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Global existence for the Hamilton-Jacobi equations in Hilbert space

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Global Existence for the Hamilton-Jacobi Equations in Hilbert Space.

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1. - Introduction.

In this paper we are concerned with the Hamilton-Jacobi equation

$$(1.1)\qquad \begin{cases} \varphi_t(t,x)+F(\varphi_x(t,x))+(Ax,\varphi_x(t,x))=g(t,x)\,; \quad x\in D(A),\ t\in[0,T] \\ \varphi(0,x)=\varphi_0(x) \end{cases}$$

in a Hilbert space H. Here F is a convex Fréchet differentiable function on H and $-A$ is the infinitesimal generator of a strongly continuous semigroup of linear continuous operators on H. The subscripts t and x denote the partial differentiation with respect to t and x and g, φ_0 are given real valued functions on $[0,T]\times H$ and H, respectively.

The contents of this paper are outlined below.

In section 2 we shall exhibit several properties of the operator $\varphi\to F(\varphi_x)$. In particular it is shown that it arises as the generator of a semigroup of a contractions on an appropriate subset K of the space $C(H)$ defined below.

In section 3 it is studied equation (1.1) with $A\equiv 0$. An explicit form of the solution in term of the semigroup $S(t)$ is given for the homogeneous equation and it is proved the existence and uniqueness of a weak solution in the class of continuous convex functions φ satisfying the conditions

$$\big(F'(\partial\varphi(x)),x\big)\geqslant 0\,,\qquad x\in H;\ 0\in\partial\varphi(0)\,.$$

Section 4 is concerned with equation

$$(1.2)\qquad \begin{cases} \varphi_t(t,x)+\frac{1}{2}|\varphi_x(t,x)|^2+(Ax,\varphi_x(t,x))=g(t,x), \qquad t\in[0,T];\ x\in D(A) \\ \varphi(0,x)=\varphi_0(x). \end{cases}$$

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The main results of this section, Theorems 3 and 4 give existence and uniqueness of weak and classical solutions in the class of continuous convex functions on H. In particular, some existence results for the operator equation

$$(1.3)\qquad \begin{cases} E_t + E_x A + A^* E + E_x E = G\,, & t \in [0, T] \\ E(0) = E_0 \end{cases}$$

are derived. Particular cases of equation (1.3) have been previously studied in [4], [5]. The situation in which G is a linear continuous self-adjoint operator on H (the Riccati equation) has been extensively studied in the past decade and we refer the reader to [12] and [15] for significant results and complete references.

In section 5 the relevance of eq. (1.1) in control theory and calculus of variations is explained. In section 6 we give some regularity properties for equation (1.3).

As far as we know, the present paper is the first attempt to study the Hamilton-Jacobi equations in Hilbert spaces. As regards the study of these equations in $\mathbb{R}^n$ the fundamental works of Kruzkov [16], Douglis [13], Fleming [14] must be cited. In [10] Crandall proposed a new method in the study of hyperbolic conservation laws equations based on the theory of nonlinear semigroup of contractions in Banach spaces (see also [6], [11]). The semigroup approach has been subsequently used in the study of Hamilton Jacobi equations in $\mathbb{R}^n$ by Aizawa [1], Burch [7], Burch and Goldstein [8] and other authors.

We conclude this section by listing briefly some definitions and notations that will be in effect throughout this paper. Let H be a real Hilbert space with norm $|\cdot|$ and inner product $(\cdot,\cdot)$.

Given a lower semicontinuous convex function $\varphi: H \to \overline{\mathbb{R}} = \mathbb{R}^1 \cup + \{\infty\}$ we shall denote by $\partial\varphi: H \to H$ the subdifferential of φ, i.e.,

$$(1.4)\qquad \partial\varphi(x) = \{x^* \in H;\ \varphi(x) \leqslant \varphi(y) + (x^*, x - y),\ \text{for all}\ y \in H\}$$

and by φ^* the conjugate of φ,

$$\varphi^*(y) = \sup\{(x, y) - \varphi(x);\ x \in H\}\,.$$

If φ is Fréchet (or more generally Gâteaux) differentiable at x then $\partial\varphi(x)$ consists of a single element, namely the gradient of φ. In the sequel we shall use either the symbol φ' or φ_x for the gradient of φ instead of the more conventional symbol $\nabla\varphi$.

For each $R > 0$ we shall denote by Σ_R the closed ball

$$\Sigma_R = \{x \in H;\ |x| \leqslant R\}.$$

$C(\Sigma_R)$ will denote the Banach space of all continuous and bounded functions $\varphi\colon \Sigma_R \to \mathbb{R}^1 =]-\infty, \infty[$ endowed with the norm

$$|\varphi|_R = \sup\left\{|\varphi(x)|;\ x \in \Sigma_R\right\}. \tag{1.5}$$

Let $C(H)$ be the space of all continuous functions $\varphi\colon H \to \mathbb{R}^1$ which are bounded on bounded subsets, topologized with the family of seminorms $\{|\varphi|_R;\ R > 0\}$. By $C^1(H)$ we shall denote the space of all Fréchet differentiable functions φ on H with Fréchet differential φ_x continuous, bounded on every bounded subset of H and with $\varphi_x(0) = 0$. $C^1(H)$ is a locally convex space endowed with the family of seminorms

$$|\varphi|_{1,R} = \sup\left\{|\varphi_x(x)|;\ x \in \Sigma_R\right\}. \tag{1.6}$$

By Lip (H) we shall denote the space of all functions $\varphi\colon H \to \mathbb{R}^1$ such that

$$|\varphi|_{\mathrm{Lip},R} = \sup\left\{\frac{|\varphi(x) - \varphi(y)|}{|x-y|};\ x \neq y;\ x, y \in \Sigma_R\right\}, \qquad \forall R > 0. \tag{1.7}$$

Further, we shall denote by $C^k(H, H)$, k a natural number or zero, the space of all continuously k times differentiable mappings $f\colon H \to H$ such that $f^{(j)}(0) = 0$ for $j = 0, 1 \dots k-1$ and

$$|f|_{k,R} = \sup\left\{\|f^{(k)}(x)\|_{L(H,H)};\ x \in \Sigma_R\right\}, \quad \forall R > 0. \tag{1.8}$$

Here $f^{(j)}$ denotes the Fréchet differential of order j of f and $\|\cdot\|_{L(H,H)}$ the norm in the space $L(H, H)$ of linear continuous operators from H into itself.

We shall denote by $C^0_{\mathrm{Lip}}(H, H)$ the space of all continuous mappings $f\colon H \to H$ which are Lipschitzian on every bounded subset, endowed with the family of seminorms

$$\|f\|_{0,R} = \sup\left\{\frac{|f(x) - f(y)|}{|x-y|};\ x, y \in \Sigma_R,\ x \neq y\right\}. \tag{1.9}$$

By $C^k_{\mathrm{Lip}}(H, H)$, where k is a natural number, we shall denote the space of all $f \in C^k(H, H)$ such that

$$\|f\|_{k,R} = \sup\left\{\frac{\|f^{(k)}(x) - f^{(k)}(y)\|_{L(H,H)}}{|x-y|};\ x, y \in \Sigma_R,\ x \neq y\right\}, \quad \forall R > 0. \tag{1.10}$$

Given a Banach space Z we shall denote by $C([0, T]; Z)$ the space of all continuous functions from $[0, T]$ to Z. If Z is one of the spaces $C(H)$, $C^1(H)$, $C^k(H, H)$ or $C^k_{\mathrm{Lip}}(H, H)$ we set

$$(1.11)\qquad C([0, T]; Z) = \Big\{f \in C([0, T]\times H);\ f(t, \cdot) \in Z \text{ for } t \in [0, T] \text{ and } |f|_{R,T} = \sup_{t[\in 0,T]} |f(t, \cdot)|_{Z,R} < \infty\Big\}$$

(here $|\cdot|_{Z,R}$ is one of the seminorms (1.5), (1.6), (1.7), (1.8), (1.9) or (1.10)) and

$$(1.12)\qquad C^1([0, T]; Z) = \{\varphi \in C([0, T]; Z);\ \varphi_t \in C([0, T]; Z)\}\,.$$

By $L^1(0, T; C(H))$ we shall denote the space of functions $f\colon [0, T] \to C(H)$ having the property that $f(t, x) \in L^1(0, T)$ for every $x \in H$ and

$$(1.13)\qquad \sup\Big\{\int_0^T |f(t, x)|\,dt;\ x \in \Sigma_R\Big\} < \infty$$

for every $R > 0$.

2. – Assumptions and auxiliary results.

To begin with let us set forth the assumptions which will be in effect throughout this paper.

(a) H is a real Hilbert space with norm $|\cdot|$ and inner product $(\cdot, \cdot)$.

(b) The function F is convex and belongs to $C^1(H)$. $F' \in C^0_{\mathrm{Lip}}(H, H)$ and

$$(2.1)\qquad \lim_{|x|\to\infty} F(x)/|x| = \infty\,.$$

(c) For every $R > 0$ there exists $\omega_R > 0$ such that

$$(2.2)\qquad (F'(x) - F'(y), x - y) \geqslant \omega_R |x - y|^2 \qquad \forall x, y \in \Sigma_R\,.$$

(d) The linear operator $-A$ is the infinitesimal generator of a strongly continuous semigroup of contractions $\exp(-At)$ on H. By A^* we shall denote the dual operator and by $D(A)$ the domain of A endowed with the graph norm.

