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1. – Introduction.

The so-called linearized stability principle is an elementary, yet powerful technique to investigate the Lyapunov stability character of solutions for a wide class of evolution equations. In the case of semilinear parabolic equations, such procedure leads to study an abstract Cauchy problem, namely

$$(1.1)\qquad \begin{cases} u'(t) = Au(t) + \sigma\big(u(t)\big) & (t\geqslant 0) \\ u(0) = u_0 \end{cases}$$

where A is the infinitesimal generator of a strongly continuous analytic semigroup $\exp[At]$ on some Banach space E and σ is a nonlinear function from E to itself $\big(\sigma(0) = 0\big)$; if the type of the semigroup is negative and σ is Fréchet differentiable at zero with $\sigma'(0) = 0$, the asymptotical stability of the trivial stationary solution with respect to solutions of (1.1) easily follows.

A naive extension of the above reasoning to the case of quasilinear parabolic equations reveals to be troublesome, as in this case the nonlinear term σ is only continuous from the domain $D(A)$ of A to the space E; the difficulty is related to the fact that the convolution with the semigroup $\exp[At]$ doesn't take continuous functions with values in E into continuous functions with values in $D(A)$—in other words, no maximal regularity result holds for the linear problem

$$(1.2)\qquad \begin{cases} v'(t) = Av(t) + f(t) & (t\geqslant 0) \\ v(0) = 0 \end{cases}$$

if f is continuous with values in E [1, 2, 7].

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As a consequence, we find it useful to work in continuous interpolation spaces intermediate between $D(A)$ and E, where such a maximal regularity result is known to hold [2]; relying upon this property, we prove by a perturbative approach a linearized stability result, which applies to any space in the above referred family. It should be obseıved that continuousin terpolation spaces between $D(A)$ and E are given a concrete characterization in several cases of interest [2, 7]: in particular, they are little-Hölder spaces when E is a space of continuous functions and A a second-order uniformly elliptic operator with regular coefficients [7]. We shall be working within such a framework to investigate a class of quasilinear parabolic problems [1], thus proving the asymptotical stability of the stationary solutions in little-Hölder spaces (see Section 3). The present results should be compared with those of [8], where similar ideas are developed starting from Sobolevskij's theory of evolution operators and used to study quasilinear parabolic problems in the case E is in L^p-space: in such case, assumptions on the space dimension are needed to prove asymptotical stability results concerning stationary solutions.

Section 2 is devoted to the statement of the results, which are proved is Section 4 and 5 and applied in Section 3 to the investigation of a class of nondegenerate initial-boundary value parabolic problems with nonlinear diffusion. Some definitions and results to be used in the following are given in the Appendix for the convenience of the reader.

2. – Statement of the results.

Let E denote a Banach space (norm $|\cdot|_E$) and $D(A) \subseteq E$ the domain of a closed linear operator A in E. For a given couple of Banach spaces E, F the Banach space $\mathcal{L}(E, F)$ of continuous linear operators from E to F will be considered (norm $\|\cdot\|_{E,F}$); we set $\mathcal{L}(E) := \mathcal{L}(E, E)$. For any $T > 0$ we denote by $C^k([0, T]; F)$ the space of k times continuously differentiable functions from $[0, T]$ to a Banach space F, endowed with the usual norm; the notation $C([0, T]; F) := C^0([0, T]; F)$ will be used.

Let E, F be two Banach spaces such that F is continuously embedded in E. By a local (classical) solution of the Cauchy problem in E:

$$(2.1)\qquad \begin{cases} u'(t) = Au(t) + N(u(t))u(t) + R(u(t)) & (t \geq 0) \\ u(0) = u_0 \end{cases}$$

we mean any $u \in C([0, T]; F) \cap C^1([0, T]; E)$ which satisfies (2.1) ($T = T(u_0) > 0$; $u_0 \in F$); the solution is said to be global if $T = +\infty$. Ex-

(1) Quasilinear parabolic problems of general form can be dealt with similarly.

istence and uniqueness results concerning classical solutions for a wide class of parabolic problems, which encompasses (2.1) as a particular case, can be found in [2]. The usual definitions of asymptotical stability or instability of the null solution with respect to solutions of (2.1) will be used in the following.

The following assumption will be made throughout:

(A) A is the infinitesimal generator of a strongly continuous analytic semigroup $\exp[At]$ on E of negative type, namely

$$\|\exp[At]\|_{E,E} \leqslant M \exp[-\omega t] \qquad (M \geqslant 1, \omega > 0; t \geqslant 0) .$$

Under the assumption (A), we can consider the following Banach spaces:

$$D_\theta := \left\{x \in E \,\Big|\, \lim_{t\to 0^+} |t^{-\theta}(\exp[At]x - x)|_E = 0\right\} \qquad (\theta \in (0,1)) ,$$

endowed with the norm $x \mapsto |x|_\theta := |x|_E + \sup_{t\in(0,1]} |t^{-\theta}(\exp[At]x - x)|_E$; respectively

$$D_{\theta+1} := \left\{x \in D(A) | Ax \in D_\theta\right\} \qquad (\theta \in (0,1)) ,$$

endowed with the norm $x \mapsto |x|_{\theta+1} := |x|_E + |Ax|_\theta$. The space D_θ (respectively $D_{\theta+1}$) can be viewed as an interpolation space between $D(A)$ and E ($D(A^2)$ and $D(A)$, respectively) [2]. It can be proved that $D_{\theta+1}$ is continuously embedded in D_θ and the restriction $A: D_{\theta+1} \to D_\theta$ is the infinitesimal generator of the restriction of the semigroup $\exp[At]$ to D_θ; in addition, such restriction is analytic [2].

We shall also use extensively the extrapolation spaces defined in [3], whose ideas and results of relevance for the present purposes can be briefly summarized as follows (see the Appendix for further details). Under the assumption (A) ([2]), a larger space $\tilde{E}$ (depending on A by construction) can be defined in which E is continuously embedded, *i.e.*, $\mathfrak{I}(E)$ is dense in $\tilde{E}$ ($\mathfrak{I}$ denoting the natural injection of E in $\tilde{E}$). Moreover, an extension $\tilde{A}$ of A can be defined in $\tilde{E}$ such that:

(α) $\tilde{A}$ is the infinitesimal generator of an analytic semigroup on $\tilde{E}$;

(β) $D(\tilde{A}) = \mathfrak{I}(E)$;

(γ) $D(\tilde{A}^2) = \mathfrak{I}(D(A))$.

Due to (β), (γ) we get $\tilde{D}_{\theta+1} = \mathfrak{I}(D_\theta)$ (with obvious notations) by classical interpolation results [12]; then it is natural to define the extrapolation

([2]) Actually, the semigroup need only be bounded [2].

spaces $D_{\theta-1} := \tilde{D}_\theta(\theta \in (0,1))$. If B is a closed linear operator on E with domain $D(B) = D(A)$, such that both $A^{-1}B$ and $B^{-1}A$ are bounded on E, an extension $\tilde{B}$ of B in $\tilde{E}$ can be defined: it follows from the closed graph theorem that the operators A and B (respectively $\tilde{A}$ and $\tilde{B}$) lead to the same interpolation spaces D_θ ($D_{\theta-1}$, respectively; $\theta \in (0,1)$).

When no confusion arises, we shall use the same notation for an operator in E and its extension in $\tilde{E}$.

