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On Nontrivial Solutions of a Semilinear Wave Equation (*).

PAUL H. RABINOWITZ

The question of the existence of nontrivial time periodic solutions of autonomous or forced semilinear wave equations has been the object of considerable recent interest [1-12]. These papers study the equation

$$u_{tt} - u_{xx} + f(x, u) = 0\,, \qquad 0 < x < l \tag{1.1}$$

(or its analogue where f also depends on t in a time periodic fashion) together with boundary conditions in x and periodicity conditions in t. In particular the following result was proved in [11, Theorem 3.37 and Corollary 4.14]:

THEOREM 1.2. *Let $f \in C([0, l] \times \boldsymbol{R}, \boldsymbol{R})$ and satisfy*

(f_1) *$f(x, 0) = 0$ and $f(x, r)$ is strictly monotone increasing in r,*

(f_2) *$f(x, r) = o(|r|)$ at $r = 0$,*

(f_t) *there are constants $\bar{r} > 0$ and $\mu > 2$ such that*

$$0 < \mu F(x, r) \leqslant r f(x, r)$$

for $|r| \geqslant \bar{r}$ and $x \in [0, l]$ where

$$F(x, r) = \int_0^r f(x, s)\, ds\,.$$

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Then for any T which is a rational multiple of l, equation (1.1) *possesses a non trivial continuous weak solution u satisfying*

$$\begin{cases} u(0, t) = 0 = u(l, t) , \\ u(x, t + T) = u(x, t) . \end{cases} \tag{1.3}$$

Furthermore $f \in C^k$ implies $u \in C^k$.

As part of the proof of Theorem 1.2, it was shown that the functional

$$I(u) = \int_0^T \int_0^l [\tfrac{1}{2}(u_t^2 - u_x^2) - F(x, u)]\,dx\,dt \tag{1.4}$$

defined on the class of functions satisfying (1.3) (and of which (1.1) is formally the Euler equation) has a positive critical value. Therefore (f_1) and the form of I imply that $u_t \not\equiv 0$ for the corresponding critical point u. Thus u is nonconstant in x and must depend explicitly on t. It was further observed in [11] (Theorem 5.24 and Remark 5.25) that if g satisfies (f_1)-(f_3) the equation

$$u_{tt} - u_{xx} - g(x, u) = 0 , \qquad 0 < x < l \tag{1.5}$$

together with (1.3) also possesses a nontrivial weak solution. Indeed the arguments of Theorem 1.2 go through with minor modifications to establish this fact. However the functional one studies for this case is

$$J(u) = \int_0^T \int_0^l [\tfrac{1}{2}(u_x^2 - u_t^2) - G(x, u)]\,dx\,dt \tag{1.6}$$

where G is the primitive of g. Again the positivity of $J(u)$ for a critical point u implies u is nonconstant but we can no longer conclude that u depends explicitly on t. In fact it is known [13, 14] that as a consequence of (f_2)-(f_3), the ordinary differential equation boundary value problem

$$-\frac{d^2u}{dx^2} = g(x, u) , \qquad u(0) = 0 = u(l) \tag{1.7}$$

has an unbounded sequence of solutions which can be characterized by the number of zeros they possess in $(0, l)$.

Our goal in this paper is to show that if (f_3) is strengthened somewhat, (1.5), (1.3) possesses infinitely many time dependent solutions. More precisely we will prove:

THEOREM 1.8. *Let $g \in ([0, l] \times \boldsymbol{R}, \boldsymbol{R})$ and suppose g satisfies (f_1)-(f_2) and $(\bar{f}_3)$. There is a constant $\mu > 0$ such that*

$$0 < \mu F(x, r) \leqslant r f(x, r)$$

for all $r \neq 0$.

Then for any $T \in l\boldsymbol{Q}$ there is a $k_0 \in \boldsymbol{N}$ such that for all $k \geqslant k_0$, (1.5), (1.3) *possesses a solution u_k which is kT periodic in t and $\partial u_k/\partial t \not\equiv 0$. Moreover infinitely many of the functions u_k are distinct.*

REMARK 1.9. We have no estimate for the size of k_0 and do not know if the result is false in general for $k = 1$. Note also that since (1.5) is an autonomous equation with respect to t, whenever $u(x, t)$ is a solution, so is $u(x, t + \theta)$ for any $\theta \in \boldsymbol{R}$. The above statement about the u_k's being distinct means in particular that they do not differ by merely a translation in time.

