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A quadratic integral equation

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A Quadratic Integral Equation.

ROGER D. NUSSBAUM (*)

Introduction.

This paper treats the integral equation

$$u(x) = f(x) + \lambda \int_x^1 u(y)u(y-x)\,dy\,, \qquad 0 \leqslant x \leqslant 1\,. \tag{0.1}$$

Generally, $f(x)$ will be assumed continuous and real-valued, λ will be real, and a continuous, real-valued solution $u(x)$ will be sought. If $f(x)$ is extended so that $f(-x) = f(x)$ for almost all x and $f(x) = 0$ for $|x| > 1$, if $u(x)$ is extended to be zero for $x \notin [0, 1]$ and if $\hat{g}(\xi) = \int_{\mathbf{R}} g(x)\, e^{i\xi x} dx$ denotes the Fourier transform of a function, then it is shown in the first section that (0.1) is equivalent (for real-valued functions) to solving

$$1 - \lambda \hat{f}(\xi) = |1 - \lambda \hat{u}(\xi)|^2\,, \qquad \xi \text{ real}\,. \tag{0.2}$$

Equation (0.2) has been studied in classical work of B. Ja. Levin and (later) M. G. Krein, who proved that if the left hand side of (0.2) is always non-negative and $f \in L^1[-1, 1]$, then there is a $u \in L^1[0, 1]$ satisfying (0.2). Our work here refines the basic Levin-Krein theorem. We shall try to answer questions like « How many positive solutions does (0.1) have? ». How do solutions of (0.1) vary with f and λ? If f is continuous, is u necessarily continuous? If F_λ indicates the nonlinear map of $C[0, 1]$ into itself determined by the right hand side of equation (0.1), what is the spectrum of the Fréchet derivative of F_λ? We shall see that a complete picture of the solution set $\{(u, \lambda)\}$ of (0.1) can be given in terms of the complex zeros of $1 - \lambda \hat{f}(z)$.

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Our immediate motivation for studying (0.1) comes from mathematical physics. The factorization result in (0.2) has played an important role as a tool in solving certain equations from statistical mechanics [3, 4, 13, 14]. In fact a parameter like λ first appears in the physics literature; Levin and Krein assume $\lambda = 1$. As we shall see in Section 3, the introduction of the parameter λ is useful mathematically in determining the number of positive solutions of (0.1).

A paper by G. Pimbley [10] and one by R. Ramalho [12] are closely related to our work here. Pimbley and Ramalho consider the equation

$$u(x) = 1 + \lambda \int_x^1 u(y)u(y-x)\,dy\,, \qquad 0 \leqslant x \leqslant 1 \tag{0.3}$$

primarily for the case $\lambda \geqslant 0$. Pimbley shows that (0.3) has no real-valued, continuous solution for $\lambda > \frac{1}{2}$ and claims to show that (0.3) has at least two positive solutions for $0 < \lambda < \frac{1}{2}$. Ramalho, building on Pimbley's work, claims to show that (0.3) has exactly two positive solutions for $0 < \lambda < \frac{1}{2}$ and exactly one for $\lambda = \frac{1}{2}$. In fact, both these results are based on Theorem 14 in [10]. As we have discussed in detail at the beginning of Section 3, there is a serious error in the proof of Theorem 14 in [10], and in fact the actual estimate which is claimed in the proof is wrong. As a result, Pimbley's paper proves only slightly more than the existence of at least one positive solution for $0 \leqslant \lambda \leqslant \frac{1}{2}$, and Ramalho's argument proves existence of at least two distinct positive solutions of (0.3) for $0 < \lambda < \frac{1}{2}$.

In fact Ramalho's original claim is correct. We prove in Section 3 that if $f(x)$ is nonnegative and continuous and $f(1) \neq 0$, then (0.1) has no real-valued, continuous solution for $\lambda > \lambda_+ = \left(2\int_0^1 f(x)\,dx\right)^{-1}$, precisely one positive solution for $\lambda = \lambda_+$ and precisely two positive solutions for $0 < \lambda < \lambda_+$. However, this result is actually quite delicate and probably inaccessible by the techniques in [10] and [12]. For example, a slight generalization of (0.3) is considered in [16], namely

$$u(x) = 1 + \lambda \int_x^1 u(y)^\alpha u(y-x)^\alpha dy\,, \qquad 0 \leqslant x \leqslant 1\,. \tag{0.4}$$

Numerical studies in [16] suggest that for each $\alpha > 1$ there is an interval J_α of positive λ such that (0.4) has only one positive solution for $\lambda \in J_\alpha$. Numerical studies suggested that this sort of behavior does not occur for $\frac{1}{2} < \alpha < 1$, and one can prove in this case that there is a number $\lambda_\alpha > 0$

such that (0.3) has no positive solutions for $\lambda > \lambda_\alpha$ and at least two positive solutions for $0 < \lambda < \lambda_\alpha$.

