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Classifying relative principal fibrations with loop space fibers

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Classifying Relative Principal Fibrations with Loop Space Fibers.

J. F. McCLENDON

The problem to be considered here is that of classifying and characterizing Relative Principal Fibrations (= RPF's) and connected Relative Principal Fibrations (= connected RPF's) up to homotopy. A typical theorem (1.4) gives a bijection

$$[X, Z]_D^C \to P(X, Z)$$

between certain homotopy classes of maps and certain equivalence classes of RPF's.

By way of motivation, let me recall a result from [5, Cor. 3.4] (see, also, [1]): if $F \to E \to B$ is any fibration with $\pi_i(F) = 0$, except possibly when $s \leqslant i < 2s - 1$, for some s, then $E \to B$ is a relative principal fibration. The fact that such a large class of fibrations can be represented as relative principal fibrations suggests that some classification theorem is desirable.

Also, RPF's play a role in obstruction theory (see [4, 8, 1]). The approach to classifying maps over D is to factor a map into basic building blocks which are RPF's. Even if D is a point, RPF's are required in the non-orientable case [8]. Theorem 1.4 of the present paper classifies these basic building blocks.

It will be shown in a separate paper that many evaluation fibrations can be represented as connected RPF's and the classification theorem proved here will be applied to obtain a classification theorem for evaluation fibrations.

In 1.1 the notion of N-principal map is defined. The set $P(X, Z)$ mentioned above is really a certain set of N-principal maps with $N = \Omega_D Z$. Thus the classifying theorem classifies N-principal maps. This might be viewed as giving a geometric interpretation of $[X, Z]_D^C$—since each element

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can now be viewed as an N-principal map. It can also be viewed as classifying or characterizing RPF's—since RPF's (originally defined as induced from a D-path-loop fibration) are identified with N-principal maps (which are defined without aid of an inducing map).

If $D = *$ ($=$ a point) then the classification theorem above is due to M. Fuchs [2]. Even if $D = *$, connected RPF's do not seem to have been discussed before and their classification may be of interest even in that case.

In section 1 the classification theorem for RPF's is proved. In section 2 connected RPF's are defined and their basic properties are developed. In section 3 connected RPF's are classified.

1. – Classifying relative principal fibrations.

First some terminology will be recalled [4]. Let $u\colon C \to D$ be a fixed map. Top $(u\colon C \to D) =$ Top $(C \to D)$ is the category whose objects are triples $(Z, \check{z}, \hat{z})$ where $\check{z}\colon C \to Z$, $\hat{z}\colon Z \to D$, and $\hat{z}\check{z} = u$. The morphisms are maps $f\colon Z \to Z'$ satisfying $f\check{z} = \check{z}'$, $\hat{z}'f = \hat{z}$. $H\colon Z\times I \to Z'$ is a homotopy in the category if each H_t is in Top $(C \to D)$. $[Z, Z']_D^C$ is the set of Top $(C \to D)$ homotopy classes of maps. Write Top (D) for Top $(id\colon D \to D)$. It has all the usual properties of Top $(*)$ $=$ the category of pointed spaces and maps.

Now let N be a monoid in Top (D). $\check{n}\colon D \to N$ is the strict unit and the associativity diagram is strictly commutative. Write $O(d) = \check{n}(d)$ and use additive notation. A map $a\colon N\times_D Y \to Y$ is an N action on $Y \in$ Top $(C \to D)$ if a is a map over D and $n'(ny) = (n' + n)y$ and $O(d)n = n$ whenever defined. Call (Y, a) (or Y) an *N-space*. These make up a category $\mathrm{Top}^N(C \to D)$ with morphisms $f\colon Y \to Y' \in$ Top $(C \to D)$ which also satisfy $f(ny) = nf(y)$. A homotopy is a Top $(C \to D)$ homotopy such that each H_t is also an N-map.

1.1 DEFINITION. (1) $p\colon E \to X \in \mathrm{Top}^N(C \to D)$ is a *numerable N-principal* map if:

(*a*) the N-action on X is trivial;

(*b*) X has a numerable cover $\{U\}$ such that each E_U is homotopically equivalent to $U\times_D N$ in $\mathrm{Top}^N(V \to D)$, $V = \check{x}^{-1}U \cap C$, $V \to U\times_D N$ defined by $c \to (c, O(uc))$;

(*c*) $\check{x}^*E = u^*N$.

(2) Two numerable N-principal maps are equivalent if they are homotopically equivalent in $\mathrm{Top}^N(C \to X)$.

In [6] relative principal fibrations were defined in terms of ordinary paths (see [1] also). For the purpose of a classification theorem, however, it will be more convenient to use Moore paths.