Concerning the problem (2.1), the following additional assumptions will be made:

(N)

(i) N is a continuous map from E to $L(D(A), E)$, respectively from $D(A)$ to $\mathfrak{L}(D(A^2), D(A))$;

(ii) there exists $\delta > 0$ such that, for any $u \in D_{\theta+1}$ with $|u|_{\theta+1} < \delta$,
$$A^{-1}N(u) \in \mathfrak{L}(E) \text{ and } \|A^{-1}N(u)\|_{E,E} \to 0 \text{ as } |u|_{\theta+1} \to 0;$$

(iii) there exists $\lambda > 0$ such that, for any $u \in D_{\theta+1}$ with $|u|_{\theta+1} < \delta$,
$$\max\{\|N(u)\|_{\theta+1,\theta}, \|N(u)\|_{\theta,\theta-1}\} \leqslant \lambda |u|_{\theta+1} \qquad (\theta \in (0,1));$$

(R) R maps $D_{\theta+1}$ into itself continuously; moreover, there exists $\beta > 0$ such that $u \in D_{\theta+1}$, $|u|_{\theta+1} < \delta$ implies
$$|R(u)|_{\theta+1} < \beta |u|^2_{\theta+1}.$$

We can now state the following asymptotical stability result.

THEOREM 1. *Let (A), (N), (R) be satisfied; moreover, assume that for any $u_0 \in D_{\theta+1}$ there exists a unique local solution $u \in C([0,T]; D_{\theta+1}) \cap C^1([0,T]; D_\theta)$ of the problem* (2.1) ($T = T(u_0) > 0; \theta \in (0,1)$). *Then:*

(*a*) *for any $u_0 \in D_{\theta+1}$ such that $|u_0|_{\theta+1}$ is small enough, the corresponding solution of* (2.1) *is global*;

(*b*) *the trivial stationary solution is asymptotically stable in $D_{\theta+1}$ with respect to solutions of* (2.1).

In order to prove the above result, we need preliminary informations about the linear nonautonomous problem in E:

$$\text{(2.2)} \qquad \begin{cases} u'(t) = Au(t) + M(t)u(t) + f(t) & (t \in [0, \bar{t}]) \\ u(0) = u_0 \end{cases}$$

where $\bar{t} > 0$ and the operator A is assumed to satisfy the assumption (A); in this respect the following result plays a central rôle [2, 9].

THEOREM 2. *Let A be the infinitesimal generator of a strongly continuous analytic bounded semigroup on E. Then:*

(*a*) $h \in C([0,\bar{t}]; D_\theta)$ *implies* $Sh \in C([0,\bar{t}]; D_{\theta+1})$, *where*

$$(Sh)(t) := \int_0^t \exp[A(t-s)]h(s)\,ds \qquad (t \in [0,\bar{t}]);$$

(*b*) *the following inequality holds*:

$$\sup_{s\in[0,t]} |(Sh)(s)|_{\theta+1} \leqslant c_\theta(t)\cdot \sup_{s\in[0,t]} |h(s)|_\theta \qquad (t\in[0,\bar{t}];\ \theta\in(0,1)), \tag{2.3}$$

where $c_\theta(t) := Mt + M_1(t^\theta + 2^{2-\theta}M_1/(1-\theta))/(\theta(1-2^{1-\theta}))$ ($M_1 > 0$ *being a constant such that* $\|tA\exp[At]\|_{E,E} \leqslant M_1$; $t>0$).

We are now in position to state the following assumption concerning the family $\{M(t)\}$:

(M) $M \in C\big([0,\bar{t}]; \mathcal{L}(D(A),E) \cap \mathcal{L}(D(A^2),D(A))\big)$; in addition,

$$\|M(t)\|_{\vartheta+1,\vartheta} \leqslant \frac{1}{2c_\theta(t)} \qquad (t\in[0,\bar{t}])$$

$c_\theta(\cdot)$ being the function of the inequality (2.3).

Due to results concerning the extrapolation spaces (see above), the maximal regularity result expressed by Theorem 2-(a) holds even if $h\in C([0,\bar{t}]; D_{\theta-1})$; moreover, the following inequality (analogous to (2.3)) holds:

$$\sup_{s\in[0,t]} |(Sh)(s)|_\theta \leqslant \tilde{c}_\theta(t) \sup_{s\in[0,t]} |h(s)|_{\theta-1} \qquad (t\in[0,\bar{t}];\ \theta\in(0,1)), \tag{2.3'}$$

the definition of $\tilde{c}_\theta(\cdot)$ being similar to that of $c_\theta(\cdot)$ above.

The following additional assumption on the family $\{M(t)\}$ can now be introduced:

(M') $M\in C\big([0,\bar{t}]; \mathcal{L}(D(A),E)\big)$; for any $t\in[0,\bar{t}]$ $A+M(t)$ is invertible, $A^{-1}M(t)$ and $[A+M(t)]^{-1}A$ are bounded on $D(A)$ (3) and the following inequality holds:

$$\|M(t)\|_{\vartheta,\vartheta-1} \in \frac{1}{2\tilde{c}_\theta(t)} \qquad (t\in[0,\bar{t}]),$$

where $\tilde{c}_\theta(\cdot)$ is the function of the inequality (2.3').

(3) We shall be dealing in the following with the extensions in E of such operators.

We can now state the following result concerning the linear problem (2.2).

THEOREM 3. *Let* (A), (M) *be satisfied; then for any* $u_0 \in D_{\theta+1}$ *there exists a unique solution* $u \in C([0, \bar{t}]; D_{\theta+1}) \cap C^1([0, \bar{t}]; D_\theta)$ *of the problem* (2.2) *with* $f \equiv 0$. *Moreover, there exists a constant* $k_\theta \geq 1$ *such that:*

$$|u(t)|_{\theta+1} \leq k_\theta \exp\left[-\frac{\omega}{2} t\right] |u_0|_{\theta+1} \qquad (t \in [0, \bar{t}];\ \theta \in (0, 1)). \tag{2.4}$$

According to Theorem 3, the solution map $G(t, 0)\colon D_{\theta+1} \to D_{\theta+1}$, $G(t, 0)u_0 := u(t)$ relative to the problem (2.2) with $f \equiv 0$ is defined for any $t \in [0, \bar{t}]$; moreover,

$$\|G(t, 0)\|_{\theta+1,\theta+1} \leq k_\theta \exp\left[-\frac{\omega}{2} t\right].$$

Similar considerations hold for the translated problem

$$\begin{cases} (u^s)'(t) = Au^s(t) + M(t)u^s(t) \\ u^s(s) = u_0 \end{cases} \tag{2.5}$$

(where $0 \leq s \leq t \leq \bar{t}$), so that the solution map $G(t, s)\colon D_{\theta+1} \mapsto D_{\theta+1}$, $G(t, s)u_0 := u^s(t)$ is defined on the triangular domain $0 \leq s \leq t \leq \bar{t}$ (observe that $u^0(t) \equiv u(t)$ in the above notations). The fact that $G(t, s)$ can be extended to D_θ so as to prove a variation-of-constants formula for the problem (2.2) is the content of the following theorem.

