real Hessian of u at x is a symmetric form corresponding to the matrix (b_{ij}). In case u is defined in $\mathbf{C}^n = R^{2n}$ we can also define formally the complex Hessian at x, in complete analogy with the smooth case. We define

$$\frac{\partial^2 u}{\partial z_k \partial \bar{z}_l} := \frac{1}{4}\left(\frac{\partial^2 u}{\partial x_k \partial x_l} + i\frac{\partial^2 u}{\partial x_k \partial x_l} - i\frac{\partial^2 u}{\partial y_k \partial x_l} + \frac{\partial^2 u}{\partial y_k \partial y_l}\right),$$

where $z = (x_1, y_1, \ldots, x_n, y_n)$ and all second order derivatives on the right-hand side of the last formula are suitable entries of the matrix (b_{ij}). The form

$$H(z, w) = \sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_l} z_k \bar{w}_l, \quad z, w \in \mathbf{C}^n,$$

is called the complex Hessian of u at x. If we set further

$$\frac{\partial^2 u}{\partial z_k \partial z_l} := \frac{1}{4}\left(\frac{\partial^2 u}{\partial x_k \partial y_l} - i\frac{\partial^2 u}{\partial x_k \partial y_l} - i\frac{\partial^2 u}{\partial y_k \partial x} - \frac{\partial^2 u}{\partial y_k \partial x_l}\right),$$

$$A(z, w) := \sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial z_k \partial z_l} z_k w_l, \qquad z, w \in \mathbf{C}^n,$$

then $B(z, w) = H(z, w) + H(z, w)^- + A(z, w) + A(z, w)^-$. Note that H is a Hermitian form, while A is bilinear. Since $H(z, z)$ is real, it holds

$$\tfrac{1}{2}B(z, z) = H(z, z) + \operatorname{Re} A(z, z). \tag{4.2}$$

THEOREM 4.1. *Let $U \subset \mathbf{C}^n$ be open, $u: U \to R$ and $0 \leqslant q \leqslant n-1$.*

(i) *If $u \in PSH_q(U)$ is second-order Peano differentiable at x, then its complex Hessian at x has at least $(n-q)$-nonnegative eigenvalues.*

(ii) *If $u \in C^1_L(U)$ for some $L \geqslant 0$ and at almost every point $x \in U$, the complex Hessian of u at x has at least $(n-q)$-nonnegative eigenvalues, then $u \in PSH_q(U)$.*

For the proof we need the following folk-lore lemma:

LEMMA 4.2. *If A and B are complex Hermitian matrices with at most q- and r-negative eigenvalues respectively, then the matrix $C = A + B$ has at most $(q + r)$ negative eigenvalues.*

PROOF. It follows from the assumptions that A and B have invariant subspaces L_1 and L_2 of dimension $(n-q)$- and $(n-r)$ respectively, such that $(Ax, x) \geqslant 0$, for $x \in L_1$, $(Bx, x) \geqslant 0$ for $x \in L_2$. Put $L = L_1 \cap L_2$. Then $\dim L \geqslant n - (q + r)$ and $(Cx, x) \geqslant 0$ for $x \in L$. By minimax formula for eigenvalues [Dunford and Schwartz, 6, Th. X. 43] the matrix C has at least $\dim L = n - q - r$ nonnegative eiganvalues. Q.E.D.

PROOF OF THEOREM 4.1. (i) (*Necessity*). Choose $r > 0$ with $B(x, r) \subset U$ and define functions $u_t: B(0, r) \to R$, $0 < t < 1$, by the formula

$$u_t(y) = t^{-2}\big(u(x + ty) - u(x) - t\,\mathrm{grad}\,(u(x), y)\big)\,.$$

By Proposition 1.2(v), (vi) all $u_t \in PSH_q\big(B(0, r)\big)$. By formula (4.1) u_t converge uniformly to $\frac{1}{2}B(y, y)$, as $t \to 0$. By Proposition 1.2 (vii) $\frac{1}{2}B(y, y)$ is q-plurisubharmonic. Since the latter is a smooth function, its complex Hessian at 0 has at least $(n-q)$-nonnegative eigenvalues. However, the complex Hessian of $\frac{1}{2}B(y, y)$ is exactly $H(z, w)$, because $\frac{1}{2}B(z, z) = H(z, z) + \mathrm{Re}\,A(z, z)$, and $\mathrm{Re}\,A(z, z)$ is a pluriharmonic function.