A Moore path [e.g., 3] in Z is a pair $m = (w, r)$ where $w\colon [0, \infty) \to Z$ is a continuous function, $r \in [0, \infty)$, and $w(t) = w(r)$ for $t \geqslant r$. Define $p_0 m = w(0)$, $p_1 m = w(r)$. If $p_1 m = p_0 m'$ then $m + m' = (w + w', r + r')$ is defined by

$$(w + w')(t) = \begin{cases} w(t) & 0 \leqslant t \leqslant r\,, \\ w'(t - r) & r \leqslant t\,. \end{cases}$$

This addition is strictly associative where defined. If $z \in Z$ define $O_z = (c_z, 0)$ where $c_z(t) = z$ all t. Then O_z is a strict unit where defined. Let WZ be the space of all Moore paths of Z, topologized as a subset of $F([0, \infty), Z) \times [0, \infty)$ where the first factor has the compact open topology. Write $m(t)$ instead of $w(t)$ for $m = (w, r)$.

$$\begin{aligned} W_D Z &= \{m \in WZ \mid \hat{z} m(t) = \hat{z} m(t') \text{ all } t, t' \in [0, \infty)\} \\ P_D Z &= \{m \in W_D Z \mid p_0 m = \check{z}\hat{z} m(t)\} \\ \Omega_D Z &= \{m \in P_D Z \mid p_0 m = p_1 m\} \end{aligned}$$

$p_1\colon P_D Z \to Z$ is a $\operatorname{Top}(D)$ fibration with $\operatorname{Top}(D)$ fiber $\Omega_D Z$. Let $f\colon X \to Z \in \operatorname{Top}(C \to D)$. Then the pullback of f and p_1 is called a *relative principal fibration* (or *RPF*) and denoted by $P(f)$ (or $P_D(f)$ if necessary). We wish to classify these.

Assume henceforth that $C \to X$ is a closed cofibration.

Note that if $Z \in \operatorname{Top}(D)$ then $\Omega_D Z$ is a monoid in $\operatorname{Top}(D)$ and if $f\colon X \to Z \in \operatorname{Top}(C \to D)$ then $P(f)$ is an $\Omega_D Z$-space under the action $\Omega_D Z \times_D P(f) \to P(f)$, $(m, (x, k)) \to (x, m + k)$.

Let $Z \in \operatorname{Top}(D)$. A map $f\colon X \to Z \in \operatorname{Top}(C \to D)$ will be called *numerable* if X has a numerable cover $\{U\}$ such that $f|U$ and $\check{z}\hat{x}|U\colon U \to Z$ are homotopic in $\operatorname{Top}(V \to D)$, $V = \check{x}^{-1} U \subset C$.

The following lemma shows that under reasonable hypotheses every f is numerable.

1.2 Theorem. *Suppose* $Z \in \operatorname{Top}(D)$ *and* $X \in \operatorname{Top}(C \to D)$ *and* X *has a numerable cover* $\{U\}$ *of* Top *contractible sets. Suppose* $\hat{z}\colon Z \to D$ *is a fibration with connected fibers. Then every* $f\colon X \to Z \in \operatorname{Top}(C \to D)$ *is numerable.*

PROOF. Let $U \in \{U\}$, $g = \hat{z}\hat{x}$. If $V \neq \emptyset$ then f and g agree on V so $f(U)$ and $g(U)$ are contained in the same path component of Z so $f \sim g\colon U \to Z$, rel (V). If $V = \emptyset$ then suppose U contractible to u, $\hat{x}u = d$. Then fu, $gu \in Z(d)$ and $Z(d)$ connected so again $f(U)$ and $g(U)$ are in the same component of Z so $f \sim g\colon U \to Z$. Now g factors through the section $\hat{z}$ of the fibration $Z \to D$ so both homotopies can be adjusted to be over D.

1.3 DEFINITION. $Z \in \mathrm{Top}\,(D)$, $X \in \mathrm{Top}\,(C \to D)$.

(1) $P(X, Z)$ = all equivalence classes of numerable $\Omega_D Z$-principal maps $E \to X$.

(2) $[X, Z]_D^C$ = the set of homotopy classes $(\mathrm{Top}\,(C \to D)$ homotopy$)$ of numerable maps $X \to Z$.

The following theorem is the main theorem of this section and the rest of the section in devoted to its proof.

1.4 THEOREM. $P\colon [X, Z]_D^C \to P(X, Z)$

$$f \quad \to \quad P(f)$$

is well defined and a bijection.

COMMENTS. (1) If C is empty and D is a point then 1.4 is a theorem of M. Fuchs [2].

(2) The theorem can be viewed as giving a « geometric » interpretation of the functor $H(X) = [X, Z]_D^C$—i.e., $H(X)$ consists of equivalence classes of $\Omega_D Z$-principal fibrations on X.

(3) Note that the above theorem does not follow from any of the various known classifying space theories or even from their (currently non-existent) Top (D) generalizations. The reason is that 1.4 classifies relative principal fibrations rather than Top (D) principal fibrations.

(4) It is convenient to view the theorem as having three parts

(*a*) $f \sim g$ in Top $(C \to D)$ implies $P(f)$ equivalent to $P(g)$;

(*b*) every numerable $\Omega_D Z$-principal map is equivalent to an induced one;

(*c*) $P(f)$ equivalent to $P(g)$ implies $f \sim g$ in Top $(C \to D)$.

Note that (*a*) and (*b*) are stronger that the corresponding results with « equivalent » replaced by « strong fiber homotopy equivalent ». On the other hand (*c*) is weaker. However there are three situations where (*c*) is easily proved assuming only strong fiber homotopy equivalence [for simplicity take $C = *$. Recall that $h: E \to E'$ is a strong fiber homotopy equivalence if it is a fiber homotopy equivalence and $hi \sim i'$ where $i: F \to p^{-1}(*) \subset E$ and $i': F \to p'^{-1}(*) \subset E'$ are the natural homeomorphisms, $F = \Omega_* \hat{z}^{-1}(*)$ here].

