ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

HIRONORI SHIGA

One attempt to the $K3$ modular function - II

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 8, nº 1 (1981), p. 157-182
<http://www.numdam.org/item?id=ASNSP_1981_4_8_1_157_0>

One Attempt to the $K3$ Modular Function - *II*.

HIRONORI SHIGA

§ **0.** In the previous paper ([9], we quote it as the part I) the author pointed out that the Picard's modular function (see [1]) coincides with the one for a certain family of elliptic $K3$ surfaces with 2 complex parameters. The elliptic modular function is characterized as a function which gives the moduli of elliptic curves. And already there are various extensions of the theory of the elliptic modular function. Perhaps one fruitfull extension is the one for the Abelian varieties. And the Abelian variety seems to be a natural extension of the concept of the elliptic curve. But the author thinks that the $K3$ surface is another extension. Then he tried to find a new modular function of several variables by considering a family of $K3$ surfaces. The part I was the first experiment for this supposition. In this paper we study several types of elliptic $K3$ surfaces and the period mapping for them. And we show that the period is given as a ratio of two solutions of Appell's hypergeometric equation F_1.

We proceed our consideration as the following. We study the elliptic $K3$ surface X with following properties (see [3] for the general theory of the elliptic surface and [4], [5] for the $K3$ surface):

(i) X has a holomorphic section,

(ii) the functional invariant $\mathfrak{f}$ is the constant function 0,

(iii) X has five singular fibres.

By a $J0K3$ *surface* we mean an elliptic $K3$ surface with the properties (i) and (ii).

In § 1 we find a representing equation (1-5) for a $J0K3$ surface with maximum number of singular fibres. Next we show that there are 9 types of $J0K3$ surfaces with the property (iii) (Proposition 1-4). It will be called *a surface of class* (j). Any $J0K3$ surface has an automorphism ϱ of order

Pervenuto alla Redazione il 31 Gennaio 1980.

three which preserves all fibres. Let Λ be a domain in $\boldsymbol{P}^2$ defined by $\{\xi = [\xi_1, \xi_2, \xi_3];\ \xi_i \neq \xi_j,\ 0 \text{ for } i \neq j\}$, where $[\xi_1, \xi_2, \xi_3]$ is a homogeneous coordinate on $\boldsymbol{P}^2$. A surface S_j of class (j) is determined by a point ξ on Λ. We denote it by $S_j(\xi)$. There uniquely exists a holomorphic 2-form φ on $S_j(\xi)$, up to constant factors, because S_j is a $K3$ surface. We determine this form in Proposition 1-5.

In § 2 we construct a basis system $\Gamma = \{C_1, \ldots, C_{22}\}$ of $H_2(S_j(\xi), \boldsymbol{Z})$ and a basis system $\Gamma^* = \{G_1, \ldots, G_{22}\}$ of $H_2(S_j(\xi), \boldsymbol{Q})$ with $C_i G_j = \delta_{ij}$ (Proposition 2-3). Among them $C_7, \ldots, C_{22}$ and $G_7, \ldots, G_{22}$ are given as divisors. The intersection matrix $A = (G_i G_j)_{1 \leq i, j \leq 6}$ is determined in Diagram 4.

In § 3 we consider a period integral

$$\eta_i(\xi) = \int_{C_i} \varphi\,, \qquad i = 1, \ldots, 22$$

for a surface $S_j(\xi)$. Because $C_7, \ldots, C_{22}$ are algebraic cycles, we have $\eta_i(\xi) = 0$ for $i = 7, \ldots, 22$. And it occurs a relation (3·4) among $\eta_1, \ldots, \eta_6$. Consequently the period mapping $\Phi = [\eta_1, \ldots, \eta_{22}]$ reduces to a mapping from Λ to $\boldsymbol{P}^2$. The image $\Phi(\Lambda)$ is contained in a domain Ω_j $(j = 1, \ldots, 9)$, where Ω_j is determined by Diagram 6 and (3-6). The domain Ω_j is biholomorphically equivalent to a hyperball (Proposition 3-1).

And we define a monodromy covering domain $\hat{\Lambda}$ over Λ at the biginning of this section. Then Φ becomes injective on $\hat{\Lambda}$ (Proposition 3-2).

In § 4 we consider the monodromy of $\eta_i(\xi)$. And we determine the Appell's hypergeometric function F_1 which coincides with $\eta_i(\xi)$ (Proposition 4-1).

In § 5 we study the behavior of Φ on the hyperplane $H_{ij} = \{\xi_i = \xi_j;\ i \neq j,\ i = 1, 2, 3 \text{ and } j = 0, 1, 2, 3\}$ in $\boldsymbol{P}^2$. And we point out the domain $\hat{\Lambda}_0$ to which the mapping Φ can be extended. As a consequence of this investigation we know that Φ induces a biholomorphic equivalence between $\bar{\Lambda} = \boldsymbol{P}^2$ and $(\Omega_j/G)^*$, where G indicates the discontinuous group induced from the monodromy and $(\Omega_j/G)^*$ is the Baily-Borel compactification of Ω_j/G, for $j = 1, 2, 3, 4$ (Proposition 5-2).

The author obtained the results in the part I and in this paper by refering the esthetic principle « Honkadori » which is found in the medieval Japanese anthology.

1. – Representing equation.

[1]. In the part I (Proposition 2-1) we already showed the necessary and sufficient condition that an elliptic surface of basic type, namely a

surface with a holomorphic section and without a multiple singular fibre, should become a $K3$ surface. According to Pjateckiĭ-Šapiro and Safarevič ([5], Corollary 2 to Theorem 1) a $J0K3$ surface (X, π, Δ) has no multiple singular fibre. Then it becomes of basic type. Hence we have $\Delta = \boldsymbol{P}$ and $\chi(X) = 24$, where χ indicates the Euler characteristic. We employ following notations:

Δ: a compact Riemann surface with genus p,

$\mathfrak{f}$: a meromorphic function on Δ,

$\xi_1, \dots, \xi_r$: points on Δ such that $\mathfrak{f} \neq 0, 1, \infty$ on $\Delta' = \Delta - \{\xi_1, \dots, \xi_r\}$ (if $\mathfrak{f} \equiv$ constant we choose arbitrary points $\xi_1, \dots, \xi_r$),

ξ': a fixed point on Δ',

$\{h_1, \dots, h_{2p}\}$: a canonical generator system of $\pi_1(\Delta, \xi')$,

g_i: a closed arc with terminal point ξ' which goes around ξ_i in the positive sense.

Then we obtain a generator system $\{h_1, \dots, h_{2p}, g_1, \dots, g_r\}$ of $\pi_1(\Delta', \xi')$ with only one relation

$$h_1 h_2 h_1^{-1} h_2^{-1} \dots h_{2p-1} h_{2p} h_{2p-1}^{-1} h_{2p}^{-1} g_1 \dots g_r = 1 .$$

Let ω be a multivalued analytic function on Δ' defined by the equality $j \circ \omega \equiv \mathfrak{f}$ (if $\mathfrak{f} \equiv 0, 1$ we set $\omega \equiv \exp(\frac{2}{3}\pi i)$ and i, respectively), where j indicates the elliptic modular function. Let $\omega(\xi')$ be a certain branch of ω at ξ'. We denote by $h_i\omega(\xi')$ and $g_i\omega(\xi')$ respectively the values of ω obtained by the continuation of $\omega(\xi')$ along h_i and g_i. These are given by modular transformations S_{h_i} and S_{g_i}, respectively. Let $S_{(h_i)}$ and $S_{(g_i)}$ be matrices in $SL(2, \boldsymbol{Z})$ which induce S_{h_i} and S_{g_i}, respectively, and suppose that these matrices give a representation of $\pi_1(\Delta', \xi')$ into $SL(2, \boldsymbol{Z})$.

This representation is a homological invariant $\mathfrak{H}$ belonging to $\mathfrak{f}$. It follows a theorem due to Kodaira ([4], Theorem 10.2).

THEOREM. *Let us consider the situation mentioned above. There exists uniquely, as a fibre surface, an elliptic surface (X, π, Δ) of basic type with the given functional invariant $\mathfrak{f}$ and the homological invariant $\mathfrak{H}$. And its singular fibres are situated over $\xi_1, \dots, \xi_r$.*

By *a critical point* we mean a point ξ_i such that $\pi^{-1}(\xi_i)$ is a singular fibre.

Here we show that the homological invariant is uniquely determined by the types of singular fibres in our situation. In case of $\mathfrak{f} \equiv 0$ the matrix

$S_{(h_i)}$ ($S_{(g_i)}$) is one of the following 6 types ([4], § 9):

(1-1)
$$\begin{cases} S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & S_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, & S_2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \\ S_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & S_4 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, & S_5 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \end{cases}$$

All matrices in (1-1) are commutative each other, then we obtain an explicit condition that $\{S_{(h_i)}, S_{(g_i)}\}$ gives a homological invariant belonging to $\mathfrak{f} \equiv 0$:

(1-2)
$$\prod_{j=1}^{r} S_{(g_i)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathfrak{H}$ be such a homological invariant. And let (X, π, Δ) be an elliptic surface of basic type with given $\mathfrak{f} \equiv 0$ and $\mathfrak{H}$ as mentioned in the above Theorem. There is a one to one correspondence between the matrices $S_{(g_i)}$ and the types of singular fibres $\pi^{-1}(\xi_i)$ as the following:

(1-3)

matrix	S_0	S_1	S_2	S_3	S_4	S_5
type of fibres	regular	II	IV	I_0^*	IV^*	II^*

By *a J0 singular fibre* we mean a singular fibre listed in (1-3). Consequently we obtain the following.

