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On the a.e. Convergence of Convolution Integrals and Related Problems.

W. B. JURKAT - G. SAMPSON (*)

0. - Introduction.

In this paper we are concerned with kernels K on R^n which satisfy the following conditions:

$$\int_{|x|\geqslant 2|y|} |K(x+y)-K(x)|\,dx \leqslant B_1\,, \qquad y\neq 0 \tag{1}$$

$$\left|\int_{\varrho_1<|x|<\varrho_2} K(x)\,dx\right| \leqslant B_2\,, \qquad 0<\varrho_1<\varrho_2 \tag{2}$$

and

$$\int_{\varrho<|x|<2\varrho} |K(x)|\,dx \leqslant B_3\,, \qquad \varrho>0. \tag{3}$$

For example in R the Hilbert kernel $H(t)=t^{-1}$, $t\neq 0$ and the fractional kernel $F(t)=t^{-1+i\beta}$, $t>0$, $F(t)=0$ for $t\leqslant 0$ ($\beta\neq 0$ and is a real number) both satisfy conditions (1), (2) and (3).

DEFINITION 1. *We say that a kernel K is well-behaved if K satisfies* (1), (2) *and* (3).

For various well-behaved kernels K, pointwise convergence of the corresponding convolution is quite dissimilar. For example in R, compare the Hilbert kernel $H(t)$ with the fractional kernel $F(t)$, [24] resp. [13]. The main result of this paper is nevertheless to show that all well-behaved kernels satisfy a common convergence criterion. The result takes the following form:

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THEOREM 1. *If K is well-behaved and $f \in L_{\mathrm{loc}}(R^n)$, then*

$$\oint\limits_{|x|\leqslant 1} K(x)(f(y-x)-f(y))\,dx = \lim_{\varepsilon\to+0} \int\limits_{\varepsilon<|x|\leqslant 1} K(x)(f(y-x)-f(y))\,dx$$

exists for almost all y.

We use Theorem 1 to define a class of well-behaved convolutions (see Theorem 3). We also use it to prove convergence of Marcinkiewicz type integrals as well as their generalizations (see Theorems 4, 5).

For finite positive absolute constants depending at most on n we use the letter c generically. If dependency on B_1, B_2, B_3 is permitted we use B generically. Finally, when dependency on another variable occurs, say like p, we indicate that by c_p or B_p.

1. – Preliminary remarks.

Here, we begin by defining maximal functions which are associated with well-behaved kernels. We shall further obtain results for these maximal functions that are needed for us to prove the main result.

DEFINITION 2. *Let $1\leqslant p<\infty$ and $f\in L^p$. If K is well-behaved then the function*

$$M(f;x) = \sup_{0<a<1<b} M_{a,b}(f;x)\,, \qquad M_{a,b}(f;y) = \sup_{\alpha,\beta}\Bigg|\int\limits_{\alpha<|x|<\beta} dx\,K(x)\,f(y-x)\Bigg|\,,$$

where $0<a<\alpha\leqslant 1<\beta<b$, denotes the maximal function of a well-behaved convolution.

For g measurable we set ($1\leqslant p<\infty$),

$$\|g\|_p^{\#} = \sup_{\lambda>0} \lambda\big|\{x\colon |g(x)|>\lambda\}\big|^{1/p}$$

the weak p « norm » of g.

By T we denote a sublinear operator defined on L_0^∞ (bounded functions with compact support). We set

$$\|T\|_p = \sup_{f\in L_0^\infty} \|Tf\|_p^{\#}/\|f\|_p,\ \|T\|_p = \sup_{f\in L_0^\infty} \|Tf\|_p/\|f\|_p$$

where $\|f\|_p\neq 0$ and $1\leqslant p<\infty$.