We shall denote by K the set of all convex functions $\varphi \in C(H)$ satisfying the following two conditions

$$(F'(y), x) \geqslant 0 \quad \forall [x, y] \in \partial\varphi;\ 0 \in \partial\varphi(0)\,. \tag{2.3}$$

LEMMA 1. *Let F satisfy assumptions (b) and (c) and let $\varphi \in K$ be a given function. Then for every $t > 0$ the equation*

$$\partial\varphi(y) - (F')^{-1}\big(t^{-1}(x-y)\big) \ni 0 \tag{2.4}$$

has a unique solution $y = y_t(x)$ satisfying

$$|y_t(x)| \leqslant |x|\,. \tag{2.5}$$

Moreover, for each $t > 0$ the mapping $x \to y_t(x)$ belongs to $C^0_{\mathrm{Lip}}(H, H)$.

PROOF. According to a well-known perturbation result due to Browder (see e.g. [2], p. 46) the operator $\Gamma y = \partial\varphi(y) - (F')^{-1}\big(t^{-1}(x-y)\big)$ is maximal monotone on H. Since $F' \in C^0(H, H)$, $(F')^{-1}$ is coercive and therefore Γ is onto H. Hence eq. (2.4) has for each $t > 0$ at least one solution $y = y_t(x)$. Writing (2.4) as

$$y + tF'\big(\partial\varphi(y)\big) \ni x$$

and using condition (2.3) it follows (2.5). The uniqueness of y as well as the Lipschitzian dependence of $y(x)$ with respect to x follows by assumption (c).

In particular if $\varphi \in C^1(H)$ then by (2.4) it follows that

$$\varphi'\big(y_t(x)\big) = (F')^{-1}\Big(t^{-1}\big(x - y_t(x)\big)\Big)\,. \tag{2.6}$$

For every $\varphi \in K$ and $t \geqslant 0$, define

$$\big(S(t)\varphi\big)(x) = (\varphi^* + tF)^*(x)\,. \tag{2.7}$$

Since $\lim\limits_{|x| \to \infty} \big(\varphi^*(x) + tF(x)\big)/|x| = +\infty$, we may infer that for each $\varphi \in K$ and $t > 0$, $S(t)\varphi$ is a continuous convex function on H as well. Moreover, by Fenchel's duality theorem (see e.g. [3], p. 188), $S(t)$ can be equivalently defined for $t > 0$, as

$$\begin{aligned}\big(S(t)\varphi\big)(x) &= \inf\big\{\varphi(y) + tF^*\big(t^{-1}(x-y)\big);\ y \in H\big\} = \\ &= \varphi\big(y_t(x)\big) + tF^*\Big(t^{-1}\big(x - y_t{}'x)\big)\Big)\,, \qquad x \in H\,.\end{aligned} \tag{2.8}$$

It turns out that $\{S(t); t \geqslant 0\}$ is a semigroup of contractions on $C(H)$. More precisely, one has

LEMMA 2. *Let* $\{S(t); t \geqslant 0\}$ *be the family of nonlinear operators on* $C(H)$ *defined by formula* (2.7). *Then*

$$S(t)K \subset K \qquad \text{for all } t \geqslant 0 \tag{2.9}$$

$$S(t+s)\varphi = S(t)S(s)\varphi \qquad \text{for all } t, s \geqslant 0,\ \varphi \in K \tag{2.10}$$

$$|S(t)\varphi - S(t)\psi|_R \leqslant |\varphi - \psi|_R \qquad \text{for all } \varphi, \psi \in K,\ t \geqslant 0 \text{ and } R > 0\,. \tag{2.11}$$

Moreover, for every $t > 0$, $S(t)$ *maps* K *into* $C^1(H)$ *and*

$$\lim_{t \downarrow 0} t^{-1}(\varphi - S(t)\varphi) = F(\varphi') \quad \text{in } C(H) \tag{2.12}$$

for each $\varphi \in K \cap C^1(H)$ *such that* φ' *is uniformly continuous on every bounded subset of* H.

PROOF. Let φ be fixed in K. As observed earlier, $S(t)\varphi$ is a continuous real valued convex function on H. By Lemma 1 and formula (2.8) it follows that $S(t)\varphi$ is bounded on every bounded subset of H (because by assumption (2.1) F^* is bounded on bounded subsets). Moreover, by (2.7) it follows that (see e.g. [3], p. 100)

$$\partial\big(S(t)\varphi\big)(x) = (\partial\varphi^* + tF')^{-1}(x)\,, \qquad \forall x \in H,\ t > 0\,. \tag{2.13}$$

Since F' is strictly monotone on H, for each $t > 0$, the map $(\partial\varphi^* + tF')^{-1}$ is single valued and Lipschitzian on every bounded subset as well. Hence $S(t)\varphi \in C^1(H)$ for all $t > 0$ and by (2.6), (2.13) reduces to

$$\begin{aligned}\big(S(t)\varphi\big)_x(x) &= (\partial\varphi^* + tF')^{-1}(x) = \\ &= (F')^{-1}\big(t^{-1}(x - y_t(x))\big)\,; \qquad \forall x \in H,\ t > 0\,.\end{aligned} \tag{2.14}$$

Hence

$$\big(F'\big(S(t)\varphi\big)_x(x), x\big) = t^{-1}(x - y_t(x), x) \geqslant 0\,; \qquad x \in H,\ t > 0$$

and again by Lemma 1 it follows that $\big(S(t)\varphi\big)_x(0) = 0$. Thus we have shown that $S(t)\varphi \in K$ for every $t > 0$.

Let φ, ψ be two elements of K. Since in virtue of Lemma 1, $y_t(x) \in \Sigma_R$

whenever $|x| \leqslant R$, it follows by (2.8),

$$\big(S(t)\psi\big)(x) - \big(S(t)\varphi\big)(x) \leqslant \\ \leqslant \psi\big(y_t(x)\big) - \varphi\big(y_t(x)\big) \leqslant |\psi - \varphi|_R \quad \text{for all } x \in \Sigma_R \text{ and } t > 0 .$$

(Here $y_t(x)$ is defined by Lemma 1). The latter implies (2.11).

Let us now prove the semigroup property (2.10). Using again the Fenchel theorem we have by (2.7) and (2.8)

$$\begin{aligned}\big(S(t+s)\varphi\big)(x) &= \inf\left\{(\varphi^* + sF)^*(y) + tF^*\big(t^{-1}(x-y)\big);\ y \in H\right\} = \\ &= \inf\left\{\big(S(s)\varphi\big)(y) \qquad + tF^*\big(t^{-1}(x-y)\big);\ y \in H\right\} = \\ &= \big(S(t)S(s)\varphi\big)(x) .\end{aligned}$$

It remains to prove equality (2.12). To this end we fix $\varphi \in K \cap C^1(H)$ and observe that by (2.8) we have

$$\varphi(x) - \big(S(t)\varphi\big)(x) = \tag{2.15}$$
$$= \varphi(x) - \varphi\big(y_t(x)\big) - tF^*\Big(t^{-1}\big(x - y_t(x)\big)\Big) \leqslant \big(\varphi'(x), x - y_t(x)\big) - tF^*\Big(t^{-1}\big(x - y_t(x)\big)\Big) .$$

Along with the well known conjugacy formula, (see e.g. [3], p. 91)

$$F^*\big(F'(y)\big) + F(y) = \big(y, F'(y)\big) , \qquad \forall y \in H , \tag{2.16}$$

relations (2.6) and (2.4) lead to

$$t^{-1}\Big(\varphi(x) - \big(S(t)\varphi\big)(x)\Big) \leqslant \Big(\varphi'(x) - \varphi'\big(y_t(x)\big), F'\Big(\varphi'\big(y_t(x)\big)\Big) + F\Big(\varphi'\big(y_t(x)\big)\Big)\Big). \tag{2.17}$$

On the other hand, since F is convex, one has the inequality

$$F'\Big(\varphi'\big(y_t(x)\big)\Big) \leqslant F\big(\varphi'(x)\big) + \Big(F'\Big(\varphi'\big(y_t(x)\big)\Big), \varphi'\big(y_t(x)\big) - \varphi'(x)\Big)$$

which along with (2.17) yields

$$t^{-1}\Big(\varphi(x) - \big(S(t)\varphi\big)(x)\Big) - F\big(\varphi'(x)\big) \leqslant 0 . \tag{2.18}$$

Similarly, by (2.8) and (2.16) it follows that

$$\begin{aligned}\big(S(t)\varphi\big)(x) - \varphi(x) &\leqslant \Big(\varphi'\big(y_t(x)\big), y_t(x) - x\Big) + tF^*\Big(t^{-1}\big(x - y_t(x)\big)\Big) = \\ &= -t\Big(\varphi'\big(y_t(x)\big), F'\Big(\varphi'\big(y_t(x)\big)\Big)\Big) + tF^*\Big(F'\Big(\varphi'\big(y_t(x)\big)\Big)\Big) = -tF\Big(\varphi'\big(y_t(x)\big)\Big) .\end{aligned}$$

Hence

$$t^{-1}\big(\varphi(x)-\big(S(t)\varphi\big)(x)\big)-F\big(\varphi'\big(y_t(x)\big)\big)\geqslant 0\,. \tag{2.19}$$

By (2.18) and (2.19) we see that

$$\begin{aligned}0\leqslant F\big(\varphi'(x)\big)-t^{-1}\big(\varphi(x)-\big(S(t)\varphi\big)(x)\big)\leqslant F\big(\varphi'(x)\big)-F\big(\varphi'\big(y_t(x)\big)\big)\leqslant\\ \leqslant\big(F'\big(\varphi'(x)\big),\,\varphi'(x)-\varphi'\big(y_t(x)\big)\big)\,;\qquad t>0,\ x\in H\,.\end{aligned} \tag{2.20}$$

Since φ' and F' are bounded on every Σ_R it follows by (2.6) that

$$|x-y_t(x)|\leqslant C_R t\qquad \forall x\in\Sigma_R,\ t>0\,.$$

Inasmuch as φ' is uniformly continuous on every Σ_R, by (2.20) we deduce (2.12) as claimed.