(ii) (*Sufficiency*). We will show that condition (iii), Proposition 1.1 cannot hold. Suppose it does and there are $z^* \in U$, $\varepsilon > 0$, $r > 0$, and $f \in PSH_{n-q-1}\big(B(z^*, r)\big)$ satisfying all requirements of Proposition 1.1 (iii). We can assume without loss of generality that $z^* = 0$. If we set $u_1(z) = u(z) + f(z)$, $|z| < r$, we get

$$u_1(0) = 0,\; u_1(z) \leqslant -\varepsilon|z|^2, \qquad |z| < r\,. \tag{4.3}$$

Observe that at almost every point x of $B(0, r)$ the real Hessian of u_1 at x has at least one nonnegative eigenvalue. Indeed, let x be any point at which the complex Hessian of u exists and has at least $(n - q)$-nonnegative eiganvalues. Clearly the Hessian of smooth function f at x has $(q+1)$-nonnegative eigenvalues. The sum of these two forms has at least one nonnegative eigenvalue (by Lemma 4.2). Thus the complex Hessian of u_1 at x, say $H(z, u)$, has at least one nonnegative eigenvalue, and so $H(z, z) \geqslant 0$ for some $z \neq 0$. Let $B(x, y)$ denote the real Hessian of u_1 at x. By Eq. (4.2) $\frac{1}{2} B(iz, iz) = H(z, z) - \operatorname{Re} A(z, z)$, and so $4H(z, z) = B(z, z) + B(iz, iz)$. Thus the symmetric form $B(z, z)$ cannot be negative-definite, and so the real Hessian of u_1 at x has at least one nonnegative eigenvalue. It is clear that the set of such points x is of full measure.

Choose $r' \in (0, r)$ and let L_1 denote the maximum of the largest eigenvalue of the real Hessian of f at x, $|x| \leqslant r'$. By Proposition 2.3 (ix) and (v), $u_1 | B(0, r) \in C^1_M(B(0, r))$, where $M = L + L_1$. By Def. 2.1 the function $v(z) = u_1(z) + \frac{1}{2} M|z|^2$, $|z| < r'$, is convex. In particular $0 = 2v(o) \leqslant v(z) + v(-z) = u_1(z) + u_1(-z) + M|z|^2$. Using this and (4.3) we get

$$\begin{aligned} -\varepsilon|z|^2 \geqslant u_1(z) &\geqslant -u_1(-z) - M|z|^2 \\ &\geqslant -M|z|^2 . \end{aligned}$$

This implies that u_1 is differentiable at 0 and $\operatorname{grad} u_1(0) = 0$, and also $\operatorname{grad} v(0) = 0$. Moreover $v(0) = 0$, $v(z) \leqslant (\frac{1}{2} M - \varepsilon)|z|^2$, $|z| \leqslant r'$. Therefore

$$K(u; 0) \leqslant M - 2\varepsilon . \quad \text{(cf. Def. 3.1)} . \tag{4.4}$$

Since the real Hessian of $\frac{1}{2} M|z|^2$ has, at every point, all eigenvalues equal to M, the real Hessian of v at almost each point has at least one eigenvalue greater or equal than M. By Proposition 3.3 (i) $K(u, x) \geqslant M$ almost everywhere. By Corollary 3.5 also $K(u, 0) \geqslant M$, which contradicts (4.4). Q.E.D.

5. – Uniqueness of solution of the generalized Dirichlet problem.

It has been already realised by Hunt and Murray [8], as well as Kalka [9] that the uniqueness of solution of the generalized Dirichlet problem in the class of q-plurisubharmonic functions follows from the following result.

