ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

H. AMANN
E. ZEHNDER

Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 7, nº 4 (1980), p. 539-603
<http://www.numdam.org/item?id=ASNSP_1980_4_7_4_539_0>

Nontrivial Solutions for a Class of Nonresonance Problems and Applications to Nonlinear Differential Equations.

H. AMANN - E. ZEHNDER

Introduction.

In this paper we study existence problems for equations of the form

$$Au = F(u)$$

in a real Hilbert space H. Here A is a self-adjoint linear operator, and F is a potential operator, mapping H continuously into itself. We suppose that there exist numbers $\alpha < \beta$, not belonging to the spectrum $\sigma(A)$ of A, such that $\sigma(A) \cap [\alpha, \beta]$ consists of at most finitely many eigenvalues of finite multiplicities. There are no restrictions whatsoever on $\sigma(A)$ outside the interval $[\alpha, \beta]$. In particular, $\sigma(A)$ can be unbounded above and below.

As for the nonlinearity F, we suppose that

$$\alpha \| u - v \|^2 \leqslant \langle F(u) - F(v), u - v \rangle \leqslant \beta \| u - v \|^2 \tag{1}$$

for all $u, v \in H$. Roughly speaking, this condition means that the nonlinearity F can only interact with the finitely many eigenvalues of A in $[\alpha, \beta]$.

The original problem is reduced to the study of critical points of a functional f, which is neither bounded above nor below, in general. Thus standard variational methods do not apply directly. Condition (1) implies that f possesses a saddle point on an appropriate subspace of H. Taking advantage of this fact, we reduce the original problem to the study of critical points of a functional a, defined on the finite-dimensional subspace Z of H, spanned by the finitely many eigenfunctions of A, belonging to the eigenvalues in $[\alpha, \beta]$. This approach has been introduced by the first author in [2].

Pervenuto alla Redazione il 30 Luglio 1979 ed in forma definitiva il 30 Agosto 1979.

In order to study the existence of critical points of a, we suppose, roughly speaking, that F has a derivative at infinity, $F'(\infty)$, such that

$$0 \notin \sigma(A - F'(\infty)) .$$

This is a nonresonance condition at infinity, and it is shown that it implies the validity of the Palais-Smale condition for a. In contrast to [2], where it has been assumed that $F'(\infty) = \nu I_H$ for some $\nu \notin \sigma(A)$, we allow now $\sigma(F'(\infty))$ to be arbitrarily distributed in $[\alpha, \beta]$. Then, given some mild additional hypotheses, which are satisfied in all of our applications, we deduce the existence of at least one solution of $Au = F(u)$. This is achieved by means of a generalized Morse theory in the sense of C. C. Conley [18]. This theory has the advantage, that it does not require the critical points of the functional a to be nondegenerate.

Then we consider the case that $F(0) = 0$, in which situation we are interested in the existence of nontrivial solutions of $Au = F(u)$, which correspond to nontrivial critical points of a. In order to deduce the existence of nontrivial critical points of the functional a, we employ two different approaches. Namely we use elementary critical point theory and, again, the generalized Morse theory of C. C. Conley. In each case, the basic idea is to compare the behavior near zero to its asymptotic behavior near infinity. Of course, each of the two approaches applies to different situations.

Our principal abstract results are contained in Section 8, namely Theorems (8.1) and (8.3), and in Section 9, Theorems (9.1) and (9.4).

In the second part of this paper we apply our general abstract results to three different kinds of problems. Namely, we prove the existence of solutions for certain nonlinear elliptic boundary value problems, the existence of periodic solutions to a class of semi-linear wave equations, and the existence of periodic solutions of Hamiltonian systems of ordinary differential equations.

In order to demonstrate the scope of our results, we now outline some of the applications in a simple setting.

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$, and consider the nonlinear Dirichlet problem

$$(2) \qquad \begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega , \end{cases}$$

where $f \in C^1(\boldsymbol{R}, \boldsymbol{R})$. Moreover we suppose that

$$f'(\infty) := \lim_{|\xi|\to\infty} f'(\xi)$$

exists. Then, by meaning by an « eigenvalue of $-\Delta$ » an eigenvalue of $-\Delta$, subject to the Dirichlet boundary condition, the following result is a very special case of Theorem (10.2).

THEOREM 1. *Suppose that $f'(\infty)$ is not an eigenvalue of $-\Delta$. Then the nonlinear Dirichlet problem* (2) *has at least one solution.*

Suppose, in addition, that $f(0) = 0$. Then the nonlinear Dirichlet problem (2) *has at least one nontrivial solution, provided there exists at least one eigenvalue λ of $-\Delta$ such that either $f'(0) < \lambda < f'(\infty)$ or $f'(\infty) < \lambda < f'(0)$.*

The existence of solutions of nonlinear boundary value problems of the prototype (2), where f is supposed to be asymptotically linear (or at least linearly bounded), has been studied by numerous authors (cf. the end of Section 10 for bibliographical remarks). In the more interesting case that $f(0) = 0$, it is a common feature of all of these results, that there exists at least one nontrivial solution, provided $f'(\xi)$ « crosses at least one eigenvalue of $-\Delta$ if $|\xi|$ goes from 0 to infinity ». However in each one of the papers known to the authors, this result has only been shown under additional restrictions, either on f', or on the eigenvalues, which are being « crossed », or on both. In our Theorem 1 and, of course, in the much more general Theorem (10.2), we establish for the first time this result in full generality, without any further restrictions besides of the nonresonance condition at infinity.

At this point it should be mentioned, that many papers on so-called Landesman-Lazer problems suggest the validity of our general result also in the case that there is resonance at infinity, provided we impose Landesman-Lazer type conditions. In fact, an analysis of these « Landesman-Lazer type proofs » shows that these additional Landesman-Lazer conditions provide appropriate a priori bounds, which we have deduced in our case from the nonresonance condition. By exploiting this observation, it should not be too difficult to replace our nonresonance condition by Landesman-Lazer type conditions, in order to extend our results to the case that resonance at infinity occurs. However, for simplicity and to avoid unnecessary length, we do not consider this somewhat more general case. A similar remark applies to our other applications. (For another interesting treatment of the resonance case we refer to the recent paper by K. Thews [42]).

Next we give an application to a nonlinear wave equation. Namely, we are looking for 2π-periodic classical solutions of the problem

$$
(3) \qquad \begin{cases} u_{tt} - u_{xx} = f(u) & \text{for } (x, t) \in (0, \pi) \times \boldsymbol{R} \\ u(0, t) = u(\pi, t) = 0\,, & t \in \boldsymbol{R}\,, \end{cases}
$$

where $f \in C^2(\boldsymbol{R}, \boldsymbol{R})$ and $|f'(\xi)| \geqslant \alpha > 0$ for all $\xi \in \boldsymbol{R}$. Moreover, we assume again that

$$f'(\infty) := \lim_{|\xi| \to \infty} f'(\xi)$$

exists.

It is known that the wave operator $\square$ under the above periodicity conditions has a pure point spectrum, extending from $-\infty$ to $+\infty$, and that every nonzero eigenvalue has finite multiplicity, whereas 0 is an eigenvalue of infinite multiplicity.

The following theorem, which is a special case of Theorem (11.2), shows again that (3) has at least one nontrivial solution if $f(0) = 0$ and $f'(\xi)$ « crosses at least one eigenvalue of $\square$ if $|\xi|$ runs from zero to infinity ». (It should be observed that, due to the monotonicity restriction $|f'| \geqslant \alpha > 0$, $f'(\xi)$ cannot cross 0.)

THEOREM 2. *Suppose that $f'(\infty)$ is not an eigenvalue of $\square$. Then problem* (3) *has at least one 2π-periodic solution.*

Suppose, in addition, that $f(0) = 0$. Then problem (3) *has at least one nontrivial solution if there exists an eigenvalue λ of $\square$ such that either $f'(0) < \lambda < f'(\infty)$ or $f'(\infty) < \lambda < f'(0)$.*

For bibliographical remarks concerning the problem of the existence of periodic solutions to the nonlinear wave equation we refer to the end of Section 11.

We finally describe some applications of our general results to the existence problem of periodic solutions of Hamiltonian systems

$$\dot{u} = J\mathcal{H}_x(t, u)\,, \tag{4}$$

where $\mathcal{H} \in C^2(\boldsymbol{R} \times \boldsymbol{R}^{2N}, \boldsymbol{R})$ is periodic in t for some period $T > 0$, and where $J \in \mathcal{L}(\boldsymbol{R}^{2N})$ is the standard symplectic structure on $\boldsymbol{R}^{2N}$. We shall assume

$$\sup_{(t,\xi)} |\mathcal{H}_{xx}(t, \xi)| < \infty\,.$$

THEOREM 3. *Assume the Hamiltonian vectorfield is asymptotically linear:*

$$J\mathcal{H}_x(t, \xi) = Jb_\infty \xi + o(|\xi|)\,, \quad \text{as } |\xi| \to \infty$$

uniformly in $t \in \boldsymbol{R}$, for a time independent symmetric $b_\infty \in \mathcal{L}(\boldsymbol{R}^{2N})$. Then the Hamiltonian system (4) *has at least one T-periodic solution, provided $\sigma(Jb_\infty) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$.*

We next assume, in addition, that the Hamiltonian vectorfield $J\mathcal{H}_x$ has an equilibrium point which we assume to be 0, $J\mathcal{H}_x(t, 0) = 0$. We consider a Hamiltonian vectorfield satisfying

$$J\mathcal{H}_x(t, \xi) = Jb_0\xi + o(|\xi|) , \qquad \text{as } |\xi| \to 0$$

and

$$J\mathcal{H}_x(t, \xi) = Jb_\infty\xi + o(|\xi|) , \qquad \text{as } |\xi| \to \infty ,$$

uniformly in $t \in \boldsymbol{R}$, for two symmetric and time independent $b_0, b_\infty \in \mathcal{L}(\boldsymbol{R}^{2N})$. The aim is to find T-periodic solutions of (4) which are not the trivial solution $u(t) = 0$. In order to describe the difference between the two linearized Hamiltonian vectorfields at 0 and at ∞, Jb_0 and Jb_∞, which will guarantee a nontrivial T-periodic solution, we introduce in section 12 an integer, $\operatorname{Ind}(b_0, b_\infty, \tau)$. This integer, which is a symplectic invariant, is defined for two symmetric $b_0, b_\infty \in \mathcal{L}(\boldsymbol{R}^{2N})$ and a frequency $\tau > 0$, and it involves only the purely imaginary eigenvalues of Jb_0 and Jb_∞ and their relation to the frequency τ. For instance $\operatorname{Ind}(b_0, b_\infty, \tau) = 0$ if $b_0 = b_\infty$, or if b_0 and b_∞ have no purely imaginary eigenvalues, while $\operatorname{Ind}(b_0, b_\infty, \tau) \neq 0$ if $b_0 > 0$ (resp. $b_0 < 0$) and $b_\infty < 0$ (resp. $b_\infty > 0$). A nonvanishing index gives rise to a nontrivial T-periodic solution of (4), as is seen from the following theorem. Here and in the following we denote by $\mathcal{L}_s(\boldsymbol{R}^{2N})$ the space of symmetric linear operators on $\boldsymbol{R}^{2N}$.

THEOREM 4. *Let $\mathcal{H}(t, x)$ be periodic in t with period $T > 0$, and assume*

$$J\mathcal{H}_x(t, \xi) = Jb_0\xi + o(|\xi|) , \qquad |\xi| \to 0$$
$$J\mathcal{H}_x(t, \xi) = Jb_\infty\xi + o(|\xi|) , \qquad |\xi| \to \infty$$

uniformly in $t \in \boldsymbol{R}$, for two time independent $b_0, b_\infty \in \mathcal{L}_s(\boldsymbol{R}^{2N})$. Assume $\sigma(Jb_0) \cap \cap\, i(2\pi/T)\boldsymbol{Z} = \emptyset$ and $\sigma(Jb_0) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$. If

$$\operatorname{Ind}\left(b_0, b_\infty, \frac{2\pi}{T}\right) \neq 0 ,$$

then the Hamiltonian system (4) *possesses at least one nontrivial T-periodic solution.*

COROLLARY. *If $b_0 > 0$ (resp. $b_0 < 0$) and $b_\infty < 0$ (resp. $b_\infty > 0$), the Hamiltonian system* (4) *has at least one nontrivial T-periodic solution provided $\sigma(Jb_0) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$ and $\sigma(Jb_\infty) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$.*

The explicit computation of the integer $\mathrm{Ind}\,(b_0, b_\infty, \tau)$ leads to other more delicate existence statements, which also are global in nature. In the time independent case we find nonconstant T-periodic solutions with prescribed period T for asymptotically linear Hamiltonian equations. For example, let $\mathcal{H}$ be a convex function on $\boldsymbol{R}^{2N}$ with $b_0, b_\infty > 0$. If the two linear Hamiltonian vectorfields Jb_0 and Jb_∞ are symplectically inequivalent one finds a T-periodic solution for every $T > 0$ belonging to some open and unbounded subset of $\boldsymbol{R}_+$. As for the results and as for bibliographical remarks we refer to Section 12.

The organization of this paper is seen from the following table of contents.

Finally we should like to thank C. C. Conley, Madison, for helpful discussions on his generalized Morse theory, and R. Stöcker, Bochum, for his advices on problems of algebraic topology. We also like to thank J. Moser, New York, for valuable discussions on Hamiltonian equations.

PART ONE

GENERAL THEORY

1. – The basic hypotheses.

Throughout Part One we use without further mention the following hypotheses and conventions.

$$H \quad \textit{is a real Hilbert space with inner product } \langle .,. \rangle\,,$$

and we identify H with its dual.

$$(A) \qquad \begin{cases} A\colon \operatorname{dom}(A) \subset H \to H \quad \textit{is a self-adjoint linear operator}. \\ \textit{There exist numbers } \alpha < \beta \textit{ such that } \alpha, \beta \notin \sigma(A), \textit{ and } \sigma(A) \cap (\alpha, \beta) \\ \textit{consists of at most finitely many eigenvalues of finite multiplicity}. \end{cases}$$

We denote by

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n$$

the eigenvalues of A in (α, β), and by $m(\lambda_j)$ the multiplicity of λ_j.

$$(F) \qquad \begin{cases} F\colon H \to H \quad \textit{is a continuous potential operator such that} \\ \alpha\|u-v\|^2 \leqslant \langle F(u) - F(v), u-v\rangle \leqslant \beta\|u-v\|^2 \quad \forall u, v \in H\,. \end{cases}$$

We denote the normalized potential of A by Φ, that is, $\Phi \in C^1(H, \boldsymbol{R})$ satisfying $\Phi(0) = 0$ and $\Phi' = F$.

We let $\{E_\lambda | \lambda \in \boldsymbol{R}\}$ be the spectral resolution of A, and we define orthogonal projections $P_\pm, P \in \mathfrak{L}(H)$ by

$$P_- := \int_{-\infty}^{\alpha} dE_\lambda\,, \qquad P_+ := \int_{\alpha}^{\infty} dE_\lambda\,, \qquad P := \int_{\alpha}^{\beta} dE_\lambda\,,$$

respectively. Moreover, we let

$$X := P_-(H)\,, \qquad Y := P_+(H)\,, \qquad Z := P(H)\,.$$

Observe that

$$H = X \oplus Y \oplus Z\,,$$

and that Z is finite-dimensional with

$$\dim Z = \sum_{j=1}^{n} m(\lambda_j)$$

(with the usual convention that the empty sum has the value 0).

Next we define self-adjoint linear operators

$$R \in \mathfrak{L}(H, X)\,, \qquad S \in \mathfrak{L}(H, Y)\,, \qquad T \in \mathfrak{L}(H, Z)$$

by

$$R := \int_{-\infty}^{\alpha} (\alpha - \lambda)^{-\frac{1}{2}} dE_\lambda\,, \qquad S := \int_{\beta}^{\infty} (\lambda - \alpha)^{-\frac{1}{2}} dE_\lambda$$

and

$$T := \int_{\alpha}^{\beta} (\lambda - \alpha)^{-\frac{1}{2}}\, dE_\lambda = \sum_{j=1}^{n} (\lambda_j - \alpha)^{-\frac{1}{2}} P_j\,,$$

respectively, where P_j denotes the orthogonal projection of H onto the eigenspace $\ker(\lambda_j - A)$ of λ_j.

It is an immediate consequence of these definitions, that R, S, and T are pairwise commuting, that $R|X$, $S|Y$, and $T|Z$ are injective, and that

$$-R^2 + S^2 + T^2 = (A - \alpha)^{-1}\,. \tag{1.1}$$

Hence

$$-R^2 = -P_-R^2 = P_-(-R^2 + S^2 + T^2) = P_-(A - \alpha)^{-1} = (A - \alpha)^{-1}P_-\,,$$

and, consequently,

$$-(A - \alpha)R^2 = P_-\,. \tag{1.2}$$

Similarly we find that

$$(A - \alpha)S^2 = P_+ \tag{1.3}$$

and

$$(A - \alpha)T^2 = P\,. \tag{1.4}$$

2. – A saddle point reduction.

Formally, the equation $Au = F(u)$ is the Euler equation of a variational problem. To be more precise, let

$$\varphi(u) := \tfrac{1}{2}\langle Au, u\rangle - \Phi(u) \qquad \forall u \in \operatorname{dom}(A).$$

Then, for $u, h \in \operatorname{dom}(A)$, the directional derivative $\delta\varphi(u; h)$ (that is, the « first variation ») of φ at u in the direction h is given by

$$\delta\varphi(u; h) = \langle Au - F(u), h\rangle .$$

Hence the solutions of $Au = F(u)$ correspond to the « critical points » of φ and, in principle, critical points could be obtained by variational methods. However, variational methods are difficult to apply directly, since φ is only defined on the dense subspace $\operatorname{dom}(A)$ of H. In addition, there is no restriction on the spectrum of A outside of the interval (α, β). Thus $\sigma(A)$ can extend from $-\infty$ to $+\infty$ and, in fact, this will be the case in some of our applications. In other words, in general the quadratic term $\langle Au, u\rangle$ will be indefinite in the strong sense, that is, it can be positive definite and negative definite on infinite-dimensional subspaces of H, respectively.

Assumption (F) implies that the nonlinearity « interacts » only with that part of the spectrum of A, which lies in (α, β). Thus the behaviour of φ on the reducing subspaces X and Y of A should be roughly the same as the behavior of the quadratic form $\langle Au, u\rangle$ on these subspaces. In fact, it can be shown that φ is strictly convex on X and strictly concave on Y. This fact can then be used to reduce the infinite-dimensional variational problem to a finite-dimensional one, which, roughly speaking, involves only $\sigma(A) \cap (\alpha, \beta)$.

To exhibit quite clearly the saddle point structure of the functional φ, we introduce now a new functional f, which is defined on all of H, and whose critical points are in a one-to-one correspondence with the solutions of the equation $Au = F(u)$.

For this purpose we let

$$\Phi_\alpha(u) := \Phi(u) - \frac{\alpha}{2}\|u\|^2 \qquad \forall u \in H ,$$

and we define

$$f \in C^1(X \times Y \times Z, \boldsymbol{R})$$

by

$$f(x, y, z) := \tfrac{1}{2}\big(\|x\|^2 - \|y\|^2 - \|z\|^2\big) + \Phi_\alpha(Rx + Sy + Tz) .$$

Then it is not difficult to verify that (x, y, z) is a critical point of f iff $Rx + Sy + Tz$ is a solution of $Au = F(u)$. Moreover, letting

$$\beta_+ := \min\{\sigma(A) \cap (\beta, \infty)\}$$

if $\sigma(A) \cap (\beta, \infty) \neq \emptyset$, fixing $\beta_+ > \beta$ arbitrarily otherwise, and setting

$$\mu := (\beta_+ - \beta)(\beta_+ - \alpha)^{-1} > 0 ,$$

it is easily verified that the maps

$$x \to D_1 f(x, y, z) - \mu x$$

and

$$y \mapsto - D_2 f(x, y, z) - \mu y$$

are monotone for every $(y, z) \in Y \times Z$ and $(x, z) \in X \times Z$, respectively.

Thus, due to an observation of Rockafellar [37], it follows that, for every $z \in Z$, the map

$$M_z\colon X \times Y \to X \times Y ,$$

defined by

$$M_z(x, y) := \big(D_1 f(x, y, z), - D_2 f(x, y, z)\big) ,$$

is μ-monotone, that is,

$$\langle M_z(x_1, y_1) - M_z(x_2, y_2), (x_1, y_1) - (x_2, y_2)\rangle \geq \mu \|(x_1, y_1) - (x_2, y_2)\|^2 \tag{2.1}$$

for all (x_1, y_1), $(x_2, y_2) \in X \times Y$ and $z \in Z$.

Let

$$F_\alpha(u) := F(u) - \alpha u \qquad \forall u \in H,$$

and observe that

$$\begin{cases} D_1 f(x, y, z) = \; x + RF_\alpha(Rx + Sy + Tz) \\ D_2 f(x, y, z) = - y + SF_\alpha(Rx + Sy + Tz) \\ D_3 f(x, y, z) = - z + TF_\alpha(Rx + Sy + Tz) \end{cases} \tag{2.2}$$

for all $(x, y, z) \in X \times Y \times Z$. Thus M_z is continuous for every $z \in Z$, and the basic existence theorem for monotone operators (e.g. [23]) implies that the equation

$$M_z(x, y) = 0$$

has a unique solution $(x(z), y(z))$ for every $z \in Z$. But this means precisely that $(x(z), y(z))$ is the unique saddle point of the functional

$$f(\cdot,\cdot, z): X \times Y \to \boldsymbol{R}$$

(e.g. [23]).