PROPOSITION 1-1. *When we choose J0 singular fibres $\pi^{-1}(q_i)$ ($i = 1, \ldots, r$) so that the corresponding matrices $S_{(\xi_i)}$ satisfy the relation* (1-2), *then there exists uniquely an elliptic surface of basic type with $\mathfrak{f} \equiv 0$ which has the appointed type of singular fibre over each ξ_i.*

REMARK 1-1. We can obtain a similar conclusion for the case $\mathfrak{f} \equiv \equiv$ constant.

REMARK 1-2. Let $\{\gamma_1, \gamma_2\}$ be a basis system of $G = H_1(\pi^{-1}(\xi'), \boldsymbol{Z})$ with the properties

$$\begin{cases} \gamma_1 \gamma_2 = -1 , \\ \int_{\gamma_1} \omega = \exp\left(\tfrac{2}{3}\pi i\right) \int_{\gamma_2} \omega , \end{cases}$$

where ω indicates the Abelian differential on $\pi^{-1}(\xi')$. And let M_i be a monodromy transformation of G induced from the arc g_i. Then the matrix in (1-3) corresponding to $\pi^{-1}(\xi_i)$ represents M_i with respect to $\{\gamma_1, \gamma_2\}$. Such a basis $\{\gamma_1, \gamma_2\}$ will be called *canonical.*

Let C be a $J0$ singular fibre and let S_i $(i = 0, 1, \ldots, 5)$ be the corresponding matrix. Then we have

$$\text{(1-4)} \qquad \begin{cases} \chi(C) = 2i\,, \\ S_i = S_1^i\,. \end{cases}$$

Next we consider r $J0$ singular fibres $C_1, \ldots, C_r$, and let $S_{i_1}, \ldots, S_{i_r}$ be corresponding matrices. Assume that we have

$$\sum_{i=1}^{r} \chi(C_i) = 24\,.$$

Because of (1-4) it follows $S_{i_1} \ldots S_{i_r} = S_1^{12} = E$. According to Proposition 2-1 in the part I and Proposition 1-1 we have the following.

PROPOSITION 1-2. *Let us appoint finite points $\xi_1, \ldots, \xi_r$ on $\boldsymbol{P}$ and the types of $J0$ singular fibres for each ξ_i so that the total sum of their Euler characteristics is equal to* 24, *then there exists uniquely a $J0K3$ surface with these singular fibres.*

REMARK 1-3. According to this proposition and the relation (1-4) we know that a $J0K3$ surface has at most 12 singular fibres, and in the maximum case any singular fibre is of type II.

[2]. Let us consider the following variety X with 12 different parameters ξ_i $(i = 1, \ldots, 12)$:

$$\text{(1-5)} \qquad \begin{cases} \eta_2^3 - \eta_1\left\{\eta_1^2 \prod_{i=1}^{12} (u - \xi_i) - \eta_0^2\right\} = 0\,, \\ \eta_2'^3 - \eta_1'\left\{\eta_1'^2 \prod_{i=1}^{12} (1 - u'\xi_i) - \eta_0'^2\right\} = 0\,, \end{cases}$$

where $[\eta_0, \varrho_1, \eta_2]$ and $[\eta_0', \eta_1', \eta_2']$ are homogeneous coordinate systems on $\boldsymbol{P}^2$, and we identify two points $([\eta_0, \eta_1, \eta_2], u)$ and $([\eta_0', \eta_1', \eta_2'], u')$ by the condition

$$u' = \frac{1}{u}, \quad \eta_2 = u'^2 \eta_2', \quad \eta_1 = u'^6 \eta_1', \quad \eta_0' = \eta_0\,.$$

If we use an affine coordinate $(u, v = \eta_1/\eta_0, w = \eta_2/\eta_0)$, we obtain an affine representation of X:

$$\text{(1-6)} \qquad w^3 - v\left\{v^2 \prod_{i=1}^{12} (u - \xi_i) - 1\right\} = 0\,.$$

Let π be a projection mapping from X to the u-sphere $\boldsymbol{P}$. And let us consider a fibre surface $(X, \pi, \boldsymbol{P})$. It is easily shown that X is nonsingular and $\pi^{-1}(u)$ is a nonsingular elliptic curve with the invariant 0 for every u except ξ_i $(i = 1, \ldots, 12)$. And also we can see that $\pi^{-1}(\xi_i)$ is a rational curve with one cusp singularity at $[\eta_0, \eta_1, \eta_2] = [0, 1, 0]$, namely it is a singular fibre of type II. Then the total sum of the Euler characteristics of $\pi^{-1}(\xi_i)$ is equal to 24. By Proposition 2-1 in the part I we obtain that $(X, \pi, \boldsymbol{P})$ is a $K3$ surface.

The curve $L = \{\eta_1 = \eta_2 = 0\} = \{\eta_1' = \eta_2' = 0\}$ gives a holomorphic section. Moreover we can show that the form $\varphi = w^{-2}\, du \wedge dv$ in (1-6) gives a holomorphic 2-form on X. Thus we obtain:

PROPOSITION 1-3. *The nonsingular variety X defined by* (1-5) *gives a representation of a $J0K3$ surface with maximum number of singular fibres. Its unique holomorphic 2-form is given by $\varphi = w^{-2}\, du \wedge dv$ using the affine representation* (1-6).

REMARK 1-4. Let us consider the above surface (1-5). And suppose that k critical points $\xi_{i_1}, \ldots, \xi_{i_k}$ coincide with a point ξ_0. Then the monodromy matrix for the arc g_0 with respect to a canonical basis $\{\gamma_1, \gamma_2\}$ in Remark 1-2 is given by S_1^k. Let q be an integer with $q \equiv k \pmod 6$ and $0 \leqq q < 6$. Because S_1 is of order 6, we have $S_1^k = S_1^q$. By the correspondence (1-3) we know that the fibre $\pi^{-1}(\xi_0)$ has to correspond to S_q.

[3]. Next we consider a $J0K3$ surface with 5 singular fibres $C_1, \ldots, C_5$. Such a surface will be called *of type F*. According to Proposition 2-1 in the part I we have

$$\sum_{i=1}^{5} \chi(C_i) = 24 . \tag{1-7}$$

In the part I we already studied a certain class of surfaces of type F, namely the surface with 4 singular fibres of type IV and one singular fibre of type IV*. We denote such a combination of singular fibres by $4\mathrm{IV} + \mathrm{IV}^*$.

PROPOSITION 1-4. *There are nine classes of the surface of type F*:

(1) $4\mathrm{IV} + \mathrm{IV}^*$	(2) $\mathrm{II} + 3\mathrm{IV} + \mathrm{II}^*$	(3) $3\mathrm{II} + \mathrm{IV}^* + \mathrm{II}^*$
(4) $2\mathrm{II} + \mathrm{IV} + \mathrm{I}_0^* + \mathrm{II}^*$	(5) $3\mathrm{IV} + 2\mathrm{I}_0^*$:	(6) $2\mathrm{II} + \mathrm{IV} + 2\mathrm{IV}^*$
(7) $\mathrm{II} + 2\mathrm{IV} + \mathrm{I}_0^* + \mathrm{IV}^*$	(8) $2\mathrm{II} + 2\mathrm{I}_0^* + \mathrm{IV}^*$	(9) $\mathrm{II} + \mathrm{IV} + 3\mathrm{I}_0^*$.

PROOF. If we consider the relation (1-4) and (1-7), we obtain the above 9 combinations by an elementary calculation. By Proposition 1-2 there exists a surface with such a combination of singular fibres. q.e.d.

By *a surface of class* (j) $(j=1, \ldots, 9)$ we mean the j-th surface in Proposition 1-4. Let us consider the following affine variety V with 5 different parameters ξ_i $(i=1, \ldots, 5)$:

$$w^3 - v\left\{v^2 \prod_{i=1}^{5} (u-\xi_i)^{\nu_i} - 1\right\} = 0\,, \tag{1-8}$$

where we assume that the values ν_i satisfy the condition

$$\begin{cases} \nu_1 + \ldots + \nu_5 = 12\,, \\ 1 \leqq \nu_i \leqq 5 \quad \text{for } i=1, \ldots, 5\,. \end{cases} \tag{1-9}$$

Let S be a minimal nonsingular model of V, and let π be a projection mapping from S to the u-sphere $\boldsymbol{P}$. Then the fibre surface $(S, \pi, \boldsymbol{P})$ is an elliptic surface with five singular fibres over $u=\xi_1, \ldots, \xi_5$, and its functional invariant $\mathfrak{f}$ is equal to 0. Because of Remark 1-4 and the relation (1-4) we have $\chi(S)=24$. This surface has a holomorphic section $L = \{v=w=0\}$. By Proposition 2-1 in the part I we obtain that $(S, \pi, \boldsymbol{P})$ is a $K3$ surface, hence it is a surface of type F.