REMARK. Let L_0 be the operator defined in $C(H)$ by

$$L_0\varphi=F(\varphi_x)\qquad \forall\varphi\in D(L_0) \tag{2.21}$$

where $D(L_0)$ consists of the set of all $\varphi\in C^1(H)\cap K$ such that φ' is uniformly continuous on every bounded subset of H.

By (2.11) and (2.12) it follows that L_0 is accretive in $C(H)$ i.e.,

$$|\varphi-\psi+\lambda(L_0\varphi-L_0\psi)|_R\geqslant|\varphi-\psi|_R\,,\qquad \forall R>0;\ \lambda>0 \tag{2.22}$$

for every pair $(\varphi,\psi)\in D(L_0)\times D(L_0)$.

On the other hand, (2.21) implies that $L_0\subset L_1$ where L_1 is the infinitesimal generator of $S(t)$.

3. - Equation (1.1) with $A\equiv 0$.

We begin with the homogeneous Cauchy problem

$$\begin{cases}\varphi_t(t,x)+F\big(\varphi_x(t,x)\big)=0\,; & t\geqslant 0,\ x\in H\\ \varphi(0,x)=\varphi_0(x)\,, & x\in H\end{cases} \tag{3.1}$$

where F satisfies assumptions (b), (c) and $\varphi_0\in K$.

Define the function $\varphi\colon R^+\times H \to R^1$,

$$(3.2)\qquad \varphi(t,x) = \big(S(t)\varphi_0\big)(x)\,, \quad t\geqslant 0,\ x\in H$$

where $S(t)$ is the semigroup defined by formula (2.7).

By Lemma 1 it follows that $\varphi(t,\cdot)\in C^1(H)$ for every $t>0$ and $\varphi(t,\cdot)\in K$ for every $t\geqslant 0$. Furthermore, it follows by (2.10) and (2.12) that for each $\varphi_0\in K$, the right derivative $(d^+/dt)S(t)\varphi_0$ exists at every $t>0$ and

$$(3.3)\qquad \frac{d^+}{dt}S(t)\varphi_0 + F\Big(\big(S(t)\varphi_0\big)_x\Big) = 0\,, \qquad \forall t>0\,.$$

We have also used the fact that for each $\varphi_0\in K$, $\big(S(t)\varphi_0\big)_x$ is uniformly continuous on bounded subsets (see (2.14)). In (3.3) d^+/dt is taken in the sense of topology of $C(H)$. If $\varphi_0\in C^1(H)$ then eq. (3.3) remains valid for $t=0$. In particular, it follows by (3.3) that φ is a strong solution to eq. (3.1) in the sense that

$$(3.4)\qquad \frac{d^+}{dt}\varphi(t,x) + F\big(\varphi_x(t,x)\big) = 0 \qquad \text{for every } t>0,\ x\in H\,.$$

It is worth noting also that by (2.14) and Lemma 1 it follows that $\varphi(t,\cdot)\in$ $\in C^1(H)$ for every $t>0$ and

$$(3.5)\qquad \sup\big\{|\varphi(t,\cdot)|_{1,R};\ t\in[\delta,T]\big\}\leqslant C_{\delta,R} \qquad \text{for every } \delta\in\,]0,T[\text{ and } R>0\,.$$

If $\varphi_0\in C^1(H)$ then by (2.6) and (2.14) it follows that $\varphi\in C\big([0,T];\,C^1(H)\big)$, i.e.,

$$(3.6)\qquad \sup\big\{|\varphi(t,\cdot)|_{1,R};\ t\in[0,T]\big\}\leqslant C_R \qquad \forall R>0\,.$$

On the other hand the accretivity of the operator $L_0(\varphi)=F(\varphi_x)$ on $C(H)$ (see (2.22)) implies via a standard argument the uniqueness of the strong solution ϱ.

Summarising, we get

THEOREM 1. *Let F satisfy assumptions* (b), (c). *Then for each $\varphi_0\in K$, the Cauchy problem* (3.1) *has a unique strong solution given by formula* (3.2). *More precisely, $\varphi(t,\cdot)\in C^1(H)\cap K$ for all $t>0$, satisfies* (3.5) *and as a function of t from $]0,+\infty[$, $\varphi(t)$ is continuous, everywhere differentiable from the right and satisfies eq.* (3.4). *Moreover the map $\varphi_0\to\varphi$ is a contraction from $C(H)$ to $C\big([0,T];\,C(H)\big)$.*

If in addition $\varphi_0\in C^1(H)$ then $\varphi\in C\big([0,T];\,C^1(H)\big)$ and equation (3.4) *is satisfied for all $t\geqslant 0$.*

REMARK. We have incidentally shown that the semigroup $S(t)$ has smoothing effect on initial data (see [3], [9] for other classes of contraction semigroups having this property).

We shall consider now the nonhomogeneous Cauchy problem

$$\left\{\begin{array}{ll}\varphi_t(t,x)+F\big(\varphi_x(t,x)\big)=g(t,x)\,; & t\in[0,T],\ x\in H\\ \varphi(0,x)=\varphi_0(x)\end{array}\right. \tag{3.7}$$

where F satisfies assumptions (*b*) and (*c*).

One assumes in addition that

(*e*) K is a closed convex cone of $C(H)$.

Further we shall assume that

$$\varphi_0\in C^1(H)\cap K\,,\qquad g\in C\big([0,T];\,C^1(H)\big)\cap\mathcal{K} \tag{3.8}$$

where $\mathcal{K}$ is the closed convex cone of $C\big([0,T];\,C(H)\big)$ defined by

$$\mathcal{K}=\big\{\varphi\in C\big([0,T];\,C(H)\big)\,;\ \varphi(t)\in K,\ \forall t\in[0,T]\big\}\,. \tag{3.9}$$

Consider the approximating equation

$$\left\{\begin{array}{l}\varphi_t(t,x)+\varepsilon^{-1}\big(\varphi(t,x)-\big(S(\varepsilon)\varphi\big)(t,x)\big)=g(t,x)\\ \varphi(0,x)=\varphi_0(x)\,,\qquad t\in[0,T],\ x\in H,\ \varepsilon>0\end{array}\right. \tag{3.10}$$

or equivalently

$$\varphi(t,x)=\exp\,(\varepsilon^{-1}t)\,\varphi_0(x)+\int_0^t\exp\,\big(\varepsilon^{-1}(t-s)\big)\,g(s,x)\,ds+ \\ +\,\varepsilon^{-1}\int_0^t\exp\,\big(\varepsilon^{-1}(t-s)\big)\big(S(\varepsilon)\varphi\big)(s,x)\,ds\,,\qquad t\in[0,T]\,\cdot \tag{3.11}$$

By assumptions (*e*), (3.8) and by Lemma 2 it follows that the operator defined by the right hand side of eq. (3.11) maps every $C\big([0,T];\,C(\Sigma_R)\big)$ into itself and is contractant. Thus for every $\varepsilon>0$, eq. (3.11) ((3.10)) has a unique solution $\varphi_\varepsilon\in\mathcal{K}\cap C^1\big([0,T];\,C(H)\big)$. Since, as proved earlier, $S(\varepsilon)\varphi\in$ $\in C^1(H)$ for every $\varphi\in K$ and $\varepsilon>0$ it follows by (3.11) that $\varphi_\varepsilon\in C\big([0,T];$ $\in C^1(H)\big)$. Furthermore, recalling that (see (2.6) and (2.14)),

$$\big(S(\varepsilon)\varphi\big)_x(x)=\varphi'\big(y_\varepsilon(x)\big)\,,\qquad x\in H,\ \varepsilon>0 \tag{3.12}$$

we see that $(\varphi_\varepsilon)_x = \varphi'_\varepsilon$ is the solution to

$$\frac{d}{dt}\varphi'_\varepsilon(t,x) + \varepsilon^{-1}\big(\varphi'_\varepsilon(t,x) - \varphi'_\varepsilon(t, y_\varepsilon(t,x))\big) = g_x(t,x) \tag{3.13}$$

where

$$y_\varepsilon(t,x) + F'\big(\varphi'_\varepsilon(t, y_\varepsilon(t,x))\big) = x\,; \qquad x\in H,\ t\in[0,T]\,. \tag{3.14}$$

Then by an easy computation involving eq. (3.13) and the Gronwall lemma it follows that

$$|\varphi_\varepsilon(t)|_{1,R} \leqslant C_R\big(|\varphi_0|_{1,R} + \sup_{0\leqslant t\leqslant T}|g(t)|_{1,R}\,, \qquad t\in[0,T]\,. \tag{3.15}$$

(By C_R we shall denote several positive constants independent of ε.) Parenthetically we notice that since by (3.14) and assumption (*e*) the mapping $x \to \varphi'_\varepsilon(x, y_\varepsilon(t,x))$ is Lipschitzian on every Σ_R, it follows by (3.13) that if $\varphi'_0 \in C^0_{\mathrm{Lip}}(H,H)$ and $g_x \in C([0,T];\ C^0_{\mathrm{Lip}}(H,H))$ then

$$(\varphi_\varepsilon)_x \in C([0,T];\ C^0_{\mathrm{Lip}}(H,H))\,. \tag{3.16}$$

Next by (2.20), (3.12) and (3.13) one has

$$\begin{aligned}
&\big|\varepsilon^{-1}\big(\varphi_\varepsilon(t,x) - (S(\varepsilon)\varphi_\varepsilon)(t,x)\big) - F(\varphi'_\varepsilon(t,x))\big| \leqslant\\
&\qquad\leqslant \big(F'(\varphi'_\varepsilon(t,x)),\, \varphi'_\varepsilon(t,x) - \varphi'_\varepsilon(t, y_\varepsilon(t,x))\big) =\\
&\qquad = -\varepsilon\left(F'(\varphi'_\varepsilon(t,x)),\, \frac{d}{dt}\varphi'_\varepsilon(t,x)\right) + \varepsilon\big(F'(\varphi'_\varepsilon(t,x)),\, g_x(t,x)\big) =\\
&\qquad = \varepsilon\big(F'(\varphi'_\varepsilon(t,x)),\, g_x(t,x)\big) - \varepsilon\frac{d}{dt}F(\varphi'_\varepsilon(t,x))\,.
\end{aligned}$$

