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Continuity of Solutions of the Porous Media Equation.

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1. – Introduction.

In a recent paper [3] Aronson and Peletier studied the large time behaviour of solutions of the Dirichlet problem for the porous media equation in an arbitrary number of space dimensions. To be explicit they considered the following problem. Let Ω denote an open bounded connected domain in $\boldsymbol{R}^N$ $(N\geqslant 1)$ and let $u_0 \in C(\bar{\Omega})$ be a given nonnegative function. Then solve the porous media equation

$$(1) \qquad u_t = \Delta(u^m) \quad \text{in } \Omega\times(0,\infty),$$

where $m>1$ is a fixed real number, subject to the boundary conditions

$$(2) \qquad u(x,0) = u_0(x) \quad \text{for } x\in\bar{\Omega},$$

$$(3) \qquad u(x,t) = 0 \quad \text{for } (x,t)\in\partial\Omega\times(0,\infty).$$

The authors proved that problem (1)-(3) has a unique solution defined in some generalized sense if $u_0^m \in C^1(\bar{\Omega})$ and $u_0 = 0$ on $\partial\Omega$, and proceeded to establish the asymptotic behaviour of this solution. As however the latter problem was their main area of interest the authors did not dwell on the regularity of their generalized solution. They conjectured, in fact, that such solutions were continuous, but did not prove so. In this note we shall prove this conjecture. In addition we shall establish the existence of a generalized solution if only $u_0 \in C(\bar{\Omega})$ and $u_0 = 0$ on $\partial\Omega$.

Throughout the remainder of this note we shall adhere to the following notation, $Q = \Omega\times(0,\infty)$, $S = \partial\Omega\times(0,\infty)$ and $\Gamma = \partial Q$.

It is now well established [8] that problem (1)-(3) admits solutions only

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in some generalized sense. Following Aronson and Peletier [3] we therefore introduce the notion of a weak solution.

DEFINITION. *A nonnegative function* $u(x, t)$ *defined in* Q *is said to be a weak solution of problem* (1)-(3) *if*:

(i) $\nabla(u^m)$ *exists in the sense of distributions in* Q *and*

$$\int_{Q'} \{u^{2m} + |\nabla(u^m)|^2\} dx\, dt < \infty \tag{4}$$

for any bounded measurable subset Q' *of* Q;

(ii) u *satisfies the identity*

$$\int_{Q} \{\nabla\varphi \cdot \nabla(u^m) - \varphi_t u\} dx\, dt = \int_{\Omega} \varphi(x, 0) u_0(x)\, dx \tag{5}$$

for any function $\varphi \in C^1(\bar{Q})$ *which vanishes on* S *and for large* t;

(iii) *given any point* $(x_0, t_0) \in S$

$$\limsup_{\substack{(x,t)\to(x_0,t_0)\\(x,t)\in Q}} u(x, t) = 0\,. \tag{6}$$

Clearly any classical solution of problem (1)-(3) satisfying (4) is a weak solution.

In the following two sections of this note we shall discuss the uniqueness and existence, respectively, of weak solutions of problem (1)-(3), and in the final section the regularity. The reader interested in rather more general aspects and « the state of the art » of problems involving the porous media equation is referred to the recent survey paper [8].

2. – Uniqueness.

THEOREM 1. *Problem* (1)-(3) *has at most one weak solution.*

PROOF. As stated in [3] the proof of this theorem is a direct extension of the proof for the one-dimensional case given in [7].

Suppose that there are two such solutions u_1 and u_2. Then by (5)

$$\int_{Q} \{\nabla\varphi \cdot \nabla(u_1^m - u_2^m) - \varphi_t(u_1 - u_2)\} dx\, dt = 0 \tag{7}$$

for all $\varphi \in C^1(\bar{Q})$ which vanish on S and for large t. It follows that (7) continues to hold for all functions φ which vanish on S and for large t such that φ_t and $\nabla\varphi$ exist in the sense of distributions in Q and

$$\int_Q \{\varphi_t^2 + |\nabla\varphi|^2\}\,dx\,dt < \infty .$$

In particular (7) continues to hold for the test function η defined for any $\tau > 0$ by

$$\eta(x, t) = \begin{cases} \int_t^\tau \{u_1^m(x, s) - u_2^m(x, s)\}\,ds & \text{for } 0 \leqslant t < \tau \\ 0 & \text{for } t \geqslant \tau . \end{cases}$$

Substituting η in (7) yields

$$\int_0^\tau \int_\Omega (u_1^m - u_2^m)(u_1 - u_2)\,dx\,dt + \frac{1}{2}\int_\Omega |\nabla\eta|^2(x, 0)\,dx = 0 .$$

Hence $u_1 = u_2$ almost everywhere in Q.

3. – Existence.