THEOREM 4. *Let* (A), (M), (M') *be satisfied. Then there exists a map* G *from the domain* $\{(t, s) | 0 \leq s \leq t \leq \bar{t}\}$ *to* $\mathcal{L}(D_\theta) \cap \mathcal{L}(D_{\theta+1})$ *such that*

(*a*) *the following equalities hold in the strong sense in* D_θ*:*

$$G(s, s) = I, \qquad G(t, s')G(s', s) = G(t, s) \qquad (0 \leq s \leq s' \leq t \leq \bar{t});$$

(*b*) *there exists* $k_\theta \geq 1$ *such that:*

$$\max\{\|G(t, s)\|_{\theta,\theta},\ \|G(t, s)\|_{\theta+1,\theta+1}\} \leq k_\theta \exp\left[-\frac{\omega}{2}(t - s)\right] (0 \leq s \leq t \leq \bar{t})\ ; \tag{2.6}$$

(*c*) *for any* $f \in C([(0, \bar{t})]; D_\theta)$ *and* $u_0 \in D_{\theta+1}$ *the unique solution* $u \in C([0, \bar{t}]; D_{\theta+1}) \cap C^1([0, \bar{t}]; D_\theta)$ *of the problem* (2.2) *is*

$$u(t) = G(t, 0)u_0 + \int_0^t G(t, s) f(s)\, ds \qquad (t \in [0, \bar{t}]). \tag{2.7}$$

The existence and uniqueness claims in Theorems 3, respectively 4 are well-known and can be proved under weaker assumptions [2]; we focus here on the proof of the estimate (2.4) and of the representation formula (2.7), which are needed in the proof of Theorem 1.

Finally, let us observe that Theorem 1 gives an asymptotical stability result in D_θ (respectively $D_{\theta+2}$) instead of $D_{\theta+1}$, if its assumptions are satisfied in the extrapolation space $\tilde{E}$ (respectively in the Banach space $D(A)$, endowed with the graph norm) instead of the space E; this argument can be iterated to give asymptotical stability results in different interpolation spaces, depending on the regularity of the solution to problem (2.1). It can also be observed that instability results concerning the problem under consideration are proved in the usual way (see [6]).

3. – Application to nonlinear diffusion problems.

In the present section we investigate stationary solutions to non-degenerate nonlinear diffusion problems by using the linearized stability result given in Theorem 1; the following subsections are devoted to the case of homogeneous Dirichlet, respectively Neumann boundary conditions.

(*a*) We are interested in the following problem:

$$\tag{3.1} \begin{cases} w_t = \Delta\varphi(w) + f(w) & \text{in } (0, +\infty)\times\Omega \\ w = 0 & \text{in } (0, +\infty)\times\partial\Omega \\ w = w_0 & \text{in } \{0\}\times\Omega , \end{cases}$$

where $\Omega \subseteq \mathbf{R}^n$ is an open bounded domain with C^∞ boundary $\partial\Omega$. The following assumptions will be made throughout the present section:

(φ) $\qquad \varphi \in C^\infty(\mathbf{R})$, $\varphi'(u) \geqslant c > 0$ for any $u \in \mathbf{R}$ and $\varphi(0) = 0$;

(f) $\qquad f \in C^\infty(\mathbf{R})$, $f(0) = 0$.

Set $v := \varphi(w)$, $\chi(v) := \varphi'(\varphi^{-1}(v))$, $g(v) := \chi(v) f(\varphi^{-1}(v))$; then the problem (3.1) can be rewritten as follows:

$$\tag{3.2} \begin{cases} v_t = \chi(v)\Delta v + g(v) & \text{in } (0, +\infty)\times\Omega \\ v = 0 & \text{in } (0, +\infty)\times\partial\Omega \\ v = v_0 := \varphi(w_0) & \text{in } \{0\}\times\Omega . \end{cases}$$

Let us denote by E the space $C_0(\bar{\Omega})$ of continuous functions on $\bar{\Omega}$ which vanish on $\partial\Omega$ and by D the domain of the Laplacian in E (*i.e.*, $D(\Delta) := \{u \in E | \Delta u \in E\}$) endowed with the graph norm. The map $\psi: D \to E$, $v \to \psi(v) := \chi(v)\Delta v + g(v)$ is easily seen to be Fréchet differentiable from D to E under the assumptions (φ), (f); if $\psi(\bar{v}) = 0$, we have for any $u \in D$

$$\psi(\bar{v} + u) = [\chi(\bar{v})\Delta + f'(\varphi^{-1}(\bar{v}))]u + \sigma(u)\,, \tag{3.3}$$

where $|\sigma(u)|_E/|u|_D \to 0$ as $|u|_D \to 0$. For the present purposes it is convenient to separate out the part of σ which is of higher order from E to itself; we get easily

$$\sigma(u) = [\chi(\bar{v} + u) - \chi(\bar{v})]\Delta u + R_1(\bar{v}, u) - f(\varphi^{-1}(\bar{v}))R_2(\bar{v}, u)\,, \tag{3.4}$$

where

$$R_1(\bar{v}, u) := g(\bar{v} + u) - g(\bar{v}) - g'(\bar{v})u$$

$$R_2(\bar{v}, u) := \chi(\bar{v} + u) - \chi(\bar{v}) - \chi'(\bar{v})u$$

and $\max_{k=1,2} (|R_k(\bar{v}, u)|_E/|u|_E) \to 0$ as $|u|_E \to 0$.

Now let $\bar{v}$ denote a (classical) stationary solution of the evolution equation in (3.2) with the given boundary conditions: the new unknown function $u := v - \bar{v}$ is seen to satisfy a Cauchy problem in E of the form (2.1), if the following definitions are introduced.

(α) Set $p(x) := \chi(\bar{v}(x))$, $q(x) := f'(\varphi^{-1}(\bar{v}(x)))$ $(x \in \bar{\Omega})$; then the operator A is defined as follows:

$$\begin{cases} D(A) := D(\Delta) \\ Au := p\Delta u + qu \qquad (u \in D(A))\,. \end{cases} \tag{3.5}$$

Observe that, due to the assumptions (φ) and (f), both Δp and Δq belong to the space $C(\bar{\Omega})$ of continuous functions on $\bar{\Omega}$. Let us also note the following equality for subsequent purposes:

$$D(A^2) := \{u \in D(A) | Au \in D(A)\} = \{u \in E | \Delta u \in E, \Delta(p\Delta u + qu) \in E\}.$$

We shall be dealing in the following with the Banach spaces $D(A)$, $D(A^2)$ endowed with the graph norm.

(β) Set $m(x, u) := \chi(\bar{v}(x) + u) - \chi(\bar{v}(x))$ $(x \in \bar{\Omega};\ u \in \mathbf{R})$. For any $u \in E$ and $z \in D(A)$ let us define:

$$(N(u)z)(x) := m(x, u(x))(\Delta z)(x) \qquad (x \in \bar{\Omega})\,; \tag{3.6}$$

observe for further reference that the map $x \to m(x, u(x))$ (respectively $x \to \Delta m(x, u(x))$) belongs to E whenever $u \in E$ ($u \in D(A)$, respectively). When no confusion arises, the notation $\mu(x) := m(x, u(x))$ ($x \in \bar{\Omega}$) will be used; clearly, μ depends on u.