Integrating the latter over $]0,T[$, we get by (3.15) the estimate

$$\begin{aligned}
&\int_0^T \big|\varepsilon^{-1}\big(\varphi_\varepsilon(t,x) - (S(\varepsilon)\varphi_\varepsilon)(t,x)\big) - F(\varphi'_\varepsilon(t,x))\big|\,dt \leqslant\\
&\leqslant \varepsilon\int_0^T |F'(\varphi'_\varepsilon(t,x))|\,|g_x(t,x)|\,dt + \varepsilon\big(F(\varphi'_0(x)) - F(\varphi'_\varepsilon(T,x))\big) \leqslant C_R\varepsilon \qquad \forall x\in\Sigma_R
\end{aligned} \tag{3.17}$$

and therefore by (3.10)

$$\begin{cases} (\varphi_\varepsilon)_t(t,x) + F\big((\varphi_\varepsilon)_x(t,x)\big) = g(t,x) + \eta_\varepsilon(t,x)\,, & t\in[0,T]\\ \varphi_\varepsilon(0,x) = \varphi_0(x) & x\in H \end{cases} \tag{3.18}$$

where

$$\int_0^T |\eta_\varepsilon(t, x)|dt \leqslant C_R \varepsilon \qquad \forall x \in \Sigma_R,\ \varepsilon > 0 .$$

Now coming back to equation (3.11) it follows by Lemma 2 (part (2.11)) and the Gronwall lemma that the mapping $(\varphi_0, g) \xrightarrow{G_\varepsilon} \varphi$ is Lipschitzian from $C(H) \times C([0, T]; C(H))$ to $C([0, T]; C(H))$. More precisely, one has

$$(3.19) \qquad |G_\varepsilon(\varphi_0, g)(t) - G_\varepsilon(\psi_0, h)|_R \leqslant \leqslant |\varphi_0 - \psi_0|_R + \int_0^T |g(s) - h(s)|_R ds \qquad \forall R > 0;\ t \in [0, T]$$

for all $\varphi_0,\ \psi_0 \in K$ and $g, h \in \mathcal{K}$. By (3.18) and (3.19) one concludes that

$$|\varphi_\varepsilon(t) - \varphi_\lambda(t)|_R \leqslant C_R T(\varepsilon + \lambda) \qquad \forall t \in [0, T];\ \varepsilon, \lambda > 0 .$$

Hence $\lim_{\varepsilon \downarrow 0} \varphi_\varepsilon = \varphi$ exists in $C([0, T]; C(H))$. Clearly $\varphi \in \mathcal{K}$ and by (3.15) we see that for each $t \in [0, T]$, $\varphi(t, \cdot)$ is Lipschitzian on every bounded subset of H and

$$|\varphi(t)|_{\mathrm{Lip},R} \leqslant C_R\Big(|\varphi_0|_{1,R} + \sup_{t \in [0,T]} |g(t)|_{1,R}\Big), \qquad t \in [0, T] .$$

Summarising, we have shown that there exists a sequence $\{\varphi_\varepsilon\} \subset$ $\subset C([0, T]; C(H))$ satisfying

$$(3.20) \qquad \varphi_\varepsilon \in C([0, T]; C^1(H)) \cap \mathcal{K} \qquad \forall \varepsilon > 0$$

$$(3.21) \qquad \varphi_\varepsilon \in C^1([0, T]; C(H)) \qquad \forall \varepsilon > 0$$

$$(3.22) \qquad \varphi_\varepsilon \to \varphi \text{ in } C([0, T]; C(H)) \qquad \text{for } \varepsilon \to 0$$

$$(3.23) \qquad \{\varphi_\varepsilon\} \text{ is bounded in } C([0, T]; C^1(H)) \text{ and } (\varphi_\varepsilon)_t \text{ in } L^1(0, T; C(H))$$

$$(3.24) \qquad \begin{cases} (\varphi_\varepsilon)_t(t, x) + F\big((\varphi_\varepsilon)_x(t, x)\big) \to g(t, x) & \text{in } L^1(0, T; C(H)) \\ \varphi_\varepsilon(0, x) = \varphi_0(x) . \end{cases}$$

Here the convergence in the space $L^1(0, T; C(H))$ is understood in the local convex topology given by the family of seminorms (1.13).

DEFINITION 1. A function φ satisfying conditons (3.20) up to (3.24) is called weak solution to the Cauchy problem (3.7). We notice that by (3.19), (3.22), (3.23) and (3.24) the weak solution $\varphi = G(\varphi_0, g)$ is unique and

$$|G(\varphi_0, g) - G(\psi_0, h)|_R \leq |\varphi_0 - \psi_0|_R + \int_0^T |g(s) - h(s)|_R ds \tag{3.25}$$

for all $\varphi_0, \psi_0 \in K$ and $g, h \in \mathcal{K}$ satisfying condition (3.8). We have therefore proved the following theorem

THEOREM 2. *Assume that hypotheses* (*a*), (*b*), (*c*) *and* (*e*) *are satisfied. Then for any pair of functions* $(\varphi_0, g) \in K \times \mathcal{K}$ *satisfying condition* (3.8), *the Cauchy problem* (3.7) *has a unique weak solution* φ *which satisfies*

$$\sup \{|\varphi(t)|_{\mathrm{Lip},R};\ t \in [0, T]\} < \infty . \tag{3.26}$$

Furthermore, the map $(\varphi_0, g) \to \varphi$ *is Lipschitzian from* $C(H) \times C([0, T]; C(H))$ *to* $C([0, T]; C(H))$.

REMARKS. 1°) It is worth noting that another way to prove Theorem 2 is to apply the Bénilan existence result (see [2], [6]) to nonlinear evolution equation

$$\frac{d\varphi}{dt} + L\varphi = g, \qquad t \in [0, T];\ \varphi(0) = \varphi_0 \tag{3.27}$$

in the space $C(H)$. Here L is the closure of L_0 (see (2.22)) in $C(H) \times C(H)$.

2°) Assumptions (*b*), (*c*) and (*e*) are verified by a large class of functions F which includes functions of the form

$$F(x) = \zeta(|x|^2) \qquad x \in H \tag{3.28}$$

where ζ is a real valued, convex and differentiable function on $[0, \infty[$ which satisfies the following conditions

$$\zeta'(0) > 0 ; \qquad \lim_{r \to +\infty} \zeta(r)/r^{\frac{1}{2}} = +\infty . \tag{3.29}$$

4. – Existence and uniqueness for equation (1.2).

We shall study here the Cauchy problem

$$\begin{cases} \varphi_t(t, x) + \frac{1}{2}|\varphi_x(t, x)|^2 + (Ax, \varphi_x(t, x)) = g(t, x) \\ \varphi(0, x) = \varphi_0(x) ; \qquad x \in D(A),\ t \in [0, T] \end{cases} \tag{4.1}$$

where

$$\varphi_0 \in C^1(H) \tag{4.2}$$

and

$$g \in C([0, T]; C^1(H)) . \tag{4.3}$$

Further we shall assume that

$$\varphi_0' \in C^0_{\mathrm{Lip}}(H, H) ; \qquad g_x \in C([0, T]; C^0_{\mathrm{Lip}}(H, H)) . \tag{4.4}$$

In this case K is the set of all convex functions $\varphi \in C(H)$ such that $\partial\varphi(0) \in 0$.

We start with the approximating equation

$$\begin{aligned} \varphi(t, x) &= \exp(-t/\varepsilon)\varphi_0(\exp[-tA)x) + \\ &+ \int_0^t \exp(-(t-s)/\varepsilon)\, g(s, \exp(-(t-s)A)\, x)\, ds + \\ &+ \varepsilon^{-1}\int_0^t \exp(-(t-s)/\varepsilon)\,(S(\varepsilon)\varphi)(s, \exp(-(t-s)A)\, x)\, ds \qquad \text{for } t \in [0, T],\ x \in H \end{aligned} \tag{4.5}$$

where by formula (2.8), $S(\varepsilon)$ is given by

$$(S(\varepsilon)\varphi)(x) = \inf\left\{\frac{|x-y|^2}{2\varepsilon} + \varphi(y);\ y \in H\right\} = \varphi(y_\varepsilon(x)) + \frac{|x - y_\varepsilon(x)|^2}{2\varepsilon} \tag{4.6}$$

and

$$y_\varepsilon(x) = (1 + \varepsilon\varphi')^{-1}x . \tag{4.7}$$

Applying the contraction principle on the closed convex cone of $C([0, T]; C(H))$

$$\mathcal{K} = \{\varphi \in C([0, T]; C(H));\ \varphi(t) \in K\ \ \forall t \in [0, T]\} \tag{4.8}$$

we see that eq. (4.5) has a unique solution $\varphi_\varepsilon \in \mathcal{K}$. Moreover, in as much as $(S(\varepsilon)\varphi)'(x) = \varphi'(y_\varepsilon(x))$, we see that $\varphi_\varepsilon \in C([0, T]; C^1(H))$ and

$$\begin{aligned} \varphi_\varepsilon'(t, x) &= \exp(-t/\varepsilon)\exp(-tA^*)\varphi_0'(\exp(-tA)x) + \\ &+ \int_0^t \exp(-(t-s)/\varepsilon)\exp(-(t-s)A^*)\, g'(s, \exp(-(t-s)A)\, x)\, ds + \\ &+ \varepsilon^{-1}\int_0^t \exp(-(t-s)/\varepsilon)\exp(-(t-s)A^*)\,\varphi_\varepsilon'(s, y_\varepsilon(s, \exp(-(t-s)A)\, x))\, ds . \end{aligned} \tag{4.9}$$