Let us consider the 2-form $\varphi = w^{-2}\, du \wedge dv$. It is holomorphic on the regular fibre. Then we investigate its behavior on the singular fibre. Let p be a projection mapping from V to the u-sphere. It is easy to see that $p^{-1}(\xi_i)$ is nonsingular on the affine part, and φ is holomorphic there. If we set $v'=1/v$ and $w'=w/v$, then we obtain a representation of V:

$$w'^3 - \prod_{i=1}^{5} (u-\xi_i)^{\nu_i} + v'^2 = 0\,. \tag{1-10}$$

From this representation we can see that $p^{-1}(\xi_i)$ has various types of singularities depending on ν_i.

We have normal forms of the isolated singularity on $p^{-1}(\xi_i)$ as Diagram 1.

These are rational double singularities. Then every curve which occurs as a consequence of the resolution of the singularity has the selfintersection number -2. Already we know that φ is holomorphic at any nonsingular point on V. It does not occur that a meromorphic form has its pole only along exceptional curves of second kind. Hence φ is holomorphic on S.

REMARK 1-5. If we have $\nu_i = 6$, the fibre $p^{-1}(\xi_i)$ has a singularity $x^2+y^3+z^6=0$ at infinity. This is a simply elliptic singularity of type $\tilde{E}_8$.

Diagram 1

ν_i	singular fibre	normal form	classification of the isolated singularity
1	II	nonsingular	—
2	IV	$x^2+y^2+z^3=0$	A_2
3	I_0^*	$x^2+y^3+z^3=0$	D_4
4	IV*	$x^2+y^3+z^4=0$	E_6
5	II*	$x^2+y^3+z^5=0$	E_8

Then it occurs a nonsingular elliptic curve E from this singularity. The form φ has its pole along E, and we have $\pi^{-1}(\xi_i)=E$.

From the above consideration we have the following.

PROPOSITION 1-5. *We have a representation of a surface of type F as the minimal nonsingular model of the variety* V *defined by* (1-8) *and* (1-9). *And its holomorphic 2-form φ is given by $w^{-2}du\wedge dv$.*

2. – Homology basis.

[1]. Let us consider a $J0K3$ surface $(S,\pi,\boldsymbol{P})$. Let $\{\xi_0,\xi_1,\ldots,\xi_r,\xi_\infty\}$ be the totality of the critical points. And we assume that $\xi_0=0$ and $\xi_\infty=\infty$. Let l_i $(i=0,1,\ldots,r)$ be an arc connecting ξ_i and ∞ so that l_i does not intersect any other l_j.

We employ the following notations:

Δ: the base Riemann surface $\boldsymbol{P}$,

$\Delta'=\Delta-\{\xi_0,\xi_1,\ldots,\xi_r,\xi_\infty\}$,

$\Delta_0=\Delta-\{l_0,l_1,\ldots,l_r\}$,

ξ': a fixed point on Δ_0,

$C=\pi^{-1}(\xi')$,

$\{\gamma_1,\gamma_2\}$: a canonical basis of $H_1(C,\boldsymbol{Z})$.

Let g_i $(i=0,1,\ldots,r,\infty)$ be the closed arc defined in § 1 [1] so that any g_i $(i=0,1,\ldots,r)$ does not intersect l_j for $i\neq j$ and that g_∞^{-1} intersects $l_0,\ldots,l_r$ in this order. And let α_i $(i=1,\ldots,r)$ be an oriented arc which starts from 0 and goes to ξ_i without intersecting any l_j $(j\neq i)$.

Let g be an element of $\pi_1(\Delta', \xi')$ so that it induces a trivial monodromy of $H_1(C, \boldsymbol{Z})$. If we make a continuation of a 1-cycle γ on C along g, we obtain a 2-cycle on S. We denote this 2-cycle by $g\times\gamma$, and g will be called *a base arc* of $g\times\gamma$ We define the orientation of $g\times\gamma$ as the ordered pair of the orientation of g and the one of γ.

[2]. Let us consider a $J0K3$ surface $(S, \pi, \boldsymbol{P})$ with 12 singular fibres $\pi^{-1}(\xi_i)$ $(i = 0, 1, \dots, \infty)$. We set

$$(2\text{-}1)\qquad \begin{cases} G_{2i-1} = g_i^{-1} g^{\infty} \times \gamma_2\,, & \\ G_{2i} = g_i g_{\infty}^{-1} \times \gamma_1 & \text{for } i = 1, \dots, 10\,, \\ G_{21} = C\,, & \\ G_{22} = C + L\,. & \end{cases}$$

The intersection matrix $M = (G_i G_j)_{1 \leqq i, j \leqq 22}$ is given as the following:

$$\begin{bmatrix}
-2 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & -2 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & -2 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & -2 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & -2 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & -2 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & 0 \\
1 & 1 & 0 & 1 & -1 & 0 & 1 & -2 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -2 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 1 & -2 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & 0 \\
-1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -2 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 1 & -2 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -2 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 0 \\
-1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 1 & -2 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -2 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 1 & -2 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -2 & 1 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -2 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

This is examined by the same method developped in the part I § 3.

Here we note that $LG_i = 0$ $(i = 1, \dots, 20)$ is obtained by the direct observation of G_i constructed by use of the representation (1-6). By an elementary calculation we obtain that M is invertible. We have $b_2 = 22$ for a $K3$ surface. Hence $\{G_1, \dots, G_{22}\}$ gives a basis system of $H_2(S, \boldsymbol{Q})$.

Because the fibre space $(\pi^{-1}(\Delta_0), \pi, \Delta_0)$ is trivial, we can determine a canonical basis $\{\gamma_1(u), \gamma_2(u)\}$ of any fibre $\pi^{-1}(u)$ over a point u of Δ_0 by making a continuation of $\{\gamma_1, \gamma_2\}$. Let us consider the representation (1-6). We can regard a general fibre $\pi^{-1}(u)$ as a three sheeted covering Riemann surface over v-sphere, and its ramified points are situated over $v = 0$ and $v = \pm\beta$, where

$$\beta = \left\{\prod_{i=1}^{12} (u - \xi_i)\right\}^{-\frac{1}{2}} .$$

We can realize $\{\gamma_1, \gamma_2\}$ as the cycles obtained by arcs connecting two ramified points over β and $-\beta$. Consequently $\gamma_1(u)$ and $\gamma_2(u)$ tend to the infinite point as u tends to a critical point ξ_i. Hence we obtain a 2-cycle on S as the continuation of a 1-cycle $\gamma(u)$ along α_i. We denote it by $\alpha_i \times \gamma$.

We set

$$\text{(2-2)} \qquad \begin{cases} C_{2i-1} = \alpha_i \times \gamma_1 , \\ C_{2i} \;\;\;\; = \alpha_i \times \gamma_2 \quad \text{for } i = 1, \dots, 10 , \\ C_{21} \;\;\; = G_{22} , \\ C_{22} \;\;\; = G_{21} , \end{cases}$$

where we define the orientation of C_i $(i = 1, \dots, 20)$ as same as for G_j. It is easily shown from the construction that we have

$$\text{(2-3)} \qquad C_i G_j = \delta_{ij} \quad \text{for } 1 \leqq i,\ j \leqq 22 .$$

Let G be an arbitrary element of $H_2(S, \boldsymbol{Z})$ and set $r_i = GG_i$. It follows from (2-3) that

$$\left(G - \sum_{i=1}^{22} r_i C_i\right) G_j = 0 \quad \text{for } j = 1, \dots, 22 .$$

This implicates $G = r_1 C_1 + \dots + r_{22} C_{22}$. Thus we have the following.

PROPOSITION 2-1. *The system $\{C_1, \dots, C_{22}\}$ defined by* (2-2) *gives a basis system of $H_2(S, \boldsymbol{Z})$. And the system $\{G_1, \dots, G_{22}\}$ defined by* (2-1) *gives a basis system of $H_2(S, \boldsymbol{Q})$. And these two systems are dual each other, namely we have the relation* (2-3).

[3]. Here we consider a surface $(S, \pi, \boldsymbol{P})$ of type F. By a fractional linear transformation we arrange the singular fibres so that they are situated

as Diagram 2.

Diagram 2

class of surfaces	types of singular fibres				
	0	ξ_1	ξ_2	ξ_3	∞
(1)	IV	IV	IV	IV	IV*
(2)	II	IV	IV	IV	II*
(3)	IV*	II	II	II	II*
(4)	I_0^*	IV	II	II	II*
(5)	IV	I_0^*	IV	IV	I_0^*
(6)	II	IV	IV*	IV*	II
(7)	I_0^*	IV*	IV	IV	II
(8)	IV*	II	I_0^*	I_0^*	II
(9)	IV	I_0^*	I_0^*	I_0^*	II

For the moment we fix the parameters $\xi = [\xi_1, \xi_2, \xi_3]$ so that we have

(2-4) $$\xi_1 < \xi_2 < \xi_3 .$$

And we assume that l_i $(i = 0, 1, 2, 3)$ is given as a line segment. We define 2-cycles G_1, G_3, G_5 on S as Diagram 3.

