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Boundary value problems for harmonic and minimal maps of surfaces into manifolds

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1. – Statement of results.

Let M, g and N, h be smooth Riemannian manifolds, with M compact and possibly with boundary. Let $\varphi\colon M \to N$ be a C^∞ map. Its energy is defined by $E(\varphi) = \int_M e(\varphi)\, V_g$, where V_g is the volume element associated with g and the energy density is given by $e(\varphi) = \frac{1}{2}|d\varphi|^2$. A map φ is called harmonic iff it is an extremal of E. Such a map satisfies the equation $\tau(\varphi) = 0$, where τ is its tension (see [5] for details).

The existence problem is primarily concerned with the presence of a harmonic map in the various homotopy classes of maps from M to N. We again refer to [5] for a list of known results and a detailed bibliography.

In the present paper, we first study this existence problem in the case of maps from a surface with boundary to a manifold, for homotopy classes relative to Dirichlet and Neumann problems. M, g will denote a compact surface with boundary, which is necessarily a connected sum of tori, projective planes and half spheres, whose boundary circles C_i ($i = 1, \ldots, b$) constitute the boundary ∂M. The interior of M is denoted by M^0. We shall always suppose ∂N empty but note that W. H. Meeks and S.-T. Yau have shown how to replace such an N by a convex manifold with boundary (see [13] for definition and proof).

For the Dirichlet problem, we shall use the following

(1.1) DEFINITION. *Let φ_0 and φ_1 be two continuous maps from M to N such that $\varphi_{0|\partial M} = \varphi_{1|\partial M} = \psi$. φ_0 and φ_1 are homotopic relatively to the Dirichlet problem induced by ψ iff they are C^0 homotopic through a family φ_t with $\varphi_{t|\partial M} = \psi$. If φ_0 and φ_1 are homotopic and C^k (resp. C^k in M^0 and C^j on M, $j < k$), then*

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the homotopy can be realised by a C^k map (resp. C^k in $M^0 \times [0, 1]$ and C^j on $M \times [0, 1]$).

In other words, the homotopy classes of—say—C^k maps relative to a given ψ are the connected components of the space of C^k extensions, with respect to any C^l topology, $0 \leqslant l \leqslant k$.

Following the basic method of C. B. Morrey [14], [15, p. 378-389], we use the Sobolev class of maps $L_1^2(M, N)$, defined by means of any Riemannian embedding of N in $\boldsymbol{R}^P$ as $L_1^2(M, N) = \{\Phi \colon M \to \boldsymbol{R}^P \colon \Phi(M) \subset N$ almost everywhere and $\Phi \in L_1^2(M, \boldsymbol{R}^P)\}$.

Φ is here seen as the composition of the map $\varphi \colon M \to N$ with the embedding.

To state the first result, recall finally that a manifold is called homogeneously regular [14] iff there exist two positive constants c and C such that any point w of N is in the domain of a coordinate chart $F \colon U \to \boldsymbol{R}^n$ whose image is the unit ball and such that $F(w) = 0$ and

$$c|dF(u)X|^2 \leqslant h_u(X, X) \leqslant C|dF(u)X|^2 \quad \text{for } u \in U$$

and $X \in T_u N$. This condition is always satisfied if N is compact.

(1.2) Theorem. *Let M, g be a compact surface with boundary and N, h a homogeneously regular manifold, such that $\Pi_2(N) = 0$, where $\Pi_2(N)$ denotes the second homotopy group. Let $\psi \colon \partial M \to N$ be the restriction to ∂M of a continous and L_1^2 map from M to N. Then the Dirichlet problem*

$$\begin{aligned} \tau(\varphi) &= 0 \quad && \text{in } M^0 \\ \varphi &= \psi && \text{on } \partial M \end{aligned}$$

admits a solution φ in cvery relative homotopy class, smooth in M^0, continuous on M and minimising the energy in the class. If $\psi \in C^{s+\alpha}(\partial M, N)$ (α-Hölder continuous derivatives up to order s) with $0 \leqslant s \leqslant \infty$, then $\varphi \in C^{s+\alpha}(M, N)$.