By (4.5) it follows that for each $x \in D(A)$, $\varphi_\varepsilon(t, x)$ is differentiable on $[0, T]$ and satisfies the equation

$$(4.10)\qquad (\varphi_\varepsilon)_t(t, x) + \varepsilon^{-1}\big(\varphi_\varepsilon(t, x) - S(\varepsilon)\varphi_\varepsilon(t, x)\big) + \big(Ax, (\varphi_\varepsilon)_x(t, x)\big) = \\ = g(t, x)\,; \qquad t \in [0, T],\ x \in D(A)\,.$$

By an easy computation involving the Gronwall lemma and eq. (4.9) it follows

$$(4.11)\qquad |\varphi_\varepsilon(t)|_{1,R} \leqslant C_R\Big(|\varphi_0|_{1,R} + \sup_{0 \leqslant t \leqslant T} |g(t)|_{1,R}\Big)\,, \qquad t \in [0, T]\,.$$

Next, since the mapping $x \to y_\varepsilon(x)$ is nonexpansive and in virtue of (2.6),

$$\varphi'_\varepsilon\big(x, y_\varepsilon(t, x)\big) = \varepsilon^{-1}\big(x - y_\varepsilon(t, x)\big)$$

it follows by (4.4) and (4.9) that $\varphi'_\varepsilon(t) \in C^0_{\mathrm{Lip}}(H, H)$ for every $t \in [0, T]$. Moreover, using once again the Gronwall lemma one finds the estimate

$$(4.12)\qquad \|\varphi'_\varepsilon(t)\|_{0,R} \leqslant C_R\Big(\|\varphi'_0\|_{0,R} + \sup_{0 \leqslant t \leqslant T} \|g'(t)\|_{0,R}\Big)$$

(we recall that $\varphi' = \varphi_x$ stands for the Fréchet derivative with respect to x.)

Next by inequality (2.20) and (4.11), (4.12)

$$(4.13)\qquad \begin{cases} \Big|\varepsilon^{-1}\Big(\varphi_\varepsilon(t, x) - \big(S(\varepsilon)\varphi_\varepsilon\big)(t, x)\Big) - F\big(\varphi'_\varepsilon(t, x)\big)\Big| \leqslant \big|F'\big(\varphi'_\varepsilon(t, x)\big)\big| \\ \|\varphi'_\varepsilon(t)\|_{0,R}\,|x - y_\varepsilon(t, x)| \leqslant C_R\,\varepsilon \end{cases}$$

for all $x \in \Sigma_R$ and $t \in [0, T]$.

Since the mapping $(\varphi_0, g) \to \varphi_\varepsilon$ is Lipschitzian from $C(H) \times C([0, T]; C(H))$ to $C([0, T]; C(H))$ (see (3.19)) we may infer by (4.13) that

$$|\varphi_\varepsilon(t) - \varphi_\lambda(t)|_R \leqslant C_R\, T(\varepsilon + \lambda)\,; \qquad t \in [0, T];\ \varepsilon, \lambda > 0$$

and therefore $\lim_{\varepsilon \to 0} \varphi_\varepsilon = \varphi$ exists in $C([0, T]; C(H))$. Clearly $\varphi \in \mathcal{K}$ and

$$(4.14)\qquad \sup_{t \in [0, T]} |\varphi(t)|_{\mathrm{Lip},R} < \infty\,, \qquad \forall R > 0\,.$$

By (4.10)-(4.12) and (4.13) we see that the function φ is a weak solution to problem (4.1) in the sense of Definition 1, i.e. there exists a sequence

$\{\varphi_\varepsilon\} \subset C([0, T]; C^1(H)) \cap \mathcal{K}$ such that for $\varepsilon \to 0$

(4.15) $\quad \varphi_\varepsilon \to \varphi \quad$ in $C([0, T]; C(H))$

(4.16) $\quad \varphi_\varepsilon \in C^1\big([0, T]; C(D(A))\big)\,; \qquad \varphi_\varepsilon(0, x) = \varphi_0(x)$

(4.17) $\quad (\varphi_\varepsilon)_t(t, x) + \frac{1}{2}|(\varphi_\varepsilon)_x(t, x)|^2 + \big(Ax, (\varphi_\varepsilon)_x(t, x)\big) \to g(t, x)$

in $C([0, T];\ C(H))$

(4.18) $\quad \{(\varphi_\varepsilon)_x\}$ is bounded in $C([0, T]; C^0_{\mathrm{Lip}}(H, H))$.

Here the space $D(A)$ is endowed with the graph norm.

Summarising, we have proved the following theorem

THEOREM 3. *Suppose that assumptions* (*a*), (*e*) *are satisfied and* φ_0, g *satisfy conditions* (4.2), (4.3) *and* (4.4). *Then the Cauchy problem* (4.1) *has a unique weak solution* $\varphi \in \mathcal{K}$ *which satisfies* (4.14). *Moreover, the map* $(\varphi_0, g) \to \varphi$ *is Lipschitzian from* $C(H) \times C([0, T]; C(H))$ *to* $C([0, T]; C(H))$ *and for every* $x \in D(A)$ *the function* $\varphi(t, x)$ *is absolutely continuous on* $[0, T]$.

Our next concern is a regularity result for the solutions to equation (4.1). To this purpose we return to approximating sequence $\{\varphi_\varepsilon\} \subset C([0, T]; C^1(H)) \cap \mathcal{K}$ and set

$$E^\varepsilon(t, x) = \varphi'_\varepsilon(t, x) \qquad t \in [0, T],\ x \in H\,.$$

As seen above $E^\varepsilon \in C([0, T]; C^0(H, H))$ is the solution to (see (4.9))

$$\begin{aligned}(4.19)\qquad E^\varepsilon(t, x) &= \exp\left(-\frac{t}{\varepsilon}\right)\exp(-tA^*)\varphi_0'(\exp(-tA)x) + \\ &+ \varepsilon^{-1}\int_0^t \exp\left(-\frac{(t-s)}{\varepsilon}\right)\exp(-(t-s)A^*)E^\varepsilon_\varepsilon\big(s, \exp(-(t-s)A)x\big)\,ds + \\ &+ \int_0^t \exp\left(-\frac{(t-s)}{\varepsilon}\right)\exp(-(t-s)A^*)g'\big(s, \exp(-(t-s)A)x\big)\,ds\end{aligned}$$

where

$$(4.20)\qquad E^\varepsilon_\varepsilon(t, x) = \big(S(\varepsilon)\varphi_\varepsilon\big)'(t, x) = E^\varepsilon(1 + \varepsilon E^\varepsilon)^{-1}(t, x)\,.$$

In addition to (4.2), (4.3) and (4.4) we shall assume that

$$(4.21)\qquad \varphi_0' \in C^1_{\mathrm{Lip}}(H, H)\,; \qquad g_x' \in C([0, T]; C^1_{\mathrm{Lip}}(H, H))\,.$$

We shall prove that under these conditions $E^{\varepsilon} \in C([0, T]; C^1_{\mathrm{Lip}}(H, H))$. To this end we introduce the following convex cone of $C^0(H, H)$

$$\Pi = \{E \in C^0(H, H);\ E \text{ monotone and } E(0) = 0\}. \tag{4.22}$$

For every $E \in \Pi$ we set $E_\varepsilon = E(1 + \varepsilon E)^{-1}$ (1 is the identity operator in H). We notice that for every $\varepsilon > 0$ the operator $(1 + \varepsilon E)^{-1}$ is well defined and nonexpansive on H. In the next lemma we gather for later use some elementary properties of E_ε.

LEMMA 3. *For all $\varepsilon > 0$ and $R > 0$ one has*

$$|E_\varepsilon|_{0,R} \leqslant |E|_{0,R} \qquad \forall E \in \Pi \tag{4.23}$$

$$|E_\varepsilon|_{1,R} \leqslant |E|_{1,R} \qquad \forall E \in \Pi \cap C^1(H, H) \tag{4.24}$$

$$\|E_\varepsilon\|_{1,R} \leqslant \|E\|_{1,R} + \varepsilon |E|_{1,R}\big(|E|_{1,R}\|E\|_{0,R} + \|E\|_{1,R}\big) \quad \forall E \in \Pi \cap C^1_{\mathrm{Lip}}(H, H). \tag{4.25}$$

Moreover, if $\|E\|_{1,R}$ and $\|\tilde{E}\|_{1,R}$ are $\leqslant \alpha$ then there exists $\eta(\alpha) > 0$ such that

$$|E - \tilde{E}|_{1,R} \leqslant (1 + \varepsilon\eta(\alpha))\,|E - \tilde{E}|_{1,R}. \tag{4.26}$$

The proof is standard and relies on the formula

$$E'_\varepsilon(x) = E'\big((1 + \varepsilon E)^{-1}x\big)\Big(1 + \varepsilon E'\big((1 + \varepsilon E)^{-1}x\big)^{-1}\Big).$$

(By E' we shall denote the Fréchet derivative of the operator E.)