THEOREM 5.1. *Let $U \subset \mathbf{C}^n$ be open and $q, r \geqslant 0$. If $u \in PSH_q(U)$ and $v \in PSH_r(U)$, then $(u + v) \in PSH_{q+r}(U)$.*

PROOF. By Theorem 2.9 and Proposition 1.2 (vii), it is enough to prove the theorem for u, v with lower bounded real Hessian, say $u \in C^1_{L'}(U)$,

$v \in C^1_{L''}(U)$, L', $L'' \geqslant 0$. By Proposition 2.3 (v), $(u+v) \in C^1_L(U)$, where $L = L' + L''$. Now we can apply Theorem 4.1. Take any point z at which both u and v are second order Peano differentiable. Let A and B denote the complex Hessians at x of u and v respectively. By Th. 4.1 (i) A and B have at most q and r negative eigenvalues respectively and by Lemma 4.2, $A+B$ has at most $(q+r)$ negative eigenvalues. Thus the complex Hessian of $u+v$ has at almost every point at most $(q+r)$ negative eigenvalues, and by Theorem 4.1 (ii) $(u+v) \in PSH_{q+r}(U)$. Q.E.D.

DEFINITION 5.2. (Hunt and Murray [8]). A function $u\colon U \to R$, $U \subset \mathbf{C}^n$ is said to be *q-Bremermann*, $0 \leqslant q \leqslant n-1$, if $u \in PSH_q(U)$ and $-u \in PSH_{n-q-1}(U)$. (Hunt and Murray use the term *q-complex Monge-Ampère* instead of q-Bremermann.) In case a q-Bremermann function is smooth, the determinant of its complex Hessian vanishes everywhere. Thus the Dirichlet problem studied in Hunt and Murray [8] and here is of homogeneous type. The uniqueness result conjectured in [8] is the following.

THEOREM 5.3. *Let $U \subset \mathbf{C}^n$ be open and bounded and let $b\colon \partial U \to R$ be continuous. Then for each $0 \leqslant q \leqslant n-1$ there is at most one q-Bremermann function u in U such that for every $z \in \partial U$*

$$\lim_{z' \to z} u(z) = b(z)\,.$$

The theorem is a special case of the following fact.

LEMMA 5.4. *Let $U \subset \mathbf{C}^n$ be open and bounded, $0 \leqslant q \leqslant n-1$ and $u, v \in C(\overline{U}) \cap PSH_q(U)$. Assume, moreover, that u is q-Bremermann in U, $u \leqslant v$ in U and $u|\partial U = v|\partial U$. Tehn $u \equiv v$.*

PROOF. We have to show that $v \leqslant u$. Since $-u \in PSH_{n-q-1}(U)$, it follows from Theorem 5.1 that the function $v - u = v + (-u)$ is $q + (n-q-1) = (n-1)$-plurisubharmonic, and by maximum principle (Proposition 1.2(viii)), $\max\limits_{\overline{U}} (v-u) = \max (v-u)|\partial U = 0$, i.e. $v \leqslant u$. Q.E.D.

As it was noticed by Hunt and Murray [8] Sec. 3, a solution to the Dirichlet-Bremermann problem of Theorem 5.3 does not always exist, even for relatively regular domains. There are, however, general positive results in this direction.

DEFINITION 5.5. [Hunt and Murray, 8, Sec. 3] A bounded domain $D \subset \mathbf{C}^n$ is said to be strictly *q-pseudoconvex* if there is an open neighbourhood U of ∂D, and a strictly q-plurisubharmonic function $\varrho\colon U \to R$, such that $D \cap U = \{z \in U\colon \varrho(z) < 0\}$.

THEOREM 5.6. *If $D \subset \mathbf{C}^n$ is strictly r-pseudoconvex, $0 \leqslant r \leqslant n-1$, and $0 \leqslant q \leqslant n-r-1$, then for every continuous function $b: \partial D \to R$ there exists a unique q-Bremermann function $u \in \mathcal{C}(\overline{D}) \cap PSH_q(U)$, such that $u|\partial D \equiv b$.*

The existential part of this theorem is due—in the case of $q = r$—to Hunt and Murray [8, Th. 3.3]. By the same method one can prove the generalization given above (formulated by Kalka [9]).