The proof of Theorem 1.8 draws on several results from [11] and ideas from [12]. For convenience we will take $l = \pi$ and $T = 2\pi$. Choosing $k \in \boldsymbol{N}$, we seek a solution u_k of (1.5) which is $2\pi k$ periodic in t and $\partial u_k/\partial t \not\equiv 0$. Making the change of time scale $\tau = t/k$, the period becomes 2π again and the problem to be solved is

$$\text{(1.10)} \qquad \begin{cases} U_{\tau\tau} - k^2(U_{xx} + g(x, U)) = 0\,, & 0 < x < \pi \\ U(0, \tau) = 0 = U(l, \tau) & \\ U(x, \tau + 2\pi) = U(x, \tau) & \end{cases}$$

with $U(x, \tau) = u(x, t)$.

For the convenience of the reader and to set the stage for a key estimate, the argument used in [11] to establish the existence of nontrivial solutions of (1.10) will be sketched quickly. Solutions are obtained by an approximation argument. To begin (1.10) is modified in two ways. The wave operator $\partial^2/\partial\tau^2 - k^2(\partial^2/\partial x^2)$ possesses an infinite dimensional null space in the class of functions satisfying the boundary and periodicity conditions of (1.10) and given by

$$N = \operatorname{span}\{\sin jx \sin kj\tau,\ \sin jx \cos kj\tau \mid j \in \boldsymbol{N}\}\,.$$

The fact that N is infinite dimensional complicates the analysis of (1.10) and to introduce some compactness to the problem in N, we perturb (1.10) by adding a βV_{tt} term to the left hand side of the equation. Here $\beta > 0$ and V is the (L^2 orthogonal) projection of U onto N. A second difficulty in treating (1.10) arises due to the unrestricted rate of growth of $g(x, r)$ as

$|r| \to \infty$. We get around this by truncating g. More precisely $g(x, r)$ is replaced by $g_K(x, r)$ which coincides with g for $|r| \leqslant K$ and grows cubically at ∞ [11]. Thus (1.10) is replaced by the modified problem

$$(1.11) \qquad \begin{cases} U_{\tau\tau} + \beta V_{\tau\tau} - k^2(U_{xx} + g_K(x, U)) = 0, & 0 < x < \pi \\ U(0, \tau) = 0 = U(l, \tau) \\ U(x, \tau + 2\pi) = U(x, \tau) \end{cases}$$

where g_K satisfies (f_1), (f_2), $(\bar{f}_3)$ with a new constant $\bar{\mu} = \min(\mu, 4)$.

Letting G_K denote the primitive of g_K, in a formal fashion (1.11) can be interpreted as the Euler equation arising from the functional

$$(1.12) \qquad J(U; k, \beta, K) = \int_0^{2\pi}\int_0^{\pi}\left[\frac{k^2}{2} U_x^2 - \frac{1}{2} U_\tau^2 - \frac{\beta}{2} V_\tau^2 - k^2 G_K(x, U)\right] dx\, d\tau .$$

Let

$$E_m \equiv \operatorname{span}\{\sin jx \sin n\tau, \sin jx \cos b\tau \,|\, 0 \leqslant j,\ n \leqslant m\} .$$

The strategy pursued in [11] was to find a critical point U_{mk} of $J|_{E_m}$, let $m \to \infty$, and then let $\beta \to 0$ to get a solution U_k of (1.10) with g replaced by g_K. Then L^∞ bounds for U_k independent of K show if we choose $K(k)$ sufficiently large $g_K(x, U_k) = g(x, U_k)$ so (1.10) obtains. A separate comparison argument is required to prove that $U_k \not\equiv 0$.