(γ) Set $\varrho(x, u) := R_1(\bar{v}(x), u) - f(\varphi^{-1}(\bar{v}(x)))R_2(\bar{v}(x), u)$ ($x \in \bar{\Omega}$; $u \in \mathbf{R}$); define for any $u \in E$

$$(R(u))(x) := \varrho(x, u(x)) \qquad (x \in \bar{\Omega}). \tag{3.7}$$

Let us consider the Hölder space $C^{k+\sigma}(\bar{\Omega})$ (k integer, $\sigma \in (0, 1)$) endowed with the usual norm, namely

$$f \to |f|_{k+\sigma} := |f|_k + \sum_{|\alpha|=k} \sup_{\substack{x,y \in \bar{\Omega} \\ x \neq y}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\sigma}$$

(where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is any multiindex); we shall denote by $h^{k+\sigma}(\bar{\Omega})$ the subspace of $C^{k+\sigma}(\bar{\Omega})$ consisting of all functions whose derivatives of the k-th order satisfy the following condition:

$$\lim_{\tau \to 0^+} \sup_{\substack{x,y \in \bar{\Omega} \\ |x-y| \leq \tau}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\tau^\sigma} = 0 \qquad (\alpha_1 + \ldots + \alpha_n = k).$$

The so-called little-Hölder space $h^{k+\sigma}(\bar{\Omega})$ is clearly a Banach space under the $|\cdot|_{k+\sigma}$-norm; its interest for the present purposes lies in the following result [7].

PROPOSITION 3.1. *Let $\Omega \subseteq \mathbf{R}^n$ be an open bounded subset with boundary $\partial\Omega$ of class C^∞. Let A be a uniformly elliptic operator in E defined as follows:*

$$\begin{cases} D(A) := \{u \in E \mid Au \in E\} \\ (Au)(x) := \sum_{i,j=1}^{n} a_{ij}(x)(\partial_{ij}^2 u)(x) + \sum_{i=1}^{n} b_i(x)(\partial_i u)(x) + c(x)u(x) \quad (x \in \Omega), \end{cases}$$

where $a_{ij}, b_i, c \in C(\bar{\Omega})$ ($i, j = 1, \ldots, n$). Then D_θ is isomorphic to the Banach space $h_0^{2\theta}(\bar{\Omega})$ of $f \in h^{2\theta}(\bar{\Omega})$ which vanish on $\partial\Omega$ ($\theta \in (0, 1)$, $\theta \neq \frac{1}{2}$). If in addition $a_{ij}, b_i, c \in h^{2\theta}(\bar{\Omega})$ and $\partial\Omega$ is of class $h^{2\theta+2}$, $D_{\theta+1}$ is isomorphic to the Banach space $h_0^{2\theta+2}(\bar{\Omega})$ of $f \in h^{2\theta+2}(\bar{\Omega})$ such that both f and Af vanish on $\partial\Omega$ ($\theta \in (0, 1)$, $\theta \neq \frac{1}{2}$).

The above concrete characterization of the interpolarion spaces D_θ, $D_{\theta+1}$ is of use in the proof of the following asymptotical stability result [4].

PROPOSITION 3.2. *Let* (φ), (f) *be satisfied and* $g''(0)=0$: *let* $\bar{v}$ *denote any stationary solution of the evolution equation in* (3.2) (*with the given boundary conditions*) *such that the real part of the spectrum of* A *is strictly negative. Then* $\bar{v}$ *is asymptotically stable in the little-Hölder space* $h_0^{2\theta+2}(\bar{\Omega})$ *with respect to solutions of the problem* (3.2).

PROOF. – According to [10], the operator A defined in (3.5) is the infinitesimal generator of a strongly continuous analytic semigroup on E; if the spectrum of A lies in the left open half-plane, the type of $\exp[At]$ is negative as A has compact resolvent. This proves the assumption (A) to be satisfied.

It is easily checked that the restriction $\psi|_{h^{2\theta+2}}$ is Fréchet differentiable from $h^{2\theta+2}(\bar{\Omega})$ to $h^{2\theta}(\bar{\Omega})$ with derivative $A|_{h^{2\theta+2}}$ $(\theta\in(0,1),\ \theta\neq\frac{1}{2})$; then the existence of a unique local solution of the problem (3.2) in $C([0,T];h^{2\theta+2}(\bar{\Omega}))\cap C^1([0,T];h^{2\theta}(\bar{\Omega}))$ for any $u_0\in h^{2\theta+2}(\bar{\Omega})$ $(T=T(u_0))$ follows from [2, Theorem 4.1]. Obviously, the same is true for the solution of the problem (2.1) in the present case.

Let us prove that the map $u\to N(u)$ defined in (2.6) satisfies the assumption (N). Due to the regularity properties of the map μ (see (β) above), it is easily seen that the property (N)-(i) is satisfied.

To check (N)-(ii), observe that the spectral properties of A imply A^{-1}, thus $(\Delta+q/p)^{-1}$, to exist as a bounded operator in E; in addition we get

$$A^{-1}N(u)=\left(\Delta+\frac{q}{p}\right)^{-1}\frac{\mu}{p}\Delta\,.$$

From the equality

$$A^{-1}N(u)z=\left(\Delta+\frac{q}{p}\right)^{-1}\left[\Delta\left(\frac{\mu}{p}z\right)-\Delta\left(\frac{\mu}{p}\right)z-z\nabla\left(\frac{\mu}{p}\right)\cdot\nabla z\right]\qquad(u,z\in D(A))$$

it follows easily that the extension of $A^{-1}N(u)$ in E satisfies the inequality

$$\|A^{-1}N(u)\|_{E,E}\leqslant c(u)\,,$$

where $c(u)\to 0$ if $|\Delta u|_E\to 0$; in particular, $c(u)\to 0$ if $u\in D_{\theta+1}$ and $|u|_{\theta+1}\to 0$, thus (N)-(ii) follows.

[4] It may be observed that also the Hölder spaces $C^{k+\sigma}(\bar{\Omega})$ can be characterized as interpolation spaces between $D(A)$ and E [2, 12]. However, $D(A)$ is not dense in such spaces, as it is required *e.g.* in the proof of Theorem 4.

To check (N)-(iii), observe that the following inequalities hold:

$$\|N(u)\|_{D(A),E} \leqslant c_1 |u|_E \qquad (u \in E,\ c_1 > 0); \tag{3.8}$$

$$\|N(u)\|_{D(A^2),D(A)} \leqslant c_2 |u|_{D(A)} \qquad (u \in D(A),\ c_2 > 0); \tag{3.9}$$

$$\|N(u)\|_{D(\tilde{A}),\tilde{E}} \leqslant c_3 |u|_{D(A)} \qquad (u \in D(A),\ c_3 > 0)\,. \tag{3.10}$$

Let us prove for instance (3.10), the proofs of (3.8) and (3.9) being similar (and easier). We have for any $z \in D(A)$ (see the Appendix):

$$|\mathfrak{I}(N(u))z|_{\tilde{E}} = |(0, N(u)z)^{\sim}|_{\tilde{E}} := \inf_{\xi \in D(A)} (|\xi|_E + |\eta|_E)\,,$$

where $\eta = N(u)z + A\xi$; the particular choice $\xi = -A^{-1}N(u)z$ gives

$$|\mathfrak{I}(N(u))z|_{\tilde{E}} \leqslant |A^{-1}N(u)z|_E \leqslant \|A^{-1}N(u)\|_{E,E} |z|_E \leqslant c_3 |u|_{D(A)} |z|_E\,,$$

whence the conclusion.

From the above inequalities (N)-(iii) follows by interpolation results [12].

Due to the characterization of $h^{2\theta+2}$ given in Proposition 3.1, the operator R defined in (3.7) is easily seen to satisfy the assumption (R): thus the result follows by Theorem 1.