(c1) $g = \check{z}\hat{x}$. Here the fiber homotopy equivalence gives a section of $P(f) \to Z$ and thus the desired homotopy $X \to P_D Z$.

(c2) $Z = L_\varphi(G, n)$ (classifying space for local coefficient cohomology). Here the classifying map is the transgression of the fundamental class so a strong fiber homotopy equivalence yields $f \sim g$ in Top $(* \to D)$.

(c3) $X = \Sigma A$. In general $P(f)$ strongly fiber homotopy equivalent to $P(g)$ gives $\Omega f \sim \Omega g$ (this follows, for example, from Cor. 2.7 below). Let $a: A \to \Omega\Sigma A$ be the adjoint of the identity, then $(\Omega f) a \sim (\Omega g) a$ but $(\Omega f) a$ is the adjoint of f—so $f \sim g$ in Top $(*)$. A simple argument with the path-loop sequence then shows $f \sim g$ in Top $(* \to D)$ (using $X = \Sigma A$).

1.5 Lemma. *Let $L = \Omega_D Z$, $f, g: X \to Z \in$ Top $(C \to D)$. If f and g are homotopic in* Top $(C \to D)$ *then the* RPF's $P(f)$ *and* $P(g)$ *are homotopy equivalent in* $\text{Top}^L(C \to X)$. *In particular, $f \sim \check{z}\hat{x}$ implies $P(f)$ is homotopy equivalent to $X \times_D L$.*

Proof. The hypothesis that $C \to X$ is a closed cofibration assures that the homotopy H between f and g can be chosen to have length zero on C so that the natural map $(x, k) \to (x, k + H(x))$ from $P(f)$ to $P(g)$ is under C. It is easily seen to be a homotopy equivalence with homotopy inverse $(x, k) \to (x, k - H(x))$. (It would suffice here to assume C a cozero subspace of X.)

1.6 Lemma. *f is numerable iff $P(f)$ is numerable.*

Proof. $f|U \sim \check{z}\hat{x}|U$ in Top $(V \to D)$ iff $P(f)_V$ is equivalent to $P(\check{z}\hat{x})_V$ (by 1.5) and the latter is $U \times_D L$.

1.5 and 1.6 show that the function P of Theorem 1.4 is well defined.

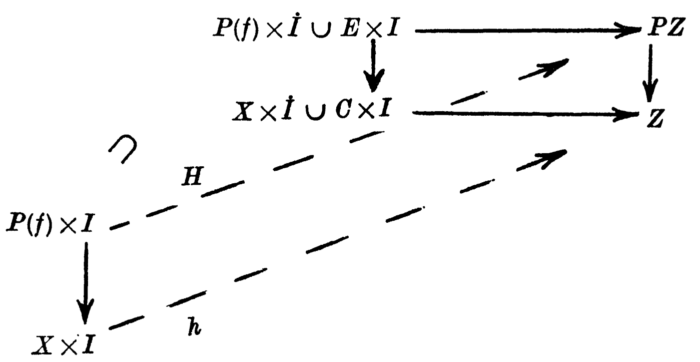
Proof that P is 1-1. Let $e: P(f) \to P(g)$ be an equivalence. By the definition of $P(f)$ and $P(g)$ there are commutative diagrams

$$\begin{array}{ccc} P(f) & \xrightarrow{F} & PZ \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Z \end{array} \qquad \begin{array}{ccc} P(g) & \xrightarrow{G} & PZ \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & Z \end{array}$$

set $E = P(\check{z}u) \subset P(f) \cap P(g)$ giving

$$\begin{array}{ccc} P(f) \times \dot{I} \cup E \times I & \xrightarrow{F \cup Ge \cup K} & PZ \\ \downarrow & & \downarrow \\ X \times \dot{I} \cup C \times I & \xrightarrow{f \cup g \cup k} & Z \end{array}$$

where $K(c, m, t) = m$, $k(c, t) = \check{z}uc$. Since $E = C \times_D L$, $L = \Omega_D Z$, and e is both a C-map and an L-map it follows that e is the identity on E so that $F \cup Ge \cup K$ is well defined. Also the diagram is commutative. This gives



The existence of the extension $\begin{pmatrix} H \\ h \end{pmatrix}$ in $\text{Top}^L(\emptyset \to D)$ follows from the following lemma. The map h then gives $f \sim g$ in $\text{Top}\,(C \to D)$, proving that P is $1-1$.

1.7 Lemma. *Suppose N a* $\text{Top}\,(D)$-monoid, $p: E \to B \in \text{Top}^N(\emptyset \to D)$, $p': P \to Z \in \text{Top}^N(D)$, *and the N action on B and Z is trivial. Suppose $B \supset A$, $F_A: p_A \to p'$ a given map in* $\text{Top}^N(\emptyset \to D)$ *and*

(1) *P contractible to $\check{p}(D)$ in* Top (D).

(2) *F_A extends to $F_V\colon p_V \to p'$, V a halo around A.*

(3) *$B - A$ has a numerable cover $\{U\}$ such that $E_U \to U$ is dominatep by $U \times_D N \to U$.*

Then F_A has an extension to $F\colon p \to p'$ in Top^N $(\emptyset \to D)$.