Diagram 3

class of surfaces	2-cycles (G_1, G_3, G_5)
(1)	$(g_1^{-1}g_\infty^{-1}, g_2^{-1}g_\infty^{-1}, g_3^{-1}g_\infty^{-1}) \times \gamma_2$
(2)	$(g_1^{-1}g_\infty^{-2}, g_2^{-1}g_\infty^{-2}, g_3^{-1}g_\infty^{-2}) \times \gamma_2$
(3)	$(g_1^{-1}g_\infty^{-1}, g_2^{-1}g_\infty^{-1}, g_3^{-1}g_\infty^{-1}) \times \gamma_2$
(4)	$(g_1^{-1}g_\infty^{-2}, g_2^{-1}g_\infty^{-1}, g_3^{-1}g_\infty^{-1}) \times \gamma_2$
(5)	$(g_1^{-1}g_\infty^{-1}, g_2^{-1}g_1^{-1}g_2^{-1}g_1^{-1}g_2^{-1}, g_3^{-1}g_2) \times \gamma_2$
(6)	$(g_1^{-1}g_\infty^{2}, g_2^{-1}g_\infty^{-2}, g_3^{-1}g_\infty^{2}) \times \gamma_2$
(7)	$(g_1^{1-}g_\infty^{-2}, g_2^{-1}g_\infty^{2}, g_3^{-1}g_\infty^{2}) \times \gamma_2$
(8)	$(g_1^{-1}g_\infty, g_2^{-1}g_\infty^{3}, g_3^{-1}g_\infty^{1}) \times \gamma_2$
(9)	$(g_1^{-1}g_\infty^{-3}, g_2^{-1}g_\infty^{-3}, g_3^{-1}g_\infty^{-3}) \times \gamma_2$

We can easily show that they are certainly 2-cycles, because we already know the monodromy transformation of $H_1(C, \boldsymbol{Z})$ induced from g_i (Remark 1-4). We set

$$G_{2i} = g^{-1} \times \gamma_1 \quad (i = 1, 2, 3), \tag{2-5}$$

where g indicates the base arc of G_{2i-1}.

Next we construct algebraic cycles on S. There are 14 components of singular fibres which does not intersect the holomorphic section L. We denote them by $G_7, \ldots, G_{20}$, and set $G_{21} = L$, $G_{22} = C$.

By the construction we can show that we have

$$G_i G_j = 0 \quad \text{for } 1 \leqq i \leqq 6,\ 7 \leqq j \leqq 22 . \tag{2-6}$$

Let A be the intersection matrix $(a_{ij}) = (G_i G_j)_{1 \leqq i, j \leqq 6}$ induced from $G_1, \ldots, G_6$. And let B be the one $(b_{ij}) = (G_{i+6} G_{j+6})_{1 \leqq i, j \leqq 16}$ induced from $G_7, \ldots, G_{22}$. We can determine B by considering the geometric figure of the $J0$ singular fibres.

REMARK 2-1. The matrix B is a direct sum of several minor intersection matrices each of them is the one induced from a $J0$ singular fibre or the one induced from G_{21} and G_{22}. But we must note that we excluded one simple component from each singular fibre to get $G_7, \ldots, G_{20}$.

As a consequence we obtain that B is invertible for any class (j) $(j = 1, \ldots, 9)$. According to the same method developped in the part I § 3 we can calculate the matrices $A = A_j$ for the surface of class (j) $(j = 1, \ldots, 9)$ as Diagram 4.

We can see that any A_j is invertible. Let M be the full size intersection matrix $(G_i G_j)_{1 \leqq i, j \leqq 22}$. From (2-6) we obtain a direct sum decomposition $M = A \oplus B$. Consequently we know that M is invertible. Then we obtain a basis system $\{G_1, \ldots, G_{22}\}$ of $H_2(S, \boldsymbol{Q})$ for a surface of type F with fixed parameters by (2-4).

Now we regard $\xi = [\xi_1, \xi_2, \xi_3]$ as a point on $\boldsymbol{P}^2$. And we use the following notations:

$H_{ij} = \{\xi = [\xi_1, \xi_2, \xi_3];\ \xi_i = \xi_j\}$, where $i \neq j$ for $i = 1, 2, 3$ and $j = 0, 1, 2, 3$,

$\Lambda = \boldsymbol{P}^2 - \{\text{the union of all } H_{ij}\}$,

$\tilde{\Lambda} =$ the universal covering of Λ,

p: the natural projection from $\tilde{\Lambda}$ to Λ,

$S_j(\xi)$: a surface of class (j) determined by parameters $\xi = [\xi_1, \xi_2, \xi_3]$ on Λ,

$\mathcal{F}_j = \{S_j(\xi);\ \xi \in \Lambda\}$.

Diagram 4

$$A_1=\begin{pmatrix}0 & H & -G\\ {}^tH & 0 & H\\ -G & {}^tH & 0\end{pmatrix},\qquad A_2=\begin{pmatrix}G & 3F & -3B\\ -3F & G & 3F\\ -3\,{}^tB & -3F & G\end{pmatrix},$$

$$A_3=\begin{pmatrix}0 & B & F\\ {}^tB & 0 & B\\ -F & {}^tB & 0\end{pmatrix},\qquad A_4=\begin{pmatrix}G & H & -{}^tH\\ {}^tH & 0 & B\\ -H & {}^tB & 0\end{pmatrix},$$

$$A_5=\begin{pmatrix}0 & 2G & 0\\ 2G & -2G & -2\,{}^tH\\ 0 & -2H & -G\end{pmatrix},\qquad A_6=\begin{pmatrix}-2G & 3B & -3F\\ 3\,{}^tB & -G & 3F\\ 3F & -3F & G\end{pmatrix},$$

$$A_7=\begin{pmatrix}-G & 3B & -3\,{}^tB\\ 3\,{}^tB & -2G & {}^tB\\ -3B & B & -2G\end{pmatrix},\qquad A_8=\begin{pmatrix}-G & 2\,{}^tB & -2\,{}^tB\\ 2B & -2G & 4B\\ -2B & 4\,{}^tB & -2G\end{pmatrix},$$

$$A_9=\begin{pmatrix}-2G & 4B & -4B\\ 4\,{}^tB & -2G & 4B\\ -4\,{}^tB & 4\,{}^tB & -2G\end{pmatrix},$$

where

$$E=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad G=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix},\quad F=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad B=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\quad H=\begin{pmatrix}1 & 2\\ -1 & 1\end{pmatrix}.$$

We can determine a trivial fibre space $\tilde{\mathfrak{F}}_j$ over $\tilde{\Lambda}$ with a fibre $S_j(\tilde{\xi}) = S_j(\xi)$ for $p(\tilde{\xi}) = \xi$. Using the trivialization of $\tilde{\mathfrak{F}}_j$ we obtain a continuation of the cycle G_i. Hence we get a basis system of $H_2(S_j(\tilde{\xi}), \boldsymbol{Q})$.

REMARK 2-2. The algebraic cycles $G_7, \ldots, G_{22}$ are invariant when the continuation is performed.

From the above argument we obtain the following.

PROPOSITION 2-3. *Let $G_i(\tilde{\xi})$ $(i = 1, \ldots, 6)$ be the 2-cycle on $S_j(\tilde{\xi})$ obtained by a continuation of G_i. Then $\Gamma^*(\tilde{\xi}) = \{G_1(\tilde{\xi}), \ldots, G_6(\tilde{\xi}), G_7, \ldots, G_{22}\}$ gives a basis system of $H_2(S_j(\tilde{\xi}), \boldsymbol{Q})$. And the intersection matrix A_j of $G_1(\tilde{\xi}), \ldots, G_6(\tilde{\xi})$ is given by Diagram 4. And algebraic cycles $G_7 \ldots, G_{22}$ are orthogonal to $G_1(\tilde{\xi}), \ldots, G_6(\tilde{\xi})$.*

Next we construct a basis system $\Gamma = \{C_1, \ldots, C_{22}\}$ of $H_2(S, \boldsymbol{Z})$ which is dual to Γ^*. Here again we consider a surface S of type F determined by fixed parameters ξ with (2-4).

When we make a continuation of a 1-cycle γ on a general fibre along α_i, it tends to a 1-cycle on $\pi^{-1}(\xi_i)$ which is homologous to zero. By this procedure we obtain a 2-cycle on S, we denote it by $\alpha_i \times \gamma$.

REMARK 2-3. In general such a 2-cycle intersects some components of the singular fibre. And the intersection multiplicity depends on the homotopy class of γ.

We determine 2-cycles $C_1, \dots, C_6$ on the surface S_j of class (j) $(j = 1, \dots, 9)$ by Diagram 5.