In [16], N. M. Rivière was able to prove the following:

THEOREM 2 (Rivière). *If* (*i*) *K is well-behaved and* (ii) *for each fixed* $\varrho_1 > 0$,

$$\lim_{\varrho\to+0} \int_{\varrho<|x|<\varrho_1} K(x)\,dx \qquad \textit{exists},$$

then

$$\|Mf\|_p^{\#} \leqslant c_p B \|f\|_p .$$

Thus, for well-behaved kernels K it follows from Rivière's theorem that

$$\|M_{a,b} f\|_p^{\#} \leqslant c_p B_{a,b} \|f\|_p \tag{4}$$

where $B_{a,b}$ is that «B» constant which is obtained from the kernel $K_{a,b}(x) = K(x)$ for $a < |x| < b$ and $K_{a,b}(x) = 0$ elsewhere. But for K well-behaved it is well-known (and easy to show) that

$$B = \sup_{a,b} B_{a,b} < \infty . \tag{5}$$

But from (4) and (5) we get for $\lambda > 0$ and $1 \leqslant p < \infty$,

$$\lambda |\{x\colon M_{a,b}(f; x) > \lambda\}|^{1/p} \leqslant c_p B \|f\|_p ,$$

and now letting $a\downarrow 0$ and $b\uparrow\infty$ we get the following:

COROLLARY 1. *The maximal operator for well-behaved convolutions is weak* (p, p) *for all* $1 \leqslant p < \infty$, i.e.,

$$\|Mf\|_p^{\#} \leqslant c_p B \|f\|_p \qquad \textit{for } f \in L^p,\ 1 \leqslant p < \infty.$$

It should be pointed out that even though the corollary is a simple extension of Rivière's Theorem (Theorem 2), it is the corollary that is useful to us. This corollary can be interpreted to mean that for well-behaved kernels one does not have to restrict their behavior about the origin (as (ii) in Theorem 2) in order to prove a weak mapping property for their associated maximal convolutions.

2. – Proof of Theorem 1.

We begin with the proof.

PROOF. Set

$$R_\varepsilon(f; y) = \int\limits_{\varepsilon<|x|<1} K(x)\big(f(y-x) - f(y)\big)\,dx$$

and consider those y's so that $|y| \leqslant N$. Then $|y-x| \leqslant N+1$ and hence,

$$R_\varepsilon(f; y) = \int\limits_{\varepsilon<|x|<1} K(x)\big(f_N(y-x) - f_N(y)\big)\,dx$$

for $|y| \leqslant N$, where $f_N(u) = f(u)$ for $|u| \leqslant N+1$ and $f_N(u) = 0$ elsewhere. Set

$$\bar{R}(f; y) = \sup_{\varepsilon>0} |R_\varepsilon(f; y)|\,, \quad \text{then for } |y| \leqslant N$$

$$\bar{R}(f; y) \leqslant M(f_N; y) + B|f_N(y)|\,.$$

Hence by the Corollary we get,

$$(6) \quad \big|\{y\colon |y| \leqslant N \text{ and } \bar{R}(f; y) > \lambda\}\big| \\ \leqslant \big|\{y\colon M(f_N; y) > \lambda/2\}\big| + \big|\{y\colon |f_N(y)| > \lambda/2B\}\big| \leqslant (cB/\lambda)\|f_N\|_1\,.$$

Set

$$\Omega(f; y) = \limsup_{\varepsilon_1,\varepsilon_2\to+0} |R_{\varepsilon_1}(f; y) - R_{\varepsilon_2}(f; y)|$$

for y fixed. Since K satisfies (3) we get for $\varphi \in C_0^\infty$ that $\Omega(f-\varphi; y) = \Omega(f; y)$. Thus, for $\varepsilon > 0$ we get

$$\big|\{y\colon |y| \leqslant N \text{ and } \Omega(f; y) > \varepsilon\}\big| = \big|\{y\colon |y| \leqslant N \text{ and } \Omega(f-\varphi; y) > \varepsilon\}\big| \\ \leqslant \big|\{y\colon |y| \leqslant N \text{ and } \bar{R}(f-\varphi; y) > \varepsilon/2\}\big| \leqslant (cB/\varepsilon)\|f_N - \varphi\|_1\,,$$

the last inequality coming from (6). Now choose $\varphi_\nu \in C_0^\infty$ so that $\varphi_\nu \to f_N$ in L^1, hence

$$\big|\{y\colon |y| \leqslant N \text{ and } \Omega(f; y) > \varepsilon\}\big| = 0$$

and this holds for each $\varepsilon > 0$, hence

$$\big|\{y\colon |y| \leqslant N \text{ and } \Omega(f; y) > 0\}\big| = 0$$

and now it follows that

$$\left|\{y\colon \Omega(f;y) > 0\}\right| = 0 .$$

Thus, $\lim\limits_{\varepsilon\to+0} R_\varepsilon(f;y)$ exists for almost all y. Hence the proof of Theorem 1 is now complete.