PROOF OF 1.7. Is similiar to that of corresponding results in [Fuchs, 2].

The next lemmas will be used to prove that P is onto. Lemmas 1.7 and 1.9 are formulated so as to be applicable in section 3 also.

1.8 DEFINITION. Suppose N is a monoid omit in a category with homotopy. Call N a hi-monoid (= monoid with homotopy inverse function) if there is an $r\colon N \to N$ such that $m(l, r)\colon N \to N \times N \to N$ is homotopic to the identity (where m is the product function for N).

1.9 LEMMA. *Suppose N is a hi-monoid in* Top (D) *and $X \in$* Top $(C \to D)$. *Suppose $g\colon E \to E'$ is a* Top^N $(C \to X)$ *map where both E and E' are homotopically equivalent to $X \times_D N$ in* Top^N $(C \to X)$. *Then g is a* Top^N $(C \to X)$ *homotopy equivalence.*

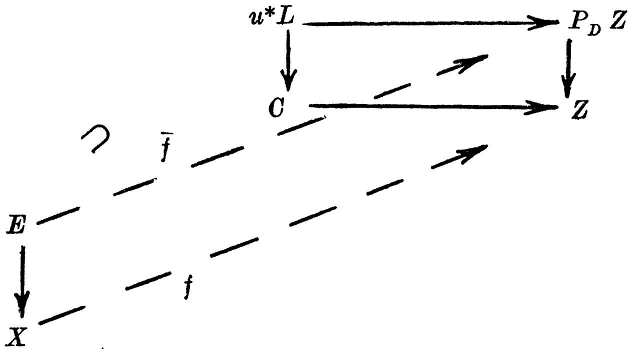
PROOF. We can assume both E and E' are $X \times_D N$ and define $h\colon X \times_D N \to$ $\to X \times_D N$ by $h(x, n) = \big(x, n(rg_2(x, 0))\big)$. Here $r\colon N \to N$ is the homotopy inverse function and g_2 is the second component of g. It is not hard to check that h is a homotopy inverse of g.

1.10 LEMMA. *Let $L = \Omega_D Z$, $p\colon E \to X \in$* Top^L $(C \to D)$ *a numerable map, and $p'\colon P_D Z \to Z$ the natural projection. If there is a* Top^L $(C \to D)$ *map F from p to p' which is f on the bottom then E is homotopically equivalent in* Top^L $(C \to X)$ *to $P(f)$ and f is a numerable map.*

PROOF OF 1.10. Since $P(f)$ is a pullback the map F gives a Top^L $(C \to X)$ map $g\colon E \to P(f)$. By a Top^L $(C \to X)$ version of Dold's theorem it will suffice to show $g(U)$ is a Top^L $(V \to U)$ homotopy equivalence for all U in some numerable cover, $V = u^{-1} U \subset C$.

By hypothesis $E(U)$ is homotopy equivalent to $U \times_D L$. The map g then gives a section of $P(f)(U) \to U$ so f and $\check{z}\hat{x}$ are homotopic on U and $P(f)(U)$ is (by 1.5) homotopically equivalent to $U \times_D L$ and f is numerable. The above lemma 1.9 shows that $g(U)$ is a homotopy equivalence as desired.

PROOF THAT P IS ONTO (IN 1.4). Let $E \to X$ be given in $P(X, Z)$. We have, $L = \Omega_D Z$,



LEMMA 1.7 gives the extension $\begin{pmatrix} \bar{f} \\ f \end{pmatrix}$ (shown) and lemma 1.10 then shows E is equivalent to $P(f)$ proving P is onto and completing the proof of 1.4.

2. – Connected relative principal fibrations.

In this section and the next it will be assumed that

(1) the fixed map $u: C \to D$ (see section 1) is pointed, so all spaces and maps will be pointed unless the contrary is stated,

(2) D is a path connected space and $\hat{z}: Z \to D$ is a Top fibration with fiber $T = \hat{z}^{-1} d_0$, and

(3) X is a path connected space.

2.1 DEFINITION. Let $f: X \to Z \in \mathrm{Top}\,(C \to D)$ and $P(f)$ be as in section 1. $P(f)$ has a natural base point $(x_0, 0)$ where $0 = 0(\check{z}d_0) \in P_D Z$. The path component of $(x_0, 0)$ in $P(f)$ will be denoted by $\overline{P}(f)$ and $\overline{P}(f)$ will be called the *connected relative principal fibration* (= *connected RPF*) induced by f.

In this section some of the basic properties of connected RPF's will be developed. In the next section some of these properties will be used to prove various classification results.

The following notation will be used

$\Gamma = \check{z}\hat{z}: Z \to Z,$

$\bar{f} = \check{z}\hat{z}f = \check{z}\hat{x}: X \to Z,$

$\Delta e = -\bar{f}e + fe$ for any path e in X,

$m \sim m'$ for paths means homotopy with ends fixed.

I want to compute the action of paths in X on the fibers of $\bar{P}(f) \to X$. Since $Z \to D$ is a Top fibration it can be easily shown that $P(f) \to X$ and hence $\bar{P}(f) \to X$ are also Top fibrations. Recall that if $F \to E \to B$ is any Top fibration then a path c from b_0 to b_1 gives a map

$$c' : F(b_1) \to F(b_0) .$$

This is not necessarily pointed and determined only up to homotopy (see Spanier [9, p. 101]). This is the action of paths in the base on the fibers.