A solution of the Dirichlet problem for maps from a plane domain to a manifold, without homotopy restriction, was obtained by C. B. Morrey [14]. The existence of harmonic elements in the homotopy classes of maps from a surface without boundary to N with $\Pi_2(N) = 0$ was proven in [11], using basically Morrey's method, and by J. Sacks and K. Uhlenbeck [16] [17] by a completely different approach. [19] provides a variation of the proof of [11] which will be used here.

(1.3) Remark. The hypothesis $\Pi_2(N) = 0$ cannot be simply omitted.

Indeed, [10] and [11] give an example of a homotopy class relative to a Dirichlet problem for maps from the 2-disk to the 2-sphere without harmonic representative. For a list of some manifolds satisfying $\Pi_2(N) = 0$, see [5, (10.9)].

Consider now the Neumann problem defined by $\partial_\nu \varphi = 0$ along ∂M, where ∂_ν is the normal derivative. If φ_0 and φ_1 satisfy this condition and are freely homotopic, then they are also relatively homotopic in the sense that they are homotopic through a family φ_t with $\partial_\nu \varphi_t = 0$ along ∂M. Indeed, it suffices to modify the parametrisation of the maps of the homotopy. We state:

(1.4) THEOREM. *Let M, g be a compact surface with boundary and N, h a compact manifold with $\Pi_2(N) = 0$. Any homotopy class of maps from M to N contains a smooth solution of the Neumann problem $\tau(\varphi) = 0$, $\partial_\nu \varphi_{|\partial M} = 0$, minimising E in the class.*

Recall that by transport of the loops, any continuous map $\varphi\colon M \to N$ induces on the first homotopy groups a conjugacy class of homomorphisms $[\varphi_*]$, the conjugacy coming from the changes of base point. The preceding result is a corollary of:

(1.5) THEOREM. *Let M, g be a surface with boundary and N, h a compact manifold. For any conjugacy class of homomorphisms $\gamma\colon \Pi_1(M) \to \Pi_1(N)$, there is a solution φ of the Neumann problem such that $[\varphi_*] = \gamma$.*

REMARK. We shall show by means of examples that a homotopy class relative to the Dirichlet or Neumann problem can contain more than one harmonic map. However, if the sectional curvature of N is negative, the harmonic maps are essentially unique in their classes.

Classically, the existence of harmonic maps is used to obtain minimal surfaces i.e. (in this framework) maps from a surface M to N which minimise or extremise the area

$$A(\varphi) = \int \left| \frac{\det \varphi^* h}{\det g} \right|^{\frac{1}{2}} V_g \, .$$

Here, the integral is independent of the choice of g, which will in general not be specified a priori.

Without going back to the history of the method (see e.g. [3] and [15]), we note that it was applied by C. B. Morrey to maps from a plane domain to N [14], and by J. Sacks and K. Uhlenbeck [18] and R. Schoen and S.-T. Yau [19] to maps of closed surfaces.

Recall [14, chap. 1] [15, (9.1)] that for two C^0 paths $\psi_0\colon I_0 \to N$, $\psi_1\colon I_1 \to N$ (where I_0 and I_1 are intervals), the distance between ψ_0 and ψ_1 is defined as the infimum over all homeomorphisms $H: I_1 \to I_0$ of $\max_{I_1} \operatorname{dist} \big(\psi_1(t) - \psi_0(H(t))\big)$. A curve in the sense of Fréchet is an equivalence class of paths at zero distance from each other. One can define the distance between two such curves as the distance defined above for any of their parametrisations, and if a sequence of curves converges to a curve in this sense, there exists a sequence of paths parametrising them converging uniformly to a parametrisation of the limit.

A closed simple curve is called a Jordan curve.

(1.6) DEFINITION. *Let Γ_i be b disjoint Jordan curves in N and φ_0, φ_1: $M \to N$ two continuous maps mapping C_i in a monotone way on Γ_i. We say that φ_0 and φ_1 are relatively homotopic if they are homotopic through a family φ_t with $\varphi_{t|C_i}$ monotone with image in Γ_i.*

Again, homotopic maps with higher differentiability are homotopic through maps with the same differentiability.