An outline of this paper may be in order. In Section 1 we prove some results which « should be » classical but do not seem to be. For example, we prove that if $f\colon [0,1] \to \mathbb{R}$ is continuous, then the L^1 solution of (0.1) which is insured by the Levin-Krein theorem is actually continuous. We also prove various results concerning continuous dependence of solutions on λ and f and the number of solutions. We prove these theorems in some detail mainly because all subsequent results depend on theorems in Section 1.

The results of Section 1 show that a deeper understanding of (0.1) depends on knowledge of the zeros of $1-\lambda\hat{f}(z)$ for z complex, so a reasonably complete analysis of the location of such zeros is given in Section 2. A discussion is also given of zeros of $\theta(z) = \hat{u}(z) - \hat{u}(-z)$, where u has support in $[0,1]$ and $u|[0,1]$ is continuously differentiable. It is shown in Section 5 that such information is essential to discuss the spectrum of the linear operator $L\colon C[0,1] \to C[0,1]$ defined by

$$(0.5) \qquad (Lh)(x) = \lambda\int_x^1 u(y)h(y-x)\,dy + \lambda\int_x^1 u(y-x)h(y)\,dy\,.$$

The operator L, of course, is the Fréchet derivative of the right hand side of (0.1).

In Section 3 we discuss positive solutions of (5.1). As we have remarked, if f is nonnegative and $f(1)\neq 0$ (somewhat less is necessary) we obtain precisely two positive solutions v_λ and u_λ for $0<\lambda<\lambda_+$. We show that $v_\lambda(x) \geqslant u_\lambda(x)$ for $0 \leqslant x \leqslant 1$ and the maps $\lambda \to v_\lambda$ and $\lambda \to u_\lambda$ can be defined continuously on $(0, \lambda_+]$. We also consider the problem of positive solutions of (0.1) when f is nonnegative and $\lambda < 0$, but our results here are far from definitive and there are many intriguing open questions. Some of these questions have subsequently been answered in [17].

In Section 4 we give an explicit formula for the « fundamental solution » u of (0, 1); the only unknown constants in the formula are the zeros of $1-\lambda\hat{f}(z)$. In Section 5 we give a complete description of the spectrum of the operator L defined by (0.5). In fact, using our results, Ramalho's argument for the existence of precisely two positive solutions could be justified. We prove that $\sigma(L)$, the spectrum of L, is given by $\sigma(L) = \{\lambda\hat{u}(z)\colon \hat{u}(z) = \hat{u}(-z)\} \cup \{0\}$. Furthermore, if $\mu \neq 0$ is an eigenvalue of L (L is compact, so it has only point spectrum aside from 0), then the algebraic multiplicity of μ is

$$(0.6) \qquad \operatorname{alg}(\mu) = \tfrac{1}{2}\sum_{z\in T} m_1(z)$$

where T is the set of z such that $\mu = \lambda\hat{u}(z)$ and $\theta(z) = \hat{u}(z) - \hat{u}(-z) = 0$ and $m_1(z)$ is the multiplicity of z as a zero of $z\theta(z) = 0$.

The key lemma in proving these results is Theorem 5.1, which discusses when the closed linear span of a set A in $L^2[-1, 1]$ is all of $L^2[-1, 1]$ and when A is minimal (in the sense of inclusion) among sets with this property. This general sort of result is classical, going back to Paley and Wiener [9]; and in fact our original proof was a generalization of ideas of Paley and Wiener. However, the results in [9] and [8] are inadequate for our purposes, and Theorem 5.1 appears to be new.

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1. – Basic theory of the equation $u(x) = f(x) + \lambda\int_x^1 u(y)u(y-x)\,dy$.

In this section we shall establish some basic facts about the equation

$$u(x) = f(x) + \lambda\int_x^1 u(y)u(y-x)\,dy\,, \qquad 0 \leqslant x \leqslant 1\,. \tag{1.1}$$

We shall recall some fundamental theorems from the literature and indicate the refinements of those theorems which will be crucial for our work. Ultimately, we shall want to assume that (at least) $f \in C[0, 1]$ and we shall seek a continuous solution $u \in C[0,1]$, but for the moment we shall assume less.

Our first lemma is implicit in the physics literature [3, 13] but we state it for completeness.