REMARK 5.7. If $D \subset \mathbf{C}^n$ is strictly pseudoconvex ($r = 0$) then each continuous function $b: \partial D \to R$ has q-Bremermann extension for every $q = 0, 1, \ldots, n-1$.

6. – The Perron method and smooth q-plurisubharmonic functions.

As it was indicated in the Introduction, the author has undertaken this study mainly to obtain the following theorem.

THEOREM 6.1. – *If u and v are respectively q- and r-plurisubharmonic, $q, r \geqslant 0$, then* $\min(u, v)$ *is $(q+r+1)$-plurisubharmonic* (in dom (u) $\cap$ dom (v)).

This result was then applied in Slodkowski [12] to prove Basener's conjecture [2]; there Theorem 5.1 played secondary role and served only to obtain Theorem 6.1. Afterwards another approach was found in [Slodkowski, 13]: Basener's conjecture was derived directly from Theorem 5.1, moreover, Theorem 6.1 is now a consequence of this conjecture; see [13] for details.

Anyway, the initial approach is not without interest. On the whole it is perhaps shorter than that of [13], and certainly more homogeneous as far as the methods are concerned. It starts from the observation that Theorem 6.1 is simple if u and v are, in addition, smooth. Something more can be obtained by the same simple methods.

LEMMA 6.2. *If $U \subset \mathbf{C}^n$ and $u \in \mathbf{C}^2(U) \cap PSH_q(U)$, $v \in PSH_r(U)$, $q, r \geqslant 0$, then* $\min(u, v) \in PSH_{q+r+1}(U)$.

(The simple proof is postponed to the Appendix.) Now our strategy is to obtain all functions in $C(\overline{U}) \cap PSH_q(U)$, at least locally, by successive application of Perron method starting from functions in $C^2 \cap PSH_q$. Then Theorem 6.1 can be obtained by a sort of induction, of which Lemma 6.2 is the first step. We will describe only the basic steps of this method, omitting the details that are similar to those in Bremermann [4], Walsh [14], and Hunt and Murray [8].

DEFINITION 6.3. Let for every open $D \subset \mathbf{C}^n$ there be given a set $A(D) \subset \subset \operatorname{usc}(D)$. We call the family $A = \{A(D)\}$ an admissible class, if it fulfils condition (i)-(iv) of Proposition 1.2, with $PSH_q(\cdot)$ replaced by $A(\cdot)$.

REMARK 6.4. PSH_q is an admissible class.

If we are given an admissible class $\{A(D): D \subset \mathbf{C}^n\}$, and a function $v \in C(\bar{D})$, we consider the set

$$A(v, D) = \{u: u \in \operatorname{usc}(\bar{D}) \cap A(D), u \leqslant v \text{ in } \bar{D}\}.$$

Denote the upper envelope of $A(v, D)$, the function $\sup\{u(z): u \in A(v, D)\}$ by $e(A, v)$. The following remark follows from condition (iv) of Def. 6.3.

REMARK 6.5. Let $D \subset \mathbf{C}^n$ be open and bounded, and $v: D \to R$ be continuous. Let A be an admissible class in $\mathbf{C}^n$. Then $e(A, v) \in \operatorname{usc}(\bar{D}) \cap A(D)$ and $e(A, v) \leqslant v$ in $\bar{D}$.

General Walsh lemma 6.6. *Let $D \subset \mathbf{C}^n$ be open and bounded and $\bar{v}: D \to R$ be continuous. Let A be an admissible class in $\mathbf{C}^n$. Assume that the envelope $e(A, v)$ is continuous at each point of ∂D. Then $e(A, v)$ is continuous in $\bar{D}$.*

The proof goes in the same way as the original one in Walsh [14]; one has to replace the class PSH_0 by A and to note, that nothing more than conditions (i)-(iv) of Def. 6.3 is needed in Walsh's proof.

We denote by AP_q, $0 \leqslant q \leqslant n-1$, the smallest admissible class in $\mathbf{C}^n$ containing all smooth q-plurisubharmonic functions defined on arbitrary open subsets of $\mathbf{C}^n$.