(1.7) THEOREM. *Let M be a compact orientable surface with boundary $C_1 \cup \ldots \cup C_b$ and N, h a homogeneously regular manifold such that $\Pi_2(N) = 0$. Let $\Gamma_1, \ldots, \Gamma_b$ be disjoint Jordan curves in N such that there exists a continuous map ψ: $M \to N$ of finite energy, mapping each C_i monotonically on Γ_i and such that $[\psi_*]: \Pi_1(M) \to \Pi_1(N)$ is injective. Then ψ is relatively homotopic to a minimal map φ, smooth in M^0 and continuous on M, mapping C_i on Γ_i in a monotone way and minimising the area among all such maps. If each Γ_i is a $C^{s+\alpha}$ curve $(2 \leqslant s \leqslant \infty)$, then $\varphi \in C^{s+\alpha}(M, N)$.*

(1.8) REMARK. The hypothesis that $[\psi_*]$ be injective is used to prevent standard degeneracies of the surface when its area decreases to an infimum, like a change of genus or the splitting into more than one connected components. The hypothesis $d < d^*$ of [3] for maps from a surface to R^n or of [14] for maps of plane domains to N has the same purpose, but is of a different nature since it restricts the possible types of surfaces M in relation with the given Γ_i's. Note also that our hypothesis excludes the case $N = \boldsymbol{R}^n$.

Theorem (1.7) is related to the Dirichlet problem for harmonic maps. One might wonder whether minimal maps could also be solutions of the Neumann problem. We shall easily observe that it is not so. Indeed:

(1.9) PROPOSITION. *Let φ: $M \to N$ be a minimal map such that $\partial_\nu \varphi = 0$ along an arc σ of ∂M (σ not reduced to a point). Then φ is constant.*

It is a pleasure to record my thanks to J. Eells and V. L. Hansen for useful conversations, in particular during the 1980 Complex Analysis Seminar at the I. C. T. P. (Trieste).

2. – The Dirichlet problem for harmonic maps.

We first justify the differentiability statements made in the definition of relative homotopy (1.1).

In the case of manifolds without boundary, it is shown in [4] chapter 16, section 26, that two C^k maps which are C^0 homotopic are in fact C^k homotopic.

In the case of definition (1.1), the proof shows that for $\varepsilon > 0$ small enough, two C^k maps φ_0 and φ_1 homotopic relatively to their boundary restriction ψ are C^k homotopic through maps φ_t such that dist $(\varphi_t(x), \psi(x)) < \varepsilon$ for $x \in \partial M$. Using the techniques of [4, (16.26.3 and 4)], one can then C^k deform the maps φ_t in a tubular neighbourhood of ∂M to make them coincide with ψ on ∂M.

When considering two maps φ_0 and φ_1 which are C^k in M^0 and C^j on M, one can first C^k deform φ_1 in such a way that it coincides with φ_0 in a tubular neighbourhood of ∂M (using a bump function and [4, (16.26.4)]). The problem is then reduced to that of C^k maps on a slightly smaller manifold with boundary

We now turn to the proof of theorem (1.2).

(2.1) LEMMA. *Let M be a compact surface with boundary, N a manifold with $\Pi_2(N) = 0$ and φ_0 and φ_1 two continuous maps from M to N equal on ∂M. If for any path β in M with endpoints on ∂M, $\varphi_0(\beta)$ is relatively homotopic to $\varphi_1(\beta)$ (i.e. homotopic with endpoints fixed), then φ_0 and φ_1 are relatively homotopic (as in definition* (1.1)).

In fact, we shall establish a more technically stated result:

(2.2) LEMMA. *With the same hypothesis on M and N, fix a point $P \in C_1$. The maps φ_0 and φ_1, equal on ∂M, are relatively homotopic if a) $\varphi_{0*} = \varphi_{1*}$: $\Pi_1(M, P) \to \Pi_1(N, \varphi_0(P))$ (we don't have conjugacy classes here since we can fix a base point P with $\varphi_0(P) = \varphi_1(P)$), and*

b) for a set of non-intersecting paths $\beta_i (2 \leqslant i \leqslant b)$ from P to points of the C_i's, $\varphi_0(\beta_i)$ and $\varphi_1(\beta_i)$ are relatively homotopic.

When $\partial M = \emptyset$, this reduces to a well-know property (see e.g. [20, proof of theorem 11, chapter 8, section 1]).