REMARK 1. By a similar argument we could show that for $f \in L^p, 1 \leqslant p < \infty$

$$\oint\limits_{|x|\geqslant 1} K(x) f(y-x)\, dx = \lim_{\omega\to\infty} \int\limits_{1\leqslant|x|<\omega} K(x) f(y-x)\, dx$$

exists for almost all y.

Here, we set

$$R_\omega(f;y) = \int\limits_{1\leqslant|x|<\omega} K(x) f(y-x)\, dx$$

$$\bar{R}(f;y) = \sup_{\omega>1} |R_\omega(f;y)|$$

and $\Omega(f;y) = \limsup\limits_{\omega_1,\varepsilon_2\to\infty} |R_{\omega_1}(f;y) - R_{\omega_2}(f;y)|$ for y fixed.

3. – Representations of well-behaved convolutions.

In this section, we will give a definition for well-behaved convolutions i.e., «$K * f$» where K is well-behaved and $f\in L^p$, $1\leqslant p<\infty$. We need to be sure that our definition for «$K * f(x)$» reduces to the classical case when $K(x)$ is the Hilbert kernel $H(x)$ or the fractional kernel $F(x)$ or when K is locally integrable at zero.

Let us begin with $K \in L_{\text{loc}}(R^n - \{0\})$ and set

$$K_{\varepsilon,\omega}(x) = \begin{cases} K(x) & \varepsilon < |x| < \omega \ \ (\varepsilon < 1 < \omega) \\ 0 & \text{elsewhere.} \end{cases}$$

Then for $f\in L^p$, $1\leqslant p < \infty$, we have

$$(7) \qquad K_{\varepsilon,\omega} * f(x) = f(x) I_\varepsilon(K) + \int\limits_{\varepsilon<|y|<1} K(y)\big(f(x-y) - f(x)\big)\, dy + \\ + \int\limits_{1<|y|<\omega} K(y) f(x-y)\, dy = \text{I} + \text{II} + \text{III},$$

where $I_\varepsilon(K) = \int\limits_{\varepsilon<|y|<1} K(y)\, dy$.

For kernels K that are well-behaved we see by Theorem 1 and Remark 1 that the terms II and III have limits for a.a.x as $\varepsilon \to +0$ and $\omega \to +\infty$. In order that this expression reduces to the usual convolution in the classical cases, we add a suitable limit for the term I. For example in the case of the Hilbert kernel one has $I_\varepsilon(H) = 0$ for each $\varepsilon > 0$, so the limit is zero.

One way of defining $I(K)$ (the limit of $I_\varepsilon(K)$) is via a summability method.

DEFINITION 3. Let $\{A_\varepsilon(t)\}$ be a fixed family of measurable functions that satisfy the following properties:

(i) $A_\varepsilon(t) \geqslant 0$ for $\varepsilon > 0$, $0 < t \leqslant 1$

(ii) $\int_0^1 A_\varepsilon(t)\,dt = 1$ for all $\varepsilon > 0$, and

(iii) for each $0 < h < 1$, $A_\varepsilon(t)$ goes uniformly to zero as $\varepsilon \to +0$ for $h \leqslant t \leqslant 1$.

For $f \in L_{\text{loc}}(R^n - \{0\})$, we set

$$\oint_{|y|\leqslant 1} f(y)\,dy = \lim_{\varepsilon \to +0} \int_0^1 dt A_\varepsilon(t) \int_{t \leqslant |u| \leqslant 1} du f(u)$$

and call this *the generalized Cauchy-Lebesgue integral* (we write CL) provided the limit exists, and in such cases we say that $f \in CL$. [This concept depends on the summability method used.] If the usual Cauchy-Lebesgue integral exists, it coincides with the generalized one, since $A_\varepsilon(t)$ defines a regular summability method. For the fractional kernel $F(x)$ in R, we could choose $A_\varepsilon(t) = t^{\varepsilon-1}$ for $0 < t \leqslant 1$.