In the space $C([0, T]; C^1(H, H))$ consider the approximating equation

$$\begin{aligned} E(t, x) = {} & \exp\Big(-\frac{\varepsilon}{t}\Big)\exp(-tA^*)E_0\big(\exp(-tA)x\big) + \\ & + \varepsilon^{-1}\int_0^t \exp\Big(-\frac{(t-s)}{\varepsilon}\Big)\exp\big(-(t-s)A^*\big)E\big(s, \exp(-(t-s)A)x\big)\,ds + \\ & + \int_0^t \exp\Big(-\frac{(t-s)}{\varepsilon}\Big)\exp\big(-(t-s)A^*\big)G\big(s, \exp(-(t-s)A)x\big)\,ds, \\ & \qquad x \in H;\ t \in [0, T] \end{aligned} \tag{4.27}$$

where $E_0 = \varphi'_0$ and $G = g'$. We consider the following closed convex cone of $C([0, T]; C^1(H, H))$

$$Q = \{E \in C([0, T]; C^1_{\mathrm{Lip}}(H, H));\ E(t) \in \Pi,\ \|E(t)\|_{1,R} \leqslant \alpha_R \text{ for every } t \in [0, T]\} \tag{4.28}$$

where $\alpha_R \to +\infty$ as $R \to +\infty$.

Let Γ be the operator defined by the right hand of equation (4.27). For the beginning we shall assume that $\|E_0\|_{1,R} \leqslant \alpha_R/2$ and $\|G(t)\|_{1,R} \leqslant \alpha_R/2$ for $t \in [0, T]$. Then for all sufficiently small T, Γ maps Q into itself and by (4.25) one has

$$|(\Gamma E)(t) - (\Gamma \tilde{E})(t)|_{1,R} \leqslant \varrho_R \sup \{|E(t) - \tilde{E}(t)|_{1,R};\ 0 \leqslant t \leqslant T\}$$

for all $E, \tilde{E} \in Q$. Here $0 < \varrho_R < 1$ for every $R > 0$. Hence eq. (4.27) (equivalently (4.19)) has a unique solution $E = E^\varepsilon \in C([0, T']; C^1_{\mathrm{Lip}}(H, H))$ where $[0, T'[$ is some subinterval of $[0, T[$. Next after some calculations involving equation (4.19), estimates (4.23), (4.25) and the Gronwall lemma it follows that

$$\|E^\varepsilon(t)\|_{1,R} \leqslant C_R \qquad \forall t \in [0, T'],\ \varepsilon > 0 \tag{4.29}$$

where C_R is independent of T'. This implies by a standard procedure that $E^\varepsilon \in C([0, T]; C^1_{\mathrm{Lip}}(H, H))$ and inequality (4.29) extends on the whole interval $[0, T]$.

On the other hand, using once again estimate (4.26) we see that

$$|\Phi_\varepsilon(t, E_0, G) - \Phi_\varepsilon(t, \tilde{E}_0, \tilde{G})|_{k,R} \leqslant C_R\Big(|E_0 - \tilde{E}_0|_{k,R} + \int_0^t |G(t) - \tilde{G}(t)|_{k,R}\, dt\Big) \tag{4.30}$$

for $t \in [0, T]$ and $k = 0, 1$, where $\Phi_\varepsilon(t, E_0, G) = E^\varepsilon$ is the solution to (4.27).

On the other hand, we have

$$\varepsilon^{-1}(E(x) - E_\varepsilon(x)) - E'(x)E(x) = \tag{4.31}$$

$$= \int_0^t \big(E'(sx + (1-s)y_\varepsilon(x))\, E(y_\varepsilon(x)) - E'(x)E(x)\big)\, ds, \qquad x \in H$$

where $y_\varepsilon(x) = (1 + \varepsilon E)^{-1}x$. Along with estimates (4.29) and (4.30) the latter yields

$$|\varepsilon^{-1}(E^\varepsilon(t) - E^\varepsilon_\varepsilon(t)) - (E^\varepsilon)'(t)E^\varepsilon(t)|_{0,R} \leqslant C_R\varepsilon, \qquad t \in [0, T]. \tag{4.32}$$

Then by (4.32) it follows that

$$|E^\varepsilon(t) - E^\lambda(t)|_{0,R} \leqslant C_R(\varepsilon + \lambda), \qquad t \in [0, T];\ \varepsilon, \lambda > 0.$$

Hence there exists $E \in C([0, T]; C_0(H, H))$ such that for $\varepsilon \to 0$,

$$E_\varepsilon \to E \quad \text{in } C([0, T]; C^0(H, H)).$$

By the uniqueness of the limit we infer that $E(t, x) = \varphi_x(t, x)$ where φ is the weak solution to equation (4.1). We have therefore proved that

$$(\varphi_\varepsilon)_x \to \varphi_x \quad \text{in } C([0, T]; C_0(H, H)) . \tag{4.33}$$

Then by (4.17) we may infer that

$$\varphi \in C^1([0, T]; D(A)) \cap C([0, T]; C^1(H, H))$$

and

$$\begin{cases} \varphi_t(t, x) + \frac{1}{2}|\varphi_x(t, x)|^2 + (Ax, \varphi_x(t, x)) = g(t, x) \\ \varphi(0, x) = \varphi_0(x) \quad \text{for } x \in D(A),\ t \in [0, T] . \end{cases} \tag{4.34}$$

This amounts to saying that φ is a classical solution to equation (4.1). We have therefore proved

THEOREM 4. *In Theorem* 3 *suppose in addition that* φ_0 *and* g *satisfy conditions* (4.21). *Then* φ *is a classical solution to equation* (4.1).

Now we notice that by (4.10), $E^\varepsilon = \varphi_\varepsilon'$ satisfy

$$\begin{aligned} E^\varepsilon(t, x) = {} & \exp(-tA^*)\varphi_0'(\exp(-tA)x) + \\ & + \int_0^t \exp(-(t-s)A^*)(E^\varepsilon)_x E^\varepsilon\big(s, \exp(-(t-s)A)x\big)\, ds + \\ & + \int_0^t \exp(-(t-s)A^*)\, g'\big(s, \exp(-(t-s)A)x\big)\, ds + \delta_\varepsilon(t, x) \end{aligned} \tag{4.35}$$

where $\delta_\varepsilon \to 0$ in $C([0, T]; C^0(H, H))$ for $\varepsilon \to 0$, while by (4.33)

$$E^\varepsilon \to E = \varphi_x \quad \text{in } C([0, T]; C^0(H, H)) . \tag{4.36}$$

Keeping in mind that the equation

$$\begin{aligned} E(t, x) = {} & \exp(-tA^*)E_0(\exp(-tA)x) + \\ & + \int_0^t \exp(-(t-s)A^*)(E_x E)\big(s, \exp(-(t-s)A)x\big)\, ds + \\ & + \int_0^t \exp(-(t-s)A^*)\, G\big(s, \exp(-(t-s)A)x\big)\, ds ; \quad t \in [0, T],\ x \in H \end{aligned} \tag{4.37}$$

is the «mild» form of equation (1.3), we may say that $E = \varphi_x$ is a *weak solution* to this equation.

We have therefore the following existence result

THEOREM 5. *Under assumptions of Theorem* 4, $E(t, x) = \varphi_x(t, x)$ *is a weak solution to operator equation* (1.3) *where* $E_0 = \varphi_0'$ *and* $G = g_x$.

5. – An example in control theory.

The relevance of the Hamilton-Jacobi equations in control theory and the calculus of variations is well-known (see e.g. [3] and [12] for recent results concerning infinite dimensional problems). Here we shall study the connection between equation (1.1) and the following optimal control problem:

Minimize

$$\int_0^T \big(g(x(t)) + h(u(t))\big)\,dt + \varphi_0(x(T)) \tag{5.1}$$

over all $u \in L^2(0, T; U)$ *and* $x \in C([0, T]; H)$ *subject to state equation*

$$\begin{cases} x' + Ax = Bu\,, & t \in [0, T] \\ x(0) = x_0\,. \end{cases} \tag{5.2}$$

Here B is a linear continuous operator from U to H, $g\colon H \to \mathbb{R}^1$, $h\colon U \to \mathbb{R}^1$, $\varphi_0\colon H \to \mathbb{R}^1$ are given lower semicontinuous convex functions and U is a real Hilbert space identified with its own dual and with inner product $\langle\cdot,\cdot\rangle$.

We shall denote by $W^{1,2}(0, T; H)$ the space

$$\{x \in L^2(0, T; H);\, x' \in L^2(0, T; H)\}$$

where x' is the derivative in the sense of distributions. We shall assume that $x_0 \in D(A)$ and $-A$ is the infinitesimal generator of an analytic semigroup of contractions on H. Then for each control $u \in L^1(0, T; U)$, system (5.2) has a unique solution $x_u \in W^{1,2}(0, T; H)$ with $Ax_u \in L^2(0, T; H)$.

We associate with problem (5.1) the equation

$$\begin{cases} \psi_t(t, x) - h^*(-B^*\psi_x(t, x)) - \big(Ax, \psi_x(t, x)\big) + g(x) = 0\,, \\ \hfill x \in D(A),\ t \in [0, T]\,. \\ \psi(T, x) = \varphi_0(x) \end{cases} \tag{5.3}$$

where h^* is the conjugate of h and B^* is the dual operator of B.

We observe that by substitution $\varphi(t, x) = \psi(T - t, x)$, eq. (5.3) can be written in the form (1.1) where $F(y) = h^*(-B^*y)$ for all $y \in H$.

By analogy with Definition 1, we say that the function $\psi \in \mathcal{K}$ is a weak solution to equation (5.3) if there exists a sequence $\{\psi_\varepsilon\} \subset C([0, T]; C^1(H)) \cap \mathcal{K}$ such that for $\varepsilon \to 0$,

(5.4) $\psi_\varepsilon \in C^1\big([0, T]; C(D(A))\big); \quad \psi_\varepsilon(T) = \varphi_0$

(5.5) $\psi_\varepsilon \to \psi$ in $C([0, T]; C(H))$

(5.6) $(\psi_\varepsilon)_t - h^*\big(-B^*(\psi_\varepsilon)_x\big) - \big(Ax, (\psi_\varepsilon)_x\big) + g \to 0$ in $C([0, T]; C(H))$.