As in [3] we shall establish conditions under which we may construct a weak solution of problem (1)-(3) as the limit of a decreasing sequence of classical solutions of (1) in Q. To do this we shall assume that $\partial\Omega \in C^{2+\alpha}$ for some $\alpha \in (0, 1)$, but, rather than assuming that $u_0^m \in C^1(\bar{\Omega})$ and $u_0(x) = 0$ for $x \in \partial\Omega$, as was done in [3], we shall merely assume that $u_0 \in C(\bar{\Omega})$ and $u_0(x) = 0$ for $x \in \partial\Omega$. Thus our results generalize those of [3].

To begin, we follow [3]. We construct a sequence of functions $u_{0p} \in C^\infty(\bar{\Omega})$, $p \geqslant 1$, with the following properties:

(8) $$u_0(x) + \frac{1}{2p} \leqslant u_{0p}(x) \leqslant u_0(x) + \frac{3}{2p} \quad \text{for all } x \in \bar{\Omega} ;$$

(9) $$u_{0p+1}(x) < u_{0p}(x) \quad \text{for all } x \in \bar{\Omega} ;$$

(10) $$u_{0p}(x) = \frac{1}{p}, \quad \Delta u_{0p}(x) = 0 \quad \text{for all } x \in \partial\Omega .$$

To show that such a sequence exists, we must first introduce some notation. For $\delta > 0$ let $D_\delta = \{x \in \boldsymbol{R}^N\colon |x - x_0| < \delta$ for some $x_0 \in \partial\Omega\}$. Define the function

$$J(x) = \begin{cases} \exp\{1/(|x|^2 - 1)\} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geqslant 1\,, \end{cases}$$

and for $\varepsilon > 0$ set

$$J_\varepsilon(x) = \varepsilon^{-N} J(x/\varepsilon) \Big/ \int_{\boldsymbol{R}^N} J(x)\, dx\,.$$

Now for fixed $p \geqslant 1$, since $u_0 \in C(\bar{\Omega})$, $u_0(x) = 0$ for $x \in \partial\Omega$ and $\partial\Omega$ is compact, there exists a $\delta_p > 0$ such that

$$u_0(x) < \frac{1}{3p} \qquad \text{for all } x \in \bar{\Omega} \cap D_{\delta_p}.$$

Hence if we define the function

$$v_p(x) = \begin{cases} \max\left\{u_0(x), \dfrac{1}{3p}\right\} + \dfrac{2}{3p} & \text{for } x \in \Omega \\ \dfrac{1}{p} & \text{for } x \notin \Omega\,, \end{cases}$$

we observe that v_p is continuous in $\boldsymbol{R}^N$ and

$$\frac{2}{3p} \leqslant v_p(x) - u_0(x) \leqslant \frac{1}{p} \qquad \text{for all } x \in \Omega$$

$$v_p(x) = \frac{1}{p} \qquad \text{for all } x \in D_{\delta_p}.$$

We then set

$$u_{0p}(x) = \int_{\boldsymbol{R}^N} J_{\varepsilon_p}(x - y)\, v_p(y)\, dy\,,$$

where ε_p is chosen so small that $0 < \varepsilon_p < \delta_p$ and

$$|u_{0p}(x) - v_p(x)| < \frac{1}{3p(p+1)} \qquad \text{for all } x \in \boldsymbol{R}^N.$$

It is easily verified, in view of the fact that $\varepsilon_p < \delta_p$ and the standard properties of mollifiers, that the functions u_{0p} so defined satisfy (8)-(10), cf. [3].

Consider now the problem

$$(11)\qquad (u_p)_t = \Delta(u_p^m) \quad \text{in } Q\,,$$

$$(12)\qquad u_p(x,0) = u_{0p}(x) \quad \text{for } x\in\bar{\Omega}\,,$$

$$(13)\qquad u_p(x,t) = \frac{1}{p} \quad \text{for } (x,t)\in S\,.$$

Since $\partial\Omega\in C^{2+\alpha}$, $\alpha\in(0,1)$ and the compatibility condition (10) is satisfied for each $p\geqslant 1$, problem (11)-(13) has a unique solution $u_p\in C^{2,1}(\bar{Q})$ [6, p. 452]. Moreover, if

$$M = \sup_{\Omega} u_0\,,$$

by the standard maximum principle

$$(14)\qquad \frac{1}{2p}\leqslant u_p(x,t)\leqslant M_p = M+\frac{3}{2p} \quad \text{for } (x,t)\in\bar{Q}$$

and

$$u_{p+1}(x,t)\leqslant u_p(x,t) \quad \text{for } (x,t)\in\bar{Q}$$

for all $p\geqslant 1$. Hence we may define the function

$$(15)\qquad u(x,t) = \lim_{p\to\infty} u_p(x,t) \quad \text{in } \bar{Q}\,.$$

We assert that the function u defined in (15) is the weak solution we seek. To show that this is indeed the case we first observe that by (14), $0\leqslant u\leqslant M$ in $\bar{Q}$. Moreover, multiplying (11) by $z_p = u_p^m - p^{-m}$ and integrating by parts over $\Omega\times(0,\tau]$, for some $\tau>0$, one obtains

$$(16)\qquad \int_0^\tau\int_\Omega |\nabla(u_p^m)|^2\,dx\,dt = \int_\Omega\left\{\frac{1}{m+1}\,u_{0p}^m(x) - p^{-m}\right\}u_{0p}(x)\,dx\,-$$
$$-\int_\Omega\left\{\frac{1}{m+1}\,u_p^m(x,\tau)-p^{-m}\right\}u_p(x,\tau)\,dx\leqslant\frac{1}{m+1}\int_\Omega\left\{u_0(x)+\frac{3}{2p}\right\}^{m+1}dx\,+$$
$$+\,p^{-m}M_p\int_\Omega dx\leqslant\left\{\frac{1}{m+1}\,(M+2)^{m+1}+M+2\right\}\int_\Omega dx\,.$$