PROOF OF LEMMA (2.2). It is well known that M can be represented by a region R bounded by a polygon, with some sides identified two by two, the others corresponding to the boundary curves (see e.g. [21, chap. 5]).

In the present situation, we can write the surface symbol (list of sides of the polygon, with identification of pairs of sides with the same name) as

$$C_1\beta_2 C_2\beta_2^{-1} \dots \beta_b C_b\beta_b^{-1} a_1 a_1 \dots a_k a_k b_1 c_1 b_1^{-1} c_1^{-1} \dots b_p c_p b_p^{-1} c_p^{-1},$$

so that all vertices except those adjacent to C_i $(2 \leqslant i \leqslant b)$ correspond to the point P, the curves β_i provide the slits from P to C_i, the a_j's represent the cross caps and the b_j, c_j's the tori.

Still denoting by φ_0 and φ_1 the maps induced from R to N, we have $\varphi_0(P) = \varphi_1(P)$ and $\varphi_{0|C_i} = \varphi_{1|C_i}$. Moreover, φ_0 and φ_1 restricted to any side of the polygon ∂R are homotopic with fixed endpoints.

We can then deform continuously φ_1 to make it equal to φ_0 on ∂R. Indeed, in a tubular neighbourhood T of ∂R (with corners, which don't matter since we work in the C^0 framework), we can first change the parametrisation of φ_1 in such a way that the half of T closer to ∂R (say S) has image on $\varphi_1(\partial R)$. The homotopy from φ_1 to φ_0 defined on ∂R can then be used to define a map from S to N, connecting $\varphi_{1|\partial R}$ and $\varphi_{0|\partial R}$.

φ_0 and φ_1 are then equal on the one-skeleton of a cellular decomposition with only one 2-cell. By a result of W. D. Barcus and M. G. Barratt [2], used in its simplest case $\Pi_2(N) = 0$, there is only one homotopy class of extensions of such maps to the 2-cell, so that φ_0 and φ_1 are relatively homotopic.

Proof of theorem (1.2). We shall use a direct method of the calculus of variations, in the framework of the space $L_1^2(M, N)$.

Recall [15, lemma 9.4.10.d] that if $\varphi \in L_1^2(M, N)$, in any chart (x, y) of M, $\varphi_{|x=x_0}$ is continuous in y except maybe for x_0 in a set of measure zero.

R. Schoen and S.-T. Yau have shown in [19] that the image of a closed contractible curve of M by φ is contractible if this image is continuous. They use this to define the action (which will mean here: up to homotopy) of an L_1^2 map on a closed curve α by the common image (up to homotopy) of almost all curves parallel to α in a tubular neighbourhood.

Since our maps are fixed on ∂M, this defines also an action of φ on any of the curves β_i that we consider, as the common homotopy class of images of curves composed of a segment of C_1, a curve parallel to β_i and a segment of C_i, the segments being necessary to build curves with same endpoints.

The proof follows then as in [14] and [19]: consider a minimising sequence for E in the class of L_1^2 maps with value ψ on ∂M and inducing on loops α_j and paths β_i the action induced by the given homotopy class. By [15, lemmas 9.4.14-15-16], a subsequence φ_k converges in $L_1^2(M, N)$ to a map φ, equal to ψ on ∂M, with $E(\varphi) \leqslant \underline{\lim}\, E(\varphi_k)$. Moreover, on almost each curve (in the sense of measure as above), a subsequence converges uniformly

[12, proof of lemma 1.4] and φ induces therefore the same action on α_i and β_i. Hence, it minimises the energy in every small disk of M^0 (for the Dirichlet problem induced by its trace on the boundary of the disk), and is C^∞ in M^0 and C^0 on M, by the regularity results of C. B. Morrey [14]. By lemma (2.2), it is in the given homotopy class.

The regularity results along ∂M are local and were obtained in [9] (see [8] for complete statements).

(2.3) Remark. As in theorem (1.5) for the Neumann problem, we could state an exitence theorem for $\Pi_2(N) \neq 0$, the homotopy classes being replaced either by the data of $[\varphi_*]$, or by that of the action on $\Pi_1(M)$ and the curves β.

3. – The Neumann problem for harmonic maps.