Let $H(z) = p^{-1}(z)$ be the fiber of $p: P_D Z \to Z$ over z. Since $Z \to D$ is a Top fibration with fiber T, p is also, with fiber $H(z_0) = \Omega T$. A path a in Z from z_0 to z gives $a' : H(z) \to \Omega T$. Since $Z \to D$ has a section, $(\Omega i)_*: [H(z), \Omega T]^\phi \to [H(z), \Omega Z]^\phi$ is monic and a' is determined by $(\Omega i) a' = a'' : H(z) \to \Omega Z$.

2.2 THEOREM. $a''(k) \sim \Gamma a + k - a$.

This will be proved by comparing $P_D Z \to Z$ with another fibration. Let

$$R = \{(d, m) \in D \times WZ \mid \check{z}(d) = m(0)\} = \{m \in Z^I \mid m(0) \in \check{z}(D)\} .$$

Then $R \to Z$, $m \to m(1)$, is the Top fibration naturally associated with the map $\check{z}$. Let $R(z)$ be the fiber over z. Let a be a path from z_0 to z so $a' : R(z) \to R(z_0)$. The following lemma is not hard to check directly from the definition of the action and the definition of R.

2.3 LEMMA. *$a''(m) = m - a$ in $R(z_0)$.*

Now consider the following commutative diagram

$$\begin{array}{ccccc} R(z) & \longrightarrow & R & \longrightarrow & Z \\ {\scriptstyle y(z)}\uparrow & & {\scriptstyle y}\uparrow & & \| \\ H(z) & \longrightarrow & P_D Z & \longrightarrow & Z \end{array}$$

where y is the inclusion. Since $P_D Z$ is contractible to D it is not hard to see that y is a fiber homotopy equivalence. Thus $y(z)$ is a homotopy equivalent with homotopy inverse $v(z)$. Let

$$w = (\Omega i) v(z_0) : R(z_0) \to \Omega Z .$$

The following lemma is proved by direct calculation.

2.4 LEMMA. *$w(m) = -\Gamma(m) + m$.*

PROOF OF THEOREM 2.2. The naturality of $a \to a'$ gives

$$\begin{array}{ccc} R(z) & \xrightarrow{a'_R} & R(z) \\ {\scriptstyle y(z)}\uparrow & & \uparrow{\scriptstyle y(z_0)} \\ H(z) & \xrightarrow[a'_H]{} & H(z) \end{array}$$

a homotopy commutative diagram. $a'' = (\Omega i)a' \sim wa'_R y$ and

$$wa'y(k) = wa'(k) = w(k - a) = -\Gamma(k - a) + k - a = \Gamma a - \Gamma k + k - a .$$

But k is a path in $H(z)$ so $\Gamma k = *$, so $wa'y(k) \sim \Gamma(a) + k - a$ proving the theorem.

Turn now to $\bar{P}(f) \to X$. Let $F(x)$ be the fiber over x and c a path from x_0 to x in X giving

$$c' \colon F(x) \to F(x_0) = \Omega T .$$

Let $c'' = (\Omega i)c'$. The naturality of the action leads to the following corollary.

2.5 COROLLARY. $c''(x, m) \sim \bar{f}c + m - fc$.

Recall that for any fibration $F \to E \to B$ there is a natural map $b \colon \Omega B \to F$ defined by $b(c) = c'(*)$. In particular, $\Omega T \to P_D Z \to Z$ gives $b \colon \Omega Z \to \Omega T$.

The fibration $T \to Z \to D$ gives $\Omega T \to \Omega Z \to \Omega D$ and the section $\check{z}$ gives a map $\varepsilon \colon \Omega Z \to \Omega T$ such that $\varepsilon(\Omega i) \sim 1$. In fact ε is defined by $i_* \colon \Omega[Z, \Omega T] \to [\Omega Z, \Omega Z]$, $i_*\varepsilon = 1 - \Gamma$.

2.6 COROLLARY. $b \sim -\varepsilon$.

PROOF. This follows from 2.2 since 2.2 shows $b \sim \Gamma - 1$.

Also $\Omega T \to P(f) \to Z$ gives $b_f \colon \Omega Z \to \Omega T$. Let $b' = (\Omega i)b_f \colon \Omega Z \to \Omega Z$. Then it follows from 2.2 (or 2.6) that:

2.7 COROLLARY. $b'c \sim \bar{f}c - fc$.

The next theorem gives a very useful characterization of all the elements of $\bar{P}(f)$. Recall that $\Delta e = -\bar{f}e + fe$ (Δ is not necessarily a homomorphism).

2.8 THEOREM. *Let $(x, m) \in P(f)$. Then $(x, m) \in \bar{P}(f)$ iff there is a path a in X from x_0 to x with $m \sim \Delta a$ in Z.*

PROOF. First take $x = x_0$ so (x_0, m) is identified with $m \in \Omega T$. There is an exact sequence

$$\to \Omega X \xrightarrow{b_f} \Omega T \xrightarrow{j} P(f) \to X$$

so $m \in \overline{P}(f)$ iff $jm \sim 0$ iff $m \sim b_f e$ for some e in ΩX. But in ΩZ, by 2.7, $b_f e$ is $b'e \sim \bar{f}e - fe$ and replacing e by $-e$ gives the desired result.