It is easily checked by a classical argument (see [5]) that the real part of the spectrum of A is negative if the map $u \to f(u)/\varphi(u)$ is decreasing; under this sufficient condition, global attractivity results can be proved by monotonicity methods [4].

(*b*) If Neumann homogeneous boundary conditions are considered, formal calculations like those of the subsection (*a*) lead us to the problem

$$\begin{cases} v_t = \chi(v)\Delta v + g(v) & \text{in } (0, +\infty) \times \Omega \\ \partial_n v = 0 & \text{in } (0, +\infty) \times \partial\Omega \\ v = v_0 & \text{in } \{0\} \times \Omega \end{cases} \tag{3.11}$$

where ∂_n denotes the outer normal derivative at $\partial\Omega$ and the other quantities are defined as above. The space E is now $C(\bar{\Omega})$, namely the space of continuous functions on $\bar{\Omega}$; as for the operator A, it is defined as follows:

$$\begin{cases} D(A) := \{u \in E \mid \Delta u \in E,\ \partial_n u = 0 \text{ on } \partial\Omega\} \\ Au := p\Delta u + qu \qquad (u \in D(A)) \end{cases} \tag{3.12}$$

It is proved in [11] that A is the infinitesimal generator of a strongly continuous analytic semigroup on E.

Clearly, the content of Proposition 3.2 still holds true in the present case, provided we have a concrete characterization of the interpolation spaces D_θ, $D_{\theta+1}$ analogous to that of Proposition 3.1. We don't know of any general result in this respect; such a characterization is given for the one-dimensional case in the following proposition, whose proof makes use of some additional Banach spaces, namely:

$$C_\# := \{u \in C(\mathbf{R}) | u(x+2) = u(x),\, x \in \mathbf{R}\};$$

$$C_\#^k := C_\# \cap C^k(\mathbf{R}) \qquad (k \geqslant 1);$$

$$h_\#^\gamma := C_\# \cap h^\gamma(\mathbf{R}) \qquad (\gamma \in (0,1));$$

$$h_N^{k+\sigma}([0,1]) := \{u \in h^{k+\sigma}([0,1]) | u^{(k)} = u^{(k-2)} = \ldots = u' = 0 \text{ in } \{0,1\}\} \qquad (k \text{ odd},\ \sigma \in (0,1)).$$

PROPOSITION 3.3. *Let A be the following uniformly elliptic operator in $C([0,1])$:*

$$\begin{cases} D(A) := \{u \in C^2([0,1]) | u'(0) = u'(1) = 0\} \\ (Au)(x) := a(x)u''(x) + b(x)u'(x) + c(x)u(x) \qquad (x \in [0,1]), \end{cases}$$

where $a, b, c \in C([0,1])$. Then D_θ is isomorphic to the Banach space $h^{2\theta}([0,1])$ (respectively $h_N^{2\theta}([0,1])$) if $\theta \in (0,\frac{1}{2})$ (if $\theta \in (\frac{1}{2},1)$, respectively).

PROOF. Let us denote by $(Y, X)_\theta$ $(\theta \in (0,1))$ the continuous interpolation spaces between two Banach spaces Y and X, Y continuously embedded in X, defined in [2]. Following the method outlined in [7], it is easily seen that $h_\#^{2\theta}$ is isomorphic to the interpolation space $(C_\#^2, C_\#)_{1-\theta}$ $(\theta \in (0,1);\, \theta \neq \frac{1}{2})$. On the other hand, we have $C_\# = F_0 \oplus F_1$, where F_0 (respectively F_1) is the space of even (odd, respectively) functions of $C_\#$; then there exists a natural isomorphism j between F_0 and $C([0,1])$, $j(u)$ denoting the restriction on the interval $[0,1]$ of any $u \in F_0$. As $j\colon F_0 \cap C_\#^2 \to D(A)$, the following isomorphisms between Banach spaces are seen to hold:

$$D_\theta \simeq (F_0 \cap C_\#^2, F_0)_{1-\theta} \simeq F_0 \cap (C_\#^2, C_\#)_{1-\theta} \qquad (\theta \in (0,1);\, \theta \neq \tfrac{1}{2});$$

thus the conclusion follows.

4. – Proof of the linear results.

Let us prove a preliminary lemma concerning the problem (2.2).

LEMMA 4.1. *Let the assumption of Theorem 2 and* (M) *be satisfied. Then for any* $f \in C([0, \bar{t}]; D_\theta)$ *and* $u_0 \in D_{\theta+1}$ *there exists a unique* $u \in C([0, \bar{t}]; D_{\theta+1}) \cap C^1([0, \bar{t}]; D_\theta)$ *which solves the problem* (2.2). *In addition, the following estimate holds*:

$$\sup_{s\in[0,t]}|u(s)|_{\theta+1} \leqslant 2c_\theta(t)\cdot\sup_{s\in[0,t]}|f(s)|_\theta + M\left(\frac{c_\theta(t)}{c_\theta(0)}+1\right)|u_0|_{\theta+1} \qquad (t\in[0,\bar{t}])\,. \tag{4.1}$$

PROOF. Consider the operator $Q\colon C([0, \bar{t}]; D_\theta) \to C([0, \bar{t}]; D_\theta)$ defined as follows:

$$(Qh)(t) := -\,M(t)(Sh)(t) \qquad (t\in[0,\bar{t}])$$

whenever $h \in C([0, \bar{t}]; D_\theta)$. For any $t \in [0, \bar{t}]$ we have:

$$\begin{aligned}\sup_{s\in[0,t]}|(Qh)(s)|_\theta &= \sup_{s\in[0,t]}|M(s)(Sh)(s)|_\theta \leqslant \sup_{s\in[0,t]}\|M(s)\|_{\theta+1,\theta}|(Sh)(s)|_{\theta+1}\\ &\leqslant \sup_{s\in[0,t]}\left\{\frac{1}{2c_\theta(s)}\,c_\theta(s)\sup_{\sigma\in[0,s]}|h(\sigma)|_\theta\right\} = \frac{1}{2}\sup_{s\in[0,t]}|h(s)|_\theta;\end{aligned}$$

here use of the inequality (2.3) and assumption (M) has been made. It follows in particular that $I + Q$ is an invertible operator from $C([0, \bar{t}]; D_\theta)$ to itself, such that

$$\|(I+Q)^{-1}\|_{C([0,\bar{t}];D_\theta),\,C([0,\bar{t}];D_\theta)} \leqslant 2\,.$$

(a) Consider first the case $u_0 = 0$. Define, for any $f \in C([0, \bar{t}]; D_\theta)$,

$$u(t) := [S(I+Q)^{-1}f](t) \qquad (t\in[0,\bar{t}]);$$

according to the maximal regularity result expressed by Theorem 2-(a), $u \in C([0, \bar{t}]; D_{\theta+1}) \cap C^1([0, \bar{t}]; D_\theta)$. To prove that u is the (unique) solution of the problem (2.2) when $u_0 = 0$, set $g := (I+Q)^{-1}f$; then $g \in C([0, \bar{t}]; D_\theta)$ and

$$\begin{aligned}(Sg)'(t) - A(Sg)(t) - M(t)(Sg)(t) &= g(t) + A(Sg)(t) - A(Sg)(t) - M(t)(Sg)(t)\\ &= [(I+Q)(I+Q)^{-1}f](t) = f(t) \qquad (t\in[0,\bar{t}])\,,\end{aligned}$$

which proves the claim. Moreover, we have:

$$\text{(4.2)}\qquad \sup_{s\in[0,t]} |u(s)|_{\theta+1} = \sup_{s\in[0,t]} |[S(I+Q)^{-1}f](s)|_{\theta+1}$$
$$\leqslant c_\theta(t)\cdot \sup_{s\in[0,t]} |[(I+Q)^{-1}f](s)|_\theta \leqslant 2c_\theta(t)\cdot \sup_{s\in[0,t]} |f(s)|_\theta \qquad \big(t\in[0,\bar t]\big)\,,$$

i.e., the inequality (4.1) for $u_0 = 0$.