LEMMA 1.1. *Assume that $u \in L^1[0, 1]$, u is real-valued, and u satisfies equation* (1.1) *where $f \in L^1[0, 1]$ f is real-valued and λ is real. Extend f to be an even map of $\mathbb{R}$ to $\mathbb{R}$ such that $f(x) = 0$ for $|x| > 1$ and extend u to be a map of $\mathbb{R}$ to $\mathbb{R}$ such that $u(x) = 0$ for $x \notin [0, 1]$. If v and w are in $L^1(\mathbb{R})$,* *define* $v * w$ *as usual*:

(1.2) $$(v * w)(x) = \int_{-\infty}^{\infty} v(y) w(x - y) dy$$

and define $\tilde{v}(x) = v(-x)$. *Then if* u *and* f *denote the extended functions, one has for almost all real* x

(1.3) $$\tilde{u}(x) + u(x) = f(x) + \lambda(u * \tilde{u})(x)$$

and

(1.4) $$1 - \lambda\hat{f}(\xi) = 1 - 2\lambda\int_0^1 f(x)\cos\xi x dx \geqslant 0$$

for all real ξ. $\Big($*Recall that if* $v \in L^1(\mathbb{R})$, $\hat{v}(\xi) = \int_{-\infty}^{\infty} v(x) e^{i\xi x} dx$ *the Fourier transform of* $\hat{v}\Big)$.

Proof. If u and f have been extended as indicated, then checking that formula (1.3) holds, is a simple exercise which we leave to the reader. Taking the Fourier transform of both sides of equation (1.3) gives

(1.5) $$\hat{u}(\xi) + \hat{u}(-\xi) = \hat{f}(\xi) + \lambda\hat{u}(\xi)\hat{u}(-\xi) .$$

Since u is real-valued, $u(-\xi)$ is the complex conjugate of $\hat{u}(\xi)$, and we obtain from (1.5) that

(1.6) $$1 - 2\hat{f}(\xi) = (1 - \lambda\hat{u}(\xi))\overline{(1 - \lambda\hat{u}(\xi))} \geqslant 0$$

which proves (1.4). ∎

Lemma 1.1 shows that (1.1) can have no real-valued solutions if inequality (1.4) fails at any real ξ, so it is important to know for what λ (1.4) will hold for all ξ. The following simple lemma answers the question.

Lemma 1.2. *Assume that* $f \in L^1(\mathbb{R})$ *and* f *is even and real-valued. There exist numbers* $\lambda_+ > 0$ *and* $\lambda_- < 0$ (*we allow* $\lambda_+ = +\infty$ *or* $\lambda_- = -\infty$) *such that*

(1.7) $$1 - \lambda\hat{f}(\xi) \geqslant 0$$

for all real ξ *if* $\lambda_- \leqslant \lambda \leqslant \lambda_+$; *inequality* (1.7) *is strict for all real* ξ *if* $\lambda_- < \lambda < \lambda_+$. *If* $\lambda > \lambda_+$ *or* $\lambda < \lambda_-$, *there exists a real* ξ *such that*

(1.8) $$1 - \lambda\hat{f}(\xi) < 0 .$$

If $f \neq 0$, *at least one of the numbers* λ_+ *and* λ_- *is finite.*

PROOF. If $1-\lambda\hat{f}(\xi)\geqslant 0$ for all λ one finds that $\hat{f}(\xi)=0$, and if $1-\lambda\hat{f}(\xi)\geqslant 0$ for all real λ and all real ξ, it follows that $\hat{f}(\xi)=0$ for all ξ. This would imply that f is identically zero. If we assume that f is not identically zero, one of λ_+ and λ_- is finite.

Define numbers λ_+ and λ_- by

$$\lambda_+ = \sup\left\{\lambda\geqslant 0:\inf_{\xi}(1-\lambda\hat{f}(\xi))>0\right\},\qquad (1.9)$$

$$\lambda_- = \inf\left\{\lambda\leqslant 0:\inf_{\xi}(1-\lambda\hat{f}(\xi))>0\right\}.$$

We shall prove that λ_+ satisfies the conditions of the lemma; the proof for λ_- is similar. Note that $\lambda_+>0$ because $\lim_{|\xi|\to\infty}\hat{f}(\xi)=0$ (true for any L^1 function). Take any finite number $\lambda_1<\lambda_+$. We have to show that $1-\lambda\hat{f}(\xi)>0$ for every real number ξ and for λ such that $0\leqslant\lambda\leqslant\lambda_1$. Because $\lim_{|\xi|\to\infty}\hat{f}(\xi)=0$, there exists a number M such that

$$1-\lambda\hat{f}(\xi)\geqslant\tfrac{1}{2}\qquad (1.10)$$

for $0\leqslant\lambda\leqslant\lambda_1$ and $|\xi|\geqslant M$. A simple compactness argument now implies that if

$$\inf_{\xi} 1-\lambda_2\hat{f}(\xi)\leqslant 0\qquad (1.11)$$

for some λ_2 with $0\leqslant\lambda_2\leqslant\lambda_1$, there exists ξ_2 with $|\xi_2|\leqslant M$ such that

$$1-\lambda_2\hat{f}(\xi_2)\leqslant 0\,.$$

If λ_3 is taken so that $\lambda_2<\lambda_3$ and

$$\inf_{\xi} 1-\lambda_3\hat{f}(\xi)>0\qquad (1.12)$$

we have a contradiction, because

$$1-\lambda_3\hat{f}(\xi_2)<0\,.\qquad (1.13)$$

It still remains to prove that $\inf 1-\lambda\hat{f}(\xi)<0$ for $\lambda>\lambda_+$, but the proof is similar to the above argument, and we leave it to the reader. ■