For K well-behaved and $K \in CL$, we set

$$I(K) = \oint_{|y|\leqslant 1} K(y)\,dy\,. \tag{8}$$

Again in the case of the Hilbert kernel H, $I(H) = 0$ and for the fractional kernel F, $I(F) = (i\beta)^{-1}$.

THEOREM 3. *Let $f \in L^p$, $1 \leqslant p < \infty$. If K is well-behaved and $K \in CL$ and if we set*

$$K * f(x) = f(x) I(K) + \oint_{|y|\leqslant 1} K(y)\big(f(x-y) - f(x)\big)\,dy + \oint_{|y|\leqslant 1} K(y) f(x-y)\,dy\,, \tag{9}$$

then $K * f(x)$ exists for almost all x.

PROOF. Since $K \in CL$ that implies $I(K)$ exists. Since K is well-behaved hence by Theorem 1 and Remark 1, it follows that $K * f(x)$ exists for almost all x.

REMARK 2. From (9) it follows for K well-behaved and $K \in CL$ that,

$$|K * f(x)| \leqslant |I(K)||f(x)| + B|f(x)| + M(f; x) . \tag{10}$$

From Corollary 1 and (10) we get that $K \in L_p^{\#}$, i.e., weak (p, p) for all $1 \leqslant p < \infty$.

We note that even for $\varphi \in C_0^\infty$ (infinitely differentiable functions with compact support) there is no simpler definition for $K * \varphi$ than (9) except that terms II and III in (9) now exist as Lebesgue integrals. The only variation that is possible is in the interpretation of $I(K)$.

Also, note that by (10),

$$|K * f - K * \varphi| \leqslant |I(K)||f(x) - \varphi(x)| + B|f(x) - \varphi(x)| + M(f - \varphi; x) .$$

Hence if $\{\varphi_\nu\}$ is a family of C_0^∞ functions that converge to f in L^p, for some $1 \leqslant p < \infty$, then $K * \varphi$ converges to $K * f$ in measure.

4. – Applications.

Marcinkiewicz has shown in R that there is a locally absolutely continuous function F (i.e., an $f \in L_{\text{loc}}$ whereby $F(x) = \int_0^x f + \text{constant}$) so that

$$\int_{+0}^{1} \frac{|F(x + t) + F(x - t) - 2F(x)|}{t^2}\, dt = + \infty \quad \text{for a.a.} x, \tag{11}$$

see [12].

It has since been shown that if $f \in L_{\text{loc}}$, $F(x) = \int_0^x f$ and $b \in L^\infty(0, 1)$, then

$$\lim_{\varepsilon \to +0} \int_{\varepsilon}^{1} b(t) \left\{\frac{F(x + t) + F(x - t) - 2F(x)}{t^2}\right\} dt \tag{12}$$

exists and is finite for a.a.x.

Plessner in [14] did the case when $b(t) = 1$ for $t \in (0, 1)$, and Ostrow and Stein in [15] did the general case. Stein [17], Walsh [20], Zygmund [23] and others have generalized some similar ideas (eg., the Marcinkiewicz function) to R^n. In this section we will give a new proof of (12) along with generalizations to R^n. Under the proper conditions on $b(t)$, we will also prove in R (along with generalizations to R^n) that

$$\lim_{\varepsilon\to+0}\int_{\varepsilon}^{1} dt\,\frac{b(t)}{t^2}\big(F(x+t)-F(x-t)-2tf(x)\big) \tag{13}$$

exists for a.a.x (see Theorem 5).

LEMMA 1. *Let U be the closed unit ball in R^n. Assume $\Omega\in L^\infty(R^n)$, $b\in L^\infty(R)$ and $b(t)=0$ for $t\notin(0,1)$, and*

$$K(x)=\frac{\Omega(x)}{|x|^{n-1}}\int_{|x|}^{1}\frac{b(t)}{t^2}\,dt\,,\qquad x\neq 0$$

and the integral is to be interpreted algebraically. Then,

$$|K(x)|\leqslant\|\Omega\|_\infty\|b\|_\infty|x|^{-n}\qquad(x\neq 0) \tag{14}$$

and

$$|K(x-y)-K(x)|\leqslant\chi_U(y)\|b\|_\infty|x|^{-n}\big(|\Omega(x)-\Omega(x-y)|+c\|\Omega\|_\infty|y||x|^{-1}\big)\quad \textit{for } |x|\geqslant 2|y|>0. \tag{15}$$