The proof of theorem (1.5) now reduces to known ingredients. We write it in a way avoiding any use of boundary estimates.

Consider the class of L_1^2 maps from M to N inducing the given class of homomorphisms γ on Π_1, without any restriction on the boundary. N being compact, a minimising sequence for E in that class is a bounded set in $L_1^2(M, N)$ and we obtain as above a minimum φ of the energy, smooth and harmonic in M^0.

Consider now any point $P \in \partial M$ and a coordinate (half) disk around P. By a conformal change of metric, we can see it as a flat half disk $D^+ = \{(x, y): x^2 + y^2 \leqslant 1,\ x \geqslant 0\}$, in such a way that $D^+ \cap \partial M = D^+ \cap \{x = 0\}$ and $P = (0, 0)$.

By conformal invariance of the energy [5, (10.2)], φ minimises E (for the flat metric) among all maps coinciding with φ on $D^+ \cap \{x^2 + y^2 = 1\}$.

We then extend φ to a map $\tilde\varphi$ defined on the whole unit disk D in $\boldsymbol{R}^2$ by setting $\tilde\varphi = \varphi$ on D^+ and $\tilde\varphi(x, y) = \varphi(-x, y)$ on $D \setminus D^+$. By standard patching properties, $\tilde\varphi \in L_1^2(D, N)$.

Moreover, $\tilde\varphi$ minimises E among all maps ψ such that $\psi_{|\partial D} = \tilde\varphi_{|\partial D}$. Indeed, if it didn't, there would exist a map ψ with $\psi_{|\partial D} = \tilde\varphi_{|\partial D}$ and $E(\psi) < E(\tilde\varphi)$. The same would be true for ψ restricted to D^+ or to $D \setminus D^+$, say to the former. The map equal to φ on $M \setminus D^+$ and ψ on D^+ would then be in the given class of $L_1^2(M, N)$ and have energy smaller than the minimum—a contradiction.

$\tilde\varphi$ is therefore smooth and harmonic in D^0. Since the normal derivative is the same along ∂M for the original metric or the conformally equivalent one (the angles and the tangent being preserved), and since $\tilde\varphi(x, y) = \tilde\varphi(-x, y)$ we get $\partial_\nu \tilde\varphi = 0$ along the boundary and φ is a solution of the Neumann problem.

If $\Pi_2(N) = 0$, the homotopy classes are in bijective correspondence with the conjugacy classes of homomorphisms on Π_1 and theorem (1.5) implies theorem (1.4).

REMARK. The fact that $\partial_\nu \varphi_{|\partial M} = 0$ for the minimising map can also be deduced from the variation formula [5, (12.11)]

$$\nabla_\chi E(\varphi) = -\int_M \langle \tau(\varphi), \chi \rangle V_g + \int_{\partial M} \langle \partial_\nu \varphi, \chi \rangle V_{g|\partial M}$$

since χ is not restricted in any way on ∂M.

4. – On the unicity of harmonic maps.

We first show by an example that the solutions of the Dirichlet and Neumann problems need not be unique in their relative homotopy classes. By using a somewhat artificial metric, we obtain in fact a continuous family of solutions.

Let M be the cylinder $[0, \Pi] \times \mathbf{R}/2\Pi\mathbf{Z}$ endowed with its flat metric and Euclidean coordinates (x, y). For $0 < b < a$, denote by $N_{a,b}$ the cylinder $[-a + b, a - b] \times \mathbf{R}/2\Pi\mathbf{Z}$ equipped in coordinates (u, v) with the metric

$$h = \begin{pmatrix} 1 & 0 \\ 0 & a^2 - u^2 \end{pmatrix}$$

and consider $N_{a,b}$ as a band isometrically embedded in a torus N.

As noted in [12, § 3], a map $\varphi \colon M \to N_{a,b}$ of the form $\varphi(x, y) = (u, v) = (F(x), y)$ is harmonic iff $d^2F/dx^2 + F = 0$ and the solutions are of course given by $\varphi(x, y) = (c \cdot \cos(x + d), y)$ where c takes any value in $[-a + b, a - b]$.

For $d = -\Pi/2$, these solutions satisfy $\varphi(0, y) = \varphi(\Pi, y) = (0, y)$, so that they are all solutions of the same Dirichlet problem.