Diagram 5

class of surfaces	construction of cycles
class (j) for $j \neq 5$	$C_{2i-1} = \alpha_i \times \gamma_1$, $C_{2i} = \alpha_i \times \gamma_2$ for $i = 1, 2, 3$
class (5)	$C_{2i-1} = \alpha_i \times \gamma_1$, $C_{2i} = \alpha_i \times \gamma_2$ for $i = 1, 3$, $C_3 = (\alpha_2 - \alpha_1) \times \gamma_1 - (C_1 + C_6)$, $C_4 = (\alpha_2 - \alpha_1) \times \gamma_2 - C_2 + C_5 + C_6$

Let $\{C_7, \dots, C_{22}\}$ be a basis system of the Abelian group of rank 16 generated by $G_7, \dots, G_{22}$ over $\boldsymbol{Z}$. As same as for the system Γ^* we obtain a system $\Gamma(\xi) = \{C_1(\xi), \dots, C_6(\xi), C_7, \dots, C_{22}\}$ of 2-cycles on $S_j(\xi)$.

By the construction we obtain that

$$C_i(\xi) G_j(\xi) = \delta_{ij} \quad \text{for } i = 1, \dots, 22 \text{ and } j = 1, \dots, 6 . \tag{2-7}$$

And we can see that $\Gamma(\xi)$ is a basis system of $H_2(S_j(\xi), \boldsymbol{Z})$ by the same argument as Proposition 2-1. Hence we obtain the following.

PROPOSITION 2-4. *The system $\Gamma(\xi)$ gives a basis system of $H_2(S_j(\xi), \boldsymbol{Z})$ which satisfies the relation* (2-7). *And $C_7, \dots, C_{22}$ are given as algebraic cycles.*

3. – Period mapping.

In the rest of this paper we consider only surfaces of type F. And we employ the notations in § 2 [1] and [3].

[1]. We already defined a holomorphic 2-form φ on a surface of type F in Proposition 1-5. Now we consider the periods

$$\eta_i = \int_{C_i} \varphi \qquad (i = 1, \dots, 22)$$

for surfaces $S_j(\xi)$ of class (j) $(j = 1, \dots, 9)$.

We have

$$\eta_7 = \dots = \eta_{22} \equiv 0 , \tag{3-1}$$

because $C_7, \dots, C_{22}$ are given as algebraic cycles. Let us identify two points ξ and ξ' on $\tilde{\Lambda}$ by the condition $\Gamma(\xi) = \Gamma(\xi')$. We obtain a covering domain $\hat{\Lambda}$ over Λ by this identification. We call $\hat{\Lambda}$ *the monodromy covering domain* of Λ for surfaces of class (j). We denote by $\hat{\xi}$ a point on $\hat{\Lambda}$ with projection ξ. Now we obtain single valued analytic functions $\eta_1(\hat{\xi}), \dots, \eta_6(\hat{\xi})$ on $\hat{\Lambda}$.

Let V be the variety defined by (1-8), and let us consider an automorphism

$$\begin{cases} w' = \exp(2\pi i/3) w \\ u' = u \\ v' = v \end{cases}$$

of V. This automorphism induces an automorphism ϱ of the nonsingular model S, and ϱ preserves each fibre $\pi^{-1}(u)$.

Now let us construct a canonical homology basis on $C = \pi^{-1}(\xi')$. We regard this fibre as a three sheeted Riemann surface over v-sphere:

$$w = \{v(v^2 - \beta^2)\}^{\frac{1}{3}} , \tag{3-2}$$

where $\beta = \left\{ \prod_{i=1}^{5} (\xi' - \xi_i)^{\nu_i} \right\}^{-\frac{1}{2}}$.

Let γ be a ∞-like closed arc on v-sphere which goes round 0 and β respectively in the negative and in the positive senses. Let γ_1 be a closed arc on C with a projection γ. And set $\gamma_2 = \varrho^2 \gamma_1$. Then we obtain a canonical basis $\{\gamma_1, \gamma_2\}$ of $H_1(C, \boldsymbol{Z})$.

By the construction of $C_1, \dots, C_6$ (Diagram 5) we have $C_{2i} = \varrho^2 C_{2i-1}$ $(i = 1, 3, 5)$. Hence it follows

$$\eta_{2i} = [\exp(4\pi i/3)] \eta_{2i-1} . \tag{3-3}$$

Set $\tilde{\eta} = (\eta_1, \dots, \eta_{22})$. And let $\{\omega_1, \dots, \omega_{22}\}$ be a basis system of $H^2(S, \boldsymbol{Z})$ given by differential 2-forms with properties

$$\int_{C_i} \omega_j = \delta_{ij} \quad \text{for } i, j = 1, \dots, 22 .$$

Such a basis system will be called *a dual basis* for $\Gamma=\{C_1,\dots,C_{22}\}$. Set

$$a_{ij}=\int_S \omega_i\wedge\omega_j$$

and let I be a matrix $(a_{ij})_{1\leqq i,j\leqq 22}$. Then we have the period relations:

$$\tilde{\eta}I^t\tilde{\eta}=0\,, \tag{3-4}$$

$$\tilde{\eta}I^t\bar{\tilde{\eta}}>0\,, \tag{3-5}$$

because S is a $K3$ surface ([5], p. 777).

According to (2-7) ω_j $(j=1,\dots,6)$ coincides with G_j as a current. Consequently we have

$$a_{ij}=G_iG_j \quad \text{for } 1\leqq i,\ j\leqq 6\,.$$

Set $\eta=(\eta_1,\eta_3,\eta_5)$. In view of (3-1) and (3-3) we can reduce the period relations to

$$\eta\tilde{A}_j\bar{\eta}>0\,, \tag{3-6}$$

where the matrices $\tilde{A}_j$ are given by Diagram 6.

Diagram 6

$$\tilde{A}_1=\begin{pmatrix}0 & -\omega^2 & -1\\ -\omega & 0 & -\omega^2\\ -1 & -\omega & 0\end{pmatrix},\qquad \tilde{A}_2=3\begin{pmatrix}1 & \sqrt{-3} & \omega^2-1\\ -\sqrt{-3} & 1 & \sqrt{-3}\\ \omega-1 & -\sqrt{-3} & 1\end{pmatrix},$$

$$\tilde{A}_3=\begin{pmatrix}0 & 1-\omega^2 & \sqrt{-3}\\ 1-\omega & 0 & 1-\omega^2\\ -\sqrt{-3} & 1-\omega & 0\end{pmatrix},\qquad \tilde{A}_4=\begin{pmatrix}3 & -3\omega^2 & 3\omega\\ -3\omega & 0 & 1-\omega^2\\ 3\omega^2 & 1-\omega & 0\end{pmatrix},$$

$$\tilde{A}_5=3\begin{pmatrix}0 & 2 & 0\\ 2 & -2 & 2\omega\\ 0 & 2\omega^2 & -1\end{pmatrix},\qquad \tilde{A}_6=3\begin{pmatrix}-2 & 1-\omega^2 & -\sqrt{-3}\\ 1-\omega & -1 & -\sqrt{-3}\\ \sqrt{-3} & -\sqrt{-3} & -1\end{pmatrix},$$

$$\tilde{A}_7=3\begin{pmatrix}-1 & 1-\omega^2 & \omega-1\\ 1-\omega & -2 & 1-\omega\\ \omega^2-1 & 1-\omega^2 & -2\end{pmatrix},\qquad \tilde{A}_8=\begin{pmatrix}-3 & 2(1-\omega) & -2(1-\omega)\\ 2(1-\omega^2) & -6 & 4(1-\omega^2)\\ -2(1-\omega^2) & 4(1-\omega) & -6\end{pmatrix},$$

$$\tilde{A}_9=2\begin{pmatrix}-3 & 2(1-\omega^2) & -2(1-\omega^2)\\ 2(1-\omega) & -3 & 2(1-\omega^2)\\ -2(1-\omega) & 2(1-\omega) & -3\end{pmatrix}.$$

Let Ω_j be a domain in $\boldsymbol{P}^2$ defined by (3-6). Because any $\tilde{A}_j$ has two negative and one positive eigen values, Ω_j is biholomorphically equivalent to a hyperball.

Let us consider a mapping

$$\Phi: \begin{cases} \zeta_0 = \eta_1(\hat{\xi}) , \\ \zeta_1 = \eta_3(\hat{\xi}) , \\ \zeta_2 = \eta_5(\hat{\xi}) \end{cases}$$

from $\hat{\Lambda}$ to $\boldsymbol{P}^2$. In view of de Rham's theorem we know that η_i $(i=1, 3, 5)$ does not vanish at a same time. The mapping Φ will be called *a period mapping for surfaces of class* (j).

From the above consideration we have:

PROPOSITION 3-1. *The period mapping Φ for surfaces of class (j) is a holomorphic mapping from $\hat{\Lambda}$ to Ω_j, and Ω_j is biholomorpfically equivalent to a hyperball.*

[2]. Here we employ the following notations:

X: an algebraic $K3$ surface,

φ: a holomorphic 2-form on X,

L: a free $\boldsymbol{Z}$-module of rank 22 with an even integer valued unimodular symmetric bilinear form (,) of signature (3, 19),

l: a fixed element of L with $(l, l) > 0$.

REMARK 3-1. Such a $\boldsymbol{Z}$-module L exists uniquely, up to isomorphisms ([9], Chap. 5).

Let ω and ω' be two elements of $H^2(X, \boldsymbol{Z})$. If we define

$$(\omega, \omega') = \int_X \omega \wedge \omega' ,$$

then $H^2(X, \boldsymbol{Z})$ is isomorphic to L ([5], p. 776).