Let $z_0 \in Z$ be fixed. Then (2.1) and the definition of $(x(z), y(z))$ imply that, for every $z \in Z$,

$$\langle M_z(x(z_0), y(z_0)), (x(z_0), y(z_0)) - (x(z), y(z))\rangle \geqslant \mu \| (x(z_0), y(z_0)) - (x(z), y(z)) \|^2 .$$

Thus

$$\| (x(z), y(z)) - (x(z_0), y(z_0)) \| \leqslant \mu^{-1} \| M_z(x(z_0), y(z_0)) \| \tag{2.3}$$

for every $z \in Z$. Since, by (2.2), the map

$$Z \to X \times Y, \qquad z \mapsto M_z(x(z_0), y(z_0))$$

is obviously continuous, it follows that

$$(x(\cdot), y(\cdot)) \in C(Z, X \times Y),$$

that is, the saddle point $(x(z), y(z))$ depends continuously on $z \in Z$.

In fact, much more is true. Namely, due to an observation of Brézis and Nirenberg [14, Proposition A.5], hypothesis (F) implies the global Lipschitz continuity of F. More precisely,

$$\|F(u) - F(v)\| \leqslant [(\beta - \alpha) + |\alpha|] \|u - v\| \qquad \forall u, v \in H .$$

This implies easily the existence of a constant $\gamma \geqslant 0$ such that

$$\| M_z(x(z_0), y(z_0)) \| \leqslant \gamma \|z - z_0\| \qquad \forall z, z_0 \in Z .$$

Consequently, (2.3) shows that

$$(x(\cdot), y(\cdot)): Z \to X \times Y$$

is globally Lipschitz continuous.

Now we define $g: Z \to \boldsymbol{R}$ by

$$g(z) := f(x(z), y(z), z) .$$

Then it can be shown [2] that $g \in C^1(Z, \boldsymbol{R})$ and that

$$g'(z) = D_3 f\big(x(z), y(z), z\big) .$$

(Observe that, in general, the map $\big(x(\cdot), y(\cdot)\big)$ is not differentiable, so that the chain rule cannot be applied.) Thus, by using the representation (2.3) of $D_3 f$ and the global Lipschitz continuity of F and $\big(x(\cdot), y(\cdot)\big)$, it follows that g' is even globally Lipschitz continuous.

In the following proposition we collect the basic facts derived above.

(2.1) Proposition. *There exists a globally Lipschitz continuous map*

$$\big(x(\cdot), y(\cdot)\big) \colon Z \to X \times Y ,$$

such that $\big(x(z), y(z)\big)$ is the unique saddle point of $f(\cdot, \cdot, z) \colon X \times Y \to \boldsymbol{R}$ for every $z \in Z$. Thus the point $\big(x(z), y(z)\big) \in X \times Y$ is characterized by the « saddle point inequalities »

$$f\big(x(z), y, z\big) \leqslant g(z) \leqslant f\big(x, y(z), z\big) \qquad \forall (x, y, z) \in X \times Y \times Z , \tag{2.4}$$

where

$$g(z) := f\big(x(z), y(z), z\big) \qquad \forall z \in Z ,$$

as well as by the fact that $\big(x(z), y(z)\big)$ is, for every $z \in Z$, the unique point $(x, y) \in X \times Y$ solving the system

$$0 = \quad x + RF_\alpha(Rx + Sy + Tz) \tag{2.5}$$

$$0 = - y + SF_\alpha(Rx + Sy + Tz) . \tag{2.6}$$

Moreover, g has a globally Lipschitz continuous derivative $g' \colon Z \to Z$, which is given by

$$g'(z) = - z + TF_\alpha\big(Rx(z) + Sy(z) + Tz\big) \qquad \forall z \in Z . \tag{2.7}$$

Finally, z is a critical point of g iff $Rx(z) + Sy(z) + Tz$ is a solution of $Au = F(u)$.

Observe that, by the above proposition, the problem of finding solutions of the equation $Au = F(u)$ is equivalent to the problem of finding critical points of the functional g. This reduction to a finite-dimensional case has been introduced in [2]. Proposition (2.1) is essentially a restatement of some of the results of [2], and we refer to that paper for further details.

It is also worthwhile to notice that, up to now, the finite-dimensionality of Z has not been used.

(2.2) REMARK. Suppose that Σ is a topological space and

$$\Sigma \times H \to H\,, \qquad (\sigma, u) \mapsto F(\sigma, u)$$

is a continuous map such that, for every $\sigma \in \Sigma$, the function $F(\sigma, \cdot)\colon H \to H$ satisfies (F) (with α and β independent of σ). Then, denoting by $\Phi(\sigma, \cdot)$ the potential of $F(\sigma, \cdot)$ and defining $f(\sigma, \cdot)\colon X \times Y \times Z \to \boldsymbol{R}$ by

$$f(\sigma, x, y, z) := \tfrac{1}{2}\big(\|x\|^2 - \|y\|^2 - \|z\|^2\big) + \Phi_\alpha(\sigma, Rx + Sy + Tz)\,,$$

an inspection of the above proof shows that, for every $(\sigma, z) \in \Sigma \times Z$, there exists a unique saddle point $\big(x(\sigma, z), y(\sigma, z)\big)$ of $f(\sigma, \cdot, \cdot, z)\colon X \times Y \to \boldsymbol{R}$, and that $\big(x(\cdot, \cdot), y(\cdot, \cdot)\big) \in C(\Sigma \times Z, X \times Y)$. Moreover,

$$\big(x(\sigma, \cdot), y(\sigma, \cdot)\big)\colon Z \to X \times Y$$

is globally Lipschitz continuous, uniformly with respect to $\sigma \in \Sigma$.

Let

$$g(\sigma, z) := f\big(\sigma, x(\sigma, z), y(\sigma, z), z\big) \qquad \forall (\sigma, z) \in \Sigma \times Z\,.$$

Then $g(\sigma, \cdot) \in C^1(Z, \boldsymbol{R})$ for every $\sigma \in \Sigma$, and $D_2 g(\sigma, \cdot)\colon Z \to Z$ is globally Lipschitz continuous, uniformly with respect to $\sigma \in \Sigma$. Finally, z is a critical point of $g(\sigma, \cdot)$ iff $Rx(\sigma, z) + Sy(\sigma, z) + Tz$ is a solution of the equation $Au = F(\sigma, u)$, $\sigma \in \Sigma$. $\square$

As an immediate corollary to Proposition (2.1) we note the following *existence and uniqueness result*, already given in [2].

(2.3) THEOREM. *If $\sigma(A) \cap (\alpha, \beta) = \emptyset$, then the equation $Au = F(u)$ has exactly one solution.*

PROOF. It suffices to observe that, in this case, $Z = \{0\}$. $\square$

Since, by the above theorem, the case $\sigma(A) \cap (\alpha, \beta) = \emptyset$ has been completely solved, *we assume henceforth that* $\sigma(A) \cap (\alpha, \beta) \neq \emptyset$.

3. – The reduced problem.

Observe that

$$T^{-1} := (T|Z)^{-1} \in \mathfrak{L}_s(Z) := \{B \in \mathfrak{L}(Z) \,|\, B = B^*\}\,,$$

and let

$$a := -\, g \circ T^{-1} \in C^1(Z, \mathbf{R}) \tag{3.1}$$

and

$$u(z) := Rx(T^{-1}z) + Sy(T^{-1}z) + z \qquad \forall z \in Z\,. \tag{3.2}$$

Then, by Proposition (2.1),

$$u(\cdot)\colon Z \to H$$

is globally Lipschitz continuous, and

$$u(z) \in \operatorname{dom}(A) \qquad \forall z \in Z \tag{3.3}$$

(cf. (2.5) and (2.6)). Moreover, a has a globally Lipschitz continuous derivative, given by

$$a' = -\, T^{-1} \circ g' \circ T^{-1}\,. \tag{3.4}$$

Thus Proposition (2.1) implies that

z is a critical point of a iff $u(z)$ is a solution of $Au = F(u)$.

Hence we have reduced the original problem of finding solutions to the equation $Au = F(u)$ to the equivalent problem of finding critical points of the functional $a \in C^1(Z, \mathbf{R})$.

In the following lemma we collect some properties of a, which will be useful for finding critical points.

(3.1) LEMMA. *For every $z \in Z$,*

$$a(z) = \tfrac{1}{2}\langle Au(z), u(z)\rangle - \Phi\big(u(z)\big) \tag{3.5}$$

and

$$a'(z) = Az - PF\big(u(z)\big) = Au(z) - F\big(u(z)\big)\,. \tag{3.6}$$

PROOF. Let

$$R^{-1} := (R|X)^{-1}\colon \operatorname{im}(R) \subset X \to X$$

and

$$S^{-1} := (S|Y)^{-1}\colon \operatorname{im}(S) \subset Y \to Y\,.$$

Then, by (1.2),

$$\|x\|^2 = \|R^{-1}Rx\|^2 = \langle R^{-2}Rx, Rx\rangle = -\,\langle (A - \alpha)Rx, Rx\rangle$$

for all $x \in X$ such that $Rx \in \operatorname{dom}(A)$. Similarly,

$$\|y\|^2 = \langle (A - \alpha) Sy, Sy \rangle$$

for all $y \in Y$ with $Sy \in \operatorname{dom}(A)$, and

$$\|z\|^2 = \langle (A - \alpha) Tz, Tz \rangle \qquad \forall z \in Z .$$

Consequently,

$$f(x, y, z) = -\tfrac{1}{2} \langle A(Rx + Sy + Tz), Rx + Sy + Tz \rangle + \Phi(Rx + Sy + Tz) \tag{3.7}$$

for all $(x, y, z) \in X \times Y \times Z$ such that $Rx, Sy \in \operatorname{dom}(A)$. Now the asserted representation of $a(z)$ follows from the definitions of a and $u(\cdot)$, and from (3.3).

The equation

$$a'(z) = Az - PF\big(u(z)\big) , \qquad z \in Z , \tag{3.8}$$

follows easily from (3.4), (2.7) and (1.4). By substituting $\big(x(z), y(z)\big)$ into the equations (2.5) and (2.6), applying R to (2.5) and S to (2.6), and by using (1.2) and (1.3), we find that $\big(Rx(T^{-1}z), Sy(T^{-1}z)\big) \in X \times Y$ is characterized by the equations

$$0 = ARx(T^{-1}z) - P_{-}F\big(u(z)\big) \tag{3.9}$$

and

$$0 = ASy(T^{-1}z) - P_{+}\big(u(z)\big) \tag{3.10}$$

for all $z \in Z$. Thus, the last part of the assertion follows by adding the equations (3.9) and (3.10) to (3.8). $\square$

We include here an invariance property of the functional a, which we will use in a later paper discussing multiplicity results.

(3.2) PROPOSITION. *Let $U \in \mathfrak{L}(H)$ be a unitary operator, which commutes with A and F, that is, $AU \supset UA$ and $F \circ U = U \circ F$, respectively. Then $a \circ U = a$.*

PROOF. Since U commutes with A, the subspaces X, Y, and Z reduce U, and U commutes with R, S, and T. Furthermore, since U commutes with F and preserves inner products

$$\Phi(Uu) = \int_0^1 \langle F(tUu), Uu \rangle \, dt = \int_0^1 \langle UF(tu), Uu \rangle \, dt = \int_0^1 \langle F(tu), u \rangle \, dt = \Phi(u)$$

for all $u \in H$. Thus, by the definition of f,

$$f(Ux, Uy, Uz) = f(x, y, z) \qquad \forall (x, y, z) \in X \times Y \times Z . \tag{3.11}$$

Since the inequalities (2.4) characterize the saddle point $(x(z), y(z))$, it follows that

$$f\big(x(Uz), y, Uz\big) \leqslant f\big(x(Uz), y(Uz), Uz\big) \leqslant f\big(x, y(Uz), Uz\big)$$

for all $(x, y) \in X \times Y$. Hence, U being unitary,

$$\begin{aligned} f\big(UU^{-1}x(Uz), Uy, Uz\big) \leqslant f\big(UU^{-1}x(Uz), UU^{-1}y(Uz), Uz\big) \leqslant \\ \leqslant f\big(Ux, UU^{-1}y(Uz), Uz\big) \end{aligned}$$

for all $(x, y) \in X \times Y$. Thus, by (3.11),

$$f\big(U^{-1}x(Uz), y, z\big) \leqslant f\big(U^{-1}x(Uz), U^{-1}y(Uz), z\big) \leqslant f\big(x, U^{-1}y(Uz), z\big)$$

for all $(x, y) \in X \times Y$, which, by the uniqueness of the saddle point, implies

$$\big(U^{-1}x(Uz), U^{-1}y(Uz)\big) = \big(x(z), y(z)\big) \qquad \forall z \in Z .$$

Consequently,

$$\big(x(Uz), y(Uz)\big) = \big(Ux(z), Uy(z)\big) \qquad \forall z \in Z ,$$

and

$$\begin{aligned} g(Uz) = f\big(x(Uz), y(Uz), Uz\big) = f\big(Ux(z), Uy(z), Uz\big) = \\ = f\big(x(z), y(z), z\big) = g(z) \end{aligned}$$

for all $z \in Z$. Now, since U commutes with T^{-1}, the assertion follows from the definition of a. $\square$

4. – Higher regularity.

For an analysis of the critical points of $a \in C^1(Z, \mathbf{R})$ it is desirable to know that $a \in C^2(Z, \mathbf{R})$. This can easily be achieved by assuming that $F \in C^1(H, H)$. However, in all of our applications H will be an L_2-space and F a substitution operator. But then it is well known (e.g. [5]), that, in general, $F \in C^1(H, H)$ iff F is an affine map. Thus it is not reasonable to assume that $F \in C^1(H, H)$. However we may well assume that F has a symmetric *Gateaux derivative*

$$F' : H \to \mathfrak{L}_s(H),$$

which, of course, will not be continuous on H, in general.

By differentiating formally the middle term in (3.6), we find that the resulting expression involves F' only at points of the form $u(z)$. In general, these points will belong to a proper subspace E of H, carrying a stronger topology than H, such that F' may well be continuous *on* E. Since this is in fact the case in our applications, we shall analyse this situation more thoroughly in this section.

First we prove the following characterization of $u(z) - z$.

(4.1) PROPOSITION. *For each* $z \in Z$, *the equation*

$$Av = (I_H - P)F(v + z) \tag{4.1}$$

has a unique solution $v(z)$, *and*

$$v(z) = u(z) - z\,. \tag{4.2}$$

PROOF. Let

$$v(z) := u(z) - z = Rx(T^{-1}z) + Sy(T^{-1}z) \qquad \forall z \in Z\,, \tag{4.3}$$

and recall that $\big(Rx(T^{-1}z), Sy(T^{-1}z)\big) \in X \times Y$ is characterized by the equations (3.9) and (3.10). From this fact the assertion follows easily. □

We introduce now the following *regularity hypothesis* (R), where we write $V \hookrightarrow W$ if V and W are Banach spaces and V is continuously imbedded in W, that is, V is a vector subspace of W and the natural injection is continuous.

$$(R)\quad\left\{\begin{array}{ll} \text{(i)} & F \in C(H, H) \textit{ possesses a symmetric Gâteaux derivative } F'. \\ \text{(ii)} & E \textit{ is a real Banach space such that} \\ & \qquad Z \hookrightarrow E \hookrightarrow H \quad \textit{and} \quad F'|E \in C\big(E, \mathfrak{L}_s(H)\big)\,. \\ \text{(iii)} & P_\pm v(\cdot) \in C(Z, E)\,. \end{array}\right.$$

Clearly, Z being finite-dimensional, $Z \hookrightarrow E$ iff $Z \subset E$. Moreover, $Z \hookrightarrow E$ and (iii) imply $u(\cdot) \in C(Z, E)$.

It should be observed that (R) is satisfied (with $E = H$), if $F \in C^1(H, H)$.

Moreover in certain applications it may be possible to avoid the relatively complicated looking condition (R) by a judicious choice of the underlying function spaces. However, we are interested in a general abstract theory which is applicable to a wide variety of problems without redoing the same arguments over and over again. For this reason we have to introduce condition (R).

(4.2) LEMMA. *If* (R) *is satisfied, then*

$$\big(x(\cdot), y(\cdot)\big) \in C^1(Z, X \times Y) .$$

PROOF. Let $\xi := (x, y) \in X \times Y =: \Xi$, and set

$$M(\xi, z) := M_z(x, y) = \big(D_1 f(x, y, z), - D_2 f(x, y, z)\big) .$$

Then it is an easy consequence of the representations (2.2) and the linearity of the operators R, S, and T, that

$$M : \Xi \times Z \to \Xi$$

has a Gâteaux derivative

$$M' = (D_1 M, D_2 M) : \Xi \times Z \to \mathfrak{L}(\Xi \times Z, \Xi) , \tag{4.4}$$

which, using matrix notation, is explicitely given by

$$D_1 M(\xi, z) = \begin{pmatrix} I_X + RF'_\alpha(p)R|X & RF'_\alpha(p)S|Y \\ SF'_\alpha(p)R|X & -I_Y + SF'_\alpha(p)S|Y \end{pmatrix} \tag{4.5}$$

and

$$D_2 M(\xi, z) = \big(RF'_\alpha(p)\,T|Z,\ SF'_\alpha(p)\,T|Z\big) , \tag{4.6}$$

where $p := Rx + Sy + Tz$.

Since $M(\cdot, z)$ is μ-monotone for every $z \in Z$ (cf. (2.1)), it follows easily that

$$\langle D_1 M(\xi, z)\eta, \eta\rangle \geqslant \mu\|\eta\|^2 \quad \forall \xi,\ \eta \in \Xi,\ \forall z \in Z .$$

Thus, μ being positive, one deduces easily that $D_1 M(\xi, z)$ has a bounded inverse

$$[D_1 M(\xi, z)]^{-1} \in \mathfrak{L}(\Xi) \quad \forall (\xi, z) \in \Xi \times Z .$$

Let $\xi(z) := \big(x(z), y(z)\big)$ denote the saddle point, which (cf. Section 2) is characterized by the equation

$$M\big(\xi(z), z\big) = 0 \quad \forall z \in Z . \tag{4.7}$$

Moreover let

$$B(z) := [D_1 M(\xi(z), z)]^{-1}$$

and observe that, due to formula (4.5), $B(z)$ depends only through $u(Tz)$ on $z \in Z$. Thus condition (R) implies that

$$B(\cdot) \in C(Z, \mathfrak{L}(\Xi)) . \tag{4.8}$$

Let $z \in Z$ be fixed, and let ε with

$$0 < \varepsilon < \tfrac{1}{2}\|B(z)\| \tag{4.9}$$

be arbitrary. Moreover, let

$$\eta(h) := \xi(z + h) - \xi(z) \qquad \forall h \in Z .$$

Then (4.7) and the mean value theorem, imply

$$\begin{aligned}
&\|D_1 M(\xi(z), z)\eta(h) + D_2 M(\xi(z), z) h\| \\
&\quad = \|M(\xi(z) + \eta(h), z + h) - M(\xi(z), z) - D_1 M(\xi(z), z)\eta(h) - D_2 M(\xi(z), z) h\| \\
&\quad \leqslant \sup_{0 \leqslant t \leqslant 1} \|M'(\xi(z) + t\eta(h), z + th) - M'(\xi(z), z)\| (\|\eta(h)\|^2 + \|h\|^2)^{\frac{1}{2}}
\end{aligned}$$

for all $h \in Z$. Since, by (4.5) and (4.6), $M'(\xi(z), z)$ depends only through $u(Tz)$ on z, the regularity hypothesis (R) implies easily that the map

$$Z \to \mathfrak{L}(\Xi \times Z, \Xi) , \qquad h \mapsto M'(\xi(z) + t\eta(h), z + th)$$

is continuous, uniformly with respect to $t \in [0, 1]$. Consequently, since $\eta(0) = 0$, there exists a number $\delta > 0$ such that

$$\|D_1 M(\xi(z), z)\eta(h) + D_2 M(\xi(z), z) h\| \leqslant \varepsilon(\|\eta(h)\| + \|h\|)$$

as soon as $h \in Z$ satisfies $\|h\| \leqslant \delta$. Thus

$$\|\eta(h) + B(z) D_2 M(\xi(z), z)\, h\| \leqslant \varepsilon \|B(z)\| (\|\eta(h)\| + \|h\|) \tag{4.10}$$

for $\|h\| \leqslant \delta$. Now, if we put

$$\gamma := 2\|B(z) D_2 M(\xi(z), z)\| + 1 ,$$

we deduce from (4.9) and (4.10) that

$$\|\eta(h)\| - \frac{\gamma - 1}{2}\|h\| \leqslant (\|\eta(h) + \|h\|)/2 ,$$

that is, $\|\eta(h)\| \leqslant \gamma\|h\|$, and therefore, that

$$\|\eta(h) + B(z) D_2 M(\xi(z), z)\, h\| \leqslant \varepsilon(\gamma + 1)\|B(z)\|\,\|h\|$$

for all $h \in Z$ satisfying $\|h\| \leqslant \delta$. By the definition of $\eta(h)$, this proves that $\xi(\cdot)$ is differentiable at z and that

$$\xi'(z) = -[D_1 M(\xi(z), z)]^{-1} D_2 M(\xi(z), z) \qquad \forall z \in Z . \tag{4.11}$$

Finally, since (R) and (4.6) imply

$$D_2 M(\xi(\cdot), \cdot) \in C(Z, \mathcal{L}(Z, E)) ,$$

it follows from (4.8) and (4.11) that $\xi'(\cdot) \in C(Z, \Xi)$. □

(4.3) Corollary. *If* (R) *is satisfied, then* $u(\cdot) \in C^1(Z, H)$. *Moreover,* $\mathrm{im}\,(u'(z)) \subset \mathrm{dom}\,(A)$ *and*

$$A v'(z) = (I - P) F'(u(z))\, u'(z) \qquad \forall z \in Z . \tag{4.12}$$

More precisely, $v'(z) = u'(z) - I_Z$ *is, for each* $z \in Z$, *the unique element* B *in* $\mathcal{L}(Z, X \oplus Y)$ *satisfying* $\mathrm{im}\,(B) \subset \mathrm{dom}\,(A)$ *and the equation*

$$AB = (I - P) F'(u(z))(B + I_Z) .$$

Proof. It is an obvious consequence of Lemma (4.2) that $u(\cdot) \in C^1(Z, H)$ and that (cf. (3.2))

$$u'(z) = Rx'(T^{-1}z)\, T^{-1} + Sy'(T^{-1}z)\, T^{-1} + I_Z \qquad \forall z \in Z .$$

Moreover, (4.5), (4.6), and (4.11) imply that $(x'(z), y'(z))$ is the unique element in $\mathcal{L}(Z, X \times Y)$ satisfying

$$x'(z) + RF'_\alpha(u(Tz))(Rx'(z) + Sy'(z) + T|Z) = 0$$

and

$$-y'(z) + SF'_\alpha(u(Tz))(Rx'(z) + Sy'(z) + T|Z) = 0$$

for every $z \in Z$. Hence, by applying R to the first and S to the second of these equations, and by using (1.1) and (1.4), it follows that

$$\operatorname{im}\big(Rx'(z)\big) \cup \operatorname{im}\big(Sy'(z)\big) \subset \operatorname{dom}(A),$$

and that

$$A\big(Rx'(z) + Sy'(z)\big) = (I - P)F'\big(u(Tz)\big)\big(Rx'(z) + Sy'(z) + T|Z\big),$$

that is,

$$Av'(Tz)T|Z = (I - P)F'\big(u(Tz)\big)\big(u'(Tz)T|Z\big)$$

for all $z \in Z$. Now the assertion follows easily. $\square$

Observe that the following lemma is not just a consequence of the chain rule, since, in general, there is no chain rule for Gâteaux differentiable maps.