For $d = 0$, they are all solutions of the Neumann problem.

In both cases, the hypothesis of the existence theorems (1.2) or (1.4) are satisfied.

On the other hand, we note the

(4.1) PROPOSITION. If the sectional curvature of N is non-positive, the Dirichlet problem of theorem (1.2) admits only one solution in each homotopy class. If the curvatures of N is strictly negative, the Neumann problem of theorem (1.4) admits only one solution with image not reduced to a point or a geodesic in each homotopy class.

As noted in [5], the first statement follows immediately from the proof of the corresponding theorem of P. Hartman in the boundaryless case [6]. For the Neumann problem, the statement can be reduced to the boundaryless case by doubling the surface M and defining an extension of the map on the double as in § 3.

5. – Minimal surfaces of given boundary.

Since the energy is invariant by conformal transformations of the surface M, it suffices to define on M a conformal—or a complex—structure in order to define its harmonic maps. To obtain a minimal map, we shall use the following version of a classical result, which can be proven exactly as in the boundaryless case [17].

(5.1) LEMMA. *Let $\varphi\colon M \to N$, h be a map extremising the energy for all variations of the conformal structure on the surface M and all deformations of the map in a relative homotopy class (in the sense of def.* (1.6)). *Then φ is harmonic and conformal in M^0 and is therefore a minimal branched immersion.*

To get such an extremal, we shall use three more lemmas.

Analogously to lemma (2.2), we have:

(5.2) LEMMA. *Let φ_0 and $\varphi_1\colon M \to N$ be two continuous maps, mapping each C_i monotonically on Γ_i, such that φ_1 restricted to the one-skeleton of a cellular decomposition is homotopic to φ_0 restricted to the same space, in such a way that along the homotopy $\varphi_t(C_i) \subset \Gamma_i$. Then φ_0 and φ_1 are relatively homotopic in the sense of definition* (1.6).

From now on, we shall suppose that the genus p of M (number of tori in the connected sum decomposition) is greater than zero or that $b \geqslant 3$. The case $p = 0$, $b = 1$ is that of a disk, and existence in the unique homotopy class (since $\Pi_2(N) = 0$) is classical. The case $p = 0$, $b = 2$ can be proven as those that we consider, by using a flat torus instead of the closed surfaces here below.

(5.3) LEMMA. *Any orientable surface with boundary, with $p \geqslant 1$ or $b \geqslant 3$, equipped with a conformal structure can be doubled as a closed Riemann surface $\tilde{M}$, endowed with a conformal involution i interchanging M and the closure* *of $\tilde{M} \setminus M$ and whose fixed points constitute ∂M. When $\tilde{M}$ is equipped with its Poincaré metric (unique metric of curvature -1 compatible with the conformal structure), i is an isometry and the C_i's are closed geodesics.*

See e.g. [1, chap. II, § 1.2].

(5.4) LEMMA. *Let N be a homogeneously regular Riemannian manifold and Γ_i $(1 \leqslant i \leqslant b)$ a family of disjoint Jordan curves in N. There is a strictly positive number K smaller than or equal to the following quantities:*

1) *the length of all non-homotopically trivial curves;*

2) *the length of all curves with an endpoint on Γ_i and the other on Γ_j, for $i \neq j$;*

3) *the length of all curves with endpoints on Γ_j and not homotopic to a point through curves with endpoints on Γ_j:*

Indeed, since N is homogeneously regular, any non-homotopically trivial curve has length greater than or equal to $2\sqrt{c}$, and in the two other classes of curves described in the lemma, a direct method of the calculus of variations yields a minimising geodesic of positive length.

PROOF OF THEOREM (1.7). Consider a minimising sequence (t_k, φ_k) for E, where each t_k is a conformal structure on M and φ_k a harmonic map with respect to t_k, such that $\varphi_{k|C_i}$ converge as Fréchet curves to Γ_i, and belonging to the given relative homotopy class, up to a small deformation in a neighbourhood of ∂M, bringing the image of C_i by φ_k on Γ_i.

We want to show that both (t_k) and (φ_k) admit converging subsequences. By continuity of E with respect to (t_k) [18, lemma 4.2] and lower semicontinuity of E with respect to (φ_k), the limit will realise a minimum of E.