The fiist step in carrying out the details of the above argument involves obtaining an upper bound M_k for $c_{mk} \equiv J(U_{mk}; k, \beta, K)$ with M_k independent of m, β, and K. For the current problem which also depends on k, it is crucial to know the behavior of M_k as a function of k. Thus we will take a closer look at c_{mk} and use a variant of an argument of [12]. By Lemma 1.13 of [11], c_{mk} can be characterized in a minimax fashion. We will not write down this characterization explicitly but will note a consequence of it which in turn provides an upper bound for c_{mk}. Set

$$\Phi_{mk} = \operatorname{span}\{\sin jx \sin n\tau,\ \sin jx \cos n\tau \,|\, 0 \leqslant j,\ n \leqslant m \text{ and } n^2 \geqslant j^2 k^2\}$$

and

$$\psi_k = a_k \sin x \sin (k-1)\tau$$

where $a_k = \sqrt{2}/\pi$ so $\|\psi_k\|_{L^2} = 1$. Set $\Psi_{mk} = \Phi_{mk} \oplus \operatorname{span} \psi_k$. Then by Lemma 1.3. of [11]

$$(1.13) \qquad 0 < c_{mk} \leqslant \max_{u \in \Psi_{mk}} J(u; k, \beta, K) .$$

Inequality (1.13) will lead to a suitable choice for M_k. Note that by $(\bar{f}_3)$ (or even (f_3)), there are constants α_1, $\alpha_2 \geqslant 0$ and independent of K such that

$$G_K(x, r) \geqslant a_1 |r|^{\bar{\mu}} - a_2 \tag{1.14}$$

for all $r \in \mathbf{R}$, $x \in [0, \pi]$. Consequently $J \to -\infty$ as $u \to \infty$ in Ψ_{mk} (under $\|\cdot\|_{L^2}$) so there is a point $z \equiv Z_{mk}$ at which the maximum in (1.13) is achieved. Writing

$$z = \|z\|_{L^2}(\gamma \xi + \delta \psi_k) \tag{1.15}$$

where $\xi \in \Phi_{mk}$, $\|\xi\|_{L^2} = 1$, and $\gamma^2 + \delta^2 = 1$ and substituting (1.15) into (1.13) gives

$$k^2 \int_0^{2\pi}\int_0^{\pi} G(x, z)\, dx\, dt \leqslant \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}(k^2 z_x^2 - z_\tau^2)\, dx\, d\tau \leqslant \tag{1.16}$$

$$\leqslant \frac{\delta^2}{2}\|z\|_{L^2}^2 \int_0^{2\pi}\int_0^{\pi}[k^2(\psi_k)_x^2 - (\psi_k)_\tau^2]\, dx\, d\tau \leqslant k\|z\|_{L^2}^2 .$$

Combining (1.14) and (1.16) shows that

$$k^2(\alpha_1\|z\|_{L^{\bar{\mu}}}^{\bar{\mu}} - \alpha_3) \leqslant k\|z\|_{L^2}^2 . \tag{1.17}$$

Applying the Hölder inequality yields

$$\|z\|_{L^2} \leqslant A \tag{1.18}$$

where A is a constant independent of m, k, β, K. Hence by (1.13), (1.18), and the form of J,

$$c_{mk} \leqslant Mk \tag{1.19}$$

for a constant M independent of m, k, β, K.

Letting $m \to \infty$ and then $\beta \to 0$, and formalizing what we have just shown gives:

LEMMA 1.20. *Under the hypotheses of Theorem* 1.8 (*with* $l = \pi$ *and* $T = 2\pi$), *for all* $k \in \mathbf{N}$, *there exists a solution* U_k *of* (1.10) *satisfying*

$$c_k \equiv J(U_k; k, 0, K) \leqslant Mk \tag{1.21}$$

with M *independent of* k *and* K.

It remains to show that for all k sufficiently large, $\partial U_k/\partial t \not\equiv 0$ and infinitely many of the functions $u_k(x, t) = U_k(x, \tau)$ are distinct. If U_k is independent of τ for any subsequence of k's tending to ∞, $U_k = U_k(x)$ is a classical solution of (1.7). Thus by (1.21) with $K = K(k)$ suitably large,

$$c_k = 2\pi k^2 \int_0^\pi \left[\frac{1}{2}\left|\frac{dU_k}{dx}\right|^2 - G(x, U_k)\right] dx . \tag{1.22}$$

By (1.7),

$$\int_0^\pi \left|\frac{dU_k}{dx}\right|^2 dx = \int_0^\pi U_k(x)\, g(x, U_k(x))\, dx . \tag{1.23}$$