Thus since τ was arbitrary, $\nabla(u_p^m)$ is uniformly bounded in $L^2(Q)$. It follows that $\{\nabla(u_p^m)\}$ has a weakly convergent subsequence in $L^2(Q)$, tending to some limit ψ.

By a standard argument [3] $\psi = \nabla(u^m)$ and therefore the whole sequence $\{\nabla(u_p^m)\}$ tends to ψ weakly in $L^2(Q)$. Hence $\nabla(u^m)$ exists in the sense of distributions in Q and by (16)

$$\int_Q |\nabla(u^m)|^2\,dx\,dt \leqslant \frac{1}{m+1}\int_\Omega u_0^{m+1}(x)\,dx\,.$$

Thus u satisfies the condition (i) $\big((4)\big)$ of a weak solution of problem (1)-(3).

To show that u satisfies the remaining conditions (ii) and (iii) required of a weak solution of problem (1)-(3) is now straightforward. The second condition follows since u_p is a classical solution of problem (11)-(13) for any $p \geqslant 1$ and $u_{0p} \to u_0$ pointwise in Ω, $u_p \to u$ pointwise in Q and $\nabla(u_p^m) \to \nabla(u^m)$ weakly in $L^2(Q)$ as $p \to \infty$. The third condition follows since for any $(x_0, t_0) \in S$, $p \geqslant 1$,

$$\limsup_{\substack{(x,t)\to(x_0,t_0)\\(x,t)\in Q}} u(x,t) \leqslant \limsup_{\substack{(x,t)\to(x_0,t_0)\\(x,t)\in Q}} u_p(x,t) = \frac{1}{p}\,.$$

Hence, as p was arbitrary, (6) holds.

Thus we have proved the following theorem.

THEOREM 2. *Suppose that Ω is a bounded open connected subset of $\mathbf{R}^N$, $N \geqslant 1$, such that $\partial\Omega \in C^{2+\alpha}$, $\alpha \in (0,1)$. Then if $u_0 \in C(\bar\Omega)$ satisfies $u_0(x) = 0$ for $x \in \partial\Omega$, problem* (1)-(3) *has at least one solution.*

By Theorem 1 this solution is unique.

4. – Regularity.

In this final section we shall show that the solution of problem (1)-(3) constructed in Theorem 2 is continuous in $\bar Q$. In the case $N = 1$ the interior continuity follows from the work of Aronson [1] and Gilding [5]. In the case $N > 1$ to prove interior continuity we shall follow an approach advocated by Caffarelli and Friedman [4] which relies on an a priori estimate of solutions of (1) of a type recently discovered for the Cauchy problem by Aronson and Bénilan [2]. We shall, for the time being, postpone the proof of continuity up to Γ. Throughout the following discussion we shall work with a solution u_p of problem (11)-(13), but for convenience we shall drop the subscript p.

To begin we derive the Aronson-Bénilan type inequality which is independent of the dimension N.

LEMMA 1. *Let u be a solution of problem* (11)-(13). *Then*

$$tu_t \geqslant -\frac{1}{(m-1)}u \qquad \text{in } Q\,. \tag{17}$$

PROOF [4]. Consider the function $z = tu_t + (1/(m-1))u$ defined in Q. Then

$$z_t = \nabla(mu^{m-1}z) \qquad \text{in } Q\,.$$

Moreover, since $u(x, t) = 1/p$ for all $(x, t) \in S$,

$$z(x, t) = \frac{1}{m-1}u(x, t) \geqslant 0 \qquad \text{for all } (x, t) \in \Gamma\,.$$

Hence, by the standard maximum principle,

$$z(x, t) \geqslant 0 \qquad \text{in } \bar{Q}\,.$$

Henceforth we shall assume that $N \geqslant 2$. Let $B(x_0; \varrho)$ denote the ball of radius ϱ centred at x_0 in $\boldsymbol{R}^N$, and for a function f defined in $B(x_0; \varrho)$ set

$$\oint_{B(x_0;\varrho)} f(x)\,dx = \frac{1}{|B(x_0; \varrho)|}\int_{B(x_0;\varrho)} f(x)\,dx\,,$$

where $|B(x_0; \varrho)|$ is the N-dimensional measure of $B(x_0; \varrho)$. We shall use the following identity.