(4.4) Lemma. *If* (R) *is satisfied, then*

$$F \circ u(\cdot) \in C^1(Z, H)$$

and

$$(F \circ u)'(z) = F'\big(u(z)\big)u'(z) \in \mathcal{L}(Z, H) \qquad \forall z \in Z.$$

Proof. By Corollary (4.3), $F'\big(u(z)\big)u'(z)$ is well defined and belongs to $\mathcal{L}(Z, H)$. Let $z \in Z$ be fixed, and let

$$w(h) := u(z + h) - u(z) \qquad \forall h \in Z.$$

Then, by the mean value theorem,

$$\begin{aligned}&\|F\big(u(z+h)\big) - F\big(u(z)\big) - F'\big(u(z)\big)u'(z)h\| \\ &\qquad \leqslant \sup_{0 \leqslant t \leqslant 1} \|F'\big(u(z) + tw(h)\big) - F'\big(u(z)\big)\|\,\|w(h)\| + \|F'\big(u(z)\big)\|\,\|w(h) - u'(z)h\|\end{aligned}$$

for all $h \in Z$. Thus, since, by (R), $w(\cdot) \in C(Z, E)$ and $F'|E \in C\big(E, \mathcal{L}(H)\big)$, we find that $F \circ u(\cdot) \in C^1(Z, H)$ and that $(F \circ u)'(z) = F'\big(u(z)\big)u'(z)$. $\square$

After these preparations we can now prove the following fundamental regularity result.

(4.5) Proposition. *If* (R) *is satisfied, then* $a \in C^2(Z, \mathbf{R})$, *and*

$$a''(z) = A|Z - PF'\big(u(z)\big)u'(z) = \big[A - F'\big(u(z)\big)\big]u'(z)$$

for all $z \in Z$.

PROOF. It follows immediately from $A|Z \in \mathfrak{L}(Z)$ and Lemmas (3.1) and (4.4), that $a \in C^2(Z, \boldsymbol{R})$ and $a''(z) = A|Z - PF'(u(z))\,u'(z)$. Finally, (4.12) implies now the second representation of $a''(z)$. $\square$

(4.6) REMARK. Let $\mathfrak{L}_s^k(H)$ be the Banach space of all k-linear symmetric continuous operators from the k-fold product of H into H. Denote by $(R)_k$, $k \geqslant 2$, the following regularity hypothesis:

$$(R)_k \quad \begin{cases} \text{(i)} & F \in C(H, H) \text{ possesses Gateaux derivatives} \\ & \qquad F^{(i)}\colon H \to \mathfrak{L}_s^i(H) \qquad \text{for } 1 \leqslant i \leqslant k\,. \\ \text{(ii)} & E \text{ is a real Banach space such that} \\ & \qquad Z \hookrightarrow E \hookrightarrow H \quad \text{and} \quad F^{(i)}|E \in C\big(E, \mathfrak{L}_s^i(H)\big) \quad \text{for } 1 \leqslant i \leqslant k\,. \\ \text{(iii)} & P_{\pm} v(\cdot) \in C(Z, E)\,. \end{cases}$$

Then it follows from the above proofs by means of easy induction arguments that $\big(x(\cdot), y(\cdot)\big) \in C^k(Z, X \times Y)$, that $u(\cdot) \in C^k(Z, H)$, that $F \circ u(\cdot) \in C^k(Z, H)$, and that $a \in C^{k+1}(Z, \boldsymbol{R})$. $\square$

5. - Asymptotic linearity.

Consider the following hypothesis concerning the asymptotic behavior of F near infinity.

$$(F_\infty) \quad \begin{cases} \textit{There exists } B_\infty \in \mathfrak{L}_s(H) \textit{ such that} \\ \qquad \sigma(B_\infty) \subset [\alpha, \beta] \quad \textit{and} \quad 0 \notin \sigma(A - B_\infty)\,, \\ \textit{and there are constants} \\ \qquad \delta_\infty \geqslant 0 \quad \textit{and} \quad 0 < \gamma_\infty < \min\{|\lambda| \,|\, \lambda \in \sigma(A - B_\infty)\} \\ \textit{such that} \\ \qquad \|F(u) - B_\infty u\| \leqslant \gamma_\infty \|u\| + \delta_\infty \quad \forall u \in H\,. \end{cases}$$

Clearly, the condition that $0 \notin \sigma(A - B_\infty)$ is kind of a « nonresonance » condition at infinity. Moreover, since B_∞ is bounded and symmetric, $A - B_\infty$

is self-adjoint (e.g. [27, Theorem V.4.3]). Hence

$$\min\{|\lambda|\,|\lambda\in\sigma(A-B_\infty)\}=\|(A-B_\infty)^{-1}\|^{-1}, \tag{5.1}$$

and the restriction

$$\gamma_\infty<\min\{|\lambda|\,|\lambda\in\sigma(A-B_\infty)\}$$

is equivalent to the condition

$$\gamma_\infty\|(A-B_\infty)^{-1}\|<1\,. \tag{5.2}$$

Recall that F is said to be *asymptotically linear*, if there exists an operator $F'(\infty)\in\mathcal{L}(H)$ such that

$$\lim_{\|u\|\to\infty}\frac{\|F(u)-F'(\infty)u\|}{\|u\|}=0\,.$$

Then $F'(\infty)$ is uniquely determined and called the *derivative* of F *at infinity*.

(5.1) LEMMA. *Suppose that F is asymptotically linear, $F'(\infty)$ is symmetric, and $0\notin\sigma(A-F'(\infty))$. Then (F_∞) is satisfied, and $\gamma_\infty>0$ can be chosen arbitrarily small.*

PROOF. We have to prove the assertion that $\sigma(F'(\infty))\subset[\alpha,\beta]$.

Observe that Hypothesis (F) implies

$$\alpha\|u\|^2\leqslant t^{-1}\langle F(tu)-F(0),u\rangle\leqslant\beta\|u\|^2$$

for all $u\in H$ and $t>0$. Hence

$$\alpha\|u\|^2\leqslant\langle t^{-1}[F(tu)-F'(\infty)tu],u\rangle-t^{-1}\langle F(0),u\rangle+\langle F'(\infty)u,u\rangle\leqslant\beta\|u\|^2$$

for all $u\in H$ and $t>0$. Thus, letting $t\to\infty$,

$$\alpha\|u\|^2\leqslant\langle F'(\infty)u,u\rangle\leqslant\beta\|u\|^2\qquad\forall u\in H\,,$$

which, by the symmetry of $F'(\infty)$, implies $\sigma(F'(\infty))\subset[\alpha,\beta]$. $\square$

In the remainder of this section we deduce some simple, but important consequences of hypothesis (F_∞).

(5.2) LEMMA. *If (F_∞) is satisfied, then*

$$\|a'(z)\|\geqslant\nu\|z\|-\delta_\infty\qquad\forall z\in Z\,,$$

where $\nu:=\|(A-B_\infty)^{-1}\|^{-1}-\gamma_\infty>0$.

PROOF. It follows from (5.2), that $\nu > 0$. Since, by Lemma (3.1),

$$a'(z) = Au(z) - F(u(z)) = (A - B_\infty)u(z) - (F - B_\infty)(u(z)),$$

condition (F_∞) implies

$$\begin{aligned}\|a'(z)\| &\geqslant \|(A - B_\infty)u(z)\| - \|(F - B_\infty)(u(z))\| \\ &\geqslant \|(A - B_\infty)^{-1}\|^{-1}\|u(z)\| - \gamma_\infty\|u(z)\| - \delta_\infty = \nu\|u(z)\| - \delta_\infty\end{aligned}$$

for all $z \in Z$. Hence the assertion follows, since, by orthogonality and (3.2), $\|u(z)\| \geqslant \|z\|$. □

(5.3) COROLLARY. *If (F_∞) is satisfied, then a satisfies the Palais-Smale condition*, that is, *every sequence (z_k) in Z, for which $(a(z_k))$ is bounded and $a'(z_k) \to 0$, possesses a convergent subsequence.*

PROOF. The assertion is an immediate consequence of Lemma (5.2) and the finite-dimensionality of Z. □

6. - Estimates near infinity.

In this section, using hypothesis (F_∞), we give qualitative estimates for the functional a near infinity.

(6.1) LEMMA. *Let (F_∞) be satisfied. Then, for every $\gamma > \gamma_\infty$, there exists a constant $\delta \geqslant 0$ such that*

$$a(z) \leqslant \tfrac{1}{2}\langle (A - B_\infty + \gamma)(P_- v(z) + z), P_- v(z) + z\rangle + \delta$$

and

$$a(z) \geqslant \tfrac{1}{2}\langle (A - B_\infty - \gamma)(P_+ v(z) + z), P_+ v(z) + z\rangle - \delta$$

for all $z \in Z$.

PROOF. By the mean value theorem and (F_∞),

$$\left|\Phi(u) - \frac{1}{2}\langle B_\infty u, u\rangle\right| \leqslant \int_0^1 |\langle F(tu) - B_\infty(tu), u\rangle|\, dt \leqslant \frac{\gamma_\infty}{2}\|u\|^2 + \delta_\infty\|u\|$$

for all $u \in H$. Since

$$\delta_\infty\|u\| \leqslant \frac{\gamma - \gamma_\infty}{2}\|u\|^2 + \frac{\delta_\infty^2}{2(\gamma - \gamma_\infty)} \qquad \forall u \in H,$$

it follows that

$$(6.1)\qquad \left|\Phi(u)-\frac{1}{2}\langle B_\infty u, u\rangle\right|\leqslant\frac{\gamma}{2}\|u\|^2+\delta \qquad \forall u\in H\,,$$

where $\delta := \delta_\infty^2/2(\gamma-\gamma_\infty)$.

Since, by the saddle point inequality of Proposition (2.1),

$$g(z)\geqslant f\big(x(z), 0, z\big) \qquad \forall z\in Z\,,$$

and since $Rx(z)\in \operatorname{dom}(A)$ (cf. (2.5)), we deduce from (3.7) that

$$g(z)\geqslant -\tfrac{1}{2}\langle A\big(Rx(z)+Tz\big), Rx(z)+Tz\rangle+\Phi\big(Rx(z)+Tz\big)$$

for all $z\in Z$. Now the first estimate of the assertion follows from (6.1), the definition of a, and the fact that $P_-v(z)=Rx(T^{-1}z)$. A similar argument based on the second half of the saddle point inequality implies the second estimate of the assertion. □

In the following we use the standard order relation between self-adjoint operators, and $B\gg 0$ means that B is positive definite. In this connection we write usually γ instead of γI_H, provided no confusion seems possible. Moreover, we let

$$\alpha_- := \max\{\sigma(A)\cap(-\infty, \alpha)\}$$

if $\sigma(A)\cap(-\infty,\alpha)\neq\emptyset$, and we fix $\alpha_-<\alpha$ arbitrarily otherwise. Similarly, we let

$$\beta_+ := \min\{\sigma(A)\cap(\beta, \infty)\}$$

if this set is nonempty, and $\beta_+>\beta$ is arbitrary otherwise.

(6.2) PROPOSITION. *Let* (F_∞) *be satisfied.*

(*a*) *Suppose that, for some* $\bar\gamma>\gamma_\infty$, *there exists an operator* $C_\infty^-\in\mathfrak{L}_s(H)$, *which commutes with* P *and* P_-, *such that*

$$\alpha_-+\bar\gamma\leqslant C_\infty^-\leqslant B_\infty\,.$$

Then there exists a number $\delta>0$, *such that*

$$a(z)\leqslant\tfrac{1}{2}\langle(A-C_\infty^-+\bar\gamma)z, z\rangle+\delta \qquad \forall z\in Z\,.$$

(*b*) *Suppose that, for some* $\hat{\gamma} > \gamma_\infty$, *there exists an operator* $C_\infty^+ \in \mathcal{L}_s(H)$, *which commutes with* P *and* P_+, *such that*

$$B_\infty \leqslant C_\infty^+ \leqslant \beta_+ - \hat{\gamma}\,.$$

Then there exists a number $\delta > 0$, *such that*

$$a(z) \geqslant \tfrac{1}{2}\langle (A - C_\infty^+ - \hat{\gamma})z, z\rangle - \delta \qquad \forall z \in Z\,.$$

PROOF. (*a*) Let $C := C_\infty^- - \bar{\gamma}$ and observe that

$$A - B_\infty + \bar{\gamma} \leqslant A - C \qquad \text{and} \qquad (A - C)|[X \cap \operatorname{dom}(A)] \leqslant 0\,.$$

Hence, by the commutativity of C with P and P_-,

$$\begin{aligned}
&\langle (A - B_\infty + \bar{\gamma})\big(P_-v(z) + z\big), P_-v(z) + z\rangle \leqslant \\
&\leqslant \langle (A - C)\big(P_-v(z) + z\big), P_-v(z) + z\rangle \leqslant \\
&\leqslant \langle (A - C)P_-v(z), P_-v(z)\rangle + \langle (A - C)z, z\rangle \leqslant \\
&\leqslant \langle (A - C)z, z\rangle
\end{aligned}$$

for all $z \in Z$. Now the assertion follows from Lemma (6.1).

(*b*) This part is proved similarly. □

(6.3) REMARK. Concerning the commutativity properties of Proposition (6.2), it should be observed that, due to the fact that $P + P_- + P_+ = I_H$, an operator $C \in \mathcal{L}(H)$ commutes with two of the projections $P_\pm$, P iff it commutes with all three of them. □

7. – Estimates near zero.

Suppose, it is already known that the equation $Au = F(u)$ has a solution u_0. Then, by replacing F by $u \to F(u + u_0) - F(u_0)$, we can assume that $u_0 = 0$. In this case we are interested in the existence of nontrivial solutions.

In this section we give qualitative estimates for a near $0 \in Z$, which will be the basis for proving existence theorems concerning nontrivial solutions of $Au = F(u)$. We begin with the following obvious consequence of the uniqueness of the saddle point and the definition of a (cf. also [2, Lemma (4.3)]).

(7.1) LEMMA. *If $F(0) = 0$, then $a(0) = 0$ and 0 is a critical point of a.*

The following lemma is the analogue to Lemma (6.1).

(7.2) LEMMA. *Suppose that $F(0) = 0$, and let (R) be satisfied. Then*

$$a(z) \leqslant \tfrac{1}{2}\langle [A - F'(0)](P_-v(z) + z), P_-v(z) + z\rangle + o(\|z\|^2) \tag{7.1}$$

and

$$a(z) \geqslant \tfrac{1}{2}\langle [A - F'(0)](P_+v(z) + z), P_+v(z) + z\rangle + o(\|z\|^2) \tag{7.2}$$

as $z \to 0$ in Z.

PROOF. The definition of a and the saddle point inequalities (2.4) imply

$$a(z) = -g(T^{-1}z) \leqslant -f(x(T^{-1}z), 0, T^{-1}z) \qquad \forall z \in Z .$$

Thus, by (3.7) and since $P_-v(z) = Rx(T^{-1}z)$,

$$a(z) \leqslant \tfrac{1}{2}\langle A(P_-v(z) + z), P_-v(z) + z\rangle - \Phi(P_-v(z) + z) \tag{7.3}$$

for all $z \in Z$. Since $\Phi(0) = 0$ and $F(0) = \Phi'(0) = 0$, the mean value theorem implies

$$\Phi(q) = \tfrac{1}{2}\langle F'(0)q, q\rangle + \int_0^1 \langle F(tq) - F'(0)tq, q\rangle\, dt ,$$

hence, applying the mean value theorem again,

$$\left|\Phi(q) - \frac{1}{2}\langle F'(0)q, q\rangle\right| \leqslant \frac{1}{2} \sup_{0 \leqslant t \leqslant 1} \|F'(tq) - F'(0)\| \, \|q\|^2$$

for all $q \in H$. Consequently, letting $q := P_-v(z) + z$, hypothesis (R) and the fact that $v(0) = 0$ imply

$$-\Phi(P_-v(z) + z) \leqslant -\tfrac{1}{2}\langle F'(0)(P_-v(z) + z), P_-v(z) + z\rangle + o(1)\|P_-v(z) + z\|^2 \tag{7.4}$$

as $z \to 0$. Thus, $v(\cdot)$ being globally Lipschitz continuous, the estimate (7.1) follows from (7.3) and (7.4). The second estimate is proved similarly. □

The proof of the following important proposition is now completely analogous to the proof of Proposition (6.2). Hence it is left to the reader.

(7.3) PROPOSITION. *Let* (R) *be satisfied and let* $F(0) = 0$.

(*a*) *Suppose that there exists an operator* $C_0^- \in \mathfrak{L}_s(H)$, *which commutes with* P *and* P_-, *such that*

$$\alpha_- \leqslant C_0^- \leqslant F'(0) .$$

Then

$$a(z) \leqslant \tfrac{1}{2}\langle (A - C_0^-)z, z\rangle + o(\|z\|^2)$$

as $z \to 0$ *in* Z.

(*b*) *Suppose that* $C_0^+ \in \mathfrak{L}_s(H)$ *commutes with* P *and* P_+ *and satisfies*

$$F'(0) \leqslant C_0^+ \leqslant \beta_+ .$$

Then

$$a(z) \geqslant \tfrac{1}{2}\langle (A - C_0^+)z, z\rangle + o(\|z\|^2)$$

as $z \to 0$ *in* Z.

8. – General existence theorems based upon elementary critical point theory.

Throughout this and the following section we presuppose hypothesis (F_∞).

We begin with an elementary existence theorem for the equation $Au = F(u)$. (Observe that condition (R) is not presupposed, and recall that α_- and β_+ are defined after Lemma (6.1).)

(8.1) THEOREM. *Suppose that each one of the operators* $C_\infty^\pm \in \mathfrak{L}_s(H)$ *commutes with* P_+ *and* P_- *and that*

$$\alpha_- + \gamma_\infty < C_\infty^- \leqslant B_\infty \leqslant C_\infty^+ < \beta_+ - \gamma_\infty .$$

Then the equation $Au = F(u)$ *has at least one solution, if either*

$$[A - C_\infty^- + \gamma_\infty]\big|Z < 0 \tag{8.1}$$

or

$$[A - C_\infty^+ - \gamma_\infty]\big|Z > 0 . \tag{8.2}$$

PROOF. It suffices to show that the functional a has a critical point. Choose $\bar{\gamma}, \hat{\gamma} > \gamma_\infty$ such that $[A - C_\infty^- + \bar{\gamma}]\big|Z < 0$ and $[A - C_\infty^+ - \hat{\gamma}]\big|Z > 0$, respectively. Then Proposition (6.2) implies $a(z) \to -\infty$ as $\|z\| \to \infty$, if (8.1) is true, and $a(z) \to \infty$ as $\|z\| \to \infty$, if (8.2) is true. Hence, Z being finite-dimensional, a possesses a global maximum or a global minimum, respectively. $\square$

For the following corollary we recall that λ_1 and λ_n have been defined in Section 1, and that $B \gg 0$ means that B is positive definite.

(8.2) COROLLARY. *Suppose that* either $B_\infty \gg \lambda_n + \gamma_\infty$ *or* $B_\infty \ll \lambda_1 - \gamma_\infty$. *Then the equation* $Au = F(u)$ *is solvable.*

PROOF. This follows from Theorem (8.1) by letting $C_\infty^- := (\lambda_n + \bar{\gamma}) I_H$ and $C_\infty^+ := (\lambda_1 - \hat{\gamma}) I_H$, where $\gamma_\infty < \bar{\gamma} < B_\infty - \lambda_n$ and $\gamma_\infty < \hat{\gamma} < \lambda_1 - B_\infty$. □

In the following theorem, by using again elementary critical point theory, we prove the existence of a nontrivial solution of $Au = F(u)$ if $F(0) = 0$.