Notice that if $\lambda_+<\infty$, the above argument shows that there will be a real number ξ_+ such that

$$1-\lambda_+ f(\xi_+)=0\,.\qquad (1.14)$$

It will be useful later to have a simple class of examples for which $\lambda_- = -\infty$. As the following proposition shows, any function $f(x)$ of the form

$$f(x) = \sum_{j=1}^{N} c_j(1 - |x|)^j \quad \text{for } -1 \leqslant x \leqslant 1$$

for nonnegative constants c_j satisfies $\lambda_- = -\infty$.

PROPOSITION 1.1. *Suppose that $f(x)$ is an even function such that $f(x) = 0$ for $|x| > 1$. Assume that $f|[0, 1]$ is continuously differentiable, $f(1) = 0$, $f'(x) \leqslant 0$ for $0 \leqslant x \leqslant 1$, and $f'(x)$ is monotonic increasing (not necessarily strictly). Then $\hat{f}(\xi) \geqslant 0$ for all real numbers ξ and $\lambda_- = -\infty$, where λ_- is defined as in Lemma 1.2.*

PROOF. The evenness of $f(x)$ and integration by parts gives

$$\int_{-1}^{1} f(x) e^{i\xi x} dx = 2\int_0^1 f(x) \cos \xi x \, dx = \frac{2}{\xi}\int_0^1 (-f'(x)) \sin \xi x \, dx \,. \tag{1.15}$$

If we define $g(x) = -f'(x)$ for $0 \leqslant x \leqslant 1$ and $g(x) = 0$ for $x > 0$, equation (1.15) becomes

$$\hat{f}(\xi) = \left(\frac{2}{\xi}\right) \int_0^\infty g(x) \sin \xi x \, dx = \left(\frac{2}{\xi}\right) \sum_{j=0}^{\infty} a_j \tag{1.16}$$

where we defined a_j by

$$a_j = \int_{j\pi\xi^{-1}}^{(j+1)\pi\xi^{-1}} g(x) \sin \xi x \, dx \,. \tag{1.17}$$

Since f is an even function, $\hat{f}$ is an even function and we can assume that $\xi > 0$. The assumptions on f imply that g is nonnegative and monotonic decreasing (not necessarily strictly) on $[0, \infty)$, and using this fact it is easy to see that (for $\xi > 0$) $(-1)^j a_j \geqslant 0$ for all j and $|a_j| \geqslant |a_{j+1}|$ for all j. It follows that (1.16) represents $\hat{f}(\xi)$ as an alternating series whose first term is nonnegative, so $\hat{f}(\xi) \geqslant 0$.

A slightly more careful examination of the proof shows that if, in addition to the other assumptions, $f'(x)$ is not constant on $[0, 1]$, then $\hat{f}(\xi) > 0$ for all ξ. It may be worth noting that the proposition is also true if $f|[0, 1]$ is C^1 on an interval $[1 - \delta, 1]$, $\delta > 0$, and piecewise C^1 on $[0, 1]$ instead of C^1 on $[0, 1]$.

Lemma 1.1 shows that one must have $1-\lambda\hat{f}(\xi)\geqslant 0$ for all ξ to have any hope of finding an $L^1[0,1]$ solution of (1.1). It turns out that this condition is also sufficient. The following result is a paraphrase of Theorem 4.4 on p. 194 in [7]; Krein attributes the theorem (for the case $1-\hat{f}_1(\xi)>0$ for all ξ) to B. Ja. Levin [8, Appendix 5].

THEOREM 1.1 (see Theorem 4.4 in [7]). *If* $a>0$ *and the function* $f_1\in L^1[-a,a]$ *is such that*

$$1-\int_{-a}^{a} f_1(t)e^{i\xi t}dt\geqslant 0 \qquad (-\infty<\xi<\infty) \tag{1.18}$$

then there is a function $u_1\in L^1[0,a]$ *such that*

$$1-\int_{-a}^{a} f_1(t)e^{i\xi t}dt=\left|1-\int_0^a u_1(t)e^{i\xi t}dt\right|^2 \qquad (-\infty<\xi<\infty) \tag{1.19}$$

and such that

$$1-\int_0^a u_1(t)e^{izt}dt\neq 0 \tag{1.20}$$

for any complex number z *with* $\operatorname{Im}(z)>0$. *If inequality* (1.18) *is strict for all* ξ, *the solution* $u_1\in L^1[0,a]$ *of* (1.19) *which also satisfies* (1.20) *is unique.*

In the statement of Theorem 1.1 we have corrected a misprint in Theorem 4.4. The statement about uniqueness is not explicitly made in Theorem 4.4 [7] but follows from the proof and the preceding results.