LEMMA 6.7. *Let $D \subset \mathbf{C}^n$ be strictly q-pseudoconvex, $0 \leqslant q \leqslant n-1$, and $v \in C(\bar{D})$. Then the envelope $e(AP_q, v)$ is of AP_q class, in particular is q-plurisubharmonic in D, continuous in $\bar{D}$ and equal to v on ∂D.*

PROOF (Sketch). By Remark 6.5 $e(AP_q, v) \in \operatorname{usc}(\bar{D}) \cap PSH_q(D)$. One can prove that $\liminf_{z' \to z} e(AP_q, v) \geqslant v(z)$, for each $z \in \partial D$, by applying Bremermann's argument [4, Proof of Th. 4.1], with obvious modifications. Since also $e(AP_q, v) \leqslant v$ (Remark 6.5), we conclude that for every $z \in \partial D$, $\lim_{z' \to z} e(AP_q, v) = v(z)$. Thus the envelope is continuous at each boundary point and, by Lemma 6.6, it is continuous in $\bar{D}$. Moreover it is equal to v on ∂D.

THEOREM 6.8. *Let $D \subset \mathbf{C}^n$ be strictly q-pseudoconvex, $0 \leqslant q \leqslant n-1$. Then $PSH_q(D) \cap C(\bar{D}) \subset AP_q(D)$.*

PROOF. Let $v \in PSH_q(D) \cap C(\bar{D})$ and denote $u := e(AP_q, v)$. It is enough to show that $u \equiv v$ (for $u|D \in AP_q(D)$). Suppose that $u \not\equiv v$ and let $H := \{z \in \bar{D}: u(z) < v(z)\}$. Since $u, v \in C(\bar{D})$ and $u|\partial D = v|\partial D$ (by Lemma 6.7), H is open in $\mathbf{C}^n$. The following holds:

$$u|\partial H = v|\partial H \tag{6.1}$$

$$u|H < v|H\,. \tag{6.2}$$

Observe that $u|H$ is q-Bremermann. If not, then $-u \notin PSH_{n-q-1}(H)$ and condition (iii) of Proposition 1.1 holds. Thus there are $z^* \in H$, $r > 0$, $\varepsilon > 0$, $f \in C^2PSH_q(\bar{B}(z^*, r))$, such that $\bar{B}(z^*, r) \subset H$,

$$f(z^*) = u(z^*)\,, \tag{6.3}$$

$$f(z) \leqslant u(z) - \varepsilon|z - z^*|^2\,, \qquad z \in \bar{B}(z^*, r)\,. \tag{6.4}$$

Choose $\delta > 0$ small enough, so that $u < v$ in $\bar{B}(z^*, r)$ and $\delta < \varepsilon r^2$. Define

$$u_1(z) = \begin{cases} u(z)\,, & z \in D \setminus B(z^*, r) \\ \max\big(u(z), f(z) + \delta\big)\,, & z \in B(z^*, r)\,. \end{cases}$$

Clearly $f(z) + \delta \in C^2PSH_q\big(\bar{B}(z^*, r)\big)$ and for $|z - z^*| = r$, $f(z) + \delta \leqslant u(z) + \delta - \varepsilon r^2 < u(z)$, by (6.4). By condition (iv) of Def. 6.3, $u_1 \in AP_q(D)$. By the choice of δ, $u_1 < v$ and so $u_1 \in AP_q(D, v)$. Finally, by (6.3), $u_1(z^*) > u(z^*)$, contrary to the assumption that u is the upper envelope of $AP_q(D, v)$. Thus $u|H$ is q-Bremermann, as required. Since $v|H \in PSH_q(H)$, by Lemma 5.4 one of relations (6.1), (6.2) cannot hold. Thus $u \equiv v$. Q.E.D.