(8.3) THEOREM. *Suppose that* $F(0) = 0$ *and that* (R) *is satisfied. Let each one of the operators* $C_0^\pm$, $C_\infty^\pm \in \mathfrak{L}_s(H)$ *commute with* P_+ *and* P_-, *and assume that*

$$\alpha_- + \gamma_\infty < C_\infty^- \leqslant B_\infty \leqslant C_\infty^+ < \beta_+ - \gamma_\infty$$

and

$$\alpha_- < C_0^- \leqslant F'(0) \leqslant C_0^+ < \beta_+ ,$$

respectively.

Then the equation $Au = F(u)$ *has at least one nontrivial solution, provided one of the following conditions is satisfied*:

(i) $(A - C_\infty^- + \gamma_\infty)|Z < 0$ *and* $(A - C_0^+)|Z \nleqslant 0$.

(ii) $(A - C_\infty^+ - \gamma_\infty)|Z > 0$ *and* $(A - C_0^-)|Z \ngeqslant 0$.

(iii) $(A - C_\infty^- + \gamma_\infty)|Z \ngeqslant 0$ *and* $(A - C_0^+)|Z > 0$.

(iv) $(A - C_\infty^+ - \gamma_\infty)|Z \nleqslant 0$ *and* $(A - C_0^-)|Z < 0$.

PROOF. By Lemma (7.1), 0 is a critical point of a. Hence we have to show that each of the hypotheses (i)-(iv) implies the existence of a nontrivial critical point of a.

(i) The proof of Theorem (8.1) shows that in this case a attains its maximum at some point $z_0 \in Z$. Since $(A - C_0^+)|Z \nleqslant 0$, there exists a nontrivial subspace Z_- of Z such that $(A - C_0^+)|Z_- > 0$. But then Proposition (7.3.b) implies that 0 is not a local maximum of a. Hence $z_0 \neq 0$.

(ii) In this case the above arguments apply to $-a$.

(iii) Since $(A - C_0^+)|Z > 0$, Proposition (7.3.b) implies that a has a local strict minimum at $0 \in Z$. Since there exists a number $\bar{\gamma} > \gamma_\infty$ such that $(A - C_\infty^- + \bar{\gamma})|Z \ngeqslant 0$, Proposition (6.2.$a$) implies easily the existence of a $z \in Z \setminus \{0\}$ satisfying $a(z) = 0$. Now, since, by Corollary (5.3), the functional a satisfies the Palais-Smale condition, and since Z is finite-dimensional, a variational lemma of Ambrosetti and Rabinowitz [6, Theorem (2.1)] implies the existence of a nontrivial critical point z_0 of a such that $a(z_0) > 0$.

(iv) In this case the arguments of the preceding paragraph apply to $-a$. $\square$

(8.4) COROLLARY. *Suppose that $F(0) = 0$ and that (R) is satisfied. Then the equation $Au = F(u)$ has at least one nontrivial solution, provided one of the following conditions is satisfied:*

(i) $B_\infty \gg \lambda_n + \gamma_\infty$ *and* $F'(0) \ll \lambda_n$.

(ii) $B_\infty \ll \lambda_1 - \gamma_\infty$ *and* $F'(0) \gg \lambda_1$.

(iii) $B_\infty \gg \lambda_1 + \gamma_\infty$ *and* $F'(0) \ll \lambda_1$.

(iv) $B_\infty \ll \lambda_n - \gamma_\infty$ *and* $F'(0) \gg \lambda_n$.

PROOF. This follows from Theorem (8.3), if we let $C_\infty^\pm := \gamma_\infty^\pm I_H$ and $C_0^\pm := \gamma_0^\pm I_H$, and if $\gamma_0^\pm$ and $\gamma_\infty^\pm$ are chosen as follows:

(i) $\lambda_n + \gamma_\infty < \gamma_\infty^- < B_\infty$ and $F'(0) < \gamma_0^+ < \lambda_n$;

(ii) $B_\infty < \gamma_\infty^+ < \lambda_1 - \gamma_\infty$ and $F'(0) > \gamma_0^- > \lambda_1$;

(iii) $B_\infty > \gamma_\infty^- > \lambda_1 + \gamma_\infty$ and $F'(0) < \gamma_0^+ < \lambda_1$;

(iv) $B_\infty < \gamma_\infty^+ < \lambda_n - \gamma_\infty$ and $F'(0) > \gamma_0^- > \lambda_n$. $\square$

(8.5) REMARK. It should be observed that the above proofs contain the additional information that there is a nontrivial critical point z_0 of a such that $a(z_0) > 0$ if (i) or (iii) are satisfied, and $a(z_0) < 0$ if (ii) or (iv) are true. $\square$

Corollary (8.4) generalizes Theorem (5.3) of [2], where it had been assumed that $B_\infty = [(\lambda_k + \lambda_{k+1})/2]I_H$ for some $k \in \{0, 1, \ldots, n\}$ with $\lambda_0 := \alpha_-$. Although the hypotheses of Corollary (8.4) are rather simple, Theorem (8.3) is much more flexible and better suited for our applications to Hamiltonian systems.

9. – Existence theorems based upon generalized Morse theory.

The definiteness assumptions of Theorem (8.1) and (8.3), needed to apply elementary critical point theory, are somewhat unnatural and rather restrictive. In this section we show that these hypotheses can be dropped, provided we impose a commutativity condition for B_∞.

The proofs of this section are based upon topological tools, namely on a generalized Morse theory for isolated invariant sets of rather general dynamical systems, due to C. C. Conley and R. W. Easton [19], in its general version given by C. C. Conley [18].

We begin with the following *general existence theorem*, which should be compared with Theorem (8.1). (Recall that we presuppose hypothesis (F_∞) throughout, and we emphasize the fact that we do not presuppose condition (R)).

(9.1) THEOREM. *Suppose that B_∞ commutes with P. Then the equation $Au = F(u)$ has at least one solution.*

PROOF. Define a continuous map $[0, 1]\times H \to H$ by

$$(\sigma, u) \mapsto F_\sigma(u) := \sigma F(u) + (1-\sigma)B_\infty u\,.$$

Since the spectrum of B_∞ is contained in $[\alpha, \beta]$, it follows that F_σ satisfies (F), uniformly with respect to $\sigma \in [0, 1]$. Hence Remark (2.2) applies, and we can define

$$a_\sigma := -\, g(\sigma, \cdot)\circ T^{-1} \qquad \forall \sigma \in [0, 1]\,.$$

Then (cf. Lemma (3.1)),

$$a_\sigma'(z) = Az - PF_\sigma\big(u(\sigma, z)\big) = Au(\sigma, z) - F_\sigma\big(u(\sigma, z)\big)$$

for all $\sigma \in [0, 1]$ and $z \in Z$, where

$$u(\sigma, z) := Rx(\sigma, T^{-1} z) + Sy(\sigma, T^{-1}z) + z\,.$$

Hence

$$a_\sigma'(z) = (A - B_\infty)u(\sigma, z) - \sigma(F - B_\infty)\big(u(\sigma, z)\big)$$

and, consequently,

$$\|a_\sigma'(z)\| \geqslant \nu\|z\| - \delta_\infty \qquad \forall z \in Z,\ \forall \sigma \in [0, 1]\,, \tag{9.1}$$

where $\nu > 0$ and $\delta_\infty \geqslant 0$ are independent of $\sigma \in [0, 1]$ (cf. Lemma (5.2)).

Since the vectorfield $a_\sigma'(\cdot)$ on Z is globally Lipschitz continuous, it defines a flow for every $\sigma \in [0, 1]$. Let S_σ denote the set of bounded solutions of the equation

$$\dot z = a_\sigma'(z)\,, \tag{9.2}$$

that is, $S_\sigma = \{z \in Z \mid$ there exists a bounded orbit of (9.2) containing $z\}$. Then (9.1) implies the existence of a compact set $K \subset Z$ containing S_σ in its interior for every $\sigma \in [0, 1]$ (cf. [18, Section II.4.3.A]). Hence K is an isolating neighborhood for the isolated invariant sets S_σ, $\sigma \in [0, 1]$, which are therefore related by continuation ([18, Section IV.1, Theorem (1.3)]). Thus, by the invariance of the homotopy index [18, Section IV.1, Theorem (1.4)], the homotopy index of S_σ is independent of $\sigma \in [0, 1]$. For $\sigma = 0$, the vector field a'_σ is given by

$$a'_0(z) = (A - B_\infty)z \qquad \forall z \in Z\,,$$

which follows immediately from the fact that B_∞ and P commute. Since $0 \notin \sigma(A - B_\infty)$, it follows that $S_0 = \{0\}$ and that 0 is a hyperbolic rest point of the flow defined by a'_0. But the homotopy index of a hyperbolic rest point is the homotopy type of a pointed sphere Σ^m, whose dimension, m, equals the number of positive eigenvalues of a'_0 (cf. [18, Section I.4.3]). Thus the homotopy index, $h(S_1)$, of S_1 is the homotopy type $[\Sigma^m]$, where

$$(9.3) \qquad m := \text{« positive » Morse index of } (A - B_\infty)|Z\,,$$

that is, m is the dimension of a maximal subspace Z_+ of Z such that $(A - B_\infty)|Z_+ > 0$.

Suppose now that the (gradient) flow defined by $\dot z = a'(z) = a'_1(z)$ does not have a rest point. Then $S_1 = \emptyset$, and the homotopy index of S_1 is the homotopy type of a pointed one-point space (cf. [18, Section I.3.3]), which is distinct from $[\Sigma^m]$. (This is also true, if $m = 0$, since Σ^0 is a pointed two-point space.) This contradiction shows that a must have a critical point, which implies the assertion. □

(9.2) Remark. If we impose the stronger assumption that B_∞ commutes with P *and* $P_\pm$, then we can give a simpler proof of Theorem (9.1), based on a recent variational lemma of P. H. Rabinowitz [34, Theorem (1.2)]. Namely, in this case, letting $C_\infty^- := C_\infty^+ := B_\infty$, Proposition (6.2) implies the existence of complementary subspaces Z_+ and Z_- such that

$$(9.4) \qquad a(z) \leqslant -\varepsilon_-\|z\|^2 + \delta_- \qquad \forall z \in Z_-$$

and

$$(9.5) \qquad a(z) \geqslant \varepsilon_+\|z\|^2 - \delta_+ \qquad \forall z \in Z_+\,,$$

where $\varepsilon_\pm$ and $\delta_\pm$ are appropriate positive constants. Indeed, for Z_- (resp. Z_+) we can take the subspace of Z spanned by the eigenfunctions belonging to the negative (resp. positive) eigenvalues of $(A - B_\infty - \gamma)|Z$, where $\gamma > \gamma_\infty$ is sufficiently close to γ_∞. On the basis of [34, Theorem (1.2)], inequalities (9.4) and (9.5), and the fact that a satisfies the Palais-Smale condition, one deduces now easily the existence of a critical point of a.

However it should be remarked that the assumption, that B_∞ commutes with P and $P_\pm$, is, of course, more restrictive than the assumption that B_∞ commutes with P. In fact, in our applications to Hamiltonian systems, we shall give examples where B_∞ commutes with P, but not with $P_\pm$ (cf. the remarks following Lemma (12.3)). $\square$

In the following we denote, for any $C \in \mathfrak{L}_s(Z)$, by

$m_Z^+(C)$ [resp. $m_Z^-(C)$] *the «positive»* [resp. *«negative»*] *Morse index of* C,

that is, $m_Z^+(C)$ [resp. $m_Z^-(C)$] is the dimension of a maximal subspace of Z on which C is positive [resp. negative] definite. Moreover, we let

$$m_Z^0(C) := \dim Z - m_Z^+(C) - m_Z^-(C) .$$

Finally, if $C \in \mathfrak{L}_s(H)$ commutes with P, which implies that $(A - C)|Z \in \mathfrak{L}_s(Z)$, we write simply

$m_Z^\pm(A - C)$ and $m_Z^0(A - C)$ instead of $m_Z^\pm((A - C)|Z)$ and $m_Z^0((A - C)|Z)$, respectively.

Using these notations we can now prove our basic existence result for nontrivial solutions of $Au = F(u)$ for the case that $F(0) = 0$ and B_∞ commutes with P.

(9.3) PROPOSITION. *Suppose that B_∞ commutes with P, that $F(0) = 0$, and that (R) is satisfied. Then the equation $Au = F(u)$ has at least one nontrivial solution, provided*

$$m_Z^+(A - B_\infty) \notin [m_Z^+(a''(0)), m_Z^+(a''(0)) + m_Z^0(a''(0))] . \tag{9.6}$$

PROOF. Let S be the set of bounded solutions of the equation

$$\dot{z} = a'(z) . \tag{9.7}$$

Then it has been shown in the proof of Theorem (9.1) that S possesses a homotopy index $h(S)$ and that

$$h(S) = [\Sigma^m], \tag{9.8}$$

where $m := m_Z^+(A - B_\infty)$.

Since $F(0) = 0$, Lemma (7.1) shows that 0 is a critical point of a. Suppose now that $\{0\}$ is an isolated invariant set of the gradient vector field a' in the sense of C. C. Conley [18, Chapter I, § 6.2]. Hence it possesses a homotopy index $h(0)$. Suppose we can show that $h(0) \neq h(S)$. Then, there must exist a bounded solution of (9.7) not containing 0 in its closure. Consequently, dealing with gradient flows, the ω-limit set of the corresponding orbit must contain a critical point of a, that is, there must exist a nontrivial critical point of a.

Suppose now that 0 is the only critical point of a'. By a linear coordinate change we can assume that a' is of the form

$$\begin{aligned} &A_+\xi + f_+(\xi, \eta, \zeta) \\ &A_-\eta + f_-(\xi, \eta, \zeta) \\ &\qquad\quad f_0(\xi, \eta, \zeta), \end{aligned}$$

where $z = (\xi, \eta, \zeta) \in Z_+ \times Z_- \times Z_0$ with $\dim(Z_+) = m^+$, $\dim(Z_-) = m^- := m_Z^-(a''(0))$, $\dim(Z_0) = m^0$, $A_\pm \in \mathfrak{L}(Z_\pm)$ with $A_+ > 0$ and $A_- < 0$, and $f := (f_+, f_-, f_0) \in C^1(Z, Z)$ such that $f(0) = 0$ and $f'(0) = 0$. Let φ_t be the flow of this vector field. Then there exists a local homeomorphism h of Z, satisfying $h(0) = 0$, such that the transformed flow $\psi_t := h \circ \varphi_t \circ h^{-1}$ has near 0 the following normal form

$$\psi_t(\xi, \eta, \zeta) = \left(\exp[tA_+]\xi, \exp[tA_-]\eta, \chi_t(\zeta)\right), \tag{9.9}$$

where χ_t is an appropriate local flow near $0 \in Z_0$. The proof of this topological normal form, which generalizes a well-known result of Hartmann and Grobmann, is implicitely contained in a paper by K. J. Palmer [32] (where the time-dependent case is treated. The same result has been announced in [10] and [38] for the case of a C^2-vector field.)

Since the flow (9.9) is a product flow on $(Z_+ \times Z_-) \times Z_0$ with isolated invariant sets $S_1 := \{0\} \subset Z_+ \times Z_-$ and $S_0 := \{0\} \subset Z_0$, respectively, it follows from [18, Chapter III, 6.D] (and the fact that the homotopy index is, of course, a topological invariant) that

$$h(0) = h(S_1) \wedge h(S_0),$$

where $\wedge$ denotes the smash product (reduced join). Since $h(S_1)$ is the index of a hyperbolic rest point,

$$h(S_1) = \Sigma^{m^+}$$

(cf. [18, Chapter I, 4.3]). The index $h(S_0)$ is the homotopy type of a space h^{m^0}, which can be obtained from a compact subset N of Z_0 and a closed subset $M \subset N$ by collapsing M to a point, that is, $h^{m^0} = N/M$ (cf. [18, Chapter III.5.1]). Consequently (cf. [18, Chapter III.6.1] or [48, Chapter III.2]),

$$h(0) = [\Sigma^{m^+} \wedge h^{m^0}] .$$

Thus it remains to show that $h(S) \neq h(0)$. To see this we compute the Alexander-Spanier cohomology $\bar{H}$ (with real coefficients) of Σ^m and of $\Sigma^{m^+} \wedge h^{m^0}$ (cf. [40, Chapter 6]). It is known (cf. [18, Section IV.4.5]) that

$$\bar{H}^m(\Sigma^m) = \mathbf{R} . \tag{9.10}$$

On the other hand, it is known that $\Sigma^{m^+} \wedge h^{m^0}$ is homeomorphic to the m^+-fold (reduced) suspension $\Sigma \wedge (\Sigma \wedge \ldots (\Sigma \wedge h^{m^0}) \ldots)$ of h^{m^0} (e.g. [48, Chapter III.2]). Let $C := h^{m^0}$ and

$$A := \{[\exp 2\pi i t] \wedge x \mid 0 \leqslant t \leqslant \tfrac{1}{2},\, x \in C\}$$

$$B := \{[\exp 2\pi i t] \wedge x \mid \tfrac{1}{2} \leqslant t \leqslant 1,\, x \in C\} ,$$

where $(s, x) \mapsto s \wedge x$ denotes the canonical projection $\Sigma \times C \to \Sigma \wedge C$, and where Σ is identified with $(S^1, (1, 0))$. Then A and B are closed subsets of $\Sigma \wedge C$ such that

$$\Sigma \wedge C = A \cup B \quad \text{and} \quad C \cong A \cap B .$$

Moreover, $A \cup B$, A, and B have the same base point. Since a closed subspace of a compact space is a «taut» subspace relative to the Alexander-Spanier cohomology theory [40, Theorem (6.6.2)], it follows from [40, Theorem (6.1.13)], that we have a long exact relative Mayer-Vietoris sequence

$$\ldots \to \bar{H}^q(A \cup B) \to \bar{H}^q(A) \oplus \bar{H}^q(B) \to \bar{H}^q(A \cap B) \to \ldots$$

(relative to the base points). Since A and B are contractible (modulo base points), $\bar{H}^q(A)$ and $\bar{H}^q(B)$ are trivial, and we obtain a short exact sequence

$$0 \to \bar{H}^q(C) \to \bar{H}^{q+1}(\Sigma \wedge C) \to 0$$

for all $q \in \boldsymbol{Z}$. Thus $\bar{H}^q(C) \cong \bar{H}^{q+1}(\Sigma \wedge C)$ and, by induction,

$$\bar{H}^q(\Sigma^{m^+} \wedge h^{m^0}) \cong \bar{H}^{q-m^+}(h^{m^0})$$

for all $q \in \boldsymbol{Z}$. Consequently (recall that h^{m^0} is obtained from a compact subset of $E_0 \approx \boldsymbol{R}^{m^0}$),

$$\bar{H}^q(\Sigma^{m^+} \wedge h^{m^0}) = 0$$

for $q < m^+$ and $q - m^+ > m^0$. Hence, by (9.10),

$$\bar{H}(\Sigma^m) \neq \bar{H}^m(\Sigma^{m^+} \wedge h^{m^0})$$

if $m \notin [m^+, m^+ + m^0]$, which implies $h(S) \neq h(0)$. □

We should like to remark that the principal ideas of the above proof are due to C. C. Conley.

By combining Propositions (9.3) and (7.3) we obtain now the following general

(9.4) THEOREM. *Suppose that B_∞ commutes with P, that $F(0) = 0$, and that (R) is satisfied. Then the equation $Au = F(u)$ has at least one nontrivial solution, provided one of the following conditions is satisfied:*

(*a*) *There exists an operator $C_0^- \in \mathfrak{L}_s(H)$, with commutes with P and P_-, such that*

$$\alpha_- \leqslant C_0^- \leqslant F'(0)$$

and

$$m_Z^-(A - C_0^-) > m_Z^-(A - B_\infty)\,. \tag{9.11}$$

(*b*) *There exists an operator $C_0^+ \in \mathfrak{L}_s(H)$, which commutes with P and P_+, such that*

$$F'(0) \leqslant C_0^+ \leqslant \beta_+$$

and

$$m_Z^+(A - C_0^+) > m_Z^+(A - B_\infty)\,. \tag{9.12}$$

PROOF. (*a*) Proposition (7.3) implies the existence of a subspace Z_- of Z of dimension $m_Z^-(A - C_0^-)$ and of a constant $\varepsilon > 0$, such that

$$a(z) \leqslant -\varepsilon \|z\|^2 + o(\|z\|^2)$$

as $z \to 0$ in Z_-. This estimate implies easily

$$m_Z^+(a''(0)) + m_Z^0(a''(0)) \leqslant \dim Z - m_Z^-(A - C_0^-)\,.$$

Thus, by (9.11),

$$m_Z^+\big(a''(0)\big) + m_Z^0\big(a''(0)\big) < \dim Z - m_Z^-(A - B_\infty) = m_Z^+(A - B_\infty)\,,$$

where the last equality follows from the fact that $0 \notin \sigma(A - B_\infty)$. Hence (9.6) is satisfied, and Proposition (9.3) implies the assertion.