(*b*) In the general case $u_0 \in D_{\theta+1}$, define $\tilde f(t) := f(t) + M(t)\exp[At]u_0$ and $u(t) := z(t) + \exp[At]u_0$, where $z\in C\big([0,\bar t]; D_{\theta+1}\big)\cap C^1\big([0,\bar t]; D_\theta\big)$ is the (unique) solution of the auxiliary problem

$$\begin{cases} z'(t) = Az(t) + M(t)z(t) + \tilde f(t) & \big(t\in[0,\bar t]\big)\\ z(0) = 0\,, \end{cases}$$

which exists due to (*a*) above. It is immediately seen that $u\in C\big([0,\bar t]; D_{\theta+1}\big)\cap C^1\big([0,\bar t]; D_\theta\big)$ is the (unique) solution of the problem (2.2); due to the inequality (4.2) we also have:

$$\sup_{s\in[0,t]} |u(s)|_{\theta+1} \leqslant 2c_\theta(t)\sup_{s\in[0,t]} |\tilde f(s)|_\theta + \sup_{s\in[0,t]} |\exp[As]u_0|_{\theta+1}$$
$$\leqslant 2c_\theta(t)\{\sup_{s\in[0,t]} |f(s)|_\theta + \sup_{s\in[0,t]} \|M(s)\|_{\theta+1,\theta} |\exp[As]u_0|_{\theta+1}\}$$
$$+ \sup_{s\in[0,t]} |\exp[As]u_0|_{\theta+1} \leqslant 2c_\theta(t)\sup_{s\in[0,t]} |f(s)|_\theta$$
$$+\left\{2c_\theta(t)\sup_{s\in[0,t]}\frac{1}{2c_\theta(s)} + 1\right\}\sup_{s\in[0,t]} |\exp[As]u_0|_{\theta+1} \qquad \big(t\in[0,\bar t]\big),$$

whence the result easily follows.

Results and estimates like those above are proved in [2] in the case $M\equiv 0$.

We can now prove Theorem 3.

Proof of Theorem 3. Existence and uniqueness follow from Lemma 4.1.

The assumption that $\exp[At]$ is of strictly negative type has not been used so far, thus we can apply Lemma 4.1 to

$$u_\omega \in C\big([0,\bar t]; D_{\theta+1}\big)\cap C^1\big([0,\bar t]; D_\theta\big)\,,\qquad u_\omega(t) := \exp[\omega t]u(t)\qquad \big(t\in[0,\bar t]\big)\,,$$

which solves the problem

$$\begin{cases} u'_\omega(t) = (A+\omega)u_\omega(t) + M(t)u_\omega(t) & \big(t\in[0,\bar t]\big)\\ u_\omega(0) = u_0\,. \end{cases}$$

In particulaı, the inequality (4.1) gives (with $f \equiv 0$)

$$|u(t)|_{\theta+1} \leqslant M\left(\frac{c_\theta(t)}{c_\theta(0)}+1\right)\exp[-\omega t]|u_0|_{\theta+1} \qquad (t\in[0,\bar t])\,,$$

whence the estimate (2.4) follows easily with $k_\theta := M\big(2+\max\{1, 2/\omega\} + M/(\omega c_\theta(0))\big)$.

In order to prove Theorem 4 we need two lemmas.

LEMMA 4.2. *Let* (A), (M), (M') *be satisfied. Then there exists a map* G *from the domain* $\{(t,s)|0\leqslant s\leqslant t\leqslant \bar t\}$ *to* $\mathcal{L}(D_\theta)\cap\mathcal{L}(D_{\theta+1})$ *such that:*

(a) $u^s(t) := G(t,s)u_0$ *is the unique solution of the problem* (2.5) *in* $C([0,\bar t]; D_{\theta+1})\cap C^1([0,\bar t]; D_\theta)$ (*respectively in* $C([0,\bar t]; D_\theta)\cap C^1([0,\bar t]; D_{\theta-1})$) *whenever* $u_0\in D_{\theta+1}$ ($u_0\in D_\theta$, *respectively*);

(b) $G(s,s)=I$, $G(t,s')G(s',s)=G(t,s)$ $(0\leqslant s\leqslant s'\leqslant t\leqslant\bar t)$ *as operators in* D_θ;

(c) $\max\{\|G(t,s)\|_{\theta,\theta}\|G(t,s)\|_{\theta+1,\theta+1}\}\leqslant k_\theta \exp[-(\omega/2)(t-s)]$ $(0\leqslant s\leqslant t\leqslant \bar t)$ *with a suitable constant* $k_\theta\geqslant 1$.

PROOF. Let (A) and (M') be satisfied; then results similar to those of Lemma 4.1 and Theorem 3 follow, whose formulation is left to the reader. As a consequence, under the present assumptions a solution map relative to the problem (2.5) can be defined in $D_{\theta+1}$ and extended to D_θ preserving the norm, due to the uniqueness of the solutions of (2.5). The property (b) is easily seen to hold, thus the proof is complete.

LEMMA 4.3. *Let the assumptions of Lemma* 4.1 *and* (M') *be satisfied. Then* $G(t,\cdot)$ *is strongly differentiable on* $D_{\theta+1}(t\in[0,\bar t])$, $\partial_s G(t,s)\colon D_{\theta+1}\to D_\theta$ *and the following holds on* $D_{\theta+1}$:

$$\partial_s G(t,s) = -G(t,s)(A+M(s)) \qquad (0\leqslant s\leqslant t\leqslant \bar t)\,. \tag{4.3}$$

PROOF. (a) For any $u_0\in D_\theta$ and $0\leqslant s_0\leqslant s\leqslant t\leqslant\bar t$ we have

$$\begin{aligned}[G(t,s)-G(t,s_0)]u_0 = u^s(t)-u^{s_0}(t) &= G(t,s)[u_0-u^{s_0}(s)]\\ &= G(t,s)[u^{s_0}(s_0)-u^{s_0}(s)]\,,\end{aligned}$$

whence the strong continuity of $G(t,\cdot)$ on D_θ as $s\to s_0^+$ immediately follows (here use of Lemma 4.2-(b), (c) is made). On the other hand, for any

$u_0 \in D_{\theta+1}$ the following inequality holds:

$$
\begin{aligned}
|(s-s_0)^{-1}[G(t,s)-G(t,s_0)]u_0 + G(t,s_0)[A+M(s_0)]u_0|_\theta \\
\leqslant |G(t,s)\{(s-s_0)^{-1}[u^{s_0}(s_0)-u^{s_0}(s)] + [A+M(s_0)]u_0|_\theta \\
+ |[G(t,s)-G(t,s_0)][A+M(s_0)]u_0|_\theta ;
\end{aligned}
$$

as a consequence, the claims are proved for the case $s \to s_0^+$.