In our case, if we take $f\in L^1[-1,1]$ to be an even, real-valued function (extended to be zero outside $[-1,1]$) and λ to be a nonzero real number such that (1.4) holds for all ξ, then Theorem 1.1 (applied to $f_1=\lambda f$) implies that there is a function $u_1=\lambda u\in L^1[0,1]$ such that (1.19) and (1.20) hold. We shall see that u_1 is real-valued, so one finds for $-\infty<\xi<\infty$

$$1-\lambda\int_{-1}^{1} f(t)e^{i\xi t}\,dt=\left(1-\lambda\int_0^1 u(t)e^{i\xi t}dt\right)\overline{\left(1-\lambda\int_0^1 u(t)e^{i\xi t}dt\right)}. \tag{1.21}$$

Working backward from the argument in Lemma 1.1, one finds that (1.1) must be satisfied for almost all x in $[0,1]$.

Unfortunately, Theorem 1.1 is not sufficient for our purposes. It is not clear that if $f \in L^2[0, 1]$ or $f \in C[0, 1]$ then the corresponding L^1 solution u of (1.1) is respectively in $L^2[0, 1]$ or in $C[0, 1]$.

Furthermore, we shall need to know that if u_λ is the unique solution of (1.1) for $\lambda_- < \lambda < \lambda_+$ such that

$$1 - \lambda \int_0^1 u_\lambda(t) e^{izt} dt \neq 0$$

for Im $(z) > 0$, then the map $\lambda \to u_\lambda$ is continuous in an appropriate Banach space and extends continuously to λ_- and λ_+.

As we shall see later, it suffices to verify these facts for $f \in L^2[0, 1]$, so we restrict attention to such f. The proofs are analogous to arguments in [7] and we refer there for more detail.

Let Y denote the complex commutative Banach algebra of functions $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with a multiplicative unit δ adjoined. Elements of Y are of the form $c\delta + g$, c a complex number. The multiplication is given by

$$(c_1\delta + g_1)(c_2\delta + g_2) = c_1c_2\delta + c_1g_2 + c_2g_1 + g_1 * g_2$$

where $g_1 * g_2$ denotes the convolution of g_1 with g_2. The norm in Y is defined by

$$\|c\delta + g\| = |c| + \max\left(\|g\|_{L^1}, \|g\|_{L^2}\right).$$

It is easy to check that with this norm Y becomes a Banach algebra. Let Y_+ and Y_- denote the subalgebras of Y given by

$$Y_+ = \{c\delta + g \in Y : g(t) = 0 \text{ for almost all } t < 0\}$$

$$Y_- = \{c\delta + g \in Y : g(t) = 0 \text{ for almost all } t > 0\}.$$

Similarly, we define Z to be the complex commutative algebra of functions of the form $c + \hat{f}(\xi)$, where c is a constant and $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The multiplication in Z is ordinary pointwise multiplication. Clearly, there is an algebra isomorphism J between Z and Y given by

$$J(c\delta + f) = c + F$$

where $F(\xi) = \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx$. We define $\|c + F\|_Z = \|c\delta + f\|_Y$, $Z_+ = J(Y_+)$ and $Z_- = J(Y_-)$.

If ψ is a nonzero, continuous linear functional on Y which preserves multiplication (so ψ comes from a maximal ideal in Y), then either $\psi(c\delta + f) = c$ for all $f \in L^1 \cap L^2$ or

$$\psi(c\delta + f) = c + \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$$

for some real number s. The proof of this fact follows the outline of the argument given on p. 170-172 in [6]. Thus Lemma 1 on p. 170 in [6] remains true, although the argument must be modified because the characteristic function of an interval of length τ is not bounded by τ in the Y norm. Similarly, property (γ) on p. 171 of [6] is true, but by a different argument.

We also need to know the maximal ideals of the Banach algebras Y_+ and Y_-. If ψ is a nonzero, continuous linear functional on Y_+ which preserves the multiplication on Y_+, then either $\psi(c\delta + f) = c$ for all $f \in L^1 \cap L^2 \cap Y_+$ or

$$\psi(c\delta + f) = c + \int_{0}^{\infty} f(x)\, e^{isx} dx$$

for some complex number s with $\operatorname{Im}(s) \geqslant 0$. A similar statement holds for Y_- except that s must satisfy $\operatorname{Im}(s) \leqslant 0$.