LEMMA 2. *Let $G(r; \varrho, N)$ denote the Green function defined by*

$$G(r; \varrho, N) = \begin{cases} r^{2-N} - \varrho^{2-N} + \frac{1}{2}(N-2)\varrho^{-N}(r^2 - \varrho^2) & \text{if } N > 2 \\ \log(\varrho/r) + \frac{1}{2}\varrho^{-2}(r^2 - \varrho^2) & \text{if } N = 2\,. \end{cases}$$

Then given any function f defined in $C^2(\overline{B(x_0; \varrho)})$

$$f(x_0) = \oint_{B(x_0;\varrho)} f(x)\,dx - a_N \int_{B(x_0;\varrho)} G(|x - x_0|; \varrho, N)\,\Delta f(x)\,dx\,,$$

where

$$a_N = \begin{cases} [2(N-2)\pi^{N/2}/\Gamma(N/2)]^{-1} & \text{if } N > 2 \\ [2\pi]^{-1} & \text{if } N = 2\,. \end{cases}$$

Observe that $G > 0$ in $B(0; \varrho)$.

The proof of the interior continuity of the weak solution in Q rests on the following two complementary inequalities derived from a combination of Lemmas 1 and 2. The first relates $u(x_0, t_0)$ to $\oint_{B(x_0;\varrho)} u^m(x, t_0)\,dx$ for some $(x_0, t_0) \in Q$ and $\varrho > 0$, and the second relates $\oint_{B(x_0;\varrho)} u^m(x, t)\,dx$ to $u(x_0, t_0)$ for some $(x_0, t_0) \in Q$, $\varrho > 0$ and $0 < t < t_0$.

LEMMA 3. *Let $(x_0, t_0) \in Q$ and suppose that ϱ is chosen so small that $B(x_0; \varrho) \subset \Omega$. Then*

$$u^m(x_0, t_0) \leqslant \frac{M\varrho^2}{2(N+2)(m-1)t_0} + \oint_{B(x_0;\varrho)} u^m(x, t_0)\,dx\,.$$

PROOF. Set $v(x) = u^m(x, t_0) + \big(M/2N(m-1)t_0\big)|x - x_0|^2$.
Then for $x \in B(x_0; \varrho)$

$$\Delta v(x) = u_t(x, t_0) + \frac{M}{(m-1)t_0} \geqslant \frac{M - u(x, t_0)}{(m-1)t_0} \geqslant 0$$

by (17). Hence by Lemma 2

$$u^m(x_0, t_0) \leqslant \oint_{B(x_0;\varrho)} v(x)\,dx \leqslant \frac{M}{2(N+2)(m-1)t_0}\varrho^2 + \oint_{B(x_0;\varrho)} u^m(x, t_0)\,dx\,.$$

LEMMA 4. *Let $(x_0, t_0) \in Q$ and suppose that ϱ is chosen so small that $B(x_0; \varrho) \subset \Omega$. Then there exists a constant b_N, which only depends on N, such that*

$$\oint_{B(x_0;\varrho)} u^m(x, t)\,dx \leqslant u^m(x_0, t_0)\left(\frac{t_0}{t}\right)^{m/(m-1)} + b_N M\varrho^2\left(\frac{t_0}{t}\right)^{m/(m-1)} \frac{1}{t_0 - t}$$

for all $t \in (0, t_0)$.

PROOF. By Lemma 2

$$\oint_{B(x_0;\varrho)} u^m(x, t)\,dx = u^m(x_0, t) + a_N \int_{B(x_0;\varrho)} G\big(|x - x_0|; \varrho, N\big)\,u_t(x, t)\,dx \tag{18}$$

for all $t > 0$. However, by (17)

$$u(x, t_1)\,t_1^{1/(m-1)} \leqslant u(x, t_2)\,t_2^{1/(m-1)} \tag{19}$$

for all $x \in \Omega$ and $0 \leqslant t_1 \leqslant t_2$. Hence we may substitute (19) in (18) and integrate with respect to t to derive

$$(m-1)t^{m/(m-1)}[t^{-1/(m-1)}-t_0^{-1/(m-1)}]\oint_{B(x_0;\varrho)} u^m(x,t)\,dx \leqslant$$
$$\leqslant (m-1)t_0^{m/(m-1)}[t^{-1/(m-1)}-t_0^{-1/(m-1)}]u^m(x_0,t_0)+$$
$$+a_N\int_{B(x_0;\varrho)} G(|x-x_0|;\varrho,N)\,u(x,t_0)\,dx$$

for all $t\in(0,t_0)$. Applying the mean value theorem, this yields

$$\oint_{B(x_0;\varrho)} u^m(x,t)\,dx \leqslant \left(\frac{t_0}{t}\right)^{m/(m-1)} u^m(x_0,t_0)+\left(\frac{t_0}{t}\right)^{m/(m-1)}\frac{1}{t_0-t}\cdot$$
$$\cdot a_N\int_{B(x_0;\varrho)} G(|x-x_0|;\varrho,N)\,u(x,t_0)\,dx \leqslant$$
$$\leqslant \left(\frac{t_0}{t}\right)^{m/(m-1)} u^m(x_0,t_0)+\left(\frac{t_0}{t}\right)^{m/(m-1)} b_N M\varrho^2\frac{1}{t_0-t}.$$

Lemmas 3 and 4 form the basis of the proof of the interior continuity of the weak solution of problem (1)-(3) following the method of Caffarelli and Friedman [4], although the Aronson-Bénilan type inequality (Lemma 1) will be used once more in the following lemma. The remainder of the proof is however rather technical and we shall need to introduce some further notation.