PROOF. The result (14) follows immediately from the hypothesis.
To show (15) we note that

$$K(x)-K(x-y)=\left(\frac{\Omega(x)-\Omega(x-y)}{|x|^{n-1}}\right)\int_{|x|}^{1}\frac{b(t)}{t^2}\,dt+ \tag{16}$$
$$+\frac{\Omega(x-y)}{|x|^{n-2}}\int_{|x|}^{|x-y|}\frac{b(t)}{t^2}\,dt+\Omega(x-y)\left(\frac{1}{|x|^{n-1}}-\frac{1}{|x-y|^{n-1}}\right)\int_{|x-y|}^{1}dt\,\frac{b(t)}{t^2}$$

and for $|x|\geqslant 2|y|$ we get that

$$|K(x)-K(x-y)|\leqslant\|b\|_\infty|x|^{-n}\big(|\Omega(x)-\Omega(x-y)|+c\|\Omega\|_\infty|y||x|^{-1}\big)\,.$$

Now since the support of K is contained in U, and the difference on the left is zero if $|x|\geqslant 2$, then the result (15) follows.

DEFINITION 4. We say that a function $\Omega(x)$ in R^n is *regular* if $\Omega \in L^\infty(R^n)$ and

$$\sup_{0<|y|\leq 1} \int_{|x|\geq 2|y|} \frac{|\Omega(x)-\Omega(x-y)|}{|x|^n}\,dx < \infty . \tag{17}$$

THEOREM 4. *Let* $f \in L_{\mathrm{loc}}(R^n)$. *If* $b \in L^\infty(R)$ *and* $b(t) = 0$ *for* $t \notin (0,1)$ *and* Ω *is regular, then*

$$\lim_{\varepsilon\to+0} \int_\varepsilon^1 dt\,\frac{b(t)}{t^2} \int_{|y|\leq t} dy\,\frac{\Omega(y)}{|y|^{n-1}}\,(f(x-y)-f(x+y)) \tag{18}$$

exists for almost all x.

REMARK 3. If $n = 1$ and $\Omega(y) = 1$ for $y > 0$, $\Omega = 0$ otherwise, then Ω satisfies the hypothesis (of Theorem 4) and hence (18) is a generalization in R^n of (12).

PROOF OF THEOREM 4. Define

$$K(x) = \frac{(\Omega(x)-\Omega(-x))}{|x|^{n-1}} \int_{|x|}^1 dt\,\frac{b(t)}{t^2}, \tag{19}$$

hence K has its support in the unit ball. We note also that K is odd and $\Omega(-x)$ is regular. We begin by showing that K is well-behaved. Since K is odd, i.e., $K(x) = -K(-x)$ we get that for each $0 < \varrho_1 < \varrho_2$

$$\int_{\varrho_1<|x|<\varrho_2} K(x)\,dx = 0 .$$

From the preceding remarks and (14) and (15) of the Lemma we get that K (as defined in (19)) is well-behaved.

$$\begin{aligned}
\int_{\varepsilon<|y|<1} (f(x-y)-f(x))\,K(y)\,dy &= \int_{\varepsilon<|y|<1} (f(x-y)-f(x))\left(\frac{\Omega(y)-\Omega(-y)}{|y|^{n-1}}\right) \int_{|y|}^1 dt\,\frac{b(t)}{t^2} \\
&= \int_\varepsilon^1 dt\,\frac{b(t)}{t^2} \int_{\varepsilon<|y|\leq t} dy\,\frac{(f(x-y)-f(x))}{|y|^{n-1}}\,(\Omega)y) - \Omega(-y)) \\
&= \int_\varepsilon^1 dt\,\frac{b(t)}{t^2} \int_{|y|\leq t} dt\,\frac{\Omega(y)}{|y|^{n-1}}\,(f(x-y)-f(x+y)) \\
&\quad - \int_\varepsilon^1 dt\,\frac{b(t)}{t^2} \int_{|y|\leq \varepsilon} dy\,\frac{\Omega(y)}{|y|^{n-1}}\,(f(x-y)-f(x+y)) .
\end{aligned}$$

But

$$\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\int\limits_{|y|\leqslant\varepsilon}\frac{|f(x+y)-f(x)|}{|y|^{n-1}}\,dy=0 \qquad \text{for a.a.}x,$$

now by Theorem 1 we are through.