(5.7) The sequence $\{(\psi_\varepsilon)_x\}$ belongs to $C([0, T]; C^0_{\mathrm{Lip}}(H, H))$ and it is bounded in $C([0, T]; C^1(H, H))$.

Here $\mathcal{K}$ is defined by (3.9) and K is the set of all convex functions $\varphi \in C(H)$ such that $0 \in \varphi(0)$ and

$$((h^*)_x(-B^*y), B^*x) \leqslant 0 \qquad \forall [x, y] \in \partial\varphi . \tag{5.8}$$

The results proved in sect. 3 and 4 give existence and uniqueness of the weak solution ψ to equation (5.3) in several situations. For instance if $A \equiv 0$ and K is a closed convex cone of $C(H)$, Theorem 2 gives existence and uniqueness of a weak solution under the following assumptions:

(5.9) $\varphi_0, \; g \in K \cap C^1(H); \qquad \varphi_0', \; g' \in C^0_{\mathrm{Lip}}(H, H)$

(5.10) $h^* \in C^1(H)$ and $F(y) = h^*(-B^*y)$ satisfies (*b*) and (*c*).

If $h(u) = |u|^2/2$ and the range of B is all of H we may apply Theorem 3 to obtain existence and uniqueness under conditions (4.2), (4.3) and (4.4).

PROPOSITION 1. *Let* $\psi \in \mathcal{K}$ *be a weak solution to equation* (5.3) *where* $(h^*)_x \in C^0_{\mathrm{Lip}}(H, H)$. *Then for every* $y \in D(A)$ *and* $t \in [0, T]$ *one has*

(5.11) $\psi(t, y) =$

$$= \inf\left\{\int_t^T \big(g(x_u(s)) + h(u(s))\big)\, ds + \varphi_0\big(x_u(T)\big); \; u \in L^2(t, T; U), \; x_u(t) = y\right\}.$$

Moreover, if u^* *is an optimal control in problem* (5.1) *then it is expressed as a* *function of the optimal state x^* by the feedback formula*

$$u^*(t) = (h^*)_x\big(-B^*\partial\psi(t, x^*(t))\big) \quad \text{a.e. } t\in[0, T]. \tag{5.12}$$

Here $\partial\psi$ is the subdifferential of $\psi(t,\cdot)$.

Formula (5.12) gives the optimal feedback law of control problem (5.1). In particular if ψ happens to be a classical solution to (5.3) (in particular this is the case if the conditions of Theorem 4 are satisfied) then $\partial\psi = \psi_x$ and it follows by (5.12) that $u^*\in C^1([0, T]; U)$.

PROOF OF PROPOSITION 1. Let $t\in[0, T]$ and $u\in L^2(t, T; U)$ be fixed. Let x_u the solution to (5.2) on $[t, T]$ such that $x_u(t) = y$. The obvious equality

$$\frac{d}{ds}\psi_\varepsilon(s, x_u(s)) = (\psi_\varepsilon)_s(s, x_u(s)) + \big((\psi_\varepsilon)_x(s, x_u(s)), x_u'(s)\big), \quad \text{a.e. } s\in\,]t, T[$$

along with (5.4) and (5.6) implies that $\psi_\varepsilon(s, x_u(s))$ is absolutely continuous on $[t, T]$ and

$$\begin{aligned}\frac{d}{ds}\psi_\varepsilon(s, x_u(s)) + g(x_u(s)) + h(u(s)) &= h^*(-B^*(\psi_\varepsilon))_x(s, x_u(s)) + \\ &+ h(u(s)) + \langle B^*(\psi_\varepsilon)_x(s, x_u(s)), u(s)\rangle + \delta_\varepsilon(s) \quad \text{a.e. } s\in\,]t, T[\end{aligned} \tag{5.13}$$

where $\delta_\varepsilon \to 0$ uniformly on $[t, T]$.

Recalling that

$$h(u) + h^*(\tilde u) \geqslant \langle u, \tilde u\rangle \quad \forall u, \tilde u\in U$$

we deduce by (5.13) that

$$\psi_\varepsilon(t, y) \leqslant \int_t^T\big(g(x_u(s)) + h(u(s))\big)\,ds + \varphi_0(x_u(T)) - \int_t^T\delta_\varepsilon(s)\,ds.$$

Therefore

$$\begin{aligned}&\psi(t, y) \leqslant \\ &\leqslant \inf\Big\{\int_t^T\big(g(x_u(s)) + h(u(s))\big)\,ds + \varphi_0(x_u(T));\ u\in L^2(t, T; U),\ x_u(t) = y\Big\}.\end{aligned} \tag{5.14}$$

Now consider the Cauchy problem

$$\begin{cases} x' + Ax = B(h^*)_x(-B^*(\psi_\varepsilon)_x(s, x)), & s\in\,]t, T[\\ x(t) = y. \end{cases} \tag{5.15}$$

For each $\varepsilon > 0$ and $y \in D(A)$, problem (5.15) has a unique solution $x_\varepsilon \in W^{1,2}(t, T; H)$. Here is the argument.

Since $(\psi_\varepsilon)_x \in C([0, T]; C^0_{\mathrm{Lip}}(H, H))$ and $(h^*)_x \in C^0_{\mathrm{Lip}}(H, H)$, we deduce by a standard argument that (5.15) has a unique continuous local solution x_ε. In as much as $\psi_\varepsilon(s, \cdot) \in K$ for all $s \in [0, T]$ it follows by (5.15) that

$$|x_\varepsilon(s)| \leqslant C \qquad s \in [t, T'[\tag{5.16}$$

where $[t, T'[$ is the maximal interval of definition for x_ε.

Estimate (5.16) then implies by a standard device that x_ε can be extended as a solution (in the « mild » sense) to (5.15) on the whole interval $[t, T]$. Clearly $x_\varepsilon \in W^{1,2}(t, T; H)$ and equation (5.15) is satisfied a.e. on $]t, T[$.

Now in (5.13) we take $u = u_\varepsilon = (h^*)_x(-B^*(\psi_\varepsilon)_x(s, x_\varepsilon))$ and obtain

$$\frac{d}{ds}\psi_\varepsilon(s, x_\varepsilon(s)) + g(x_\varepsilon(s)) + h(u_\varepsilon(s)) = \delta_\varepsilon(s) \qquad \text{a.e. } s \in]t, T[$$

and therefore

$$\psi_\varepsilon(t, y) - \int_t^T \big(g(x_\varepsilon(s)) + h(u_\varepsilon(s))\big)\,ds + \varphi_0(x_\varepsilon(T)) \to 0 \qquad \text{for } \varepsilon \to 0 .$$

Along with (5.14) the latter implies (5.11) as claimed.

Let $u^* \in L^2(0, T; U)$ be any optimal control of the problem and let $x^* \in W^{1,2}(0, T; H)$ be the corresponding optimal state. By (5.11) it follows that

$$\psi(t, x^*(t)) = \int_t^T \big(g(x^*(s)) + h(u^*(s))\big)\,ds + \varphi_0(x^*(T)) , \qquad t \in [0, T] . \tag{5.17}$$

Next, in (5.13) we take $u = u^*$ and $x_u = x^*$. We get

$$\psi_\varepsilon(t, x^*(t)) = \int_t^T \Big(g\big(x^*(s)\big) + h\big(u^*(s)\big)\Big)\,ds + \varphi_0(x^*(T)) -$$
$$- \int_t^T \Big(h^*\big(-B^*(\psi_\varepsilon)_x(s, x^*(s))\big) + h\big(u^*(s)\big) + \langle B^*(\psi_\varepsilon)_x(s, x^*(s)), u^*(s)\rangle + \delta_\varepsilon(s)\Big)\,ds$$

and by (5.6), (5.17) we see that

$$\int_t^T \Big(h\big(u^*(s)\big) + h^*\big(-B^*(\psi_\varepsilon)_x(s, x^*(s))\big) + \langle B^*(\psi_\varepsilon)_x(s, x^*(s)), u^*(s)\rangle\Big)\,ds \to 0$$

uniformly on $[0, T]$

which implies in particular that

$$(5.18)\qquad \int_t^T \big(h^*(-B^*(\psi_\varepsilon)_x(s, x^*(s)) - h^*(v)\big)\,ds \leqslant \\ \leqslant \int_t^T \langle -B^*(\psi_\varepsilon)_x(s, x^*(s)) - v(s), u^*(s)\rangle\, ds + \eta_\varepsilon\,, \qquad \forall v \in L^2(0, T; U)$$

where $\eta_\varepsilon \to 0$ for $\varepsilon \to 0$.

On the other hand, it follows by (5.7) that $\{(\psi_\varepsilon)_x(t, x^*(t))\}$ is bounded in $L^\infty(0, T; H)$. Thus we may assume that

$$(5.19)\qquad (\psi_\varepsilon)_x(t, x^*) \to q \quad \text{weak star in } L^\infty(0, T; H)$$

and letting ε tend to zero in (5.18) we get (because the convex integrand is weakly lower semicontinuous),

$$\int_0^T \big(h^*(-B^*q(t)) - h^*(v)\big)\,dt \leqslant \int_0^T \langle -B^*q(t) - v(t), u^*(t)\rangle\, dt\,, \qquad \forall v \in L^2(0, T; U)\,.$$

Equivalently,

$$(5.20)\qquad u^*(t) \in (h^*)_x\big(-B^*q(t)\big) \quad \text{a.e. } t \in \,]0, T[\,.$$

Similarly by (5.19) it follows that

$$q(t) \in \partial\psi\big(t, x^*(t)\big) \quad \text{a.e. } t \in \,]0, T[$$

which along with (5.20) implies (5.12) thereby completing the proof.