To begin we pick a function $\sigma\in C^1(-\infty,\infty)$ with the following properties:

(i) $\sigma'(x)<0$ for all $x\in\boldsymbol{R}$,

(ii) $\sigma(x)\to 0$ as $x\to\infty$,

(iii) $2^{-2x}=o(\sigma^{N+2}(x))$ as $x\to\infty$,

(iv) $\sigma^2(x+1)/\sigma^2(x)=1-o(\sigma^N(x))$ as $x\to\infty$.

An example of such a function is given by [4]:

$$\sigma(x)=x^{-\lambda},\qquad 0<\lambda<\frac{1}{N},$$

for large x.

Let $d>0$ and

$$\Omega'=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>d\}.$$

Furthermore, let

$$Q'=\Omega'\times(d^2,\infty).$$

Fix a point $(x_0, t_0) \in Q'$. For any positive integer k such that $2^{-k} \leqslant d$ define the set

$$R_k = B(x_0; 2^{-k}) \times (t_0 - 2^{-2k}, t_0 + 2^{-2k}] .$$

Note that $R_k \subset Q$. Finally set

$$\mu_k = \sup_{R_k} u^m .$$

Throughout the following discussion we shall keep $(x_0, t_0) \in Q'$ fixed.

LEMMA 5. *There exists a positive integer k_0 and positive constants C_0 and C_1, all three depending only on m, M, N and d, such that if*

$$u^m(x_0, t_0) \leqslant \sigma^2(k_0) ,$$

then

$$\mu_{k+1} \leqslant A_k\{1 - C_0 \sigma^N(k)\} ,$$

where

$$A_k = \max\{\mu_k, C_1 \sigma^2(k)\}$$

for all $k \geqslant k_0$ such that

$$u^m(x_0, t_0) \leqslant \sigma^2(k) .$$

PROOF. To begin with choose k_1 so large that $2^{-k_1} < \frac{1}{2} d$. Then $R_k \subset Q$ and moreover $t_0 - 2^{-2k} > \frac{1}{2} t_0 > \frac{1}{2} d^2$ for all $k \geqslant k_1$. Secondly choose $k_2 \geqslant k_1$ such that $\sigma(k_2) \leqslant \frac{1}{3}$. Then we observe that for all $k \geqslant k_2$ and $x \in B(x_0; \frac{1}{3} 2^{-k})$:

$$B\big(x_0; \sigma(k) 2^{-k}\big) \subset B(x; \tfrac{2}{3} 2^{-k}) \subset B(x_0; 2^{-k}) \subset \Omega .$$

Thus by Lemma 3 and the definition of μ_k

$$u^m(x, t_0 - 2^{-2k}) \leqslant \frac{M(\frac{2}{3} 2^{-k})^2}{(N+2)(m-1) d^2} + \oint_{B(x; \frac{2}{3} 2^{-k})} u^m(x, t_0 - 2^{-2k})\, dx \leqslant \tag{20}$$

$$\leqslant \frac{4M 2^{-2k}}{9(N+2)(m-1) d^2} + \mu_k\big(1 - \sigma^N(k)\big) + \left(\frac{3}{2}\right)^N \sigma^N(k) \oint_{B(x_0; \sigma(k) 2^{-k})} u^m(x, t_0 - 2^{-2k})\, dx$$

for all $x \in B(x_0; \frac{1}{3}2^{-k})$ and $k \geq k_2$. However by Lemma 4

$$\oint_{B(x_0;\sigma(k)2^{-k})} u^m(x, t_0 - 2^{-2k})\,dx \leq u^m(x_0, t_0)\{t_0/(t_0 - 2^{-2k})\}^{m/(m-1)} + \tag{21}$$
$$+ b_N M\sigma^2(k)\{t_0/(t_0 - 2^{-2k})\}^{m/(m-1)} \leq u^m(x_0, t_0)2^{m/(m-1)} + b_N M 2^{m/(m-1)}\sigma^2(k)$$

for all $k \geq k_2$. Thus combining (20) and (21) we find that if $u^m(x_0, t_0) \leq \sigma^2(k)$, then

$$u^m(x; t_0 - 2^{-2k}) \leq \frac{4M}{9(N+2)(m-1)d^2}\,2^{-2k} + \left(\frac{3}{2}\right)^N (1 + b_N M)2^{m/(m-1)}\sigma^{N+2}(k) +$$
$$+ \mu_k(1 - \sigma^N(k)).$$