Combining (1.21)-(1.23) yields

$$\int_0^\pi [\tfrac{1}{2} U_k g(x, U_k) - G(x, U_k)]\, dx \to 0 \tag{1.24}$$

as $k \to \infty$ along this subsequence. Moreover by $(\bar{f}_3)$,

$$\int_0^\pi \left[\frac{1}{2} U_k g(x, U_k) - G(x, U_k)\right] dx \geqslant \int_0^\pi \left(\frac{1}{2} - \frac{1}{\bar{\mu}}\right) U_k g(x, U_k)\, dx . \tag{1.25}$$

Thus $U_k g(x, U_k) \to 0$ in L^1. From (1.23) again we conclude that $dU_k/dx \to 0$ in L^2 which easily implies $U_k \to 0$ in L^∞. By (f_2), for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|r| \leqslant \delta$ implies $|g(x, r)| \leqslant \varepsilon r$. Choosing $\varepsilon < 1/\pi$ and k large enough so that $\|U_k\|_{L^\infty} \leqslant \delta$, (1.23) then shows

$$\left\|\frac{dU_k}{dx}\right\|_{L^2}^2 \leqslant \varepsilon \|U_k\|_{L^2}^2 \leqslant \pi\varepsilon \left\|\frac{dU_k}{dx}\right\|_{L^2}^2 < \left\|\frac{dU_k}{dx}\right\|_{L^2}^2 , \tag{1.26}$$

a contradiction. Consequently U_k depends on τ for all large k.

To prove the second assertion of Theorem 1.8, suppose two functions $U_k(x, \tau)$, $U_j(x, \tau)$ correspond to the same function of x, t modulo a translation in time (keeping Remark 1.9 in mind). Thus $U_k(x, \tau) = U_k(x, t/k) \equiv U(x, t)$ and $U_j(x, \tau) = U_j(x, t/j) = U(x, t + \theta)$ for some $\theta \in \mathbf{R}$ or $U_k(x, \tau) = U(x, k\tau)$, $U_j(x, \tau) = U(x, j\tau + \theta)$. Since U must be both $2\pi k$ and $2\pi j$ periodic in t, letting σ denote the greatest common divisor of j and k, we have $j = \sigma\bar{j}$,

$k = \sigma\bar{k}$ and U has period $2\pi\sigma$ in t. Furthermore

$$c_k = \int_0^{2\pi}\int_0^{\pi}\left[\frac{k^2}{2}U_{kx}^2 - \frac{1}{2}U_{k\tau}^2 - k^2 G(x, U_k)\right]dx\,d\tau = \tag{1.27}$$
$$= k\int_0^{2\pi k}\int_0^{\pi}\left[\frac{1}{2}(U_x^2 - U_t^2) - G(x, U)\right]dx\,dt =$$
$$= \frac{k^2}{\sigma}\int_0^{2\pi\sigma}\int_0^{\pi}\left[\frac{1}{2}(U_x^2 - U_t^2) - G(x, U)\right]dx\,dt \equiv \frac{k^2}{\sigma}b$$

and similarly

$$c_j = \frac{j^2}{\sigma}b\,. \tag{1.28}$$

Consequently if there were a sequence of solutions U_{k_i} of (1.10) corresponding to the same function U (up to a translation in t), by (1.27)-(1.28) we have

$$c_{k_i} = \frac{k_j^2}{\sigma}b \tag{1.29}$$

and $c_{k_i} \to \infty$ like k_i^2 along this sequence contrary to (1.19). Thus at most finitely many functions $U_k(x, \tau)$ correspond to the same solution $u_k(x, t)$ of (1.5), (1.3) and infinitely many of the functions u_k must be time dependent solutions of (1.5), (1.3). The proof of Theorem 1.8 is complete.