(*b*) In this case Proposition (7.3) implies the existence of a subspace Z_+ of Z of dimension $m_Z^+(A - C_0^+)$ and of a constant $\varepsilon > 0$, such that

$$a(z) \geqslant \varepsilon \|z\|^2 + o\big(\|z\|^2\big)$$

as $z \to 0$ in Z_+. From this estimate it follows that

$$m_Z^+(A - C_0^+) \leqslant m_Z^+\big(a''(0)\big)\,.$$

Hence (9.12) implies the validity of (9.6), and the assertion follows again from Proposition (9.3). $\square$

We add a simple corollary which will suffice for some of our applications.

(9.5) COROLLARY. *Suppose that $B_\infty = \nu_\infty I_H$ for some $\nu_\infty \in [\alpha, \beta]$. Moreover, let $F(0) = 0$ and let (R) be satisfied. Then the equation $Au = F(u)$ has at least one nontrivial solution, provided either*

(9.13) $$B_\infty < \lambda_k \ll F'(0)$$

or

(9.14) $$F'(0) \ll \lambda_k < B_\infty$$

for some $k \in \{1, \ldots, n\}$.

PROOF. Fix $\varepsilon > 0$ such that $\varepsilon < |\lambda_k - \nu_\infty|$, $(\lambda_k - \varepsilon, \lambda_k + \varepsilon) \cap \sigma\big(F'(0)\big) = \emptyset$, and $(\lambda_k - \varepsilon, \lambda_k + \varepsilon) \cap \sigma(A) = \{\lambda_k\}$. Moreover, let $C_0^- := (\lambda_k + \varepsilon) I_H$ and $C_0^+ := (\lambda_k - \varepsilon) I_H$. Then

$$m_Z^-(A - C_0^-) = \sum_{j=1}^{k} m(\lambda_j)$$

and

$$m_Z^+(A - C_0^+) = \sum_{j=k}^{n} m(\lambda_j)\,.$$

Furthermore, $\alpha_- \leqslant C_0^- \leqslant F'(0)$, if (9.13) is true, and $F'(0) \leqslant C_0^+ \leqslant \beta_+$ if (9.14) is satisfied. Now the assertion follows from Theorem (9.4), since

$$m_Z^-(A - B_\infty) \leqslant \sum_{j=1}^{k-1} m(\lambda_j)$$

and

$$m_Z^+(A - B_\infty) \leqslant \sum_{j=k+1}^{n} m(\lambda_j) . \qquad \square$$

It seems worthwhile to point out that in the proof of Proposition (9.3) we have obtained the following topological result.

(9.6) PROPOSITION. *Suppose that the vector field $v \in C^1(\boldsymbol{R}^N, \boldsymbol{R}^N)$ has 0 as an isolated rest point. Let m^+ or m^0, respectively, denote the dimension of the unstable- or center-manifold, respectively, of the rest point 0. Then, if $\{0\}$ is an isolated invariant set of $\dot{x} = v(x)$, its homotopy index $h(0)$ is given by*

$$h(0) = [\Sigma^{m^+} \wedge h^{m^0}] ,$$

where Σ^{m^+} is a pointed m^+-sphere and h^{m^0} is obtained from a compact index pair (N_1, N_2) in $\boldsymbol{R}^{m^0}$ by collapsing N_2 into a point (cf. [18, Chapter III.5]).

PART TWO

APPLICATIONS

10. – Elliptic boundary value problems.

In this section we consider the semilinear elliptic boundary value problem (BVP)

$$\text{(10.1)} \qquad \begin{cases} A(x, D)u = f(x, u) & \text{in } \Omega , \\ B(x, D)u = 0 & \text{on } \partial\Omega , \end{cases}$$

where $\Omega \subset \boldsymbol{R}^N$ is a bounded domain with smooth boundary, $\partial\Omega$, lying locally on one side of Ω. Moreover we suppose that

(i) $A(x, D)u := \sum_{|\alpha|, |\beta| \leqslant m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta} D^\beta u)$ is a strongly uniformly elliptic differential operator of order $2m$, with real smooth coefficients.

(ii) $B(x, D)$ is a family of m smooth boundary operators.

(iii) $(A(x, D), B(x, D))$ induces a self-adjoint linear operator A in $H := L_2(\Omega)$, which is bounded below.

(iv) For every $p \in (1, \infty)$, there exists a constant $c > 0$ such that

$$\|u\|_{W_p^{2m}(\Omega)} \leqslant c\big(\|A(x, D)u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)}\big)$$

for all $u \in W_p^{2m}(\Omega)$ satisfying $B(x, D)u = 0$ on $\partial\Omega$.

(v) $f \in C^1(\bar{\Omega} \times \boldsymbol{R}, \boldsymbol{R})$ and $\sup_{(x,\eta)\in\Omega\times\boldsymbol{R}} |f_\xi(x, \eta)| < \infty$, where f_ξ denotes the partial derivative of f with respect to the second variable.

By a solution of (11.1) we mean a classical solution.

It is well known that the hypotheses (i)-(iv) are satisfied, for example, if $B(x, D)$ is the family of boundary operators describing Dirichlet boundary conditions, or if $m = 1$ and

$$B(x, D) = \sum_{i,j=1}^{N} a_{ij}(x)\, {}^i D_j u + b(x)u\,,$$

where $b \in C^\infty(\partial\Omega, \boldsymbol{R})$, $\nu = (\nu^1, \ldots, \nu^N)$ is the outer normal on $\partial\Omega$, and (a_{ij}) is the symmetric coefficient matrix of $A(x, D)$ (cf. [23, 29]).

Standard elliptic regularity theory implies that the BVP (11.1) is equivalent to the equation

$$Au = F(u) \tag{10.2}$$

in H, where F is the Nemytskii operator of f, that is,

$$F(u)(x) := f\big(x, u(x)\big) \qquad \forall x \in \bar{\Omega},\ \forall u \in H\,.$$

It follows from assumption (v) that F is a continuous potential operator on H, which is everywhere Gateaux differentiable, the derivative F' being given by

$$[F'(u)h](x) = f_\xi\big(x, u(x)\big)h(x) \qquad \forall x \in \bar{\Omega}\,, \tag{10.3}$$

for all $u, h \in H$ (cf. [2, Section 6]).

It is a consequence of Sobolev type imbedding theorems that A has a compact resolvent. Thus A has a pure point spectrum consisting of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots$$

of finite multiplicities, having no finite accumulation-point. Moreover, the eigenvalue problem $Au = \lambda u$ is equivalent to the elliptic eigenvalue problem

$$(10.4) \qquad \begin{cases} A(x, D)u = \lambda u & \text{in} \quad \Omega\,, \\ B(x, D)u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

In the following we let $\lambda_0 := -\infty$.

(10.1) LEMMA: (*a*) *Problem* (10.2) *satisfies conditions* (*A*), (*F*), *and* (*R*).

(*b*) *If there exist positive numbers* ε *and* ϱ *and an integer* $k \in \boldsymbol{N}$ *such that*

$$\lambda_k + \varepsilon \leqslant f(x, \xi)/\xi \leqslant \lambda_{k+1} - \varepsilon \qquad \forall (x, \xi) \in \bar{\Omega} \times \boldsymbol{R}$$

satisfying $|\xi| \geqslant \varrho$, *then condition* (F_∞) *with*

$$B_\infty := 2^{-1}(\lambda_k + \lambda_{k+1}) I_H$$

is satisfied.

PROOF: (*a*) Condition (v) implies the existence of constants $\alpha < \beta$ such that $\alpha \leqslant f_\xi(x, \eta) \leqslant \beta$ for all $(x, \eta) \in \bar{\Omega} \times \boldsymbol{R}$. Since we can assume that $\alpha, \beta \notin \sigma(A)$ this estimate and the spectral properties of A imply easily the validity of the hypotheses (*A*) and (*F*).

Since Z is spanned by finitely many eigenfunctions of (10.4), which are smooth, it is obvious that $Z \hookrightarrow C(\bar{\Omega})$. It follows from Proposition (4.1) that, for each $z \in Z$, the functions $P_\pm v(z)$ satisfy the equations

$$AP_\pm v(z) = P_\pm F\big(u(z)\big)\,,$$

where $u(z) = P_+ v(z) + P_- v(z) + z$.

It is not difficult to verify that

$$(10.5) \qquad \operatorname{dom}(A) = \{u \in W_2^{2m}(\Omega) | B(x, D)u = 0 \text{ on } \partial\Omega\}$$

(cf. [29, Section II.8.3]). Hence (10.5) and the L_2-estimate (iv) imply

$$\|P_\pm v(z) - P_\pm v(z_0)\|_{W_2^{2m}} \leqslant c\{\|F\big(u(z)\big) - F\big(u(z_0)\big)\|_{L_2} + \|P_\pm v(z) - P_\pm v(z_0)\|_{L_2}\}$$

for all $z, z_0 \in Z$ (where, of course, one has to take everywhere the same subscript $+$ or $-$, respectively). Since $u(\cdot) \in C(Z, H)$ and $F \in C(H, H)$, it follows that $P_\pm v(\cdot) \in C(Z, W_2^{2m}(\Omega))$. Thus, by a Sobolev type imbedding theorem, $P_\pm v(\cdot) \in C(Z, L_p(\Omega))$, hence $u(\cdot) \in C(Z, L_p(\Omega))$, for an appropriately chosen $p > 2$.

Since the eigenfunctions of (10.4) are smooth, it follows easily that P_- and P, and hence $P_\pm \in \mathfrak{L}(L_q(\Omega))$ for every $q \in (1, \infty)$. Thus, by the L_p-estimate (iv),

$$\|P_\pm v(z) - P_\pm v(z_0)\|_{W_p^{2m}} \leqslant c\{\|F(u(z)) - F(u(z_0))\|_{L_p} + \|P_\pm v(z) - P_\pm v(z_0)\|_{L_p}\}$$

for all $z, z_0 \in Z$. Since, by (v),

$$|f(x, \xi)| \leqslant a + b|\xi| \qquad \forall (x, \xi) \in \bar{\Omega} \times \boldsymbol{R}$$

and appropriate constants $a, b > 0$, it is well known that $F \in C(L_p(\Omega), L_p(\Omega))$. Hence it follows that $P_\pm v(\cdot) \in C(Z, W_p^{2m}(\Omega))$, hence $u(\cdot) \in C(Z, W_p^{2m}(\Omega))$. By repeating this bootstrapping argument a finite number of times, it follows finally that $P_\pm v(\cdot) \in C(Z, C(\bar{\Omega}))$.

Finally, the fact that $f_\xi \in C^1(\bar{\Omega})$ implies easily that $F' | C(\bar{\Omega}) \in C(C(\bar{\Omega}), \mathfrak{L}(H))$. Moreover it follows that condition (R) with $E = C(\bar{\Omega})$ is satisfied.

(b) For a proof of this fact we refer to [2, Lemma (6.3)]. □

After these preparations we can now prove the following general existence theorem.

(10.2) THEOREM. *Let conditions* (i)-(v) *be satisfied and suppose, in addition, that there exist positive constants* ε *and* ϱ *such that*

$$\lambda_k + \varepsilon \leqslant f(x, \xi)/\xi \leqslant \lambda_{k+1} - \varepsilon$$

for all $(x, \xi) \in \bar{\Omega} \times \boldsymbol{R}$ *satisfying* $|\xi| \geqslant \varrho$, *and some* $k \in \boldsymbol{N}$ (*where* $\lambda_0 := -\infty$). *Then the semilinear elliptic* BVP (10.1) *has at least one solution.*

Suppose, in addition, that $f(x, 0) = 0$ *for all* $x \in \bar{\Omega}$. *Then the* BVP (10.1) *possesses at least one nontrivial solution if* either

$$f_\xi(x, 0) \leqslant \lambda_k - \varepsilon \quad \text{or} \quad f_\xi(x, 0) \geqslant \lambda_{k+1} + \varepsilon \tag{10.6}$$

for all $x \in \Omega$.

PROOF. Due to Lemma (10.1), the first assertion follows immediately from Theorem (9.1).

It is an obvious consequence of the representation (10.3), that the inequalities (10.6) imply $F'(0) \ll \lambda_k$ or $F'(0) \gg \lambda_{k+1}$, respectively. Since, by Lemma (10.2), $\lambda_k < B_\infty < \lambda_{k+1}$, the second part of the assertion is a consequence of Corollary (9.5). □

The problem of the existence of nontrivial solutions to nonlinear elliptic boundary value problems has attracted numerous authors (cf. [1-4, 14, 15, 20, 26, 34, 41, 42]. The bibliographies of these papers should also be consulted.) In order to describe the qualitative feature of the results so far known, let us consider the simple case of the boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{10.7}$$

where f is smooth, asymptotically linear, and $f(0) = 0$. (Clearly, in almost all of the above mentioned papers there are considered more general situations as far as the differential operator and the boundary conditions, the regularity hypotheses for f, and the asymptotic behavior is concerned. It is our purpose to exhibit only the qualitative features of the hypotheses.) The best results so far known are due to K. Thews [41] and P. Hess [26]. In [41] it is shown that (10.7) has a nontrivial solution if *either*

$$f'(\xi) \leqslant \lambda_{k+1} \quad \text{and} \quad f'(0) < \lambda_k < f'(\infty) < \lambda_{k+1}$$

or

$$f'(\xi) \geqslant \lambda_{k-1} \quad \text{and} \quad \lambda_{k-1} < f'(\infty) < \lambda_k < f'(0)$$

for some $k \in \boldsymbol{N}^*$ and all $\xi \in \boldsymbol{R}$. Hess [26] obtains the existence of at least one nontrivial solution if *either*

$$\lambda_{k-1} \leqslant f'(0) < \lambda_k \leqslant \lambda_l < f'(\infty) < \lambda_{l+1}$$

and

$$\big(f(\xi) - \lambda_{k-1}\xi\big)\xi \geqslant 0 \qquad \forall \xi \in \boldsymbol{R} \tag{10.8}$$

or

$$\lambda_{k-1} < f'(\infty) < \lambda_k \leqslant \lambda_l < f'(0) \leqslant \lambda_{l+1}$$

and

$$\big(f(\xi) - \lambda_{l+1}\xi\big)\xi \leqslant 0 \qquad \forall \xi \in \boldsymbol{R}\,, \tag{10.9}$$

for some $k, l \in \boldsymbol{N}^*$ satisfying $k \leqslant l$. Observe that our results imply that neither of the restrictions $f' \leqslant \lambda_{k+1}$, $f' \geqslant \lambda_{k-1}$, (10.8), or (10.9) is necessary.

Finally, concerning the case that resonance at infinity occurs, we refer to our remarks in the Introduction about Landesman-Lazer problems.

11. – Periodic solutions of a semilinear wave equation.

In this section we prove the existence of classical T-periodic solutions of the semilinear wave equation

$$(11.1)\qquad \begin{cases} u_{tt} - u_{xx} = f(x, t, u) & \text{in } (0, \pi)\times\boldsymbol{R}, \\ u(0, t) = u(\pi, t) = 0 & \text{for } t\in\boldsymbol{R}, \end{cases}$$

where we impose the following assumptions:

(i) $T = 2\pi/\tau$ for some $\tau\in\boldsymbol{Q}$, $\tau > 0$.

(ii) $f\in C^2([0, \pi]\times\boldsymbol{R}\times\boldsymbol{R}, \boldsymbol{R})$ and $f(x, t + T, \xi) = f(x, t, \xi)$ for all $(x, t, \xi)\in[0, \pi]\times\boldsymbol{R}\times\boldsymbol{R}$.

(iii) There exist constants $\alpha < \beta$ such that *either* $\alpha > 0$ *or* $\beta < 0$, and such that

$$\alpha \leqslant f_\xi(x, t, \eta) \leqslant \beta \qquad \forall (x, t, \eta)\in[0, \pi]\times\boldsymbol{R}\times\boldsymbol{R},$$

where f_ξ denotes the partial derivative with respect to the third variable.

We let $\Omega := (0, \pi)\times(0, T)$ and $H := L_2(\Omega)$, and we define

$$\Box\colon \operatorname{dom}(\Box)\subset H\to H \qquad \text{by} \qquad \Box u := u_{tt} - u_{xx},$$

where $\operatorname{dom}(\Box)$ consists of all $u\in C^2(\bar\Omega)$ satisfying $u(0,\cdot) = u(\pi,\cdot) = 0$, $u(\cdot, 0) = u(\cdot, T)$, and $u_t(\cdot, 0) = u_t(\cdot, T)$. Then it can be shown that $A := \Box^*$, the adjoint of $\Box$, is self-adjoint. Moreover, A has a pure point spectrum, given by

$$\sigma(A) = \{j^2 - \tau^2 k^2 | (j, k)\in\boldsymbol{N}^*\times\boldsymbol{Z}\},$$

and every $\lambda\in\sigma(A)\setminus\{0\}$ is an eigenvalue of finite multiplicity, whereas 0 has infinite multiplicity. Hence we can assume that $\alpha, \beta\notin\sigma(A)$.

In the following we denote by F the Nemytskii operator of f, that is,

$$F(u)(x, t) := f\big(x, t, u(x, t)\big) \qquad \forall (x, t)\in\Omega$$

and all $u\in H$. Then condition (iii) implies that F is a continuous potential operator on H, possessing everywhere a Gateaux derivative F', given by

$$[F'(u)h](x, t) = f_\xi\big(x, t, u(x, t)\big)h(x, t) \qquad \forall (x, t)\in\Omega$$

and all $u, h \in H$. It follows now from regularity results in [33] that the problem of finding T-periodic classical solutions of (11.1) is equivalent to the problem of finding solutions to the equation $Au = F(u)$ in H (cf. also [2, Lemma (8.2)]).

(11.1) LEMMA. *(a) The problem* $Au = F(u)$ *satisfies conditions* (A), (F), *and* (R).

(b) If there exist positive numbers ε *and* ϱ *and consecutive eigenvalues* $\bar{\lambda} < \hat{\lambda}$ *of* A *such that*

$$\bar{\lambda} + \varepsilon \leqslant f(x, t, \xi)/\xi \leqslant \hat{\lambda} - \varepsilon \qquad \forall (x, t, \xi) \in \Omega \times \mathbf{R}$$

satisfying $|\xi| \geqslant \varrho$, *then condition* (F_∞) *with*

$$B_\infty := 2^{-1}(\bar{\lambda} + \hat{\lambda}) I_H$$

is satisfied.

PROOF. (a) Hypotheses (ii) and (iii) and the above information on $\sigma(A)$ imply easily the validity of (A) and (F).

We claim that condition (R) with $E := C(\bar{\Omega})$ is true. Indeed, since Z is being spanned by finitely many smooth eigenfunctions of A, it follows that $Z \hookrightarrow C(\bar{\Omega})$. Moreover, the regularity assumption upon f implies, as in the preceding section, that $F' \in C(Z, C(\bar{\Omega}))$. Hence it remains to show that

$$P_\pm v(\cdot) \in C(Z, C(\bar{\Omega})),$$

where $P_\pm v(z)$ is the unique solution of

$$AP_\pm v(z) = P_\pm F(u(z)) \qquad \forall z \in Z, \tag{11.2$_\pm$}$$

and $u(z) = P_+ v(z) + P_- v(z) + z$ (cf. Proposition (4.1)).

For this purpose we let

$$A^{-1} := [A|(\operatorname{dom}(A) \cap \ker(A)^\perp)]^{-1}.$$

Then it is known (cf. [33, Formula (1.3)]), that

$$A^{-1} \in \mathcal{L}(\ker(A)^\perp, C^{\frac{1}{2}}(\bar{\Omega}) \cap H^1(\Omega)). \tag{11.3}$$

We assume now, for definiteness, that $\ker(A) \subset X = P_-(H)$. Then, using the facts that $u(\cdot) \in C(Z, H)$ and $F \in C(H, H)$, (11.3) and equation $(11.2)_+$

imply

$$P_+v(\cdot) \in C(Z, C(\bar{\Omega}) \cap H^1(\Omega)) . \tag{11.4}$$

In the following we denote by P_0 the orthogonal projection of H onto $\ker(A)$, and we let $Q := P_- - P_0$. Then equation $(11.2)_-$ is equivalent to the system

$$AQv(z) = QF(u(z)) \tag{11.5}$$

$$0 = P_0F(v_0(z) + r(z)) , \tag{11.6}$$

where $v_0(z) := P_0v(z)$ and $r(z) := u(z) - v_0(z) = Qv(z) + P_+v(z) + z$ for all $z \in Z$. Thus, similarly as above, (11.3) and (11.5) imply

$$Qv(\cdot) \in C(Z, C(\bar{\Omega}) \cap H^1(\Omega)) .$$

Consequently, recalling (11.4),

$$r(\cdot) \in C(Z, C(\bar{\Omega})) . \tag{11.7}$$

Now (11.6) and the regularity results of [33] (cf. also [2, Lemma (8.1)]) imply that

$$v_0(z) \in C(\bar{\Omega}) \qquad \forall z \in Z .$$

Finally, in order to show that $v_0(\cdot) \in C(Z, C(\bar{\Omega}))$, we employ some ideas of P. H. Rabinowitz [33].

By means of Fourier series it is easily seen that $\ker(A)$ consists of the closure in H of the set of functions φ of the form $\varphi(x, t) = \psi(t + x) - \psi(t - x)$, where ψ is smooth and periodic with periods 2π and T (cf. [13, 33]). Suppose that $\varphi \in \ker(A)$ has the representation $\varphi(x, t) = \psi(t + x) - \psi(t - x)$ such that

$$[\psi] := \int_{\Omega} \psi(t + x)\, dx\, dt = 0 .$$

In this case, which can always be achieved by adding a suitable constant to ψ, we let $\varphi^{\pm}(x, t) := \psi(t \pm x)$ for all $(x, t) \in \Omega$.