We now choose $k_3 \geq k_2$ so large that

$$2^{-2k} \leq \sigma^{N+2}(k) \qquad \text{for all } k \geq k_3$$

and

$$C_1 > 2\left\{\frac{4M}{9(N+2)(m-1)d^2} + \left(\frac{3}{2}\right)^N (1 + b_N M)2^{m/(m-1)}\right\}.$$

Summarizing we have then obtained

$$u^m(x; t_0 - 2^{-2k}) \leq A_k(1 - \tfrac{1}{2}\sigma^N(k)) \tag{22}$$

for all $x \in B(x_0; \frac{1}{3}2^{-k})$, and $k \geq k_3$ such that $u^m(x_0, t_0) \leq \sigma^2(k)$.

For $k \geq k_3$ such that $u^m(x_0, t_0) \leq \sigma^2(k)$ define the variables

$$y = (x - x_0)2^k, \qquad s = (t - t_0)2^{2k}$$

and consider the function

$$z(y, s) = 1 - u^m(x, t)/A_k$$

in the domain $(y, s) \in D = B(0; 1) \times (-1, 1]$. Observe that

$$z(y, s) \geq 0 \qquad \text{in } D \tag{23}$$

and by (22)

$$z(y, -1) \geq \tfrac{1}{2}\sigma^N(k) \qquad \text{for } y \in B(0; \tfrac{1}{3}). \tag{24}$$

Moreover, if we define the operator

$$\mathfrak{L}w = \Delta w - \frac{1}{mM^{m-1}}\,w_s \qquad \text{in } D,$$

then

$$\mathcal{L}z = -\frac{2^{-2k}}{A_k}\left\{\Delta u^m - \frac{1}{mM^{m-1}}(u^m)_t\right\} =$$

$$(25)\qquad = -\frac{2^{-2k}}{A_k}\left\{1-\left(\frac{u}{M}\right)^{m-1}\right\}u_t \leqslant$$

$$\leqslant \frac{2^{-2k}}{(m-1)A_k t}\left\{1-\left(\frac{u}{M}\right)^{m-1}\right\}u \leqslant$$

$$\leqslant \frac{2^{-2k}}{(m-1)A_k t}M \quad \text{in } D,$$

where we have used Lemma 1 once again.

If we now represent z in terms of the Green function for $\mathcal{L}$ in D, we obtain, as in [4] with (23)-(25), that

$$z(y', s') \geqslant \frac{1}{2}C_2\sigma^N(k) - \frac{2C_3 2^{-2k}M}{(m-1)A_k d^2}$$

for $(y', s') \in B(0; \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}]$, where C_2 and C_3 are positive constants which only depend on $\mathcal{L}$ and D, i.e. only on m, M and N. Noting that $C_1\sigma^2(k) \leqslant A_k$, it follows that if we choose $k_0 \geqslant k_3$ so large that

$$\frac{2C_3 2^{-2k}M}{(m-1)C_1 d^2\sigma^2(k)} \leqslant \frac{1}{4}C_2\sigma^N(k) \qquad \text{for all } k \geqslant k_0$$

and set $C_0 = \frac{1}{4}C_2$ then

$$z(y', s') \geqslant C_0\sigma^N(k)$$

for all $(y', s') \in B(0; \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}]$ and $k \geqslant k_0$ such that $u^m(x_0, t_0) \leqslant \sigma^2(k)$. Returning to the variables x and t completes the proof of Lemma 5.

As a corollary to Lemma 5 we have the following result.

LEMMA 6. *There exists a positive integer k^* and a constant C, both depending only on m, M, N and d, such that if*

$$u^m(x_0, t_0) \leqslant \sigma^2(k^*)$$

then

$$(26)\qquad \mu_k \leqslant C\sigma^2(k)$$

for all k such that

$$(27)\qquad u^m(x_0, t_0) \leqslant \sigma^2(k).$$

PROOF. Continuing from Lemma 5, choose $k^* \geqslant k_0$ so large that

$$\frac{\sigma^2(k+1)}{\sigma^2(k)} \geqslant 1 - C_0 \sigma^N(k) \tag{28}$$

for all $k \geqslant k^*$. Then set $C = \max\{C_1, M^m \sigma^{-2}(k^*)\}$. Plainly then

$$\mu_k \leqslant C\sigma^2(k) \qquad \text{for } k \leqslant k^* .$$

In particular $\mu_{k^*} \leqslant C\sigma^2(k^*)$. If however (26) holds for some $\bar{k} \geqslant k^*$ such that $u^m(x_0, t_0) \leqslant \sigma^2(\bar{k})$, then by definition

$$A_{\bar{k}} \;=\; \max\; \{\mu_{\bar{k}}, C_1\sigma^2(\bar{k})\} \leqslant C\sigma^2(\bar{k})$$

and therefore by Lemma 5

$$\mu_{\bar{k}+1} \leqslant C\sigma^2(\bar{k})\big(1 - C_0\sigma^N(\bar{k})\big) \leqslant C\sigma^2(\bar{k}+1)$$

by (28). Thus by induction (26) is true for all k for which (27) holds.