For a general (x, m), let c be a path from x_0 to x so $c' : F(x) \to F(x_0) = \Omega T$ and $(x, m) \in \overline{P}(f)$ iff $c'(x, m)$ is iff $c'(x, m) \sim -\bar{f}e + fe$ for some loop e of X. By 2.5 this says (in ΩZ) that $\bar{f}c + m - fc \sim -\bar{f}e + fe$. Then 2.8 follows by setting $a = e + c$.

2.9 DEFINITION. Define $N = Nf \in \mathrm{Top}\,(D)$, $N \subset \Omega_D Z$, by (for each $d \in D$).

$$N(d) = \left\{ k \in \Omega_d Z_d \;\middle|\; \begin{array}{l} \text{for every } x \text{ with } \hat{x}x = d, \text{ for every path } e \text{ from } x_0 \text{ to } x, \\ \text{there is a path } w \text{ from } x_0 \text{ to } x \text{ with } k \sim \Delta w - \Delta e \text{ in } Z. \end{array} \right\}$$

2.10 THEOREM. (1) *N is a sub-monoid of $\Omega_D Z$.*

(2) *The action of $\Omega_D Z$ on $P(f)$ gives an action*

$$N \times_D \overline{P}(f) \to \overline{P}(f)$$

and N is the largest subset of $\Omega_D Z$ giving such an action.

PROOF. (1) Suppose $k, k' \in N(d)$ and e from x_0 to x is given. First, select w' with $k' \sim \Delta w' - \Delta e$, then, select w with $k \sim \Delta w - \Delta w'$. Then $k + k' \sim \Delta w - \Delta e$ as desired.

(2) $k \in N(d)$ iff for every m with $m \sim \Delta e$, some e, there is a w with $k + m \sim \Delta w$ iff (by 2.8) $(x, k + m) \in \overline{P}(f)$ all $(x, m) \in \overline{P}(f)$ and this proves (2).

Some other representations of $N = N_1$ will be needed later. Define N_2, N_3, N_4 by

$$N_2(d) = \left\{ k \in \Omega_d Z_d \;\middle|\; \begin{array}{l} \text{for any } x \text{ with } \hat{x}x = d, \text{ for some paths } w, e \text{ from } x_0 \text{ to } x, \\ k \sim \Delta w - \Delta e \text{ in } Z. \end{array} \right\}$$

$N_3(d)$ same as N_2 but « for some x ».

$$N_4(d) = \left\{ k \in \Omega_d Z_d \;\middle|\; \begin{array}{l} k \sim \check{z}c + \Delta a - \check{z}c \text{ in } Z \text{ for some } a \text{ from } x_0 \text{ to } x_0, \text{ for} \\ \text{some } c \text{ from } d \text{ to } d_0. \end{array} \right\}$$

Note first that it is always true that

$$N_1 \subset N_2 \subset N_3 \subset N_4 .$$

The only non-obvious inclusion is the last. Let $k \in N_3$, $k \sim \Delta w - \Delta e \sim$ $\sim -\bar{f}e + \bar{f}e - \bar{f}w + fw - fe + \bar{f}e = -\bar{f}e - \bar{f}(w-e) + f(w-e) + \bar{f}e$ and let $w = -\hat{x}e$, $a = w - e$, to get $k \in N_4$.

Consider now two hypotheses that will be used in the next theorem and later as well.

HYP. A: (1) The elements of $\check{z}\pi_1(D, d_0)$ commute with those of the image of Δ in $\pi_1(Z, z_0)$.

(2) The elements of $f_*\pi_1(X, x_0)$ commute with those of the image of Δ in $\pi_1(Z, z_0)$.

HYP. B: The elements of $f_*\pi_1(X, x_0)$ commute with those of $\check{z}_*\pi_1(D, d_0)$ in $\pi_1(Z, z_0)$.

2.11 THEOREM. *Assume* Hyp. A *or* Hyp. B. *Then*

(1) $N = N_2 = N_3 = N_4$.

(2) N *is a hi-monoid.*

PROOF OF (1). (i.e., $N_4 \subset N$). First assume Hyp. A. It is helpful to note that under Hyp. A1 the representation in $k \in N_4$ is valid for any c from d to d_0 since $\check{z}c + \Delta a - \check{z}c \sim \check{z}c + \Delta a - \check{z}c + \check{z}c' - \check{z}c' \sim \check{z}c' + \Delta a - \check{z}c'$ $(-\check{z}c + \check{z}c'$ is in $\check{z}\pi_1(Z, z_0))$.

Now given e from x_0 to x (in the definition of N) set $c = \hat{x}(-e)$ so $k \in N_4$, $k \sim -\bar{f}e - \bar{f}a + fa + (fe - fe) + \bar{f}e = -\bar{f}(a+e) + f(a+e) - f(e) + \bar{f}(e)$ and $w = a + e$ shows $k \in N$. (Only Hyp. A1 was used here.)