By means of a Fourier series development it is easily verified that

$$\int_{\Omega} f(t + x) g(t - x)\, dx\, dt = 0 , \tag{11.8}$$

whenever f and g are 2π-periodic and square integrable, and $[f][g] = 0$.

Finally, by multiplying (11.6) by -1, if necessary, we can assume that $\alpha > 0$.

Now let $z, z_0 \in Z$ be arbitrarily fixed, and let $w := v_0(z) - v_0(z_0) \in \ker(A)$. Moreover, let

$$M := \tfrac{1}{2}\|w^+\|_{C(\bar{\Omega})} = \tfrac{1}{2}\|w^-\|_{C(\bar{\Omega})},$$

define $q\colon \boldsymbol{R} \to \boldsymbol{R}$ by

$$q(t) := \begin{cases} t - M & \text{if } t \geqslant M, \\ 0 & \text{if } |t| \leqslant M, \\ t + M & \text{if } t \leqslant -M, \end{cases}$$

and observe that

$$tq(t) \geqslant M|q(t)| \quad \forall t \in \boldsymbol{R}. \tag{11.9}$$

Then $\varphi := q(w^+) - q(w^-) \in \ker(A)$, and, due to (11.8) and (11.9),

$$\begin{aligned} \langle \varphi, w\rangle &= \langle q(w^+) - q(w^-), w^+ - w^-\rangle \\ &= \langle q(w^+), w^+\rangle + \langle q(w^-), w^-\rangle \\ &\geqslant M\int_\Omega \big(|q(w^+)| + |q(w^-)|\big)\,dx\,dt. \end{aligned} \tag{11.10}$$

Now (11.6), the positivity of $\langle w, \varphi\rangle$, the assumption (iii), and the mean value theorem imply

$$\begin{aligned} 0 &= \big\langle F\big(v_0(z) + r(z)\big) - F\big(v_0(z_0) + r(z_0)\big), \varphi\big\rangle \\ &= \big\langle F\big(v_0(z_0) + r(z)\big) - F\big(v_0(z_0) + r(z_0)\big), \varphi\big\rangle \\ &\quad + \big\langle F\big(v_0(z) + r(z)\big) - F\big(v_0(z_0) + r(z)\big), \varphi\big\rangle \\ &\geqslant \big\langle F\big(v_0(z_0) + r(z)\big) - F\big(v_0(z_0) + r(z_0)\big), \varphi\big\rangle + \alpha\langle w, \varphi\rangle. \end{aligned}$$

Consequently, by (11.10),

$$\begin{aligned} \alpha M\int_\Omega \big(|q(w^+)| + |q(w^-)|\big)\,dx\,dt &\leqslant \\ &\leqslant \big\|F\big(v_0(z_0) + r(z)\big) - F\big(v_0(z_0) + r(z_0)\big)\big\|_{C(\bar{\Omega})}\|\varphi\|_{L_1(\Omega)}. \end{aligned} \tag{11.11}$$

Since $\|\varphi\|_{L_1(\Omega)} \leqslant \int_\Omega \big(|q(w^+)| + |q(w^-)|\big)\,dx\,dt$, and since the latter integral is positive, if $w \neq 0$, we obtain from (11.11) and the definition of M,

$$\tfrac{1}{2}\|w^\pm\|_{C(\bar{\Omega})} \leqslant \alpha^{-1}\big\|F\big(v_0(z_0) + r(z)\big) - F\big(v_0(z_0) + r(z_0)\big)\big\|_{C(\bar{\Omega})}.$$

Consequently, since $\|w\|_{C(\bar{\Omega})} \leqslant 2\|w^+\|_{C(\bar{\Omega})}$, it follows that

$$\|v_0(z) - v_0(z_0)\|_{C(\bar{\Omega})} \leqslant \alpha^{-1}\|F(v_0(z_0) + r(z)) - F(v_0(z_0) + r(z_0))\|_{C(\bar{\Omega})}$$

for all $z, z_0 \in Z$. Hence (11.7) and the obvious fact that $F|C(\bar{\Omega}) \in C(C(\bar{\Omega}), C(\bar{\Omega}))$, imply $v_0(\cdot) \in C(Z, C(\bar{\Omega}))$. Consequently,

$$P_-v(\cdot) = v_0(\cdot) + Qv(\cdot) \in C(Z, C(\bar{\Omega})),$$

and the validity of (R) is shown.

(*b*) For a proof of this fact we refer again to [2, Lemma (6.3)]. □

After these preparations we can now prove the following general existence theorem for T-periodic solutions of (11.1).

(11.2) THEOREM. *Let the assumptions* (i)-(iii) *be satisfied. Suppose that* $\bar{\lambda} < \hat{\lambda}$ *are two consecutive numbers of the (discrete) set* $\{j^2 - \tau^2 k^2 | (j, k) \in \boldsymbol{N}^* \times \boldsymbol{Z}\}$, *and that there exist positive numbers* ε *and* ϱ *such that*

$$\bar{\lambda} + \varepsilon \leqslant f(x, t, \xi)/\xi \leqslant \hat{\lambda} - \varepsilon \tag{11.12}$$

for all $(x, t, \xi) \in [0, \pi] \times \boldsymbol{R} \times \boldsymbol{R}$ *satisfying* $|\xi| \geqslant \varrho$. *Then the semilinear wave equation* (11.1) *possesses at least one T-periodic solution.*

Suppose, in addition, that $f(x, t, 0) = 0$ *for all* $(x, t) \in [0, \pi] \times \boldsymbol{R}$. *Then the equation* (11.1) *has at least one nonzero T-periodic solution if* either

$$f_\xi(x, t, 0) \leqslant \bar{\lambda} - \varepsilon \quad \text{or} \quad f_\xi(x, t, 0) \geqslant \hat{\lambda} + \varepsilon \tag{11.13}$$

for all $(x, t) \in [0, \pi] \times \boldsymbol{R}$.

REMARK. Suppose that $\bar{\lambda} = 0$. Then it is an easy consequence of (11.12), condition (iii), and the mean value theorem, that $\alpha > 0$. Consequently, in this case only the second alternative of (11.13) is possible. A similar remark applies if $\hat{\lambda} = 0$.

PROOF OF THEOREM (11.2). The first assertion follows immediately from Lemma (11.1) and Theorem (9.1). Since the inequalities (11.13) imply $F'(0) \ll \bar{\lambda}$ or $F'(0) \gg \hat{\lambda}$, respectively, and since, by Lemma (11.1), $\bar{\lambda} < B_\infty < \hat{\lambda}$, the second part of the assertion follows, again on the basis of Lemma (11.1), from Corollary (9.5). □

The problem of the existence of periodic solutions to the nonlinear wave equation (11.1) has been studied by many authors under the assumption that f is of the form $\varepsilon g(x, t, \xi)$ and $\varepsilon > 0$ is small (*e.g.* [13, 33, 43]). In addition the bibliographies of these papers should be consulted.) There are only few papers studying the global problem (cf. [2, 13, 20, 30, 35]). Rabinowitz [35] treats the case of superlinear nonlinearities, to which our results are not applicable. In the other papers the case of linearly bounded nonlinearities, to which the techniques of [35] do not seem to apply, has been treated. As far as the qualitative behavior is concerned, the best results for the latter case are contained in [2]. Namely it has been shown that there exists at least one nonzero T-periodic solution if, given the assumptions (i)-(iii) and assuming for simplicity that f is independent of (x, t) and asymptotically linear, *either*

$$f' < \hat{\lambda} \quad \text{and} \quad f'(0) < \bar{\lambda} < f'(\infty) < \hat{\lambda}$$

or

$$f' < \hat{\lambda} \quad \text{and} \quad f'(\infty) < \bar{\lambda} < f'(0) < \hat{\lambda}$$

or

$$f' > \bar{\lambda} \quad \text{and} \quad \bar{\lambda} < f'(\infty) < \hat{\lambda} < f'(0)$$

or

$$f' > \bar{\lambda} \quad \text{and} \quad \bar{\lambda} < f'(0) < \hat{\lambda} < f'(\infty)$$

for two consecutive eigenvalues of the wave operator. In addition, it is always presupposed that $f'(\infty)$ is not an eigenvalue of $\Box$. (We refer to [2] for a comparison of these results with the above mentioned work of the other authors.) Theorem (11.2) shows that neither of the assumption $f' < \hat{\lambda}$ and $f' > \bar{\lambda}$ is necessary.

In a forthcoming paper [2*a*] we shall prove the existence of multiple periodic solutions for a class of autonomous nonlinear wave equations.

12. – Periodic solutions of Hamiltonian systems.

In this section we consider the existence problem of periodic solutions of Hamiltonian equations:

$$(12.1) \qquad \dot{p} = -\mathcal{H}_q(t, p, q), \quad \dot{q} = \mathcal{H}_p(t, p, q),$$

where the dot denotes the derivative with respect to the independent variable t. The Hamiltonian function $\mathcal{H}$ is assumed to depend periodically on t. More precisely, denoting a generic point of $\boldsymbol{R}^{2N} = \boldsymbol{R}^N \times \boldsymbol{R}^N$ by $x := \{p, q\}$, where $p, q \in \boldsymbol{R}^N$, we shall assume in the following for the function $\mathcal{H}: \boldsymbol{R}\times\boldsymbol{R}^{2N}\to\boldsymbol{R}$:

(i) $\mathcal{H}(t+T,\cdot)=\mathcal{H}(t,\cdot)$ for all $t\in\boldsymbol{R}$ and some $T>0$.

(ii) $\mathcal{H}$ possesses a second partial derivative $\mathcal{H}_{xx}$ with respect to $x\in\boldsymbol{R}^{2N}$ such that $\mathcal{H}_{xx}\in C(\boldsymbol{R}\times\boldsymbol{R}^{2N}, \mathcal{L}(\boldsymbol{R}^{2N}))$, and moreover

$$\sup_{(t,\eta)}|\mathcal{H}_{xx}(t,\eta)|<\infty\,. \tag{12.2}$$

Without loss of generality we normalize the Hamiltonian function assuming $\mathcal{H}(t,0)=0$, $t\in\boldsymbol{R}$. We denote by $J\in\mathcal{L}(\boldsymbol{R}^{2N})$,

$$J:=\begin{pmatrix}0 & -I_N\\ I_N & 0\end{pmatrix} \tag{12.3}$$

the standard symplectic structure on $\boldsymbol{R}^{2N}$, where I_N is the identity on $\boldsymbol{R}^N$. We then can rewrite (12.1) as

$$\dot{u}=J\mathcal{H}_x(t,u)\,. \tag{12.4}$$

The aim is to find T-periodic solutions $u\in C^1(\boldsymbol{R},\boldsymbol{R}^{2N})$ of (12.4).

We first formulate the problem in our abstract set up. For the remainder of this section we let

$$\tau:=\frac{2\pi}{T}\,,$$

and we consider the real Hilbert space $H:=L_2(0,T;\boldsymbol{R}^{2N})$. We define a linear operator

$$A:\operatorname{dom}(A)\subset H\to H \qquad \text{by} \qquad \operatorname{dom}(A):=\{x\in H^1(0,T;\boldsymbol{R}^{2N})|x(0)=x(T)\}\,,$$

and

$$Ax:=-J\dot{x}=\{\dot{q},-\dot{p}\}\,.$$

Finally $F: H\to H$ is defined by

$$F(u)(t):=\mathcal{H}_x(t,u(t))\,,\qquad \forall t\in[0,T]\,,\qquad \forall u\in H\,.$$

The assumptions (i) and (ii) imply that F is a continuous potential operator on H, the potential Φ being given by

$$\Phi(u)=\int_0^T\mathcal{H}(t,u(t))\,dt\,,\qquad \forall u\in H\,.$$

Moreover, F possesses a symmetric Gateaux derivative F' on H, and

$$[F'(u)h](t) = \mathcal{H}_{xx}(t, u(t))h(t), \qquad \forall u, h \in H, \ t \in [0, T]. \tag{12.5}$$

Therefore, by the mean value theorem,

$$\alpha\|u - v\|^2 \leqslant \langle F(u) - F(v), u - v\rangle \leqslant \beta\|u - v\|^2, \qquad \forall u, v \in H,$$

provided $\alpha, \beta \in \boldsymbol{R}$ satisfy

$$\alpha \leqslant \mathcal{H}_{xx}(t, \xi) \leqslant \beta, \qquad \forall (t, \xi) \in [0, T] \times \boldsymbol{R}^{2N} \tag{12.6}$$

that is $\alpha|\eta|^2 \leqslant (\mathcal{H}_{xx}(t, \xi)\eta, \eta) \leqslant \beta|\eta|^2$, $t \in [0, T]$, $\xi, \eta \in \boldsymbol{R}^{2N}$, where $(\cdot, \cdot)$ denotes the Euclidean inner product in $\boldsymbol{R}^{2N}$. Observe that (12.6) is equivalent to $\sigma(\mathcal{H}_{xx}(t, \xi)) \subset [\alpha, \beta]$, $\forall (t, \xi) \in [0, T] \times \boldsymbol{R}^{2N}$, and that condition (ii) implies the existence of constants $\alpha, \beta \in \boldsymbol{R}$ satisfying (12.6).

Clearly, every solution $u \in \operatorname{dom}(A)$ of

$$Au = F(u) \tag{12.7}$$

defines (by T-periodic continuation) a (classical) T-periodic solution of the Hamiltonian system (12.4). Conversely, every T-periodic (classical) solution of (12.4) defines (by restriction to the interval $[0, T]$) a solution of (12.7). Thus the problem of finding T-periodic solutions of the Hamiltonian system (12.4) is equivalent to the problem of finding solutions of the equation $Au = F(u)$. Observe that, for $u \in \operatorname{dom}(A)$, the equation $Au = F(u)$ is the Euler equation of the variational problem:

$$\delta \int_0^T \{(p(t), \dot{q}(t)) - \mathcal{H}(t, p(t), q(t))\}\, dt = 0,$$

subject to the periodicity conditions $(p(0), q(0)) = (p(T), q(T))$.

The following properties of the operator A are readily verified (cf. also [2]).

(12.1) LEMMA. (i) *A is self-adjoint, has closed range and a compact resolvent.*

(ii) *$\sigma(A) = \tau\boldsymbol{Z}$, and each $\lambda \in \sigma(A)$ is an eigenvalue of multiplicity $2N$.*

(iii) *For each $\lambda \in \sigma(A)$, the eigenspace* $\ker(\lambda - A)$ *is spanned by the orthogonal basis*

$$t \to \exp[\lambda t J]e_k = (\cos \lambda t)e_k + (\sin \lambda t)Je_k, \qquad k = 1, \ldots, 2N,$$

where $\{e_k | 1 \leqslant k \leqslant 2N\}$ *is the standard basis of* $\boldsymbol{R}^{2N}$. *In particular,* $\ker(A) = \boldsymbol{R}^{2N}$, *that is, it consists of the constant functions.*

On the basis of Lemma (12.1) and the remarks preceding it, it is now easy to prove the following

(12.2) LEMMA. *The problem* $Au = F(u)$ *satisfies conditions* (A), (F), *and* (R).

PROOF. It is obvious that conditions (A) and (F) are true, if we fix $\alpha, \beta \in \boldsymbol{R} \setminus \tau \boldsymbol{Z}$ such that (12.6) is satisfied. As for condition (R) we observe first that, due to (12.5), $F' | C([0, T], \boldsymbol{R}^{2N}) \in C\big(C([0, T], \boldsymbol{R}^{2N}), \mathcal{L}(H)\big)$. Moreover, Lemma (12.1.iii) implies $Z \hookrightarrow C\big([0, T], \boldsymbol{R}^{2N}\big)$, where Z is the subspace of H spanned by the finitely many eigenfunctions of A belonging to the eigenvalues $\tau \boldsymbol{Z} \cap [\alpha, \beta]$. Finally, observe that, for every $\lambda \notin \tau \boldsymbol{Z}$ and $v \in H$,

$$[(\lambda - A)^{-1} v](t) = \exp[t\lambda J] u_0 - \int_0^t \exp[\lambda(t-s)J] v(s)\, ds\,, \qquad 0 \leqslant t \leqslant T\,,$$

where

$$u_0 := -\big[I_{2N} - \exp[T\lambda J]\big]^{-1} \int_0^T \exp[\lambda(T-s)J] v(s)\, ds\,.$$

This implies

$$(\lambda - A)^{-1} \in \mathcal{L}\big(H, C([0, T], \boldsymbol{R}^{2N})\big) \qquad \forall \lambda \notin \tau \boldsymbol{Z}\,.$$

Consequently, since Proposition (4.1) implies

$$P_{\pm} v(z) = (\lambda - A)^{-1} P_{\pm} \big[v(z) - F\big(v(z) + z\big)\big] \qquad \forall z \in Z\,,$$

provided $\lambda \notin \tau \boldsymbol{Z}$, and since $v(\cdot) \in C(Z, H)$, it follows that

$$P_{\pm} v(\cdot) \in C\big(Z, C([0, T], \boldsymbol{R}^{2N})\big)\,.$$

Hence condition (R) with $E = C\big([0, T], \boldsymbol{R}^{2N}\big)$ is satisfied. $\square$

In order to formulate the asymptotic behaviour of $\mathcal{H}$ in the abstract framework, we introduce some special linear operators in H. Let $b \in \mathcal{L}_s(\boldsymbol{R}^{2N})$ be a symmetric matrix, then $B \in \mathcal{L}(H)$ is defined by

$$Bu(t) := bu(t)\,, \qquad \forall t \in [0, T],\ \forall u \in H\,.$$

The following Lemma summarizes some properties of B needed later on. Here and in the following $\sigma_p(\cdot)$ denotes the point spectrum.

(12.3) LEMMA. (i) *B is symmetric and $\sigma(B) = \sigma_p(B) = \sigma(b)$. The operator $A - B$ is selfadjoint and has compact resolvent, hence $\sigma(A - B) = \sigma_p(A - B)$.*

(ii) *$\lambda \in \sigma(A - B)$ iff $\sigma(J(b + \lambda)) \cap i\tau\boldsymbol{Z} \neq \emptyset$.*

(iii) *Let $\alpha = -\beta$, then B commutes with $P := \int_\alpha^\beta dE_\lambda$, the orthogonal projection onto the subspace Z of H, spanned by the eigenfunctions of A belonging to the eigenvalues in $[\alpha, \beta]$.*

PROOF. (i) It is obvious that B is symmetric and $\sigma(B) = \sigma_p(B) = \sigma(b)$. Standard arguments (cf. [27]) and Lemma (12.1.i) imply that $A - B$ is self-adjoint and has compact resolvent.

(ii) From (i) we conclude that $\lambda \in \sigma(A - B)$ iff the equation $(A - B)u = \lambda u$ has a nontrivial solution $u \in \operatorname{dom}(A)$. From $-J\dot{u} - bu = \lambda u$ we find $u(t) = \exp[tJ(b + \lambda)]u(0)$. Since $u(0) = u(T)$ for $u \in \operatorname{dom}(A)$, we conclude that $\lambda \in \sigma(A - B)$ iff $1 \in \sigma(\exp[TJ(b + \lambda)]) = \exp[T\sigma(J(b + \lambda))]$, by the spectral mapping theorem. Now the assertion (ii) is obvious.

(iii) For every $\lambda \in \sigma(A) = \tau\boldsymbol{Z}$, let $E(\lambda) := \ker(\lambda - A)$ be the eigenspace of λ. Then, by Lemma (12.1):

$$E(\lambda) + E(-\lambda) = \left\{ \sqrt{\frac{2}{T}} \cos(\lambda t)x + \sqrt{\frac{2}{T}} \sin(\lambda t)y \,\middle|\, t \in [0, T], \ x, y \in \boldsymbol{R}^{2N} \right\}. \tag{12.8}$$

Obviously, B maps $E(\lambda) + E(-\lambda)$ into itself. This implies the last part of the assertion. $\square$

As a technical sideremark we observe that if $b \in \mathcal{L}_s(\boldsymbol{R}^{2N})$ is of the form $b = \operatorname{Diag}(a, -a)$, for $a \in \mathcal{L}_s(\boldsymbol{R}^N)$, then $B(\ker(\lambda - A)) \subset \ker(-\lambda - A)$. Hence $B(X) \subset Y := P_+(H)$ and $B(Y) \subset X := P_-(H)$, provided $\alpha = -\beta$. Thus B commutes with P, but it does not commute with P_+ or P_-.

After these technical preparations we are ready to prove the following existence statement for T-periodic solutions of the Hamiltonian system (12.4).

(12.4) THEOREM. *Let $\mathcal{H}(t, x)$ be periodic in t with period $T > 0$. Assume the Hamiltonian vectorfield is asymptotically linear:*

$$J\mathcal{H}_x(t, \xi) = Jb_\infty \xi + o(|\xi|), \quad \text{as } |\xi| \to \infty, \tag{12.9}$$

uniformly in $t \in \boldsymbol{R}$, for a time independent $b_\infty \in \mathcal{L}_s(\boldsymbol{R}^{2N})$. Then the Hamiltonian system

$$\dot{u} = J\mathcal{H}_x(t, u)$$

has at least one T-periodic solution, provided $\sigma(Jb_\infty) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$.