A marked $K3$ surface is defined as a triple (X, ψ, F) satisfying the conditions:

(1) ψ is an isomorphism from L to $H^2(X, \boldsymbol{Z})$,

(2) F is a line bundle on X such that $c(F) = \psi(l)$, c indicates the Chern class, and $FD \geqq 0$ for any effective divisor D.

Two marked $K3$ surfaces (X_1, ψ_1, F_1) and (X_2, ψ_2, F_2) are identified if there exists a biholomorphic mapping f from X_1 to X_2 with $\psi_1 = f^* \circ \psi_2$.

Let $\{e_1, \ldots, e_{22}\}$ be a fixed basis system of L. And let (X, ψ, F) be a marked $K3$ surface.

And set a basis system $C_1, \ldots, C_{22}$ of $H_2(X, \boldsymbol{Z})$ with a property

$$\int_{C_i} \psi(e_j) = \delta_{ij} \quad \text{for } i,\ j = 1, \ldots, 22 .$$

Let $M(l)$ be a family of all marked $K3$ surfaces (X, ψ, F) with fixed l. Then we obtain a mapping Ψ from $M(l)$ to $\boldsymbol{P}^{21}$ with homogeneous coordinate $[\eta_1, \ldots, \eta_{22}]$ by setting

$$\eta_i = \int_{C_i} \varphi .$$

According to Pjateckiĭ-Šapiro and Safarevič we have the following Torelli type theorem ([5], see also [7]).

THEOREM. *The period mapping Ψ is injective.*

Let us define a marking on $S_j(\hat{\xi})$. For the convenience we write $S(\hat{\xi})$ for $S_j(\hat{\xi})$. Set

$$L = H^2(S(\hat{\xi}'), \boldsymbol{Z}) , \qquad L(\hat{\xi}) = H^2(S(\hat{\xi}), \boldsymbol{Z}) , \qquad D = G_{21} + 2G_{22}$$

and $l = c(D)$.

We have $(l, l) = D^2 = 2$, because $G_{21}^2 = -2$, $G_{22}^2 = 0$ and $G_{21} G_{22} = 1$. Let $\{\omega_1(\hat{\xi}), \ldots, \omega_{22}(\hat{\xi})\}$ be a dual basis of $L(\hat{\xi})$ for $\Gamma(\hat{\xi})$. Take an element $\omega = x_1\omega_1(\hat{\xi}') + \ldots + x_{22}\omega_{22}(\hat{\xi}')$ of L. We obtain an isomorphism ψ from L to $L(\hat{\xi})$ by defining $\psi(\omega) = x_1\omega_1(\hat{\xi}) + \ldots + x_{22}\omega_{22}(\hat{\xi})$. Hence we obtain a marked $K3$ surface $(S(\hat{\xi}), \psi, F)$. We note that $F = c^{-1} \circ \psi(l) = [G_{21} + 2G_{22}]$ is induced from a divisor independent of the parameter $\hat{\xi}$.

REMARK 3-2. Because S is a $K3$ surface, the Chern mapping c is injective.

LEMMA 3-1. *Let D be a divisor on an algebraic $K3$ surface S. Suppose that there exists a divisor D' with only simple components which is linearly equivalent to D. Then we have*

$$H^1(S, \mathcal{O}([D])) = 0 .$$

PROOF. We consider the following sheaf exact sequence:

$$0 \to \mathcal{O}_S[-D] \to \mathcal{O}_S \to \mathcal{O}_S/\mathcal{O}_S[-D] \to 0 .$$

By the assumption we have

$$\mathcal{O}_S/\mathcal{O}_S[-D] = \mathcal{O}_S/\mathcal{O}_S[-D'] = \mathcal{O}_{D'} .$$

Hence we obtain the long exact sequence:

$$0 \to H^0(S, \mathcal{O}_S[-D]) \to H^0(S, \mathcal{O}_S) \to H^0(D', \mathcal{O}_{D'})$$
$$\to H^1(S, \mathcal{O}_S[-D]) \to H^1(S, \mathcal{O}_S) \to .$$

And we have

$$H^0(S, \mathcal{O}_S[-D]) = 0 ,$$
$$H^0(S, \mathcal{O}_S) \cong H^0(D', \mathcal{O}_{D'}) \cong \boldsymbol{C} .$$

For a $K3$ surface we have

$$H^1(S, \mathcal{O}_S) = 0 .$$

Hence we have $H^1(S, \mathcal{O}_S[-D]) = 0$. In view of Serre duality and the triviality of the canonical bundle K of S, we have

$$H^1(S, \mathcal{O}_S[-D]) \cong H^1(S, \mathcal{O}_S(K + [D])) = H^1(S, \mathcal{O}_S[D]) = 0 \qquad \text{q.e.d.}$$

LEMMA 3-2. *Let $(S_j(\hat{\xi}_1), \psi_1, F_1)$ and $(S_j(\hat{\xi}_2), \psi_2, F_2)$ be two marked $K3$ surfaces of class (j). They coincide each other if and only if $\hat{\xi}_1 = \hat{\xi}_2$.*

PROOF. The sufficiency is trivial. Then we show the necessity. We write S and S' respectively for $S_j(\hat{\xi}_1)$ and $S_j(\hat{\xi}_2)$. We have $c_1 = 0$, $c_2 = 24$ for a $K3$ surface. Hence, by Riemann-Roch theorem, we obtain

$$\dim H^0(S, \mathcal{O}[D]) - \dim H^1(S, \mathcal{O}[D]) + \dim H^2(S, \mathcal{O}[D]) = D^2/2 + 2 .$$

By Lemma 3-1 the second term of the left hand side vanishes. By Serre duality the third term also vanishes. As we have $D^2 = 2$ for $D = G_{21} + 2G_{22}$, then it follows

$$\dim H^0(S, \mathcal{O}[D]) = 3 .$$

Take a coordinate covering $\bigcup_{i \in I} U_i = S$, and let f_i be a representing equation of D on U_i. Then we have

$$[D] = \{f_{ij}\} \equiv \{f_i/f_j\} .$$

Let t be a parameter of the base curve $\Delta = \boldsymbol{P}$ such that $\pi^{-1}(0)$ is a general fibre. Set

$$\varphi_i^{(1)} = f_i\,,$$
$$\varphi_i^{(2)} = f_i/t\,,$$
$$\varphi_i^{(3)} = f_i/t^2\,.$$

Then $\varphi^{(k)} = \{\varphi_i^{(k)}\}$ $(k = 1, 2, 3)$ gives a basis system of $H^0(S, \mathcal{O}[D])$. Suppose that we have $(S, \psi_1, F_1) = (S', \psi_2, F_2)$. Then we can regard the latter as the surface S equipped with another elliptic fibring with a holomorphic section G_{21} and a general fibre G_{22}. We denote this elliptic surface by (S', π_1, Δ_1). Let t_1 be a parameter on $\Delta_1 = \boldsymbol{P}$. The parameter t was given as a ratio of two holomorphic sections of $\mathcal{O}[D]$. Then also t_1 must be so. Hence t_1 is a rational function $R(t)$ of t with the degree at most 2. But if the degree of R is exactly two, (S', π_1, Δ_1) must have a disconnected fibre. This is absurd. Hence $R(t)$ is a fractional linear transformation.

Let $\{\xi_0, \xi_1, \xi_2, \xi_3, \xi_\infty\}$ and $\{\xi'_0, \xi'_1, \xi'_2, \xi'_3, \xi'_\infty\}$ be critical points of S and S' respectively. Because the projection π_1 differs from π only by a coordinate transformation of $\boldsymbol{P}$, $\{\xi_0, \dots, \xi_\infty\}$ coincides with $\{\xi'_0, \dots, \xi'_\infty\}$ as a set of points. Hence the transformation R carries ξ_i $(i = 0, 1, 2, 3, \infty)$ to some ξ'_j. The 2-cycles C_1, C_3, C_5 are constructed over the arcs between ξ_0 and ξ_1, ξ_2, ξ_3, respectively (§ 2, Diagram 4). If we have a point ξ_i such that $R(\xi_i) \neq \xi'_i$, then the marking of (S', π_1, Δ_1) can not coincide with that of (S, π, Δ). Then R must be the identity. Hence we have a same projection ξ of $\hat{\xi}_1$ and $\hat{\xi}_2$. Reviewing the assumption $(S_j(\hat{\xi}_1), \psi_1, F_1) = (S_j(\hat{\xi}_1), \psi_2, F_2)$ we have $\Gamma(\hat{\xi}_1) = \Gamma(\hat{\xi}_2)$. Thus we obtain $\hat{\xi}_1 = \hat{\xi}_2$. q.e.d.

As a consequence of the above lemma we obtain:

PROPOSITION 3-2. *The period mapping Φ for surfaces of class* (j) $(j = 1, \dots, 9)$ *is injective.*

4. – Differential equation.

As already shown in § 3 Proposition 3-2 the dimension of the vector space Σ generated by $\eta_1(\hat{\xi}), \dots, \eta_6(\hat{\xi})$ is three. Then we investigate the monodromy transformation of Σ induced from a closed arc on Λ.