PROOF OF THEOREM 6.1. Let us introduce an auxiliary class A by setting for each open $D \subset \mathbf{C}^n$

$$A(D) = \{u \in \operatorname{usc}(D): \max(u, v) \in PSH_{q+r+1}(D)\} \qquad \text{for every } v \in PSH_r(D)\,.$$

Since both PSH_r and PSH_{q+r+1} are admissible classes, it is obvious that the conditions (i)-(iv) of Def. 6.3 hold and $\{A(D)\}$ is admissible as well. By Lemma 6.2 $C^2PSH_q \subset A$ therefore $AP_q(D) \subset A(D)$, for every $D \subset \mathbf{C}^n$. Thus Theorem 6.1 holds for $u \in AP_q$, and—by Theorem 6.8—for $u \in C(\bar{B}) \cap PSH_q(B)$, where B denotes an arbitrary ball, and finally—by approximation (Remark 2.10)—for $u \in \operatorname{usc}(\bar{B}) \cap PSH_q(B)$. This concludes the proof, for Theorem 6.1 is of local character.

Appendix.

PROOF OF PROPOSITION 3.3 (iii). Let us assume that the ball $B((c, t), r)$, $c \in R^N$, supports graph (u) from the above at $(x, u(x))$ and is differentiable at x. Let $G = \operatorname{grad} u(x)$ and $g = |G|$. Define $d\colon B(x, R) \to R$ to be the function whose graph is the lower (open) hemisphere of $S(c, t), r)$. Of course

$$d(x) = u(x)\,, \qquad d(x') \geqslant u(x')\,, \qquad x' \in B(x, r)\,. \tag{A.1}$$

Let us express first $d(x+h)$. The vector $(c, t) - (x, u(x))$, of length r, is proportional to the upward pointing normal to the graph of u at $(x, u(x))$, which is equal to $(1+g^2)^{-\frac{1}{2}}(-G, 1)$. Therefore

$$c - x = -(1+g^2)^{-\frac{1}{2}}G\,, \qquad t - u(x) = r(1+g^2)^{-\frac{1}{2}}\,.$$

The equation for $d(x+h)$ is

$$|(x+h) - c|^2 + |d(x+h) - t|^2 = r^2\,.$$

From these two equations we get

$$\begin{aligned} d(x+h) &= u(x) + r(1+g^2)^{-\frac{1}{2}} - \left(r^2(1+g^2)^{-1} - 2(G, h)r(1+g^2)^{-\frac{1}{2}} - |h|^2\right)^{-\frac{1}{2}} \\ &= u(x) + \left(2(G, h) + |h|^2\gamma\right)\left(1 + \left(1 - 2(G, h)\gamma - |h|^2\gamma^2\right)^{\frac{1}{2}}\right)^{-1}, \end{aligned}$$

where $\gamma = (1+g^2)^{\frac{1}{2}}/r$.

We have to estimate $K(u, x) = \limsup \{\max k(u, x, h, \varepsilon)\colon |h| \leqslant \varepsilon\}$, where $k(u, x, h, \varepsilon) := 2\varepsilon^{-2}\left(u(x+h) - u(x) - \varepsilon(G, h)\right)$. One checks that $\operatorname{grad} d(x) = G$. By this and (A.1), $k(u, x, h, \varepsilon) \leqslant k(d, x, h, \varepsilon)$ and so

$$K(u, x) \leqslant K(d, x)\,,$$

$$\begin{aligned} &k(d, x, h, \varepsilon) \\ &\quad = 2\varepsilon^{-2}\Big(\big(2(G, \varepsilon h) + |\varepsilon h|^2\gamma\big)/\big(1 + (1 - 2(G, \varepsilon h) - |\varepsilon h|^2\gamma^2)^{\frac{1}{2}}\big) - (G, \varepsilon h)\Big) \\ &\quad = 2|h|^2\gamma\big(1 + (1 - 2(G, \varepsilon h)\gamma - |\varepsilon h|^2\gamma^2)^{\frac{1}{2}}\big)^{-1} + 2\varepsilon^{-1}(G, h)\,q(h, \varepsilon)\,, \end{aligned} \tag{A.2}$$

where

$$q(h, \varepsilon) = \frac{1 - (1 - 2(G, \varepsilon h)\gamma - |\varepsilon h|^2\gamma^2)^{\frac{1}{2}}}{1 + (1 - 2(G, \varepsilon h)\gamma - |\varepsilon h|^2\gamma^2)^{\frac{1}{2}}} = \frac{2\varepsilon(G, h)\gamma^2 + \varepsilon^2|h|^2\gamma^2}{\big(1 + (1 - 2(G, \varepsilon h)\gamma - |\varepsilon h|^2\gamma^2)^{\frac{1}{2}}\big)^2}\,.$$