For each k, consider first the double of M, t_k with its conformal structure $\tilde{t}_k$ and Poincaré metric as in lemma (5.3). Define an L_1^2 map $\tilde{\varphi}_k$ on $\tilde{M}$ by $\tilde{\varphi}_k = \varphi_k$ on M and $\tilde{\varphi}_k = \varphi_k \circ i$ on $\tilde{M} \setminus M$. $\tilde{\varphi}_k \in L_1^2(\tilde{M}, N)$ and $E(\tilde{\varphi}_k)$ is bounded independently of k.

By $\tilde{\varphi}_k$, the image of almost all non-homotopically trivial curve is continuous, L_2^1 and of length bounded below by $K > 0$. Indeed, observe that any such curve in $\tilde{M}$ has one of the following properties:

1) it is contained and non-homotopically trivial in M^0 or $\tilde{M} \setminus M$,

2) it intersects two C_i's,

3) it is homotopically equivalent to a multiple of C_i,

4) it contains an arc with endpoints on C_i, non homotopic to a point through arcs with endpoints on C_i.

In each case, the injectivity of $[\varphi_{k*}]$ and lemma (5.4) imply that the length of the image of the curve is $\geqslant K$.

From lemma 3.1 and the proof of theorem 3.1 of [19], it follows that the sequence of conformal structures $\tilde{t}_k$ is contained in a compact subset of the moduli space of $\tilde{M}$, so that a subsequence converges to a conformal structure on $\tilde{M}$, inducing one on M by restriction.

From now on, we shall calculate the energies with respect to this limit t, which, by compacity, will simply modify the inequalities in consideration by fixed constants.

The sequence (φ_k) is a bounded set in $L_1^2(M, N)$, so that a subsequence converges weakly to an L_1^2 limit φ, with $E(\varphi) \leqslant \underline{\lim}\, E(\varphi_k)$, and such that $\varphi(C_i) \subset \Gamma_i$. On almost each curve, a subsequence converges uniformly.

Following the proofs of lemmas 9.3.2 and 9.3.3 of [15], we shall show that the restriction of the maps to each C_i converge uniformly to a continuous and monotone limit.

For a boundary curve $C_i \equiv C$ parametrised by a coordinate $\eta + 2\Pi k$, and a Jordan curve $\Gamma_i \equiv \Gamma$, there exist by the properties recalled in § 1 two sequences of maps $f_k\colon \boldsymbol{R} \to \boldsymbol{R}$ and $F_k\colon \boldsymbol{R} \to N$ such that $\varphi_{k|C}(\eta) = F_k \circ f_k(\eta)$, (F_k) is a sequence of continuous periodic maps converging uniformly to F, (f_k) a sequence of monotone continuous functions such that $f_k(\eta) - \eta$ is periodic of period 2Π and a subsequence (f_k) converges in L^2 and almost everywhere to a function f in such a way that $\varphi_{|C} = F \circ f$.

We first show that f is continuous.

Suppose, on the contrary, that for a value of η, $f^+(\eta) \neq f^-(\eta)$ (where f^+ and f^- denote right and left limits).

Let D_r denote the disk of radius r centred at η, γ_r' its boundary in M and γ_r'' the arc $C \setminus D_r$. For any φ_k and a.e. r small enough, $\varphi_k(\gamma_r' \cup \gamma_r'')$ has length $\geqslant K$ since it is homotopically non-trivial in N.

If $f^+(\eta) - f^-(\eta) \neq 2\Pi$, $Ff^+(\eta) \neq Ff^-(\eta)$ and for suitable r, the length of arcs $\varphi_k(\gamma_r')$ must tend to a positive value.