Given the above facts, the general theory of Banach algebras implies that an element $u = c\delta + f$ in Y has a multiplicative inverse in Y if and only if $c \neq 0$ and

$$c + \int_{-\infty}^{\infty} f(x)\, e^{isx} dx \neq 0\,, \qquad -\infty < s < \infty\,.$$

An element $u = c\delta + f$ in Y_+ has a multiplicative inverse in Y_+ if and only if $c \neq 0$ and

$$c + \int_{0}^{\infty} f(x)\, e^{isx} dx \neq 0 \tag{1.22}$$

for every complex number s such that $\operatorname{Im}(s) \geqslant 0$. Translating these facts to the isomorphic algebra Z implies that a (uniformly continuous) function $u = c + F$ in Z (where F is the Fourier transform of a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$) has a multiplicative inverse in Z if and only if $c \neq 0$ and $u(\xi) \neq 0$ for any real number ξ. Since multiplication is pointwise in Z, $(u^{-1})(\xi) = (u(\xi))^{-1}$. Similarly, if $u \in Z_+$, then u has a multiplicative inverse in Z_+ if and only if (1.22) holds.

We need to recall one more general fact before we can return to equation (1.1). Recall that if B is any complex, commutative Banach algebra, with unit I, $u \in B$ and f is a complex-valued function which is defined and analytic on some open neighborhood of the spectrum $\sigma(u)$ of u, then one can define in a natural way $f(u) \in B$, and the functional calculus defined in this way has all the properties one would expect (see [6]). To be precise, let D be a bounded open neighborhood of $\sigma(u)$ such that f is analytic on D and continuous on $\bar{D}$ and such that Γ, the boundary of D, consists of a finite number of simple closed rectifiable curves. Then

$$f(u) = \frac{1}{2\pi i}\int_{\Gamma} f(z)(zI - u)^{-1}\,dz$$

where Γ is oriented positively and I is the multiplicative unit.

The next lemma is a standard result whose proof we include for completeness.

LEMMA 1.3. *Let B be a complex commutative Banach algebra with unit. Suppose that $u \in B$, $u_n \in B$ is a sequence such that $u_n \to u$ and f is a complex valued function which is analytic on an open neighborhood of the spectrum of u. Then $f(u_n)$ is defined for n large enough and $f(u_n) \to f(u)$.*

PROOF. Let D be a bounded open neighborhood of $\sigma(u) =$ spectrum of u such that $\Gamma = \partial D$ consists of a finite number of simple closed rectifiable curves and f is analytic on a neighborhood of $\bar{D}$. It is known that $\sigma(u_n) \subset D$ for n large enough. Define $M = \max_{z\in\Gamma} \|(zI - u)^{-1}\|$. For $n \geqslant N$ we can assume $\sigma(u_n) \subset D$ and $\|u_n - u\| < \varepsilon$, where $\varepsilon < M^{-1}$. It follows that we can write for $z \in \Gamma$

$$(1.23)\qquad \begin{aligned}(z - \varphi_n)^{-1} &= \big[(z-\varphi)\big(I - (z-\varphi)^{-1}(\varphi_n - \varphi)\big)\big]^{-1}\\ &= [I - (z-\varphi)^{-1}(\varphi_n - \varphi)]^{-1}(z-\varphi)^{-1}\\ &= \Big[I + \sum_{k=1}^{\infty}[(z-\varphi)^{-1}(\varphi_n - \varphi)]^k\Big](z-\varphi)^{-1}.\end{aligned}$$

Equation (1.23) implies that for $z \in \Gamma$ and $n \geqslant N$ one has

$$(1.24)\qquad \|(z-\varphi_n)^{-1} - (z-\varphi)^{-1}\| \leqslant \varepsilon M^2(1 - \varepsilon M)^{-1}$$

and (1.24) implies the lemma. ■

We can now start to modify Theorem 1.1 to give the form we shall need.

LEMMA 1.4. *Suppose that g is an even, real-valued function such that $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and*

$$1-\hat{g}(\xi) > 0 \quad -\infty < \xi < \infty \tag{1.25}$$

where

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x)\, e^{i\xi x}\, dx .$$

Then there is a unique real-valued function $v \in L^1[0, \infty) \cap L^2[0, \infty)$ (extended to be zero on $(-\infty, 0]$) such that

$$1-\hat{g}(\xi) = \big(1-\hat{v}(\xi)\big)\big(1-\hat{v}(-\xi)\big), \quad -\infty < \xi < \infty \tag{1.26}$$

and

$$1-\int_0^{\infty} v(x)\, e^{isx}\, dx \neq 0 , \quad s \text{ complex}, \ \operatorname{Im}(s) > 0 . \tag{1.27}$$