(*b*) Assume now $0 \leqslant s \leqslant s_0 \leqslant t \leqslant \bar{t}$; for any $u_0 \in D_{\theta+1}$ we have

$$
\begin{aligned}
(4.4)\qquad & (s-s_0)^{-1}[G(t,s)-G(t,s_0)]u_0 + G(t,s_0)[A+M(s_0)]u_0 \\
&= G(t,s_0)\{(s-s_0)^{-1}[u^s(s_0)-u^s(s)] + [A+M(s_0)]u_0\} \\
&= G(t,s_0)\Big\{(s-s_0)^{-1}\int_s^{s_0}[A+M(\sigma)]\,u^s(\sigma)\,d\sigma + [A+M(s_0)]u_0\Big\} \\
&= G(t,s_0)\Big\{(s-s_0)^{-1}\int_s^{s_0}[A+M(\sigma)]\,[u^s(\sigma)-u_0]\,d\sigma \\
&\quad + (s-s_0)^{-1}\int_s^{s_0}[M(\sigma)-M(s_0)]\,u_0\,d\sigma\Big\}.
\end{aligned}
$$

For the first integral in the right-hand side the following estimate holds due to the assumption (*M*):

$$
\begin{aligned}
\int_s^{s_0}|[A+M(\sigma)][u^s(\sigma)-u_0]|_\theta\,\mathrm{d}\sigma &\leqslant \int_s^{s_0}\|A+M(\sigma)\|_{\theta+1,\theta}|u^s(\sigma)-u^s(s)|_{\theta+1}\,d\sigma \\
&\leqslant \left(1+\frac{1}{2c_\theta(s)}\right)\int_s^{s_0}|u^s(\sigma)-u^s(s)|_{\theta+1}\,d\sigma ;
\end{aligned}
$$

the second one can be similarly estimated as follows:

$$
\int_s^{s_0}|[M(\sigma)-M(s_0)]u_0|_\theta\,d\sigma \leqslant \int_s^{s_0}\|M(\sigma)-M(s_0)\|_{\theta+1,\theta}\,d\sigma\,|u_0|_{\theta+1}\,.
$$

Now observe that u^s (respectively M) is a continuous, thus uniformly continuous map from $[s,\bar{t}]$ (respectively $[0,\bar{t}]$) to $D_{\theta+1}$ ($\mathcal{L}(D_{\theta+1}, D_\theta)$, respectively); namely, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $s_0 - s < \delta$ implies for any $\sigma \in [s, s_0]$

$$\max\left\{|u^s(\sigma) - u^s(s)|_{\theta+1}, \|M(\sigma) - M(s_0)\|_{\theta+1,\theta}\right\} < \varepsilon .$$

Then from (4.4) and the above inequalities we get

$$\begin{aligned}&|(s - s_0)^{-1}[G(t, s) - G(t, s_0)]u_0 + G(t, s_0)[A + M(s_0)]u_0|_\theta \\ &\quad\leq k_\theta\left(1 + \frac{1}{2c_\theta(0)}\right)\int_s^{s_0}|u^s(\sigma) - u^s(s)|_{\theta+1}\, d\sigma \\ &\quad+\int_s^{s_0}\{\|M(\sigma) - M(s_0)\|_{\theta+1,\theta}\, d\sigma |u_0|_{\theta+1}\}(s_0 - s)^{-1} < 2k_\theta\left\{\left(1 + \frac{1}{2c_\theta(0)}\right) + |u_0|_{\theta+1}\right\}\varepsilon\end{aligned}$$

whenever $s_0 - s < \delta$; thus the claims follow also when $s \to s_0^-$ and the proof is complete.

We can now prove Theorem 4.

PROOF OF THEOREM 4. Claims (a) and (b) are the content of Lemma 4.2; as for (c), choose $f \in C([0, \bar t]; D_{\theta+1})$, $u_0 \in D_{\theta+1}$ and denote by u the corresponding unique solution of the problem (2.2) (which exists by Lemma 4.1). For $0 \leq s \leq t \leq \bar t$ we get, according to Lemma 4.3,

$$\partial_s\big(G(t, s)u(s)\big) = - G(t, s)[A + M(s)]u(s) + G(t, s)u'(s) = G(t, s)f(s);$$

then the equality (2.7) follows by integration on $[0, t]$. A standard extension argument proves the result for a general $f \in C([0, \bar t]; D_\theta)$; this comtes the proof.

5. - Proof of theorem 1.

Set $\varepsilon_0 := \min\left\{1/(2\lambda\bar c(0)), \omega/(2k\beta), \delta\right\}$ where $\bar c(t) = \bar c_\theta(t) := \max\left\{c_\theta(t), \tilde c_\theta(t)\right\}$ ($c_\theta(\cdot)$ and $\tilde c_\theta(\cdot)$ being the functions which appear in the inequalities (2.3) and (2.3′), respectively) and $k = k_\theta := M\big(2 + \max\{1, 2/\omega\} + M/(\omega c_\theta(0))\big)$. It is easily seen that there exists $\bar\varepsilon \in (0, \varepsilon_0)$ such that, for any $\varepsilon \in (0, \bar\varepsilon)$, the quantity

$$\tilde\tau = \tilde\tau(\varepsilon) := \bar c^{-1}\big((2\lambda\varepsilon)^{-1}\big)$$

has the following properties:

$$\lambda\varepsilon < \frac{1}{2\bar{c}(t)} \quad \text{for any } t \in [0, \tilde{\tau});$$

$$\tilde{\tau} > \frac{2}{\omega - 2k\beta\varepsilon} \log k .$$

In addition, denote by τ $(\tau \in (0, T))$ the supremum of the interval (containing the origin) of times such that $|u_0|_{\theta+1} < \varepsilon k$ implies $|u(t)|_{\theta+1} < \varepsilon$.

(*a*) Let us first prove that $|u_0|_{\theta+1} < \varepsilon k$ implies $\tau \geqslant \tilde{\tau}$ whenever $\varepsilon \in (0, \bar{\varepsilon})$. Otherwise, we have $|u(t)|_{\theta+1} < \varepsilon$ on the interval $[0, \tau) \subset [0, \tilde{\tau})$, which entails

$$\max\left\{\|N(u(t))\|_{\theta+1,0},\ \|N(u(t))\|_{\theta,\theta-1}\right\} \leqslant \lambda |u(t)|_{\theta+1} < \lambda\varepsilon < \frac{1}{2\bar{c}(t)} \qquad (t \in [0, \tau)),$$

due to (N) and the definition of $\bar{\varepsilon}$. Then the variation-of-constants formula (2.7) can be used to represent on $[0, \tau)$ the solution of problem (2.1); here $G(t, s)$ is the solution map relative to problem (2.5) with $M(t) := N(u(t))$ $(0 \leqslant s \leqslant t \leqslant \tau)$. Due to (2.6) and (R) we get:

$$\begin{aligned}
|u(t)|_{\theta+1} &\leqslant k \exp\left[-\frac{\omega}{2}t\right] |u_0|_{\theta+1} + k\int_0^t \exp\left[-\frac{\omega}{2}(t-s)\right] |R(u(s))|_{\theta+1}\, ds \\
&\leqslant k \exp\left[-\frac{\omega}{2}t\right] |u_0|_{\theta+1} + k\beta\int_0^t \exp\left[-\frac{\omega}{2}(t-s)\right] |u(s)|^2_{\theta+1}\, ds \\
&\leqslant k \exp\left[-\frac{\omega}{2}t\right] |u_0|_{\theta+1} + k\beta\varepsilon\int_0^t \exp\left[-\frac{\omega}{2}(t-s)\right] |u(s)|_{\theta+1}\, ds \qquad (t \in [0, \tau)),
\end{aligned}$$

whence, by Gronwall's lemma:

$$|u(t)|_{\theta+1} \leqslant k|u_0|_{\theta+1} \exp\left[-\left(\frac{\omega}{2} - k\beta\varepsilon\right)t\right] \qquad (t \in [0, \tau)). \tag{5.1}$$

This in turn implies $u(\tau) < \varepsilon$, thus $\tau = +\infty$, contrary to the assumption $\tau < \tilde{\tau}$.