Now assume Hyp. B. $k \in N_4$ and e from x_0 to x are given. $k \sim \check{z}c - \bar{f}a +$ $+ fa - \check{z}c - \bar{f}e + \bar{f}e$, $-\check{z}c - \bar{f}e$ is in $\check{z}\pi_1(D, d_0)$ so $k \sim -\bar{f}e - \bar{f}a + fa +$ $+ (fe - fe) + \bar{f}e = -\bar{f}(a+e) + f(a+e) - fe + \bar{f}e$ so again $w = a + e$ serves.

PROOF OF (2). Let $k \in N(d)$. Suppose e given from x_0 to x and w from x_0 to x such that $k \sim \Delta w - \Delta e$, so $-k \sim \Delta e - \Delta w$.

Assume Hyp. A2. $-k \sim -\bar{f}e + fe - fw + \bar{f}w - fw + fw$. Now $\bar{f}w - fw =$ $= \Delta(-w)$ so $-k \sim -\bar{f}e + \bar{f}w + (-\bar{f}e + \bar{f}e) - fw + fe \sim -\bar{f}(e - w + e) +$ $\bar{f}e - fw + fe \sim -\bar{f}(e - w + e) + \bar{f}e - fe + fe - fw + fe$. Now commute $\bar{f}e - fe$ and $fe - fw$, $-k \sim -\bar{f}(e - w + e) + fe - fw + \bar{f}e \sim -\bar{f}(e - w + e) +$ $+ f(e - w + e) - fe + \bar{f}e$ and so $\bar{w} = e - w + e$ works. The proof using Hyp. B is similar.

NOTES. (1) $\pi_1(Z, *)$ is always a semi-direct product of $\pi_1(T, *)$ and $\pi_1(D, *)$ by means of $\check{z}_*$. Hyp. A1 will follow if this is a direct product representation since $\operatorname{Im} \dot{\Delta} \subset \operatorname{Ker}(\hat{z}_*) = \pi_1(T, *)$.

(2) Let $\bar{\Omega}_d Z(d)$ be all loops of $Z(d)$ at $\check{z}d$ which are pointed null-homotopic in $Z(d)$. Let $\bar{\Omega}_D Z$ be all elements of $\Omega_D Z$ which are freely homotopic in Z to the constant map: $S^1 \to z_0$. It is not hard to check [see, 7, 4.1] that $\bar{\Omega}_D Z = \cup \bar{\Omega}_D Z$.

(3) If $f_* = (\check{z}\hat{x})_*$ on $\pi_1(X, *)$ then $N = \bar{\Omega}_D Z$. If $f \sim \check{z}\hat{x}$ then $\bar{P}(f) = \bar{\Omega}_D Z = Nf$.

(4) If f is freely homotopic to g over D then $f_*, g_*: \pi_1(X, *) \to \pi_1(Z, *)$ are conjugate by an element of $\pi_1(T, *)$. If we assume A1 and also that the elements of $f_*\pi_1(X, x_0)$ commute with those of Ker $(\hat{z}_*)$ (a bit more than A2) then $N(f) = N_4(f) = N_4(g) = N(g)$. This observation will be useful in connection with evaluation fibrations [7].

2.12 THEOREM. *If* $f \sim g$ *in* Top $(C \to D)$ *then* $Nf = Ng$ *and* $\bar{P}(f)$, $\bar{P}(g)$ *are homotopically equivalent in* $\mathrm{Top}^N (C \to X)$, $N = Nf = Ng$.

PROOF. The definition of N shows that $Nf = Ng$ and the maps (see proof of 1.5) giving Pf homotopically equivalent to Pg restrict to the desired equivalence here.

The final theorem of this section will be a local representation theorem for $\bar{P}(f)$. Perhaps first it should be pointed out that there is something to prove. Recall (1.6) that $(Pf)_U$ is homotopically equivalent to $U \times_D \Omega_D Z$ for appropriate U. This shows that the path component of a nice base point in $(Pf)_U$ will be homotopically equivalent to $U \times_D \bar{\Omega} Z$. However, this is not $(\bar{P}f)_U$ and so is not relevant here.

2.13 DEFINITION. Let $f: X \to Z \in$ Top $(C \to D)$. An open subset U of X will be called *f-regular* if

(1) U is path connected.

(2) For some $\bar{x} \in U$ and any path $\bar{m}$ from $\check{z}\hat{x}\bar{x}$ to $f\bar{x}$ over $\hat{x}\bar{x}$, there is a homotopy $F: \check{z}\hat{x} \sim f: U \to Z$ in Top $(V \to D)$, $V = u^{-1}U$, such that $F(\bar{x}) = \bar{m}$.

2.14 THEOREM. *Suppose* Hyp. A1 *or* Hyp. B *and suppose* U *is f-regular. Then* $\bar{P}(f)_U$ *and* $U \times_D N$ *are homotopy equivalent in* $\mathrm{Top}^N (V \to D)$, $V = u^{-1}U$.

PROOF. Let $F: U \to P_D Z$ be the given homotopy so that $F(\bar{x}) = \bar{m}$. There are maps $\alpha: P(f)_U \to U \times_D \Omega_D Z$, $\alpha(x, m) = (x, m - F(x))$, and $\beta: U \times_D \Omega_D Z \to P(f)_U$, $\beta(x, k) = (x, k + F(x))$.