THEOREM 5. *Let $f\in L_{\text{loc}}(R^n)$. Suppose that $b\in L^\infty(R)$, $b(t)=0$ for $t\notin(0,1)$ and Ω is regular. Also assume that*

$$\sup_{0<\varepsilon}\left|\int\limits_{\varepsilon}^{1}dt\,\frac{b(t)}{t^2}\int\limits_{|y|\leqslant t}dy\,\frac{\Omega(y)}{|y|^{n-1}}\right|<\infty. \tag{20}$$

Then,

$$\lim_{\varepsilon\to+0}\int\limits_{\varepsilon}^{1}dt\,\frac{b(t)}{t^2}\int\limits_{|y|\leqslant t}dy\,\frac{\Omega(y)}{|y|^{n-1}}\,\big(f(x+y)+f(x-y)-2f(x)\big) \tag{21}$$

exists for a.a.x.

REMARK 4. If $n=1$ and $\Omega(y)=1$, then Ω is regular and hence (21) is a generalization to R^n of (13), condition (20) reduces to $\int\limits_{\varepsilon}^{1}b(t)/t\,dt=0(1)$ in this case.

PROOF OF THEOREM 5. Define

$$K(x)=\left(\frac{\Omega(x)+\Omega(-x)}{|x|^{n-1}}\right)\int\limits_{|y|}^{1}\frac{b(t)}{t^2}\,dt \tag{22}$$

hence K is even and has support in the unit ball. Therefore, in order to show (2) we may assume $\varrho_2\leqslant 1$.

Then

$$\int\limits_{\varrho<|x|<\varrho_2}dx\left(\frac{\Omega(x)+\Omega(-x)}{|x|^{n-1}}\right)\left(\int\limits_{|x|}^{\varrho_2}dt\,\frac{b(t)}{t^2}+\int\limits_{\varrho_2}^{1}dt\,\frac{b(t)}{t^2}\right)=\int\limits_{\varrho_1<|x|<\varrho_2}\left(\frac{\Omega(x)+\Omega(-x)}{|x|^{n-1}}\right)\int\limits_{\varrho_2}^{1}dt\,\frac{b(t)}{t^2}$$
$$+\int\limits_{\varrho_1}^{\varrho_2}dt\,\frac{b(t)}{t^2}\int\limits_{|x|\leqslant t}dx\left(\frac{\Omega(x)+\Omega(-x)}{|x|^{n-1}}\right)-\int\limits_{\varrho_1}^{\varrho_2}dt\,\frac{b(t)}{t^2}\int\limits_{|x|\leqslant\varrho_1}dx\left(\frac{\Omega(x)+\Omega(-x)}{|x|^{n-1}}\right),$$

and using (20) it follows that K satisfies (2). Now by (14) and (15) of the Lemma, since Ω is regular, we get that K in (22) is well-behaved.

But since K in (22) is even, we get

$$\int_{\varepsilon<|y|\leqslant 1} K(y)\big(f(x+y)-f(x)\big)\,dy = \int_{\varepsilon<|y|\leqslant 1} dy\,\frac{\Omega(y)}{|y|^{n-1}}\Bigg(\int_{|y|}^{1} dt\,\frac{b(t)}{t^2}\Bigg)\big(f(x+y)+f(x-y)-2f(x)\big)$$

$$=\int_{\varepsilon}^{1} dt\,\frac{b(t)}{t^2}\int_{|y|\leqslant t} dy\,\frac{\Omega(y)}{|y|^{n-1}}\big(f(x+y)+f(x-y)-2f(x)\big)$$

$$-\int_{\varepsilon}^{1} dt\,\frac{b(t)}{t^2}\int_{|y|\leqslant \varepsilon} dy\,\frac{\Omega(y)}{|y|^{n-1}}\big(f(x+y)+f(x-y)-2f(x)\big)$$

and now by Theorem 1 we are through, since the last term goes to zero as $\varepsilon \to +0$.

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