Let $\xi' = [\xi'_1, \xi'_2, \xi'_3]$ be a fixed point on Λ. Let us make a closed arc β_{ij} as following:

β_{ij} starts at ξ' and vary only one parameter ξ_i so that the moving point ξ goes around $H_{ij}=\{[\xi_1,\xi_2,\xi_3]\colon \xi_i=\xi_j$ for $i\neq j$, $i=1,2,3$ and $j=0,1,2,3,\infty\}$ in the positive sense.

Set

$$\bar{C}_1=(\alpha_j-\alpha_i)\times\gamma_1\,,$$
$$\bar{C}_2=(\alpha_j-\alpha_i)\times\gamma_2\,.$$

Let k and l be two indices in $\{0,1,2,3\}-\{i,j\}$, and let r_1 and r_2 be oriented arcs on Δ_0 starting at ξ_i and goes to ξ_k and ξ_l, respectively. Set

$$\bar{C}_3=r_1\times\gamma_1\,,\qquad \bar{C}_4=r_1\times\gamma_2\,,$$
$$\bar{C}_5=r_2\times\gamma_1\,,\qquad \bar{C}_6=r_2\times\gamma_2\,.$$

Then Σ is generated by

$$\bar{\eta}_i=\int_{\bar{C}_j}\varphi\qquad(i=1,3,5)\,.$$

And also we have the relation (3-3) for $\bar{\eta}_i$. Let I and J be the types of the singular fibres $\pi^{-1}(\xi_i)$ and $\pi^{-1}(\xi_j)$, respectively ($I, J=$ II, IV, I_0^*, IV*, II*). Let S_i and S_j be the corresponding matrices determined by (1-3) ($i,j=$ $=1,\dots,5$). And let M_{ij} be the monodromy transformation induced from β_{ij}.

We can determine M_{ij} as the following:

(*Case* 1). If we have $j\neq\infty$,

$$M_{ij}\begin{pmatrix}\bar{C}_1\\ \bar{C}_2\end{pmatrix}=(\alpha_j-\alpha_i)\times S_iS_j\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix},$$
$$\text{(4-1)}\qquad M_{ij}\begin{pmatrix}\bar{C}_3\\ \bar{C}_4\end{pmatrix}=\begin{pmatrix}\bar{C}_3\\ \bar{C}_4\end{pmatrix}+(\alpha_j-\alpha_i)\times(S_j-E)\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix},$$
$$M_{ij}\begin{pmatrix}\bar{C}_5\\ \bar{C}_6\end{pmatrix}=\begin{pmatrix}\bar{C}_5\\ \bar{C}_6\end{pmatrix}+(\alpha_j-\alpha_i)\times(S_j-E)\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}.$$

(*Case* 2). If we have $j=\infty$,

$$M_{ij}\begin{pmatrix}\bar{C}_1\\ \bar{C}_2\end{pmatrix}=(\alpha_j-\alpha_i)\times S_j\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix},$$
$$\text{(4-2)}\qquad M_{ij}\begin{pmatrix}\bar{C}_3\\ \bar{C}_4\end{pmatrix}=\{r_1\times S_i^{-1}+(\alpha_j-\alpha_i)\times(S_i^{-1}S_j-S_i^{-1})\}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix},$$
$$M_{ij}\begin{pmatrix}\bar{C}_5\\ \bar{C}_6\end{pmatrix}=\{r_2\times S_i^{-1}+(\alpha_j-\alpha_i)\times(S_i^{-1}S_j-S_i^{-1})\}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}.$$

By using (4-1), (4-2) and (3-3) we can represent the monodromy M_{ij} with respect to the basis $\{\bar{\eta}_1, \bar{\eta}_3, \bar{\eta}_5\}$. Set

$$\lambda(k) = 1 - k/6\ , \qquad \varrho(k) = \exp\left((-\pi\sqrt{-1}k)/3\right).$$

LEMMA 4-1. *The monodromy M_{ij} is of order infinite if and only if $\varrho(i)\varrho(j) = 1$. And we have the Jordan's normal form of M_{ij} as the following*:

$$(a)\quad M_{ij} \sim \begin{pmatrix} \varrho(i)\varrho(j) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } j \neq \infty \text{ and } \varrho(i)\varrho(j) \neq 1\ ,$$

$$(b)\quad M_{ij} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } j \neq \infty \text{ and } \varrho(i)\varrho(j) = 1,$$

$$(c)\quad M_{ij} \sim \begin{pmatrix} \varrho(j) & 0 & 0 \\ 0 & \varrho(i)^{-1} & 0 \\ 0 & 0 & \varrho(i)^{-1} \end{pmatrix} \quad \text{if } j = \infty \text{ and } \varrho(i)\varrho(j) \neq 1\ ,$$

$$(d)\quad M_{ij} \sim \begin{pmatrix} \varrho(i)^{-1} & 1 & 0 \\ 0 & \varrho(i)^{-1} & 0 \\ 0 & 0 & \varrho(i)^{-1} \end{pmatrix} \quad \text{if } j = \infty \text{ and } \varrho(i)\varrho(j) = 1\ .$$

Let q be a point on some H_{ij} $(i = 2, 3,\ j = 0, 1, 2, 3, \infty)$ and suppose q is not an intersection point with other H_{ij}. Let us choose a local coordinate (x, y) in the neighborhood of q so that we have $x = (\xi_i - \xi_j)/\xi_1$.

By Lemma 4-1 we can choose the following forms of basis system of Σ in the neighborhood of q:

(a) $F_1(x, y),\ F_2(x, y),\ x^{\lambda(i)+\lambda(j)-1}F_3(x, y)$,

(b) $F_1(x, y),\ F_2(x, y),\ F_3(x, y) + (\log x)F_2(x, y)$,

(c) $x^{1-\lambda(i)}F_1(x, y),\ x^{1-\lambda(i)}F_2(x, y),\ x^{\lambda(j)}F_3(x, y)$,

(d) $x^{1-\lambda(i)}F_1(x, y),\ x^{1-\lambda(i)}F_2(x, y),\ x^{1-\lambda(i)}\{F_3(x, y) + (\log x)F_2(x, y)\}$,

where F_i $(i = 1, 2, 3)$ indicates a single valued holomorphic function in the neighborhood of q.

Let us denote by $\lambda(\xi_i)$ the value λ induced from the matrix corresponding to the singular fibre $\pi^{-1}(\xi_i)$. Then, in view of (1-9), we have

$$\lambda(\xi_0) + \lambda(\xi_1) + \lambda(\xi_2) + \lambda(\xi_3) + \lambda(\xi_\infty) = 3 \,. \tag{4-3}$$

If we consider the function η in Σ as a function of x with parameter y, η has singularities at $x = 0, 1, \infty$ and ξ_j/ξ_1, where $j = \{2, 3\} - \{i\}$. The relation (4-3) indicates that the sum of the exponents of η satisfies the Fuchs' relation. We can proceed the same argument as for $y = \xi_j/\xi_1$, because our functions are symmetric with respect to ξ_2 and ξ_3.

According to the theorem of Picard-Terada ([1] and [6]) we can determine the Appell's hypergeometric differential equation which has the solution Σ.

PROPOSITION 4-1. *Let $\eta_i(\xi)$ $(i = 1, \ldots, 6)$ be a period for surfaces of class (j). Set $x = \xi_2/\xi_1$ and $y = \xi_3/\xi_1$. Then $\eta_i(x, y)$ is a solution of the Appell's hypergeometric equation $F_1(\alpha, \beta, \beta', \gamma; x, y)$ with parameters indicated in Diagram* 7.

Diagram 7

class (j)	λ_0	λ_1	λ_2	λ_3	$\alpha = \lambda_\infty$	$\beta = \beta' = 1 - \lambda_1$	$\gamma = \alpha + \lambda_3$
(1)	2/3	2/3	2/3	2/3	1/3	1/3	1
(2)	5/6	2/3	2/3	2/3	1/6	1/3	5/6
(3)	1/3	5/6	5/6	5/6	1/6	1/6	1
(4)	1/2	2/3	5/6	5/6	1/6	1/3	1
(5)	2/3	1/2	2/3	2/3	1/2	1/2	7/6
(6)	5/6	2/3	1/3	1/3	5/6	1/3	7/6
(7)	1/2	1/3	2/3	2/3	5/6	2/3	3/2
(8)	1/3	5/6	1/2	1/2	5/6	1/6	4/3
(9)	2/3	1/2	1/2	1/2	5/6	1/2	4/3

5. – Continuation of the period mapping.

Let $\mathcal{F} = \{S(\xi)\}$ be a certain class of surfaces of type F. From Lemma 4-1 we obtain that Λ has algebraic ramifications over H_{ij} if and only if $\lambda(\xi_i) + \lambda(\xi_j) \neq 1$, namely $\chi(\pi^{-1}(\xi_i)) + \chi(\pi^{-1}(\xi_j)) \neq 12$. When the parameter ξ shifts to a point on H_{ij}, we get an elliptic surface with 4 singular fibres. By identifying ξ_i and ξ_j this surface is represented by (1-8).