REMARK 1.30. Both the existence assertions from [11] and the arguments given above use hypothesis (f_2) which requires that g vanish more rapidly than linearly at 0. However this condition can be weakened. The simplest such generalization would be to replace $g(x, r)$ by $\alpha r + g(x, r)$ with α a constant and for this case we have:

THEOREM 1.31. *Let g satisfy (f_1), (f_2), $(\bar{f}_3)$ and let $\alpha > 0$. Then for all $T \in l\boldsymbol{Q}$, there exists a $k_0 \in \boldsymbol{N}$ such that for all $k \geqslant k_0$, the problem*

$$\begin{cases} u_{tt} - u_{xx} - \alpha u - g(x, u) = 0 & 0 < x < l \\ u(0, t) = 0 = u(l, t) \\ u(x, t + kT) = u(x, t) \end{cases} \tag{1.32}$$

has a continuous weak solution u_k which is kT periodic in t and $\partial u_k/\partial t \not\equiv 0$. Moreover infinitely many of these functions are distinct.

PROOF. For convenience we again take $l = \pi$, $T = 2\pi$. It was shown in [11] that Theorem 1.2 carries over to (1.32) for $\alpha > 0$. It is also easy to see that the argument of Lemma 1.20 will give (1.21) for this setting. Likewise (1.27)-(1.29) are unaffected by the α term. Thus we get Theorem 1.31 provided that we can show $U_k(x, \tau)$ depends on τ for all large k. If not, the analogues of (1.22)-(1.23) here are

$$c_k = 2\pi k^2 \int_0^\pi \left[\frac{1}{2}\left|\frac{dU_k}{dx}\right|^2 - \frac{1}{2}\alpha U_k^2 - G(x, U_k)\right] dx \tag{1.33}$$

and

$$\int_0^\pi \left|\frac{dU_k}{dx}\right|^2 dx = \int_0^\pi (\alpha U_k^2 + U_k g(x, U_k))\, dx\,. \tag{1.34}$$

Thus (1.19), (1.33)-(1.34), and $(\bar{f}_3)$ show that $U_k g(x, U_k) \to 0$ in L^1 as $k \to \infty$ as in (1.24)-(1.25). Since

$$\|g(x, U_k)\|_{L^1} \leqslant \pi \max_{0 \leqslant x \leqslant \pi, |r| \leqslant 1} |g(x, r)| + \|U_k g(x, U_k)\|_{L^1} \tag{1.35}$$

and the right hand side of (1.35) is uniformly bounded in k, it follows from (1.7) that the functions d^2U_k/dx^2 are uniformly bounded in L^1. The boundary conditions $U_k(0) = 0 = U_k(\pi)$ imply that there is $x_k \in (0, \pi)$ such that $(dU_k/dx)(x_k) = 0$. Hence

$$\frac{dU_k}{dx} = \int_{x_k}^{x} \frac{d^2U_k(\xi)}{d\xi^2}\, d\xi$$

which implies that

$$\left\|\frac{dU_k}{dx}\right\|_{L^\infty} \leqslant \left\|\frac{d^2U_k}{dx^2}\right\|_{L^1}. \tag{1.36}$$

Thus the functions U_k, dU_k/dx are bounded in L^∞ and by (1.7) again, so are d^2U_k/dx^2. It follows that a subsequence of U_k converges (in $\|\cdot\|_{C^1}$) to a solution U of (1.7) as $k \to \infty$. But (f_1) and $\|U_k g(x, U_k)\|_{L^1} \to 0$ as $k \to \infty$ imply $U \equiv 0$.

Next observe that (1.7) can be written as

$$U_k(x) = \int_0^\pi H(x, y)\big(\alpha U_k(y) + g(y, U_k(y))\big)\, dy \tag{1.37}$$

where H is the Green's function for $-d^2/dx^2$ under the boundary condition $U(0)=0=U(\pi)$. Dividing (1.37) by $\|U_k\|_{C^1}$ gives

$$\frac{U_k(x)}{\|U_k\|_{C^1}}=\int_0^\pi H(x,y)\left(\alpha\frac{U_k(y)}{\|U_k\|_{C^1}}+\frac{g(x,U_k(y))}{\|U_k\|_{C^1}}\right)dy\,. \tag{1.38}$$