If $f^+(\eta) - f^-(\eta) = 2\Pi$, $Ff^+(\eta) = Ff^-(\eta)$, but as the length of $\varphi_k(\gamma_r'')$ tends to zero, the length of $\varphi_k(\gamma_r')$ tends again to a positive value. More precisely, for any $\delta > 0$ small enough, there is an integer k_0 such that for $k \geqslant k_0$, the length of $\varphi_k(\gamma_r')$ is greater than $d > 0$ for a.e. r such that $\delta^2 < r < \delta$. Therefore, in polar coordinates (r, θ) in D_δ, we have, using the flat metric:

$$E(\varphi_k) \geqslant \int_{\delta^2}^{\delta} r^{-1} \int_{\gamma_r'} \left| \frac{\partial \varphi}{\partial \theta} \right|^2 d\theta\, dr \geqslant \int_{\delta^2}^{\delta} r^{-1} \frac{1}{2\Pi} \left(\int_{\gamma_r'} \left| \frac{\partial \varphi}{\partial \theta} \right| d\theta \right)^2 dr \geqslant \frac{d^2}{2\Pi} \log \frac{1}{\delta}\,.$$

Since $E(\varphi_k)$ is bounded, this is impossible for $\delta \to 0$ and the limit f is continuous.

The sequence (f_k) of continuous and monotone functions converges a.e. to a continuous and monotone function f. This convergence must be uniform.

Indeed, $\forall \varepsilon > 0$, $\exists \delta$ such that $0 < \mu < \delta$ implies $0 \leqslant f(\eta + \mu) - f(\eta) < \varepsilon$. We can then choose $[4\Pi/\delta]$ numbers η_i in an interval containing $[0, 2\Pi]$ (in which we want to show convergence) such that $\eta_i < \eta_{i+1} < \eta_i + \delta$ and $f_k(\eta_i) \to f(\eta_i)$. For the given ε, there is therefore a number k_0 such that $k \geqslant k_0$ implies $|f_k(\eta_i) - f(\eta_i)| < \varepsilon$. Then, for any η, say such that $\eta_i < \eta < < \eta_{i+1}$, we have

$$\begin{aligned} f_k(\eta) - f(\eta) &\leqslant f_k(\eta_{i+1}) - f(\eta_i) \\ &= f_k(\eta_{i+1}) - f(\eta_{i+1}) + f(\eta_{i+1}) - f(\eta_i) < 2\varepsilon \end{aligned}$$

and similarly for $f(\eta) - f_k(\eta)$.

The sequence (φ_k) converges therefore uniformly on each C_i. With the interior convergence as above, this implies that the characterising properties of lemma (5.2) are preserved at the limit φ. As φ minimises E among all maps satisfying those properties, it is smooth in M^0 and continuous on M. By lemma (5.2), it belongs to the given homotopy class.

The regularity results along ∂M are due to E. Heinz and S. Hildebrandt [7].

To prove that φ minimises the area, we can essentially follow [15] or [19]. Let $e: M \to \boldsymbol{R}^Q$ be a Riemannian embedding and εe its composition with the multiplication by ε in $\boldsymbol{R}^Q$.

Suppose that there is a map ψ, relatively homotopic to φ with $A(\psi) < A(\varphi)$.

The map $\Psi \oplus \varepsilon e: M \to N \times \boldsymbol{R}^Q \hookrightarrow \boldsymbol{R}^P \times \boldsymbol{R}^Q$ is an embedding for $\varepsilon > 0$, and for the induced conformal structure $\bar{t}$ on M we have $E_{\bar{t}}(\Psi \oplus \varepsilon e) = = A(\Psi \oplus \varepsilon e)$ [5, (10.3)]. For any $\delta < A(\varphi) - A(\psi)$, we can choose ε such that $A(\Psi \oplus \varepsilon e) \leqslant A(\psi) + \delta$ so that $E_{\bar{t}}(\psi) \leqslant E_{\bar{t}}(\Psi \oplus \varepsilon e) \leqslant A(\psi) + \delta < A(\varphi) = = E_t(\varphi)$, contradicting the minimising property of φ for the energy.

(5.5) REMARK. If $\Pi_2(N) \neq 0$, the hypothesis of lemma (5.2) describe classes of maps containing a minimal surface, as in remark (2.3).

6. – Minimal surfaces and the Neumann problem.

Suppose that $\varphi: M \to N$ is smooth, harmonic, conformal and satisfies $\partial_\nu \varphi_{|\sigma} = 0$. Then, along σ, the tangential derivative is zero by conformality and φ is constant. Steps 2 and 3 of the proof of theorem (3.2) of [11] show then precisely that φ must be constant, as successive derivations of the conformality conditions show that all its derivatives vanish at a point.

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