(1) $\alpha(\bar{P}(f)_U) \subset N$. *Proof*: let $(x, m) \in \bar{P}(f)$, $x \in U$ and b a path from $\bar{x}$ to x in U. By 2.8, $m \sim \Delta a$ and $\bar{m} \sim \Delta \bar{a}$ for paths $a, \bar{a}$ from x_0 to $x, \bar{x}$ (resp.). For any path n let $n^s(t) = n(st)$. Then $(\bar{f}b)^s + F(b(s)) - f(b)^s$ gives $\bar{f}b +$

$+ F(x) - f(b) \sim F(\bar{x}) \sim \Delta\bar{a}$. So $F(x) \sim \Delta e$ where $e = \bar{a} + b$. Thus $m - F(x) \sim \sim \Delta a - \Delta e$ showing $\alpha(x, m) \in N_3 = N$.

(2) $\beta(U \times_D N) \subset \bar{P}(f)_U$. *Proof*: given $(x, k) \in U \times_D N$. As above $\bar{m} \sim \Delta\bar{a}$ so $F(x) \sim \Delta e$. The definition of N gives w from x_0 to x so that $k \sim \Delta w - \Delta e$. Hence $k + F(x) \sim \Delta w - \Delta e + \Delta e \sim \Delta w$ so $\beta(x, m) \in \bar{P}(f)$.

Similar arguments show that the natural homotopies $\alpha\beta \sim 1$, $\beta\alpha \sim 1$ have the correct image spaces. This proves 2.14.

Note that if $D = *$, Hyp. B is fulfilled. Here

$$N = \{k \in \Omega Z | k \sim fa \text{ some loop } a \text{ in } X\}$$

and 2.14 says that for appropriate U, $(\bar{P}f)_U$ is equivalent to $U \times N$.

3. – Classifying connected relative principal fibrations.

The assumptions at the beginning of section 2 are still made here.

3.1 DEFINITION. $Z \in \text{Top}(D)$. A map $f\colon X \to Z \in \text{Top}(C \to D)$ is connected-numerable if X has a numerable cover by f-regular sets (see 2.13 for « f-regular »).

3.2 THEOREM. *Suppose $Z \in \text{Top}(D)$, $X \in \text{Top}(C \to D)$, and X has a numerable cover by sets each of which is contractible to a non degenerate base point relative to that base point. Suppose the fiber of $Z \to D$ is connected. Then any $f\colon X \to Z \in \text{Top}(C \to D)$ is connected-numerable.*

PROOF. Similar to the proof of 1.2.

3.3 DEFINITION. Let $Z \in \text{Top}(D)$, $X \in \text{Top}(C \to D)$, N a sub-hi–monoid of $\Omega_D Z$.

(1) $\bar{P}(X, Z, N)$ = all equivalence classes of connected numerable N-principal maps $E \to X$ (in def. 1.1, E is now assumed to be path connected).

(2) $[X, Z]_D^{C,N}$= the set of all homotopy classes of connected numerable maps $f\colon X \to Z$, with $N = Nf$.

3.4 THEOREM. *Assume* Hyp. A *or* Hyp. B. *Then*

$$\begin{aligned} \bar{P}.\ [X, Z]_D^{C,N} &\to \bar{P}(X, G; N) \\ f \quad &\to \quad \bar{P}(f) \end{aligned}$$

is well defined and a bijection.

PROOF. I'll point out the modifications necessary in the proof of 1.4.

2.14 and 2.12 show that $\overline{P}$ is well defined. In the proof that P is 1-1, restrict F and G to $\overline{P}(f)$ and $\overline{P}(g)$. Since $P_D Z \to Z$ can be viewed as having an N action (rather than a $\Omega_D Z$-action) 1.7 can be applied (using 2.14 to check hypothesis (3)). This shows $\overline{P}$ is 1-1.

Since we assume E is connected, lemma 1.10 can be modified to apply to $\overline{P}(f)$ as follows (since 1.9 applies directly). In the proof of 1.10, $g: E \to \to P(f)$ gives $g: E \to \overline{P}(f)$. Now the remaining part of the proof is valid for $\overline{P}$. The proof that $\overline{P}$ is onto is now just the same as the proof that P is onto.

NOTES. (1) Note that both sets in 3.4 will be empty if N is not Nf for some f.

(2) Suppose $f_* = (\check{z}\hat{x})_* : \pi_1(X, x_0) \to \pi_1(Z, z_0)$ for all $f: X \to Z \in \mathrm{Top}\,(C \to D)$ (e.g. $\pi_1 X$ finite, $\pi_1 Z$ free abelian). Then $N = Nf = \bar{\Omega}_D Z$, for all f, so

$$[X, Z]_D^C \to \overline{P}(X, G; N)$$

is a bijection (Hyp. A is automatic here). So elements of $[X, Z]_D^C$ can be « geometrically » represented either by RPF's or by connected RPF's.

(3) If $D = *$, $[X, Z]_D^{C,N} = \{[f] | f_* \pi_1(X) = \pi_0(N) \subset \pi_1(Z)\}$.

(4) Suppose Hyp. A or Hyp. B and $N = N(f) = N(g)$. Then the following are equivalent

(1) $\overline{P}(f) \sim \overline{P}(g)$ in $\mathrm{Top}^N (C \to X)$

(2) $P(f) \sim P(g)$ in $\mathrm{Top}^L (C \to X)$

since both are equivalent to $f \sim g$ in $\mathrm{Top}\,(C \to D)$.

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