6. – Additional regularity properties for the equation (1.3).

LEMMA 4. *For all $\varepsilon > 0$ and $R > 0$ one has*

$$(6.1)\qquad \|E_\varepsilon\|_{2,R} \leqslant (1 + \alpha\|E\|_{0,R})(\|E\|_{2,R} + 3\alpha\|E\|_{1,R}^2) \qquad \forall E \in \Pi \cap C^2_{\mathrm{Lip}}(H, H)$$

$$(6.2)\qquad |E_\varepsilon - \tilde{E}_\varepsilon|_{2,R} \leqslant (1 + BR^2)\{1 + 2\varepsilon BR(1 + R) + \\ + \varepsilon^2 B^2 R^2 \cdot (1 + BR + 3R)\}\,|E - \tilde{E}|_{2,R} \qquad \forall E, \tilde{E} \in \Pi \cap C^2(H, H)$$

where $B = \sup\{\|E\|_{2,R}, \|\tilde{E}\|_{2,R}\}$.

PROOF. The proof is standard and relies on the formula:

$$E_\varepsilon''(x) = E''(J_\varepsilon(x))(J_\varepsilon'(x), J_\varepsilon'(x)) + E'(J_\varepsilon(x))J_\varepsilon''(x) \tag{6.3}$$

where

$$\begin{cases} J_\varepsilon(x) = (1+\varepsilon E)^{-1}(x) \\ J_\varepsilon'(x) = \big(1+\varepsilon E'(J_\varepsilon(x))\big)^{-1} \\ J_\varepsilon''(x) = -\varepsilon J_\varepsilon'(x) E''(J_\varepsilon(x))(J_\varepsilon'(x), J_\varepsilon'(x)) . \end{cases} \tag{6.4}$$

LEMMA 5. *Assume that* E_0, $\tilde{E}_0 \in \Pi \cap C^2_{\mathrm{Lip}}(H, H)$ *and* G, $\tilde{G} \in C([0, T]; C^2_{\mathrm{Lip}}(H, H))$ *with* $G(t) \in \Pi$, $\forall t \in [0, T]$; *then equation* (4.27) *has unique solutions* E, $\tilde{E} \in C([0, T]; C^2_{\mathrm{Lip}}(H, H))$ *and it is:*

$$\|E(t,\cdot)\|_{2,R} \leqslant \exp(c_R t)\|E_0\|_{2,R} + \int_0^t \exp(c_R(t-s))\|G(s,\cdot)\|_{2,R}\, ds \tag{6.5}$$

$$|E(t,\cdot) - \tilde{E}(t,\cdot)|_{1,R} \leqslant \exp(c_R t)|E_0 - \tilde{E}_0|_{1,R} + \int_0^t \exp(c_R(t-s))|G(s,\cdot) - \tilde{G}(s,\cdot)|_{1,R}\, ds \tag{6.6}$$

where c_R *is independent from* α.

PROOF. The proof is quite similar to that of Theorem 4 (using estimates (6.1) and (6.2)).

THEOREM 6. *Assume that* $E_0 \in C^2_{\mathrm{Lip}}(H, H) \cap \Pi$ *and* $G \in C([0, T]; C^2_{\mathrm{Lip}}(H, H))$ *with* $G(t) \in \Pi$, $\forall t \in [0, T]$. *Then there exists a unique solution* $E \in C([0, T]; C^1(H, H))$ *to equation* (4.37).

PROOF. Consider the approximating equation

$$E^\varepsilon(t, x) = \exp(-tA^*)E_0(\exp(-tA)x) + \int_0^t \exp(-(t-s)A^*) \cdot \Big[\gamma_\varepsilon(E^\varepsilon)\big(s, \exp(-(t-s)A)x\big) + G\big(s, \exp(-(t-s)A)x\big)\Big]\, ds$$

where

$$\gamma_\varepsilon(f) = (f - f_\varepsilon)/\varepsilon \qquad \forall f \in \Pi .$$

We write γ_ε in the following form:

$$\gamma_\varepsilon(f) = f_x f + R_\varepsilon(f) \tag{6.7}$$

where

$$R_\varepsilon(f)(x) = \int^t \Big\{ f_x\big(\xi x + (1-\xi)(1 + J_\varepsilon(x)) f_\varepsilon(x) - f_x(x) f(x)\big)\Big\} d\xi . \tag{6.8}$$

If $f \in C^2_{\mathrm{Lip}}(H, H)$ it is:

(6.9) $$\lim_{\varepsilon\to 0} |R_\varepsilon(f)|_{1R} = 0 .$$

Let now $\mu > 0$, it is:

$$E^\mu(t, x) = \exp(-tA^*)E_0(\exp(-tA)x) + \int_0^t \exp(-(t-s)A^*) \cdot$$
$$\cdot[\gamma_\varepsilon(E^\mu) + R_\varepsilon(E^\mu) - R_\mu(E^\mu)]\cdot\big(s, \exp(-(t-s)A)x\big)\,ds .$$

Recalling (6.6), (6.9) and estimating $|E^\varepsilon(t,\cdot) - E^\mu(t,\cdot)|_{1,R}$ via the Gromwall lemma we get:

$$\lim_{\varepsilon\to 0} E^\varepsilon = E \qquad \text{in } C\big([0, T];\ C^1(H, H)\big)$$

and from (6.5) it follows that E is a solution to equation (4.37). To prove uniqueness consider two solutions E_1 and E_2, then for every $\beta > 0$ it is:

$$E_i(t, x) = \exp(-tA^*)E_0(\exp(-tA)x) + \int_0^t \exp(-(t-s)A^*) \cdot$$
$$\cdot[\gamma_\beta(E_i) + G - R_\beta(E_i)]\big(s, \exp(-(t-s)A)x\big)\,ds \qquad i = 1, 2 .$$

Using again (6.6), (6.9) and the Gromwall lemma we get $E_1 = E_2$.

Equation (4.37) is a «mild» form of equation (1.3). To find classical solution we consider the semigroup in $C_0(H, H)$ defined by

(6.10) $$G_t(f)(x) = \exp(-tA)f(\exp(-tA)x) \qquad \forall f \in C^0(H, H) .$$

We remark that G_t applies Π in itself; we put

$$\begin{cases} D(\mathcal{A}) = \left\{f \in C^0(H, H);\ \exists \lim\limits_{h\to 0^+} \dfrac{1}{h}\big(G_h(f)(x) - f(x)\big) \in C^0(H, H)\right\} \\ \mathcal{A}(f)(x) = \lim\limits_{h\to 0^+} \dfrac{1}{h}\big(G_h(f)(x) - f(x)\big) \end{cases}$$

LEMMA 6. – *Assume that $f \in D(\mathcal{A}) \cap C^1(H, H)$ and $x \in D(A)$; then it is $f(x) \in D(A^*)$ and moreover*:

(6.11) $$\mathcal{A}(f)(x) = A^* f(x) + f_x(x)Ax .$$

PROOF. Put $g = \mathcal{A}(f)$ and take $x, y \in D(A)$; it is:

$$\langle g(x), y\rangle = \frac{d}{dh}\langle \exp(-hA^*) f(\exp(-hA)x), y\rangle|_{h=0} =$$
$$= \frac{d}{dh}\langle f(\exp(-hA)x), \exp(-hA)y\rangle|_{h=0} = -\langle f_x(x)Ax, y\rangle - \langle f(x), Ay\rangle,$$

therefore the linear mapping

$$y \to \langle f(x), Ay\rangle = \langle g(x), y\rangle - \langle f_x(x)Ax, y\rangle$$

is continuous; consequently it is $f(x) \in D(A^*)$ and (6.11) is fulfilled.

We write now equation (1.3) in the following form

$$\begin{cases} E_t + \mathcal{A}(E) + E_x E = G \\ E(0) = E_0 . \end{cases} \tag{6.12}$$

THEOREM 7. *Assume that* $E_0 \in C^2_{\mathrm{Lip}}(H, H) \cap \Pi \cap D(\mathcal{A})$, $\mathcal{A}(E_0) \in C^1(H, H)$, $G, G_t \in C([0, T]; C^2_{\mathrm{Lip}}(H, H))$ *with* $G(t, \cdot) \in \Pi$.
Then the equation (1.3) *has a unique classical solution.*

PROOF. We can solve (in the « mild » form) the following linear problem:

$$\begin{cases} V_t + \mathcal{A}(V) + V_x E + E_x V = G_t \\ V(0) = G(0, \cdot) - \mathcal{A}(E_0) - E_{0x}E_0 \end{cases} \tag{6.13}$$

where $E \in C([0, T]; C^1(H, H))$ is the solution to (4.37).

Let us consider the approximating problems:

$$\begin{cases} E^n_t + \mathcal{A}_n(E^n) + E^n_x E^n = G \\ E^n(0) = E_0 \\ V^n_t + \mathcal{A}_n(V^n) + V^n_x V^n + E^n_x V^n = G_t \\ V^n(0) = G(0, \cdot) - \mathcal{A}_n(E_0) - E_{0x}E_0 \end{cases}$$

where $\mathcal{A}_n(E) = A^*_n E + E_x A_n$ and $A_n = n - n^2\ (n + A)^{-1}$: it is clear that $V^n = E^n_t$ and it is easy to show that

$$V^n \to V, \quad E^n \to E \qquad \text{in } C([0, T]; C^1(H, H))$$

this implies $V = E_t$ and the thesis follows.

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