Let $\mu = [\mu_1, \mu_2]$ be a projective parameter on H_{ij}. Let $S(\mu)$ denote the above surface. If we have $\lambda(\xi_i) + \lambda(\xi_j) > 1$, by Proposition 2-1 in the part I and Remark 1-4, $S(\mu)$ is a $K3$ surface. And if we have $\lambda(\xi_i) + \lambda(\xi_j) \leqq 1$, we get a $J0$ singular fibre of Euler characteristic

$$\chi(\pi^{-1}(\xi_i)) + \chi(\pi^{-1}(\xi_j)) - 12$$

over $\xi_i = \xi_j$. By the study of Kodaira ([4], § 12) we know that $S(\mu)$ is rational. In the former case $\varphi = w^{-2}\, du \wedge dv$ is a holomorphic 2-form on $S(\mu)$, and $\eta(\xi)$ can be continued to H_{ij}. The periods of $S(\mu)$ generate a 2-dimensional vector space. And these are Gauss' hypergeometric functions. Also in this case the Fuchs' relation is induced from the fact $\chi(S(\mu)) = 24$.

The above conclusions are obtained by the same argument as developped in § 4.

Next we consider an intersection point P_{ijk} of H_{ik} and H_{jk}. Let $S(P_{ijk})$ denote the surface occurs by shifting ξ_i and ξ_j to ξ_k. If we have $\lambda(\xi_i) + \lambda(\xi_j) + \lambda(\xi_k) > 2$, $S(P_{ijk})$ is a $K3$ surface and φ is a holomorphic 2-form on $S(P_{ijk})$. This is obtained by the same argument as for $S(\mu)$.

Hence the period $\eta(\xi)$ can be continued to this point, and $\Phi(P_{ijk})$ is an interior point of Ω. If we have $\lambda(\xi_i) + \lambda(\xi_j) + \lambda(\xi_k) = 2$, $S(P_{ijk})$ is rational and Φ is extended to P_{ijk}. In this case $\Phi(P_{ijk})$ is a boundary point of Ω, this is examined by observing the relation between η_1, η_3 and η_5.

Here again we consider the case that ξ is situated on H_{ij}. In case of $\lambda(\xi_i) + \lambda(\xi_j) < 1$ we have $\lambda(\xi_k) + \lambda(\xi_l) + \lambda(\xi_h) > 2$, where $\{k, l, h\} = \{0, 1, 2, 3, \infty\} - \{i, j\}$, then $S(P_{klh})$ is a $K3$ surface. And the period of the surface $S(\mu)$ is given by $\Phi(P_{klh})$. Hence the period $\Phi(\mu)$ is a fixed point on Ω for any μ on H_{ij}. In case of $\lambda(\xi_i) + \lambda(\xi_j) + \lambda(\xi_k) < 2$ we can see that P_{ijk} is an indefinite point of Φ by the inverse procedure.

Hence we have:

PROPOSITION 5-1. *We obtain a continuation of the period mapping Φ to H_{ij} and to P_{ijk} as* Diagram 8.

Diagram 8

(*a*-1)	$\lambda(\xi_i) + \lambda(\xi_j) > 1$	$\Phi(H_{ij})$ = a hypersurface on Ω
(*a*-2)	$\lambda(\xi_i) + \lambda(\xi_j) = 1$	$\Phi(H_{ij})$ = a boundary point of Ω
(*a*-3)	$\lambda(\xi_i) + \lambda(\xi l) < 1$	$\Phi(H_{ij})$ = an interior point of Ω
(*b*-1)	$\lambda(\xi_i) + \lambda(\xi_j) + \lambda(\xi_k) > 2$	$\Phi(P_{ijk})$ = an interior point of Ω
(*b*-2)	$\lambda(\xi_i) + \lambda(\xi_j) + \lambda(\xi_k) = 2$	$\Phi(P_{ijk})$ = a boundary point of Ω
(*b*-3)	$\lambda(\xi_i) + \lambda(\xi_j) + \lambda(\xi_k) < 2$	P_{ijk} is an indefinite point of Φ and $\Phi(P_{ijk})$ = a hypersurface of Ω

Let us consider a projective plane $X = \boldsymbol{P}^2$ with homogeneous coordinate $[\xi_1, \xi_2, \xi_3]$. If the monodromy covering $\hat{\Lambda}$ for class (j) has algebraic ramification over H_{ij}, namely the case (a-1) or (a-3), we can attach ramified curves to $\hat{\Lambda}$. If we have (b-1) or (b-2) for a point P_{ijk}, we can attach ramified points over P_{ijk}. In case of (a-3) we make the blow down of the curve over H_{ij}: And in case of (b-3) we make the blow up of the point over P_{ijk}. Thus we obtain an analytic space as an extension of $\hat{\Lambda}$. We denote it by $\hat{\Lambda}_0$. By Proposition 5-1 we get an extension Φ_0 of Φ to $\hat{\Lambda}_0$. And Φ_0 is everywhere nondegenerate on $\hat{\Lambda}_0$. Hence, in view of Proposition 3-2, Φ_0 is an injective mapping from $\hat{\Lambda}_0$ to Ω.

According to Terada ([6], Lemma 3) Φ_0 is surjective. In general we need the blow up and the blow down process to get $\hat{\Lambda}_0$, but only for $j = 1, 2, 3, 4$ it does not occur the case (a-3) and (b-3). Let G be the discontinuous transformation group of Ω generated by M_{ij}. And let $(\Omega/G)^*$ be the Satake-Baily compactification. Then Φ_0 induces a birational equivalence between $\boldsymbol{P}^2 = \bar{\Lambda}$ and $(\Omega/G)^*$. By the above consideration we obtain:

PROPOSITION 5-2. *The extended period mapping Φ_0 gives a biholomorphic equivalence between $\hat{\Lambda}_0$ and Ω. And Φ_0 induces a biholomorphic equivalence between $\boldsymbol{P}^2 = \bar{\Lambda}$ and $(\Omega/G)^*$ for the class $j = 1, 2, 3, 4$.*

6. – Coda.

Finally we mention about the period mapping for the surfaces represented by (1-5). Let $\mathcal{F}$ be the totality of those surfaces. If we fix the parameters ξ_{11} and ξ_{12} at 0 and ∞ respectively, then the element of $\mathcal{F}$ is determined by a point $\xi = [\xi_1, \dots, \xi_{10}]$ of the domain $\Lambda = \{\xi | \xi_i \neq \xi_j$ for $i = 1, \dots, 10$ and $j = 1, \dots, 11\}$ in $\boldsymbol{P}^9$, where $[\xi_1, \dots, \xi_{10}]$ is a homogeneous coordinate. We denote it $S(\xi)$.

Set

$$\eta_i(\xi) = \int_{G_{2i}} \varphi \qquad (i = 1, \dots, 10)\,,$$

where G_{2i} is a 2-cycle on $S(\xi)$ given in § 2 [2] and φ is the holomorphic 2-form in Proposition 1-3. Then we get a period mapping

$$\Phi\colon \Lambda \to \boldsymbol{P}^9$$

by defining $\Phi(\xi) = [\eta_1(\xi), \dots, \eta_{10}(\xi)]$. According to (3-4) and (3-5) we know that the image $\Phi(\Lambda)$ is contained in a domain Ω which is biholomorphically equivalent to the 9-dimensional hyperball in $\boldsymbol{P}^9$. And by the argument similar to Proposition 3-2 we get the injectivity of Φ.

If we set $x_i = \xi_i/\xi_{10}$ $(i = 1, \ldots, 9)$, then $\eta_i(\xi)$ becomes an analytic function of 9 variables. According to Terada we know that $\eta_i(\xi)$ satisfies the Lauricella's differential equation given in [6] (***), where any parameter λ_i takes the value 5/6.

And the surface $S(\xi)$ is a deformation of an isolated singularity defined by $x^2 + y^3 + z^{12} = 0$.

REFERENCES

[1] E. Picard, *Sur les fonctions de deux variables indépendentes analogues aux fonctions modulaires*, Acta Math., **2** (1883), pp. 114-135.
[2] E. Picard, *Sur une extension aux fonctions de deux variables du problème de Riemann*, Ann. de l'Ecole Normale, **10** (1881), pp. 305-322.
[3] K. Kodaira, *On compact analytic surfaces, II-III*, Ann. of Math., **77** (1963), pp. 563-626; **78** (1963), pp. 1-40.
[4] K. Kodaira, *On the structure of compact complex analytic surfaces, I*, Amer. J. Math., **86** (1964), pp. 751-798.
[5] Pjateckiĭ-Šapiro - Šafarevič, *A Torelli theorem for algebraic surfaces of type K3*, Math. USSR Isvestija, **5** (1971), pp. 547-588.
[6] T. Terada, *Problème de Riemann et fonctions automorphes provenant des fonctions hypergéométriques de plusieurs variables*, J. Math. Kyoto Univ., **13** (1973), pp. 557-578.
[7] T. Shioda, *The period map of Abelian surfaces*, J. Fac. Science Univ. Tokyo, **25** (1978), pp. 47-59.
[8] J. P. Serre, *Cours d'arithmetique*, Presses universitaires de France.
[9] H. Shiga, *One attempt to the K3 modular function I*, Ann. Scuola Norm. Sup. Pisa, **6** (1979), pp. 609-635.

Chiba University
Faculty of Science
Department of Mathematics,
1-33 Yayoi-cho
Chiba-city, Giappone