We are now at the heart of the matter. Define the modulus of continuity function ω by

$$\omega(r) = \begin{cases} \sigma^2(-\log_2 r) & r > 0 \\ 0 & r = 0 . \end{cases}$$

Then we can state the following result.

LEMMA 7. *There exists a constant* $\mathcal{C}$, *which only depends on* N, m, M *and* d, *such that*

$$u^m(x, t) \leqslant \mathcal{C}\omega\big(|x - x_0| + |t - t_0|^{\frac{1}{2}}\big) + \mathcal{C}u^m(x_0, t_0) \tag{29}$$

for all $(x, t) \in Q$.

PROOF. Let k^* denote the positive integer and C the positive constant defined in Lemma 6. Then choose $\mathcal{C}$ so large that

$$\mathcal{C} > C \sup\{\sigma^2(k)/\sigma^2(k+1) \colon k \geqslant k^*\} . \tag{30}$$

Note that this implies by the definition of C that

$$\mathcal{C} > M^m \sigma^{-2}(k^*) = M^m/\omega(2^{-k^*}) . \tag{31}$$

Now if $u^m(x_0, t_0) \geqslant \sigma^2(k^*)$, (29) holds by (31). On the other hand if $u^m(x_0, t_0) < < \sigma^2(k^*)$ and $(x, t) \notin R_{k^*}$ then (29) also holds by (31). It remains therefore to show that (29) holds when $u^m(x_0, t_0) < \sigma^2(k^*)$ and $(x, t) \in R_{k^*}$. In this case we define $\hat{k} \geqslant k^*$ such that

$$\sigma^2(\hat{k} + 1) \leqslant u^m(x_0, t_0) < \sigma^2(\hat{k}) .$$

Then by Lemma 6 and (30) we obtain for $(x, t) \in R_{\hat{k}}$

$$u^m(x, t) \leqslant \mu_{\hat{k}} \leqslant C\sigma^2(\hat{k}) \leqslant \mathcal{C}\sigma^2(\hat{k} + 1) \leqslant \mathcal{C}u^m(x_0, t_0) .$$

Finally, suppose $\hat{k} > k^*$, and $\hat{k} > k \geqslant k^*$. Then for $(x, t) \in R_k \setminus R_{k+1}$,

$$u^m(x, t) \leqslant \mu_k \leqslant C\sigma^2(k) \leqslant \mathcal{C}\sigma^2(k + 1) \leqslant \mathcal{C}\omega\big(|x - x_0| + |t - t_0|^{\frac{1}{2}}\big) .$$

Since $\mathcal{C}$ does not depend on k, it follows that this inequality in fact holds in $R_{k^*} \setminus R_{\hat{k}}$. This completes the proof.

The proof of the continuity of the weak solution of problem (1)-(3) in Q' may now be continued following Caffarelli and Friedman [4]. However these authors were particularly concerned with deriving a global modulus of continuity and the proof is extremely technical. Instead, as we are concerned only in proving continuity in the most general sense, we shall take a short cut proceeding directly from Lemma 7.

We reintroduce now the subscripts p in the solution of problem (11)-(13). Lemma 7 states that given any points (x_1, t_1), $(x_2, t_2) \in Q'$ and $p \geqslant 1$ there exists a constant $\mathcal{C}$ depending only on m, M, N and d such that

$$u_p^m(x_1, t_1) \leqslant \mathcal{C}\omega\big(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}\big) + \mathcal{C}u_p^m(x_2, t_2) \tag{32}$$

and

$$u_p^m(x_2, t_2) \leqslant \mathcal{C}\omega\big(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}\big) + \mathcal{C}u_p^m(x_1, t_1) . \tag{33}$$

Thus taking the limit, (32) and (33) also hold for the weak solution u of problem (1)-(3). Now if $u(x_1, t_1) = 0$ the continuity of u at (x_1, t_1) follows immediately from (33). On the other hand if $u(x_1, t_1) > 0$ then by (32)

$$u^m(x_2, t_2) \geqslant \frac{1}{2\mathcal{C}} u^m(x_1, t_1)$$

for all $(x_2, t_2) \in Q'$ such that

$$\omega\big(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}\big) \leqslant \frac{1}{2\mathcal{C}} u^m(x_1, t_1) .$$

In other words u is uniformly bounded away from zero in a neighbourhood $\mathcal{H}$ of (x_1, t_1). Particularly the decreasing sequence of functions u_p is uniformly bounded away from zero in $\mathcal{H}$. Thus in fact, by the standard theory for uniformly parabolic equations there exists a second neighbourhood of (x_1, t_1), $\mathcal{H}_0 \subset \mathcal{H}$ such that u_p is uniformly bounded in $C^{2+\beta}(\mathcal{H}_0)$ for some $\beta \in (0, 1)$. Subsequently $u \in C^{2+\beta}(\mathcal{H}_0)$. Finally then u is continuous at (x_1, t_1) in either case.