Actually we prove a more general statement. Instead of requiring (12.9) we merely make the following assumptions on the asymptotic behaviour of $\mathcal{H}_x$: there exist two constants γ_∞ and δ_∞, such that

$$(12.9') \qquad |\mathcal{H}_x(t, x) - b_\infty x| \leqslant \gamma_\infty |x| + \delta_\infty$$

for all $(t, x) \in \boldsymbol{R} \times \boldsymbol{R}^{2N}$, where γ_∞ satisfies $0 < \gamma_\infty < \min\{|\lambda| \,|\, \lambda \in \sigma(A - B_\infty)\}$; the operator $B_\infty \in \mathcal{L}(H)$ being defined as

$$B_\infty u(t) := b_\infty u(t), \quad t \in [0, T] \text{ and } u \in H.$$

PROOF. In view of the general assumption (12.2) we fix $\alpha := -\beta$, $\beta > 0$, such that $\alpha \notin \tau \boldsymbol{Z}$ and $\sigma(b_\infty)$, $\sigma(\mathcal{H}_{xx}(t, \xi)) \subset [\alpha, \beta]$ for all $(t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^{2N}$. Since by assumption $\sigma(Jb_\infty) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$, we know by Lemma (12.3.(ii)), that $0 \notin \sigma(A - B_\infty)$. Due to (12.9′), condition (F_∞) is met. By Lemma (12.3.(iii)), B_∞ commutes with P, hence, recalling Lemma (12.2), the assertion follows immediately from Theorem (9.1). □

In the autonomous case, that is, if $\mathcal{H}$ is independent of t, Theorem (12.4) is not of much interest. Indeed, in this case the first part of the following proposition implies that, in general, $\mathcal{H}$ possesses a critical point hence, (12.4) has a constant solution, which is clearly T-periodic.

(12.5) PROPOSITION. *Let $f \in C^2(\boldsymbol{R}^n, \boldsymbol{R})$, and suppose that there exists a nonsingular matrix $b_\infty \in \mathcal{L}_s(\boldsymbol{R}^n)$ such that*

$$|f'(x) - b_\infty x| \leqslant \gamma_\infty |x| + \delta_\infty, \qquad \forall x \in \boldsymbol{R}^n,$$

where $\delta_\infty > 0$ and $0 < \gamma_\infty < \|b_\infty^{-1}\|^{-1}$ are constants. Then f has at least one critical point. Suppose, in addition, that $f'(0) = 0$. Then f has at least one nontrivial critical point provided

$$m^-(b_\infty) \notin [m^-(f''(0)), m^-(f''(0)) + m^0(f''(0))],$$

where $m^-(\cdot)$ denotes the «negative» Morse index.

PROOF. The assertion follows easily from Theorem (9.1) and Proposition (9.3), letting $H := \boldsymbol{R}^n$, $A := 0$, and $F := f'$. □

In view of this Proposition we shall assume in the following that the Hamiltonian vectorfield $J\mathcal{H}_x$ possesses an equilibrium point, which we assume to be 0, hence $J\mathcal{H}_x(t, 0) = 0$. We consider a Hamiltonian vectorfield satisfying

$$J\mathcal{H}_x(t,\xi) = Jb_0\xi + o(|\xi|), \quad \text{as } |\xi| \to 0$$

and

$$J\mathcal{H}_x(t,\xi) = Jb_\infty\xi + o(|\xi|), \quad \text{as } |\xi| \to \infty,$$

uniformly in $t \in \boldsymbol{R}$, for two symmetric time independent matrices b_0, $b_\infty \in \mathcal{L}_s(\boldsymbol{R}^{2N})$. Our aim now is to find T-periodic solutions of $\dot{u} = J\mathcal{H}_x(t,u)$, which are not the trivial solution $u(t) = 0$. In order to describe the difference between the two linearized systems at 0, and at ∞, Jb_0 and Jb_∞, which will guarantee a nontrivial T-periodic solution, we shall introduce next an integer, $\operatorname{Ind}(b_0, b_\infty, \tau)$.

For a fixed symmetric $b \in \mathcal{L}_s(\boldsymbol{R}^{2N})$ we define the quadratic forms Q_μ, $\mu \in \boldsymbol{R}$, on $\boldsymbol{R}^{2N} \times \boldsymbol{R}^{2N}$ as follows:

$$Q_\mu(z) := 2\mu(Jx, y) - (bx, x) - (by, y), \tag{12.10}$$

with $z := \{x, y\} \in \boldsymbol{R}^{2N} \times \boldsymbol{R}^{2N}$. The matrix of this form is given by

$$\mu\begin{pmatrix} 0 & J^T \\ J & 0 \end{pmatrix} - \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \tag{12.11}$$

observing $-J = J^T$. With $m^+(\cdot)$, $m^0(\cdot)$ and $m^-(\cdot)$ we denote in the following the positive, the zero and the negative Morse index, respectively, of a quadratic form or of the symmetric matrix defining it.

(12.6) LEMMA. (i) $m^+(Q_\mu) = 2N$ *if* $\mu > \max\{\alpha \in \boldsymbol{R} | i\alpha \in \sigma(Jb)\}$. (ii) *Assume Jb has no purely imaginary eigenvalues* (*except possibly* 0) *then* $m^+(Q_\mu) = 2N$ *for all* $\mu > 0$. *If, in addition, b is invertible, then* $m^+(Q_\mu) = 2N$ *for all* $\mu \geqslant 0$, *and* $m^+(b) = N = m^-(b)$. (iii) $m^0(Q_\mu) = 0$ *iff* $i\mu \notin \sigma(Jb)$.

PROOF. If $\mu > 0$ and sufficiently large, then $m^+(Q_\mu) = 2N$, the positive index of the form (Jx, y) in (12.10). If μ decreases, the index $m^+(Q_\mu)$ can change only at those values of μ, for which the matrix (12.11) is singular, that is $m^0(Q_\mu) \neq 0$. This happens precisely for those values of $\mu \in \boldsymbol{R}$, for which $i\mu$ is a purely imaginary eigenvalue of Jb. Indeed, assume $z := \{x, y\} \in \boldsymbol{R}^{2N} \times \boldsymbol{R}^{2N}$ is an eigenvector of (12.11) with eigenvalue 0. Then, since $J^T = -J$,

$$bx + \mu Jy = 0$$

$$by - \mu Jx = 0,$$

therefore $b(x+iy)=\mu J(ix-y)=i\mu J(x+iy)$, hence

$$Jb(x+iy)=-i\mu(x+iy)\,,$$

therefore $\pm i\mu\in\sigma(Jb)$. From these remarks the assertion is immediate. $\square$

We now define $\operatorname{Ind}(b_0, b_\infty, \tau)\in \boldsymbol{Z}$ for $b_0, b_\infty \in \mathcal{L}_s(\boldsymbol{R}^{2N})$ and for $\tau>0$ as follows:

$$\text{(12.12)}\quad \operatorname{Ind}(b_0, b_\infty, \tau):=m^-(b_0)-m^-(b_\infty)+\sum_{j=1}^{\infty}\{m^+(Q^0_{j\tau})-m^+(Q^\infty_{j\tau})\}\,.$$

Here $Q^0_{j\tau}$ (resp. $Q^\infty_{j\tau}$) is the quadratic form (12.10) with $\mu:=j\tau$ and $b:=b_0$ (resp. $b:=b_\infty$). In view of Lemma (12.6) the sum is finite. The following properties of the integer (12.12) follow immediately from the definition and from Lemma (12.6). Observe, 0 is also considered as purely imaginary in the next Lemma.

(12.7) LEMMA. (i) *$\operatorname{Ind}(b_0, b_\infty, \tau)$ is a symplectic invariant, that is $\operatorname{Ind}(s^Tb_0s, s^Tb_\infty s, \tau)=\operatorname{Ind}(b_0, b_\infty, \tau)$ for all $s\in \operatorname{Sp}(2N)$.*

(ii) *$\operatorname{Ind}(b_0, b_\infty, \tau)=0$ if either (1) $b_0=b_\infty$, or (2) Jb_0 and Jb_∞ have no purely imaginary eigenvalues.* $\square$

We are ready to prove

(12.8) THEOREM. *Let $\mathcal{H}(t,x)$ be periodic in t with period $T>0$. Assume:*

$$J\mathcal{H}_x(t,\xi)=Jb_0\xi+o(|\xi|)\,,\qquad |\xi|\to 0$$
$$J\mathcal{H}_x(t,\xi)=Jb_\infty\,\xi+o(|\xi|)\,,\qquad |\xi|\to \infty\,,$$

uniformly in $t\in\boldsymbol{R}$, for two time independent symmetric $b_0, b_\infty\in\mathcal{L}_s(\boldsymbol{R}^{2N})$. Assume $\sigma(Jb_\infty)\cap i(2\pi/T)\boldsymbol{Z}=\emptyset$ and $\sigma(Jb_0)\cap i(2\pi/T)\boldsymbol{Z}=\emptyset$. If

$$\operatorname{Ind}\left(b_0, b_\infty, \frac{2\pi}{T}\right)\neq 0\,,$$

there exists at least one nontrivial T-periodic solution of $\dot u=J\mathcal{H}_x(t,u)$.

In view of Lemma (12.7) the occurrence of purely imaginary eigenvalues of Jb_0 or Jb_∞ is necessary in order to have $\operatorname{Ind}(b_0, b_\infty, 2\pi/T)\neq 0$.

PROOF. Define B_0, $B_\infty\in\mathcal{L}(H)$ by $B_0u(t)=b_0u(t)$ and $B_\infty u(t)=b_\infty u(t)$, $t\in[0,T]$, $u\in H$. Fix $\alpha:=-\beta$ for $\beta>0$ sufficiently large, $\alpha\notin\sigma(A)$. By assumption $\sigma(Jb_\infty)\cap i(2\pi/T)\boldsymbol{Z}=\emptyset$, hence by Lemma (12.3), $0\notin\sigma(A-B_\infty)$, and (F_∞) is satisfied. The statement follows from Proposition (9.3). Indeed, B_∞ and B_0 commute with P by Lemma (12.3). From Proposition (4.5) and formula (3.2) we find

$$a''(0) = (A - B_0)|Z .$$

Since $\sigma(Jb_0) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$, we have $0 \notin \sigma(A - B_0)$, hence $m_Z^0\big(a''(0)\big) = 0$. It remains to prove that

$$m_Z^+(A - B_\infty) \neq m_Z^+(A - B_0) .$$

Let $\lambda = j\tau \in \sigma(A), j \geqslant 1$, with $\tau = 2\pi/T$, and let $E(\lambda) \subset H$ be the corresponding eigenspace. The restriction of $A - B_i$, $i = 0, \infty$, onto the subspace $E(-\lambda) + E(\lambda) \subset H$ defines a quadratic form. In view of (12.8) this form is given by (12.10), with $\mu = \lambda$ and with $b = b_i$, $i = 0, \infty$. Therefore, by (12.12), the definition of $\operatorname{Ind}(b_0, b_\infty, \tau)$,

$$m_Z^+(A - B_0) - m_Z^+(A - B_\infty) = \operatorname{Ind}(b_0, b_\infty, \tau) .$$

The statement now follows from Proposition (9.3). $\square$

We next compute $\operatorname{Ind}(b_0, b_\infty, \tau)$ in terms of the purely imaginary eigenvalues of Jb_0 and Jb_∞. To simplify the presentation we do not consider the most general case, for which we refer to [2*b*].

Let $b \in \mathcal{L}_s(\boldsymbol{R}^{2N})$ be symmetric. The quadratic Hamiltonian function

$$h(x) = \tfrac{1}{2}(bx, x) , \qquad x \in \boldsymbol{R}^{2N} ,$$

defines the linear Hamiltonian vectorfield $\dot{x} = Jbx$ on $\boldsymbol{R}^{2N}$. Clearly, every purely imaginary eigenvalue $i\alpha$, $\alpha \in \boldsymbol{R}$, of the infinitesimally symplectic matrix Jb gives rise to a periodic solution of $\dot{x} = Jbx$ with period $2\pi/|\alpha|$. The purely imaginary eigenvalues of Jb occur in pairs $\pm i|\alpha|$, $\alpha \in \boldsymbol{R}$, that is, if $i\alpha$ is an eigenvalue of multiplicity l, then $-i\alpha$ is an eigenvalue of multiplicity l. It is well known (cf. [12]) that the eigenspace belonging to a pair $\pm i\alpha$ of purely imaginary eigenvalues is a symplectic subspace of $\boldsymbol{R}^{2N}$, the restriction of Jb onto this subspace is infinitesimally symplectic with quadratic Hamiltonian. Let $\pm i|\alpha_k|$, $k = 1, 2, \ldots$, be the pairs of purely imaginary eigenvalues of Jb counted with their multiplicities. For simplicity we shall assume, that on the invariant subspace which belongs to these eigenvalues, there exists a symplectic transformation $s \in \operatorname{Sp}(2r)$, which puts the corresponding Hamiltonian function into the following form:

$$\tag{12.13} \tfrac{1}{2}\sum_{k=1}^{r} \alpha_k(p_k^2 + q_k^2) ,$$

which is a symplectic normal form. We shall rephrase this assumption as follows: the imaginary part of Jb is symplectically diagonalizable. It is well known that this assumption is met, if the imaginary eigenvalues are simple, or if the restrictions of the Hamiltonian function onto the eigenspaces of the pairs $\pm i\alpha_k$ are positively or negatively definite. The symplectic normal form (12.13) allows to choose the signs of the pairs $\pm i|\alpha_k|$, $k = 1, 2, \ldots, r$ in a symplectically invariant way; we call the (unordered) set

$$S := \{i\alpha_1, i\alpha_2, \ldots, i\alpha_r\}$$

the set of positively oriented imaginary eigenvalues of Jb. For example, let the multiplicity of the pair $\pm i\alpha$, $\alpha \in \boldsymbol{R}$, be 3, with normal form $\frac{1}{2}\alpha \sum_{i=1}^{2} (p_k^2 + q_k^2) - \frac{1}{2}\alpha(p_3^2 + q_3^2)$, then the set of positively oriented eigenvalues is $\{i\alpha, i\alpha, -i\alpha\}$. For an intrinsic definition of the symplectically invariant « orientation », induced by the symplectic structure in the set of imaginary eigenvalues, we refer to [31*a*].

In the following let $[S]$ denote the cardinality of a finite set S.

(12.9) LEMMA. *Assume the imaginary part of Jb is symplectically diagonalizable, and let*

$$S = \{i\alpha_1, i\alpha_2, \ldots, i\alpha_r\}$$

be the set of positively oriented imaginary eigenvalues. Then:

$$m^+(Q_\mu) = 2N - 2[i\alpha \in S | \alpha > \mu] + 2[i\alpha \in S | \alpha < -\mu]$$

provided $\mu > 0$ and $\mu \neq |\alpha_k|$, $k = 1, 2, \ldots, r$.

If, in addition, b is invertible, then:

$$m^-(b) = N - [i\alpha \in S | \alpha > 0] + [i\alpha \in S | \alpha < 0] .$$

PROOF. In the proof of Lemma (12.6) we have seen that, for $\mu > 0$ and sufficiently large, $m^+(Q_\mu) = 2N$, and that if μ decreases, the index $m^+(Q_\mu)$ can change only if $i\mu \in \sigma(Jb)$. Let now $i\alpha$, $\alpha \in \boldsymbol{R}$ be a simple, positively oriented eigenvalue of Jb. Putting, by means of a symplectic transformation, the restriction of the Hamiltonian belonging to Jb onto the eigenspace of the pair $\pm i\alpha$ into its symplectic normalform (12.13), that is into $\frac{1}{2}\alpha(x_1^2 + x_2^2)$, we find for the restriction of $-Q_\mu$ onto these eigenspaces:

$$(12.14) \qquad 2\mu(x_2 y_1 - x_1 y_2) + \alpha(x_1^2 + x_2^2 + y_1^2 + y_2^2) =$$
$$= \{2\mu x_2 y_1 + \alpha(x_2^2 + y_1^2)\} + \{-2\mu x_1 y_2 + \alpha(x_1^2 + y_2^2)\} ,$$

where $\{x, y\} \in \boldsymbol{R}^2 \times \boldsymbol{R}^2$. Assume $\alpha > 0$, then the positive index of the quadratic form (12.14) changes by $+2$ if μ crosses α from above. On the other hand if $\alpha < 0$, then the positive index of the form changes by -2 if μ crosses $-\alpha = |\alpha| > 0$ from above. In case the eigenvalues of the pair $\pm i\alpha$ are not simple, the restriction of Q_μ onto the eigenspaces is a sum of quadratic forms of the type (12.14) according to the corresponding normal-form which by assumption does exist. The Lemma now follows. □

We are ready to express the integer $\operatorname{Ind}(b_0, b_\infty, \tau)$ in terms of the positively oriented imaginary eigenvalues of Jb_0 and Jb_∞. By means of Lemma (12.9), Lemma (12.6) and Definition (12.12) we easily find:

(12.10) LEMMA. *Let $b_0, b_\infty \in \mathcal{L}_s(\boldsymbol{R}^{2N})$ and $\tau > 0$. Assume the imaginary parts of Jb_0 and Jb_∞ are symplectically diagonalizable and let*

$$S^0 := \{i\alpha_1^0, i\alpha_2^0, \ldots, i\alpha_{r_0}^0\},$$

and

$$S^\infty := \{i\alpha_1^\infty, i\alpha_2^\infty, \ldots, i\alpha_{r_\infty}^\infty\},$$

be the sets of positively oriented imaginary eigenvalues of Jb_0 and Jb_∞. Assume $\sigma(Jb_0) \cap i\tau\boldsymbol{Z} = \emptyset$ and $\sigma(Jb_\infty) \cap i\tau\boldsymbol{Z} = \emptyset$. Then

$$\begin{aligned}(12.15)\qquad \operatorname{Ind}(b_0, b_\infty, \tau) &= m^-(b_0) - m^-(b_\infty) + \\ &+ 2\sum_{j=1}^{\infty}\big([i\alpha^0 \in S^0 | \alpha^0 < -j\tau] - [i\alpha^0 \in S^0 | \alpha^0 > j\tau]\big) - \\ &- 2\sum_{j=1}^{\infty}\big([i\alpha^\infty \in S^\infty | \alpha^\infty < -j\tau] - [i\alpha^\infty \in S^\infty | \alpha^\infty > j\tau]\big),\end{aligned}$$

where

$$\begin{aligned}m^-(b_0) - m^-(b_\infty) = {} & [i\alpha^0 \in S^0 | \alpha^0 < 0] - [i\alpha^0 \in S^0 | \alpha^0 > 0] - \\ & - [i\alpha^\infty \in S^\infty | \alpha^\infty < 0] + [i\alpha^\infty \in S^\infty | \alpha^\infty > 0].\end{aligned}$$

In particular, $\operatorname{Ind}(b_0, b_\infty, \tau) \neq 0$ *if either* (1) $\alpha^0 > 0$ *and* $\alpha^\infty < 0$, *or* (2) $\alpha^0 < 0$ *and* $\alpha^\infty > 0$, *or* (3) Jb_0 *(resp. Jb_∞) has no purely imaginary eigenvalues and the restriction of the form b_∞ (resp. b_0) onto the eigenspace of the purely imaginary eigenvalues of Jb_∞ (resp. Jb_0) is positively or negatively definite.*

(12.11) THEOREM. *Let $\mathcal{H}(t, x)$ be periodic in t with period $T > 0$, and let*

$$\mathcal{H}_x(t, \xi) = b_0\,\xi + o(|\xi|), \qquad |\xi| \to 0$$

$$\mathcal{H}_x(t, \xi) = b_\infty \xi + o(|\xi|), \qquad |\xi| \to \infty$$

uniformly in $t \in \boldsymbol{R}$, *for two time independent symmetric* $b_0, b_\infty \in \mathcal{L}_s(\boldsymbol{R}^{2N})$. *Assume* $\sigma(Jb_0) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$ *and* $\sigma(Jb_\infty) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$, *and assume that* Jb_0 *and* Jb_∞ *are symplectically diagonalizable. If* $\operatorname{Ind}(b_0, b_\infty, 2\pi/T) \neq 0$, *there exists at least one nontrivial T-periodic solution of* $\dot{u} = J\mathcal{H}_x(t, u)$. *Here* $\operatorname{Ind}(b_0, b_\infty, 2\pi/T)$ *is explicitely given by* (12.15).

(12.12) COROLLARY 1. *If the restriction of* b_0 *onto the eigenspace of the purely imaginary eigenvalues of* Jb_0 *is positively definite* (*resp. negatively definite*) *and if the restriction of* b_∞ *onto the eigenspace of the purely imaginary eigenvalues of* Jb_∞ *is negatively definite* (*resp. positively definite*) *then there exists at least one nontrivial T-periodic solution of* $\dot{u} = J\mathcal{H}_x(t, u)$ *provided* $\sigma(Jb_\infty) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$ *and* $\sigma(Jb_0) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$. □

In particular, if $b_0 > 0$ (resp. $b_0 < 0$) and $b_\infty < 0$ (resp. $b_\infty > 0$), there exists at least one nontrivial T-periodic solution provided the nonresonance conditions of Corollary 1 are satisfied.

(12.13) COROLLARY 2. *If* Jb_0 (*resp.* Jb_∞) *has no purely imaginary eigenvalues, and if the restriction of* b_∞ (*resp.* b_0) *onto the eigenspace of the imaginary eigenvalues of* Jb_∞ (*resp.* Jb_0) *is definite, then there exists at least one nontrivial T-periodic solution of* $\dot{u} = J\mathcal{H}_x(t, u)$ *provided* $\sigma(Jb_\infty) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$ *and* $\sigma(Jb_0) \cap i(2\pi/T)\boldsymbol{Z} = \emptyset$.

In particular, if Jb_0 (resp. Jb_∞) has no imaginary eigenvalues and Jb_∞ (resp. Jb_0) has only one pair $\pm i\alpha$, $\alpha \in \boldsymbol{R}$, of imaginary eigenvalues, which, in addition, are simple, then there exists at least one nontrivial T-periodic solution, provided $\alpha \notin (2\pi/T)\boldsymbol{Z}$.