Furthermore, if g_n is a sequence of even, real-valued functions in $L^1(\mathbb{R}) \cap \cap L^2(\mathbb{R})$ such that $\|g_n - g\|_{L^1} \to 0$ and $\|g_n - g\|_{L^2} \to 0$, and if $v_n \in L^1[0, \infty) \cap \cap L^2[0, \infty)$ denotes the corresponding unique solution of

$$1-\hat{g}_n(\xi) = \big(1-\hat{v}_n(\xi)\big)\big(1-\hat{v}_n(-\xi)\big) \quad -\infty < \xi < \infty \tag{1.28}$$

such that $1-\hat{v}_n(s) \neq 0$ for $\operatorname{Im} s \geqslant 0$ (where $v_n(x) = 0$ for $x \leqslant 0$), then

$$\lim_{n\to\infty} \|v_n - v\|_{L^1} = 0 , \qquad \lim_{n\to\infty} \|v_n - v\|_{L^2} = 0 . \tag{1.29}$$

PROOF. Let Y, Z, Y_+ and Z_+ be the Banach algebras previously defined. Define $u \in Z$ by $u(\xi) = 1-\hat{g}(\xi)$ for real ξ and note that u is real-valued. By our assumptions, there exist positive constants ε and M such that

$$0 < \varepsilon \leqslant u(\xi) \leqslant M , \quad -\infty < \xi < \infty .$$

By our previous remarks, the spectrum of u lies in the interval $[\varepsilon, M]$. If $f(z) = \log(z)$, where $\log(z)$ agrees with the standard logarithm for $z > 0$ and is undefined for $z \leqslant 0$, then $f(z)$ is analytic on an open neighborhood of $\sigma(u)$ and $f(u)$ is defined and $f(u) \in Z$. In fact, if Γ is the boundary of the rectangle whose vertices are (cyclically) $\varepsilon/2 - i$, $M + 1 - i$, $M + 1 + i$,

$\varepsilon/2 + i$, then

$$f(u) = \frac{1}{2\pi i}\int_{\Gamma} \log[\zeta](\zeta - u)^{-1} d\zeta \,. \tag{1.30}$$

Since we are working in Z, where multiplication is ordinary pointwise multiplication, one can see (for a general analytic function f defined on a neighborhood of the spectrum of a general element $u \in Z$) that

$$\big(f(u)\big)(\xi) = f\big(u(\xi)\big) \tag{1.31}$$

so $f(u)$ is a real-valued function and $\lim_{|\xi|\to\infty} \big(f(u)\big)(\xi) = 0$. Since u is an even function, we also see that $f(u)$ is an even function. The above remarks show that there is a function $w \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ such that

$$\log\big(1 - \hat{g}(\xi)\big) = \int_{-\infty}^{\infty} w(x)\, e^{i\xi x} dx \tag{1.32}$$

and since $\log\big(1 - \hat{g}(\xi)\big)$ is even and real-valued, w is real-valued and even. If we define $w_1(x) = w(x)$ for $x \geqslant 0$ and $w_1(x) = 0$ for $x < 0$, then (1.32) becomes

$$\log\big(1 - \hat{g}(\xi)\big) = \hat{w}_1(\xi) + \hat{w}_1(-\xi)\,, \qquad -\infty < \xi < \infty\,. \tag{1.33}$$

Since $\hat{w}_1 \in Z_+$ and the function given by $\hat{w}_1(-\xi)$ is an element of Z_-, by taking the exponential of both sides of (1.33) we get

$$\begin{aligned} 1 - \hat{g}(\xi) &= \exp\big(\hat{w}_1(\xi)\big) \exp\big(\hat{w}_1(-\xi)\big)\,, \\ &= \exp\big(\hat{w}_1(\xi)\big) \overline{\exp\big(\hat{w}_1(\xi)\big)}\,. \end{aligned} \tag{1.34}$$

By our previous remarks we know that

$$\exp\big(\hat{w}_1(\xi)\big) \in Z_+ \quad \text{and} \quad \lim_{\xi\to\infty} \exp\big(\hat{w}_1(\xi)\big) = 1\,,$$

so

$$\exp\big(\hat{w}_1(\xi)\big) = 1 - \int_0^{\infty} v(x)\, e^{i\xi x} dx\,, \qquad -\infty < \xi < \infty \tag{1.35}$$

where $v \in L^1[0, \infty] \cap L^2[0, \infty]$. The left and right hand sides of (1.35) extend to [functions which are analytic on the upper half of the complex plane just by letting ξ be a complex variable with $\operatorname{Im}(\xi) \geqslant 0$.

Since both the left and right hand sides of (1.35) approach 1 as $|\xi| \to \infty$ (with $\operatorname{Im}(\xi) \geqslant 0$) and since they are equal for real ξ, the maximum modulus principle implies they are equal for all complex ξ with $\operatorname{Im}(\xi) \geqslant 0$. Since the left hand side of (1.35) is never zero, we have a v which satisfies (1.26) and (1.27).