Since $\Omega' \subset \Omega$ and $d > 0$ were arbitrary this shows that u is continuous in Q when $N \geqslant 2$. When $N = 1$ the continuity follows from [1] and [5]. The continuity of u up to S is a consequence of the definition of a weak solution. To show that u is continuous in $\bar{Q}$ it therefore remains to show that u is continuous down to the time $t = 0$, and to do this it is clearly sufficient to show that given any $x_0 \in \bar{\Omega}$

$$\limsup_{\substack{(x,t)\to(x_0,0)\\(x,t)\in Q}} u(x, t) \leqslant u_0(x_0) \tag{34}$$

and

$$\liminf_{\substack{(x,t)\to(x_0,0)\\(x,t)\in Q}} u(x, t) \geqslant u_0(x_0)\,. \tag{35}$$

Now, plainly for any $p \geqslant 1$,

$$\limsup_{\substack{(x,t)\to(x_0,0)\\(x,t)\in Q}} u(x, t) \leqslant \limsup_{\substack{(x,t)\to(x_0,0)\\(x,t)\in Q}} u_p(x, t) = u_{0p}(x_0) \leqslant u_0(x_0) + \frac{3}{2p}\,.$$

Thus (34) holds. On the other hand since u is nonnegative (35) holds at those points $x_0 \in \bar{\Omega}$ for which $u_0(x_0) = 0$. It remains therefore to prove that (35) holds at those points $x_0 \in \Omega$ such that $u_0(x_0) > 0$.

Let $x_0 \in \Omega$ be such that $u_0(x_0) > 0$ and pick $\varepsilon \in \big(0, u_0(x_0)\big)$. Then by the continuity of u_0 there exists a $\delta > 0$ such that

$$u_0(x) > u_0(x_0) - \varepsilon \qquad \text{for all } x \in B(x_0; \delta)\,.$$

Now we define an instantaneous point source solution of (1) by

$$w(x, t) = (t + \tau)^{-n} \left\{\left[a^2 - \frac{n(m-1)}{2Nm} \frac{|x - x_0|^2}{(t+\tau)^{2n/N}}\right]_+\right\}^{1/(m-1)},$$

where $[f]_+ = \max\{0, f\}$, $n = \{m - (N-2)/N\}^{-1}$, and a and τ are chosen such that

$$a^{2/(m-1)} \tau^{-n} = u_0(x_0) - \varepsilon\,,$$

and

$$\left(\frac{2Nm}{n(m-1)}\right)^{\frac{1}{2}} a\tau^{n/N} < \delta ;$$

and we choose $T > 0$ such that

$$B\left(x_0; \left(\frac{2Nm}{n(m-1)}\right)^{\frac{1}{2}} a(\tau + T)^{n/N}\right) \subset \Omega .$$

For $p \geqslant 1$ set

$$w_p(x, t) = \max\left\{\frac{1}{2p}, w(x, t)\right\} \qquad \text{for } (x, t) \in \bar{\Omega} \times [0, T] .$$

Now w_p is the maximum of two smooth solutions of (1). Moreover

$$u_p(x, t) \geqslant \frac{1}{2p} = w_p(x, t) \qquad \text{for } (x, t) \in \partial\Omega \times [0, T] ,$$

$$u_p(x, 0) \geqslant u_0(x) + \frac{1}{2p} \geqslant u_0(x_0) - \varepsilon + \frac{1}{2p} \geqslant w_p(x, 0) \qquad \text{for } x \in B(x_0; \delta) ,$$

and

$$u_p(x, 0) \geqslant \tfrac{1}{2}p = w_p(x, 0) \qquad \text{for } x \in \bar{\Omega} \setminus B(x_0; \delta) .$$

Hence by the standard maximum principle

$$u_p(x, t) \geqslant w_p(x, t) \qquad \text{in } \bar{\Omega} \times [0, T] .$$

It follows that

$$u(x, t) \geqslant w(x, t) \qquad \text{in } \bar{\Omega} \times [0, T]$$

and hence that

$$\liminf_{\substack{(x,t)\to(x_0,0)\\(x,t)\in Q}} u(x, t) \geqslant \liminf_{\substack{(x,t)\to(x_0,0)\\(x,t)\in Q}} w(x, t) = w(x_0, 0) = a^{2/(m-1)}\tau^{-n} = u_0(x_0) - \varepsilon .$$

Since $\varepsilon > 0$ was arbitrary this completes the proof of (35) and thus of the following theorem.

THEOREM 3. *Suppose that Ω is a bounded open connected subset of $\boldsymbol{R}^N$, $N \geqslant 1$, such that $\partial\Omega \subset C^{2+\alpha}$, $\alpha \in (0, 1)$. Then if $u_0 \in C(\bar{\Omega})$ satisfies $u_0(x) = 0$ for $x \in \partial\Omega$, the weak solution of problem* (1)-(3) *is continuous in $\bar{Q}$.*

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