In case the Hamiltonian function is independent of t, Theorem (12.8) guarantees a nonzero $(2\pi/\tau)$-periodic solution of $\dot{u} = J\mathcal{H}_x(u)$, for every frequency τ for which $\operatorname{Ind}(b_0, b_\infty, \tau) \neq 0$. The periodic solution so found may however be a constant, $u(t) := \gamma \in \boldsymbol{R}^{2N}$, $\gamma \neq 0$, namely if γ is an equilibrium point, that is $\mathcal{H}_x(\gamma) = 0$. Our next aim is to find nonconstant T-periodic solutions of the time independent Hamiltonian system with prescribed period T. We clearly have to impose additional assumptions on $\mathcal{H}$. We first study the case of a convex function $\mathcal{H}$, such that

$$J\mathcal{H}_x(\xi) = Jb_0\xi + o(|\xi|), \qquad |\xi| \to 0$$
$$J\mathcal{H}_x(\xi) = Jb_\infty\xi + o(|\xi|), \qquad |\xi| \to \infty$$

for two positively definite $b_0, b_\infty \in \mathcal{L}_s(\boldsymbol{R}^{2N})$. In this case, as it is well known, all the eigenvalues of Jb_0 and Jb_∞ are purely imaginary, $\pm i\alpha_k^\sigma$, $\sigma = 0, \infty$, $1 \leqslant k \leqslant N$ (counted with their multiplicities), and Jb_0, Jb_∞ are symplectically diagonalizable. There are symplectic transformations which put $\frac{1}{2}(b_\sigma x, x)$, $\sigma = 0, \infty$ into their symplectic normal forms

$$\frac{1}{2}\sum_{k=1}^{N} \alpha_k^\sigma (p_k^2 + q_k^2)\,, \qquad \alpha_k^\sigma > 0,\; k = 1, 2, \ldots, N \tag{12.16}$$

for $\sigma = 0, \infty$. Assume $\sigma(Jb_0) \cap i\tau\boldsymbol{Z} = \emptyset$ and $\sigma(Jb_\infty) \cap i\tau\boldsymbol{Z} = \emptyset$. It then follows from Lemma (12.10):

$$\operatorname{Ind}(b_0, b_\infty, \tau) = 2\sum_{j=1}^{\infty} \big([\alpha^\infty|\alpha^\infty > j\tau] - [\alpha^0|\alpha^0 > j\tau]\big)\,. \tag{12.17}$$

(12.14) THEOREM. *Assume* $\mathcal{H}(x)$ *to be a convex function, such that*

$$\begin{cases} \mathcal{H}_x(\xi) = b_0\,\xi + o(|\xi|)\,, & |\xi| \to 0 \\ \mathcal{H}_x(\xi) = b_\infty \xi + o(|\xi|)\,, & |\xi| \to \infty \end{cases} \tag{12.18}$$

with $b_0, b_\infty \in \mathcal{L}_s(\boldsymbol{R}^{2N})$, $b_0 > 0$, $b_\infty > 0$. *Then for every* $\tau > 0$, *such that* $\sigma(Jb_0) \cap$ $\cap\, i\tau\boldsymbol{Z} = \emptyset$ *and* $\sigma(Jb_\infty) \cap i\tau\boldsymbol{Z} = \emptyset$, *and such that* $\operatorname{Ind}(b_0, b_\infty, \tau) \neq 0$ *there is a nonconstant* $(2\pi/\tau)$*-periodic solution of* $\dot{u} = J\mathcal{H}_x(u)$. *Here* $\operatorname{Ind}(b_0, b_\infty, \tau)$ *is given by* (12.17).

PROOF. Since by the convexity of $\mathcal{H}$, 0 is the only equilibrium point of the Hamiltonian vectorfield $J\mathcal{H}_x$, the statement is an immediate consequence of theorem (12.8). $\square$

(12.15) COROLLARY. *Let* $\mathcal{H}$ *be convex and satisfy* (12.18). *Then the Hamiltonian system* $\dot{u} = J\mathcal{H}_x(u)$ *possesses at least one nonconstant* $(2\pi/\tau)$*-periodic solution for every* $\tau > 0$ *satisfying one of the following conditions*:

(*a*) $\max\{\alpha_k^0\} < \tau < \max\{\alpha_k^\infty\}$ *and* $\alpha_k^\infty \notin \tau\boldsymbol{Z}$, $1 \leqslant k \leqslant N$.

(*b*) $\max\{\alpha_k^\infty\} < \tau < \max\{\alpha_k^0\}$ *and* $\alpha_k^0 \notin \tau\boldsymbol{Z}$, $1 \leqslant k \leqslant N$.

PROOF. From (12.17) we read off:

(*a*) $\operatorname{Ind}(b_0, b_\infty, \tau) = \quad 2\sum_{j=1}^{\infty} [\alpha^\infty|\alpha^\infty > j\tau] > 0$

(*b*) $\operatorname{Ind}(b_0, b_\infty, \tau) = -\,2\sum_{j=1}^{\infty} [\alpha^0|\alpha^0 > j\tau] < 0\,.$ $\square$

We point out, that in many of the previous statements the nonresonance condition on b_0 is not necessary. This requires a more careful study of

$\operatorname{Ind}(b_0, b_\infty, \tau)$ and the application of Proposition (9.3) in its full generality (cf. [2b]). We mention another consequence of Theorem (12.14). Assume that S^∞, the set of positively oriented eigenvalues of Jb_∞, is different from S^0, the set of positively oriented eigenvalues of Jb_0; then there is a sequence $\{\tau_k\}$, $\tau_k > 0$, such that $\lim\limits_{k\to\infty} \tau_k = 0$ and $\operatorname{Ind}(b_0, b_\infty, \tau_k) \neq 0$. Therefore, if the two linear Hamiltonian vectorfields Jb_0 and Jb_∞ are symplectically inequivalent, there is an open and unbounded set $U \subset \boldsymbol{R}$, such that $\dot{u} = J\mathcal{H}_x(u)$ possesses a nonconstant T-periodic solution for every $T \in U$. In addition, it can be shown [2b], that the integer $|\operatorname{Ind}(b_0, b_\infty, \tau)|$ is a lower bound of the number of geometrically distinct $2\pi/\tau$-periodic solutions.

In Theorem (12.14) $\mathcal{H}$ is assumed to be a convex function. We finally present an existence statement which does not assume that the autonomous Hamiltonian is convex. Instead of giving rather general conditions, we restrict ourselves to a simple situation, and we leave it to the reader to deduce more general results along the lines of this proof.

(12.16) PROPOSITION. *Let $\mathcal{H}(t, x)$ be periodic in t with period $T > 0$, and let $\mathcal{H} \geqslant 0$. Suppose that, for some constants $\alpha < 0 < \beta$, $\sigma(\mathcal{H}_{xx}(t, x)) \subset [\alpha, \beta]$, $\forall (t, x) \in \boldsymbol{R} \times \boldsymbol{R}^{2N}$. Assume*

$$J\mathcal{H}_x(t, \xi) = o(|\xi|)\,, \qquad |\xi| \to 0$$

$$J\mathcal{H}_x(t, \xi) = \beta J\xi + o(|\xi|)\,, \qquad |\xi| \to \infty$$

uniformly in $t \in \boldsymbol{R}$. Then the Hamiltonian system $\dot{u} = J\mathcal{H}_x(t, u)$ possesses at least one nontrivial T-periodic solution, provided $T > 2\pi/\beta$ and $T \notin (2\pi/\beta)\boldsymbol{Z}$.

If, in addition, $\mathcal{H}$ is independent of t, then the system $\dot{u} = J\mathcal{H}_x(u)$ possesses at least one nonconstant T-periodic solution for every $T > 0$ satisfying $T > 2\pi/\beta$ and $T \notin (2\pi/\beta)\boldsymbol{Z}$.

PROOF. Since $B_\infty = \beta I_H$ commutes with A and $\sigma(B_\infty) = \{\beta\} \notin \tau\boldsymbol{Z} = \sigma(A)$, it follows that condition (F_∞) is satisfied and that $\gamma_\infty > 0$ can be chosen arbitrarily small. Since $\beta > \tau$, there exists a largest positive number j such that $\lambda_n = j\tau$. Consequently, since $B_0 = 0$, it follows that $(A - B_0)|Z \nleqslant 0$. On the other hand, $(A - B_\infty)|Z = (A - \beta)|Z < 0$. Consequently, letting $C_0^+ := B_0 = 0$ and $C_\infty^- = B_\infty$ and choosing $\gamma_\infty > 0$ sufficiently small, Theorem (8.3.i) implies the existence of a nontrivial solution u of $Au = F(u)$.

Suppose now that $\mathcal{H}$ is independent of $t \in \boldsymbol{R}$ and that $u(t) = \gamma \in \boldsymbol{R}^{2N}$ for all $t \in [0, T]$. Then $u \in \ker(A)$ and $\Phi(u) = T\mathcal{H}(\gamma)$. Hence, by Lemma (3.1) and since $\mathcal{H} \geqslant 0$, $a(z) = -T\mathcal{H}(\gamma) \leqslant 0$, which contradicts the fact that $a(z) > 0$ by Remark (8.5). Now the assertion follows. $\square$

The problem of finding periodic solutions of Hamiltonian equations $\dot{u} = J\mathcal{H}_x(t, u)$ is an old one. Many papers are devoted to the study of periodic solutions near an equilibrium point, say $x = 0$, of the (usually autonomous) Hamiltonian vectorfield $J\mathcal{H}_x$. Clearly, every purely imaginary eigenvalue of the linearization $J\mathcal{H}_{xx}(0)$ gives rise to a periodic solution of the linearized equation, and it is well known that the presence of purely imaginary eigenvalues is necessary in order to find periodic solutions near $x = 0$. In the case of purely imaginary eigenvalues $\pm i\alpha_1, \ldots, \pm i\alpha_N$, which are nondegenerate in the sense that, say $\alpha_k \notin \alpha_1 \boldsymbol{Z}$, $2 \leqslant k \leqslant N$, Lyapunov established a one parameter family of periodic solutions close to $x = 0$, having periods close to $2\pi/\alpha_1$ (cf. [28, 39]; see also [7-9]). More recently, A. Weinstein [46, 47] removed the additional nonresonance condition. He proved for $\mathcal{H} \in C^2(\boldsymbol{R}^{2N}, \boldsymbol{R})$, satisfying $\mathcal{H}(0) = 0$, $\mathcal{H}'(0) = 0$, and having a positive definite Hessian $\mathcal{H}''(0)$, that, for sufficiently small $\varepsilon > 0$, the energy surface $\mathcal{H}^{-1}(\varepsilon)$ contains at least N periodic orbits, whose periods are close to those of the linearized system $\mathcal{H}''(0)$. Subsequently this result was generalized by J. Moser [31], R. Bottkol [11], A. Weinstein [45], and by E. R. Faddell and P. H. Rabinowitz [24]. The existence proofs of these bifurcation results depend on topological arguments. If $\mathcal{H}$ is sufficiently smooth, that is, $\mathcal{H} \in C^k(\boldsymbol{R}^{2N}, \boldsymbol{R})$ for $k > 3N + 2$, one finds in general an abundance of periodic solutions near a nondegenerate elliptic equilibrium point. Indeed, under finitely many inequalities involving the coefficients of the 4-th order jet of $\mathcal{H}$ at 0, there is, in every open neighborhood of 0, a set of positive Lebesgue measure, consisting of the closure of the set of periodic solutions in this neighborhood. The periods of the solutions so found are however very large.

All the results described so far involve small perturbations and are, in this respect, not global. As for more global results, P. H. Rabinowitz [36] found on every regular energy surface, which is radially homeomorphic to the $(2N-1)$-sphere, a periodic orbit. Related results for convex Hamiltonians are due to A. Weinstein [44], I. Ekeland [21], and I. Ekeland and J.-P. Lasry [22].

In contrast to these existence results for periodic orbits, whose periods are either not at all or only approximately known, we are interested in the existence of periodic solutions whose periods are prescribed. Results in this direction are due to Rabinowitz [36], Clarke and Ekeland [17], and J. Coron [20]. In [36] T-periodic orbits are found (in the autonomous case) for every $T > 0$, provided $\mathcal{H}$ grows superquadratically (like $|x|^\alpha$, $\alpha > 2$) at infinity and grows slowly near the origin. As far as the growth condition is concerned, the results of [17] are related to ours. Namely, if $\mathcal{H}(x) \leqslant (k/2)|x|^2$ for large values of $|x|$, and $\mathcal{H}(x) \geqslant (K/2)|x|^2$ for $|x|$ near zero, where $K > \sqrt{2}k$, the authors derive the existence of a T-periodic orbit with minimal period for every T satisfying $2\pi/K < T < \sqrt{2}/k$, provided $\mathcal{H} \geqslant 0$ and $\mathcal{H}$ is convex. The solutions in question are characterized as the solutions of a specific minimization problem following an idea of F. Clarke [16]. In addition estimates for the energy levels of the solutions are given. A related approach has been used by Coron [20].

Our results are different in nature. We assume the vectorfield to be asymptotically linear. The comparison of the linearized systems at the equilibrium point $x = 0$ and near infinity yields (in the autonomous case) the periods T for which we find periodic orbits. Related results for special cases are due to D. C. Clark [15a]. We do not necessarily have to assume that the spectra of these linearized systems are separated from each other, neither that the Hamiltonians are convex. In addition we emphasize the fact that we can handle without difficulties the nonautonomous case, which is not true for most of the above mentioned papers.

REFERENCES

[1] S. Ahmad - A. C. Lazer - J. L. Paul, *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. I., **25** (1976), pp. 933-944.

[2] H. Amann, *Saddle points and multiple solutions of differential equations*, Math. Z., **169** (1979), pp. 127-166.

[2a] H. Amann - E. Zehnder, *Multiple periodic solutions for a class of nonlinear autonomous wave equations*, to appear in Houston J. of Math.

[2b] H. Amann - E. Zehnder, *Periodic solutions of asymptotically linear Hamiltonian equations*, to appear.

[3] A. Ambrosetti - G. Mancini, *Theorems of existence and multiplicity for nonlinear elliptic problems with noninvertible linear part*, Ann. Scuola Norm. Sup. Pisa, **5** (1978), pp. 15-28.

[4] A. Ambrosetti - G. Mancini, *Existence and multiplicity results for nonlinear elliptic problems with linear part at resonance. The case of the single eigenvalue*, J. Differential Equations, **28** (1978), pp. 220-245.

[5] A. Ambrosetti - G. Prodi, *Analisi non Lineare*, Scuola Norm. Sup. Pisa, 1973.

[6] A. Ambrosetti - P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis, **14** (1973), pp. 349-381.

[7] M. Berger, *On a family of periodic solutions of Hamiltonian systems*. J. Differential Equations, **10** (1971), pp. 17-26.

[8] M. Berger, *On periodic solutions of second order Hamiltonian systems*, J. Math. Anal. Appl., **29** (1970), pp. 512-522.

[9] M. Berger, *Periodic solutions of second order dynamical systems and isoperimetric variational problems*, Amer. J. Math., **93** (1971), pp. 1-10.

[10] D. Blackmore, *On local normal forms for diffeomorphisms and flows*, Notices Amer. Math. Soc., (1977), A-313.

[11] M. Bottkol, *Bifurcation of periodic orbits on manifolds, and Hamiltonian systems*. Thesis N.Y.U. (1978).

[12] N. Bourgoyne - R. Cushman, *Normal forms for real linear Hamiltonian systems*, in *Lie Groups: History, Frontiers, and Applications*, vol. VII, editors: C. Martin and R. Hermann, Math. Sci. Press, Brookline Mass., 1977, pp. 483-529.

[13] H. Brezis - L. Nirenberg, *Forced vibrations for a nonlinear wave equation*. Comm. Pure Appl. Math., **31** (1978), pp. 1-30.

[14] H. Brezis - L. Nirenberg, *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems*, Ann. Scuola Norm. Sup. Pisa, Ser. IV, **5** (1978), pp. 225-326.

[15] A. Castro - A. C. Lazer, *Critical point theory and the number of solutions of a nonlinear Dirichlet problem*, Ann. Mat. pura e appl., (IV) **120** (1979), pp. 113-137.

[15*a*] D. C. Clark, *Periodic solutions of variational systems of ordinary differential equations*, J. Differential Equations, **28** (1978), pp. 354-368.

[16] F. H. Clarke, *Periodic solutions to Hamiltonian inclusions*, Preprint, Vancouver, 1978.

[17] F. H. Clarke - I. Ekeland, *Hamiltonian trajectories having prescribed minimal period*, Cahiers de mathématiques de la Decision N. 7822, Université de Paris IX (1978).

[18] C. C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Math., **38** (1978), AMS, Providence, R.I.

[19] C. C. Conley - R. W. Easton, *Isolated invariant sets and isolating blocks*, Trans. Amer. Math. Soc., **158** (1971), pp. 35-61.

[20] J. M. Coron, *Résolution de l'équation $Au + Bu = f$ où A est linéaire auto-adjoint et B déduit d'un potential convexe*, C. R. Acad. Sci. Paris Sér. A-B, **288** (1979), pp. A805-A808.

[21] I. Ekeland, *Periodic solutions of Hamiltonian equations and a theorem of P. Rabinowitz*, Cahiers de Mathématiques de la Decison N. 7827, Université de Paris IX (1978).

[22] I. Ekeland - J.-M. Lasry, *Nombre de solutions périodiques des équations de Hamilton*, Preprint, Paris (1978).

[23] I. Ekeland - R. Temam, *Analyse convexe et problèmes variationels*, Dunod, Paris (1974).

[24] E. R. Fadell - P. Rabinowitz, *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math., **45** (1978), pp. 139-174.

[25] A. Friedman, *Partial Differential equations*, Holt, Rinehart and Winston, Inc., New York, 1969.

[26] P. Hess, *Solutions nontriviales d'un problème aux limites elliptique non linéaire*, C.R. Acad. Sci. Paris, to appear.

[27] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.

[28] A. Liapunoff, *Problème générale de la stabilité du mouvement*, Ann. Fac. Sci. Toulouse (2) (1907), pp. 203-474.

[29] J. L. Lions - E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications - I*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[30] G. Mancini, *Periodic solutions of some semilinear autonomous wave equations*, Boll. Un. Mat. Ital., (5), **15**-B (1978), pp. 649-672.

[31] J. Moser, *Periodic orbits near an equilibrium and a theorem by Alan Weinstein*, Comm. Pure Appl. Math., **29** (1976), pp. 727-747.

[31*a*] J. Moser, *New aspects in the theory of stability of Hamiltonian systems*, Comm. Pure Appl. Math., **11** (1958), pp. 81-114.
[32] K. J. Palmer, *Linearization near an integral manifold*, J. Math. Anal. Appl., **51** (1975), pp. 243-255.
[33] P. Rabinowitz, *Periodic solutions of nonlinear hyperbolic partial differential equations*, Comm. Pure Appl. Math., **20** (1967), pp. 145-205.
[34] P. Rabinowitz, *Some minimax theorems and applications to nonlinear partial differential equations*, Nonlinear Analysis, A Collection of Papers in Honor of Erich H. Rothe, pp. 161-177, Academic Press, 1978.
[35] P. Rabinowitz, *Free vibrations for a semilinear wave equation*, Comm. Pure Appl. Math., **31** (1978), pp. 31-68.
[36] P. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math., **31** (1978), pp. 157-184.
[37] R. T. Rockafellar, *Monotone operators associated with saddle-functions and minimax theorems*, in *Nonlinear Functional Analysis*, Part I, Proc. Symp. Pure Math., **18** (1970), pp. 241-250.
[38] A. N. Shoshitaishvili, *Bifurcations of topological type at singular points of parametrized vector fields*, Functional Anal. Appl., **6** (1972), pp. 169-170.
[39] C. L. Siegel - J. Moser, *Lectures on Celestial Mechanics*, Springer-Verlag, New York, 1971.
[40] E. H. Spanier, *Algebraic Topology*, McGraw-Hill Book Co., Inc., New York, 1966.
[41] K. Thews, *A reduction method for some nonlinear Dirichlet problems*, J. Nonlinear Analysis. Theory, Methods, Appl., **3** (1979), pp. 795-813.
[42] K. Thews, *Nontrivial solutions of elliptic equations at resonance*, Proc. Roy. Soc. Edinburgh, **85** A (1980), pp. 119-129.
[43] O. Vejvoda, *Periodic solutions of nonlinear partial differential equations of evolution*, Proc. Symp. Diff. Eqs. Appl. at Bratislava, 1966, Acta Fac. Rerum Natur. Univ. Comenian. Math., **17** (1967), pp. 293-300.
[44] A. Weinstein, *Periodic orbits for convex Hamiltonian systems*, Ann. of Math., **108** (1978), pp. 507-518.
[45] A. Weinstein, *Bifurcations and Hamilton's principle*, Math. Z., **159** (1978), pp. 235-248.
[46] A. Weinstein, *Normal modes for nonlinear Hamiltonian systems*, Invent. Math., **20** (1973), pp. 47-57.
[47] A. Weinstein, *Lagrangian submanifolds and Hamiltonian systems*, Ann. of Math., **98** (1973), pp. 377-410.
[48] G. W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, New York, Heidelberg, Berlin, 1978.
[49] J. Williams, *On the algebraic problem concerning the normal form of a linear dynamical system*, Amer. J. Math., **58** (1936), pp. 141-163.

Universität Zürich
Mathematisches Institut
CH 8032 Zürich (Switzerland)

Ruhr-Universität
Mathematisches Institut
D-4630 Bochum (Germany)