By (f_2), the arguments of the integral operator are uniformly bounded in C^1. Hence since this operator is compact from C^1 to C^2, by (f_2) again a subsequence of $U_k/\|U_k\|_{C^1}$ converge to V satisfying $\|V\|_{C^1}=1$ and

$$V(x)=\alpha\int_0^\pi H(x,y)V(y)\,dy \tag{1.39}$$

or equivalently

$$-V''=\alpha V\qquad 0<x<\pi;\qquad V(0)=0=V(\pi)\,. \tag{1.40}$$

If α is not an eigenvalue of $-d^2/dx^2$ under these boundary conditions we have a contradiction and the proof is complete. Thus suppose α is an eigenvalue. Consider the eigenvalue problems:

$$-z''=\lambda\alpha z\,,\qquad 0<x<\pi;\qquad z(0)=0=z(\pi) \tag{1.41}$$

$$-y''=\mu\left(\alpha+\frac{g(x,\varphi)}{\varphi}\right)y\,,\qquad 0<x<\pi;\qquad y(0)=0=y(\pi) \tag{1.42}$$

where φ is C^1 on $[0,\pi]$. Let λ_j (resp. $\mu_j(\varphi)$) denote the j-th eigenvalue of (1.41) (resp. (1.42)), the eigenvalues being ordered according to increasing magnitude. As is well known any eigenfunction corresponding to λ_m or $\mu_m(\varphi)$ belongs to

$$S_m=\{\varphi\in C^1([0,\pi],\boldsymbol{R})\,|\,\varphi(0)=0=\varphi(\pi),\ \varphi \text{ has exactly } m-1 \text{ zeros in } (0,\pi), \text{ and } \varphi'\neq 0 \text{ at all zeros of } \varphi \text{ in } [0,\pi]\}\,.$$

(Indeed the eigenvalues of (1.41) are $\lambda_m=m^2\alpha^{-1}$ and corresponding eigenfunctions are multiples of $\sin mx$). Since $g(x,\varphi)\varphi^{-1}\geqslant 0$ via (f_1), we have $\lambda_j\geqslant\varphi_j(\varphi)$ for all $j\in\boldsymbol{N}$ and $\varphi\in C^1$, $\varphi\not\equiv 0$ via a standard comparison theorem [15, Chapter 6]. By (1.40), 1 is an eigenvalue of (1.41), say $1=\lambda_m$ and $V\in S_m$. Thus $\mu_m(\varphi)\leqslant 1$ and since S_m open (in the C^1 topology) and $U_k/\|U_k\|_{C^1}\to V$ in C^1 along some subsequence, it follows that $U_k/\|U_k\|_{C^1}$ and therefore U_k belongs to S_m for all large k in this subsequence. Writing (1.7) as

$$(1.43)\qquad -U_k'' = \left(\alpha + \frac{g(x, U_k)}{U_k}\right)U_k\,, \qquad 0 < x < \pi; \qquad U_k(0) = 0 = U_k(\pi)\,,$$

we see $\mu_m(U_k) = 1$. By (f_1) again, $g(x, U_k)U_k^{-1} > 0$ except at the $m+1$ zeros of U_k. An examination of the proof of the Sturm Comparison Theorem [16, pp. 208-209] then shows U_k has a zero between each pair of successive zeros of V. Consequently $U_k \in S_{m+1}$, a contradiction. Thus Theorem 1.31 is established.

REMARK 1.44. In [5], Brezis, Coron and Nirenberg study (1.1), (1.3) replacing (f_3) by

(f_4) $\frac{1}{2}rf(r) - V(r) \geq \beta|f(r)| - \gamma$

and

(f_5) $f(r)/r \to \infty$ as $|r| \to \infty$

(and with no analogue of (f_2)). If we use (f_4)-(f_5) with x dependent f in place of (f_3), it is not difficult to see that the proof of [11] carries over for this case as does Lemma 1.20 and (1.27)-(1.29). Thus we obtain a variant of Theorem 1.8 for this case once it is established that $U_k(x, \tau)$ depends on τ for large k. To do this, we argue as in the proof of Theorem 1.8. Assume (f_4) holds with $\gamma = 0$. Then by (1.25) and (f_4), $\|g(x, U_k)\|_{L^1} \to 0$ as $k \to \infty$. This in turn implies $\|U_k\|_{L^\infty} \to 0$ via (1.7) and (1.36). Hence (1.26) again provides a contradiction.

It is also possible for us to drop (f_2) and even the requirement that $f(x, 0) = 0$ in (f_1) but then a new existence mechanism is required and we shall not carry out the details here.

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