(*b*) Let us assume $|u_0|_{\theta+1} < \varepsilon/k$ with $\varepsilon \in (0, \bar{\varepsilon})$; then it follows $\tau > \tilde{\tau}$, according to (*a*): the inequality (5.1) is now satisfied on $[0, \tilde{\tau})$ and we have

$$|u(\tilde{\tau})|_{\theta+1} < |u_0|_{\theta+1},$$

due to the definition of $\tilde{\tau}$ and $\bar{\varepsilon}$. Were $\tau < +\infty$, there would exist a positive integer n such that $\tau = n\tilde{\tau} + s$ $(s \in [0, \tilde{\tau}))$; this in turn would imply, iterating the above procedure,

$$\begin{aligned} |u(\tau)|_{\theta+1} &\leqslant k|u(n\tilde{\tau})|_{\theta+1} \exp\left[-\left(\frac{\omega}{2} - k\beta\varepsilon\right)s\right] \\ &\leqslant k\left[k^n \exp\left[-\left(\frac{\omega}{2} - k\beta\varepsilon\right)n\tilde{\tau}\right]|u_0|_{\theta+1}\right] \exp\left[-\left(\frac{\omega}{2} - k\beta\varepsilon\right)s\right] \\ &< k|u_0|_{\theta+1} \exp\left[-\left(\frac{\omega}{2} - k\beta\varepsilon\right)s\right] < \varepsilon\,, \end{aligned}$$

which is absurd. From the contraction the global existence of the solution of (2.1) and the asserted $D_{\theta+1}$-stability of the trivial stationary solution follow.

In order to prove that $|u(t)|_{\theta+1}$ is infinitesimal as $t \to +\infty$, observe that

$$\max_{t \in [n\tilde{\tau}, (n+1)\tilde{\tau}]} |u(t)|_{\theta+1} \leqslant k^{n+1}|u_0|_{\theta+1} \exp\left[-\left(\frac{\omega}{2} - k\beta\varepsilon\right)n\tilde{\tau}\right]$$

for any positive integer n; as the right-hand side of the above inequality is infinitesimal when $n \to \infty$, the result follows.

6. – Appendix.

Let E denote a Banach space and A the infinitesimal generator of a strongly continuous analytic semigroup $\exp[At]$ on E, with domain $D(A) \subseteq E$. Let $G(A)$ denote the graph of A, namely

$$G(A) := \{(u, v) \in D(A) \times E | Au = v\}\,;$$

$G(A)$ is a Banach space when endowed with the norm of the product $E \times E$, since A is closed.

Let us introduce the space $\tilde{E}$ defined as follows:

$$\tilde{E} := (E \times E)/G(A)\,;$$

if we denote by $(u, v)^{\sim}$ the coset of (u, v) in $E \times E$, namely

$$(u, v)^{\sim} := (u, v) + G(A)\,,$$

the space $\tilde{E}$ is a Banach space when endowed with the norm

$$(u, v)^{\sim} \to |(u, v)^{\sim}|_{\tilde{E}} := \inf\{|\xi|_E + |\eta|_E\}\,,$$

where the infimum is taken with respect to any $(\xi, \eta) \in (E \times E)$ such that $(\xi - u, \eta - v) \in G(A)$.

The natural injection $\mathfrak{J}$ of E into $\tilde{E}$ is defined as follows:

$$\mathfrak{J}(u) := (0, u)^{\sim} \qquad (u \in E)\,.$$

The following results can be proved [3].

THEOREM A.1. (i) $\mathfrak{J}(E)$ *is dense in* $\tilde{E}$. (ii) *Let* $\tilde{A}$ *be the operator defined as follows*:

$$\text{(A.1)} \qquad \left|\begin{array}{l} D(\tilde{A}) := \mathfrak{J}(E) \\ \tilde{A}(0, u)^{\sim} := -(u, 0)^{\sim} \quad ((0, u)^{\sim} \in D(\tilde{A}))\,; \end{array}\right.$$

then $\tilde{A}$ *has the same spectral properties as* A *and* $\tilde{A}\mathfrak{J} = \mathfrak{J}A$. (iii) $D(\tilde{A}^2) = \mathfrak{J}(D(A))$.

The operator $\tilde{A}$ is called the extrapolation of A to $\tilde{E}$.

According to the above results, it is natural to set by definition

$$\text{(A.2)} \qquad D_{\theta-1} := \tilde{D}_\theta \qquad (\theta \in (0, 1))\,,$$

where

$$\tilde{D}_\theta := \left\{ x \in \tilde{E} \,\Big|\, \lim_{t \to 0^+} \left| t^{-\theta}(\exp[\tilde{A}t]x - x)\right|_{\tilde{E}} = 0 \right\} \qquad (\theta \in (0, 1))\,.$$

The spaces $D_{\theta-1}$ will be referred to as the extrapolation spaces relative to the operator A (obviously, $D_{\theta-1} \subseteq \mathfrak{J}(E)$ by definition).

The extrapolation $\tilde{A}$ of A to $\tilde{E}$ having been defined, let us consider a different linear operator B in E and look for conditions which allow us to define its extrapolation $\tilde{B}$ to $\tilde{E}$; observe that the dependence of $\tilde{E}$ on the operator A makes the problem nontrivial. Sufficient conditions for extrapolating B in an important case are the content of the following theorem.

THEOREM A.2. *Let* B *denote an unbounded closed operator in* E *such that* $D(B) = D(A)$ *and* $(\lambda - B)^{-1}$ *exists (as a bounded operator in* E) *for any* $\lambda > 0$. *Assuming that*

$$\text{(A.3)} \qquad \text{both } A^{-1}B \text{ and } B^{-1}A \text{ are bounded in } E\,,$$

define the extrapolation $\tilde{B}$ *of* B *to* $\tilde{E}$ *as follows*:

$$\text{(A.4)} \qquad \left\{\begin{array}{l} D(\tilde{B}) := \mathfrak{J}(E) \\ \tilde{B}(0, u)^{\sim} := -(\overline{A^{-1}Bu}, 0)^{\sim} \quad ((0,u)^{\sim} \in D(\tilde{B})) \end{array}\right.$$

(*where* $\overline{A^{-1}B}$ *denotes the closure of* $A^{-1}B$ *in* E). *Then* $(\lambda - \tilde{B})^{-1}$ *exists* (*as a bounded operator in* $\tilde{E}$) *for any* $\lambda > 0$. *In addition,*

$$|(\lambda - B)^{-1}|_{E,E} \leqslant M_B/\lambda \qquad (\lambda > 0)$$

implies

$$|(\lambda - \tilde{B})^{-1}|_{\tilde{E},\tilde{E}} \leqslant M_B/\lambda \qquad (\lambda > 0)\,.$$

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