Therefore $\lim\limits_{\varepsilon\to 0} q(h, \varepsilon) = (G, h)^2$, and by (A.2),

$$\lim_{\varepsilon\to 0} k(d \quad x \quad h, \varepsilon) = |h|^2\gamma + (G, h)^2\gamma\,,$$

and so $K(u, x) \leqslant K(d, x) = \gamma + |G|^2\gamma = r^{-1}(1 + g^2)^{\frac{1}{2}}$. Q.E.D.

PROOF OF LEMMA 6.2. Suppose, on the contrary, that for some $u \in C^2PSH_q(D)$ and $v \in PSH_r(D)$, where $D \subset \mathbf{C}^n$, $w := \min(u, v) \notin PSH_{q+r+1}(D)$. By definition w is not $(q + r + 1)$-plurisubharmonic on some $(q + r + 2)$-dimensional hyperplane L. We assume, without loss of generality, that $L = \mathbf{C}^n$, i.e. $n = q + r + 2$. By Proposition 1.1 (iii) there is $z^* \in D$, $\varepsilon > 0$, $r > 0$, with $B(z^*, r) \subset D$ and a function f, strictly 0-plurisubharmonic in $B(z^*, r)$ such that

$$\text{(A.3)}\quad w(z^*) + f(z^*) = 0\,, \qquad w(z) + f(z) \leqslant -\varepsilon|z - z^*|^2\,, \qquad z \in B(z^*, r)\,.$$

Set $u_1 = u + f$, $v_1 = v + f$ in $B = B(z^*, r)$. Then v_1 is r-plurisubharmonic and u_1 is strictly q-plurisubharmonic (for Hess $u_1(z) =$ Hess $u(z)$ + a positive definite form, and by Dunford and Schwartz [6, Th. X. 43], the eigenvalues of Hess $u_1(z)$ are greater than corresponding eigenvalues of Hess $u(z)$). By (A.3), $\min\big(u_1(z^*), v_1(z^*)\big) = 0$ and

$$\text{(A.4)}\qquad \min\big(u_1(z), v_1(z)\big) \leqslant -\varepsilon|z - z^*|^2\,, \qquad z \in B\,.$$

If $u_1(z^*) \neq v_1(z^*)$ then $\min(u_1, v_1)$ is identically equal to one of the functions u_1, v_1 near z^* and (A.3) is impossible by maximum principle (Proposition 1.2 (viii)). Thus $u_1(z^*) = v_1(z^*) = 0$.

Take the complex hyperplane X such that $X - z^*$ is the subspace spanned by eigenspaces corresponding to positive eigenvalues of Hess $u(z^*)$; since u is strictly q-plurisubharmonic, $\dim X \geqslant n - q = r + 2$. Set $u_2 = u_1|X \cap B$, $v_2 = v_1|X \cap B$. Then u_2 has positive-definite Hessian at z^*, and so is strictly plurisubharmonic near z^*. By Gunning and Rossi [7, Th. 9.B.2] there is a quadratic polynomial p on X and $r' \in (0, r)$ such that $p(z^*) = 0$, $B_{r+2}(z, r') \cap$ $\cap \{p = 0\} \subset \{u_2 > 0\} \cup \{z^*\}$. Then $v_2|\{p = 0\}$ has strict maximum at z^*, therefore the function $v_2(z) - C|p(z)|^2$, $z \in X$, does not have local maximum property in $B \cap X$, if $C > 0$ is big enough, and so is not $(r + 1)$-plurisubharmonic in X ($\dim X = r + 1$). On the other hand this function is $(r + 1)$-plurisubharmonic in $X \cap B$, for $v_2 \in PSH_r(X \cap B)$ and $-|p|^2 \in PSH_1(X \cap B)$ (by Th. 5.1, a direct elementary proof can be given as well). Q.E.D.

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