To prove uniqueness, we just argue as in [7]. Suppose that

$$(1.36) \qquad \begin{aligned} 1 - \hat{g}(\xi) &= \big(1 - \hat{v}(\xi)\big)\big(1 - \hat{v}(-\xi)\big) \\ &= \big(1 - \hat{w}(\xi)\big)\big(1 - \hat{w}(-\xi)\big) \end{aligned}$$

where $v, w \in L^1 \cap L^2$, $v(x) = w(x) = 0$ for $x < 0$ and v and w are real-valued. Define $\theta_+(s) = 1 - \hat{v}(s)$ and $\psi_+(s) = 1 - \hat{w}(s)$ for complex s with $\operatorname{Im}(s) \geqslant 0$. Observe that θ_+ and ψ_+ are analytic on the upper half plane π_+ and continuous on $\bar{\pi}_+$. We assume that $\theta_+(s)$ and $\psi_+(s)$ do not vanish on $\bar{\pi}_+$ and we want to show they are identically equal. Define $\theta_-(s) = \theta_+(-s)$ and $\psi_-(s) = \psi_+(s)$. For s real we have

$$(1.37) \qquad \theta_+(s)\big(\psi_+(s)\big)^{-1} = h(s) = \psi_-(s)\big(\theta_-(s)\big)^{-1}.$$

For complex s, if we define

$$(1.38) \qquad \begin{aligned} h(s) &= \theta_+(s)\big(\psi_+(s)\big)^{-1}, \quad \operatorname{Im}(s) \geqslant 0 \\ h(s) &= \psi_-(s)\big(\theta_-(s)\big)^{-1}, \quad \operatorname{Im}(s) \leqslant 0 \end{aligned}$$

$h(s)$ is continuous for all s and analytic for non-real s, hence analytic everywhere. Since $\lim_{s \to \infty} h(s) = 1$, Liouville's theorem implies that $h(s)$ is a constant and $\theta_+(s) = \psi_+(s)$ for all s.

It remains to show that $v_n \to v$. Recall that $J\colon Y \to Z$ denotes the natural Banach algebra isomorphism. Let P denote the natural projection of Y onto Y_+ defined by $P(c\delta + h) = c\delta + h_1$, where $h_1(x) = 0$ for $x < 0$ and $h_1(x) = h(x)$ for $x \geqslant 0$. Define a continuous projection Q of Z onto Z_+ by $Q = JPJ^{-1}$. An examination of the previous construction shows that if g and v are as before, then

$$(1.39) \qquad 1 - \hat{v} = \exp\big(Q(\log(1 - \hat{g}))\big).$$

According to Lemma 1.3, this is just the composition of three continuous maps on Z, and Lemma 1.3 implies that $1 - \hat{v}_n = \exp\big(Q(\log(1 - \hat{g}_n))\big)$ approaches $1 - \hat{v}$ in the Z topology, which is the desired result. ∎

In the notation of Lemma 1.4, we will be particularly interested in the case in which $g(x) = 0$ almost everywhere for $|x| > a$. If $v(x)$ is the function whose existence is insured by Lemma 1.4, the next lemma shows that $v(x) = 0$ for almost all $x > a$.

LEMMA 1.5. *Suppose that g is an even, real-valued function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $g(x) = 0$ almost everywhere for $|x| > a$. If v is the unique real-valued function in $L^1[0, \infty) \cap L^2[0, \infty)$ which satisfies equations* (1.26) *and* (1.27), *then $v(x) = 0$ almost everywhere for $x > a$.*

PROOF. The function $v(x)$ is understood to be zero for $x < 0$. For complex numbers z such that $\operatorname{Im}(z) \geqslant 0$, define

$$\theta_+(z) = 1 - \int_0^\infty v(x) e^{izx} dx$$

and for $\operatorname{Im}(z) \leqslant 0$ define

$$\theta_-(z) = 1 - \int_0^\infty v(x) e^{-izx} dx .$$

Equation (1.26) gives

$$1 - \int_{-a}^{a} g(x) e^{izx} dx = \theta_+(z)\theta_-(z) , \qquad -\infty < z < \infty .$$

The function $\Psi(z)$ defined by the left hand side of the preceding equation makes sense and is analytic for all complex z. Since $\theta_-(z) \neq 0$ for $\operatorname{Im}(z) \leqslant 0$ we can define

$$\theta_+(z) \overset{\text{def}}{=} \Psi(z)\big(\theta_-(z)\big)^{-1} , \qquad \operatorname{Im}(z) \leqslant 0 . \tag{1.40}$$

With this definition $\theta_+(z)$ is analytic for all z. The defining equation for $\theta_+(z)$ implies that $\theta_+(z)$ is bounded by $1 + \|v\|_{L^1}$ for all z with $\operatorname{Im}(z) \geqslant 0$. Since we know

$$\lim_{z \to \infty} \theta_-(z) = 1 , \qquad \operatorname{Im} z \leqslant 0 \tag{1.41}$$

uniformly in z with $\operatorname{Im} z \leqslant 0$, equation (1.40) implies that

$$|\theta_+(z)| \leqslant c \exp\big(a |\operatorname{Im} z|\big) , \qquad \operatorname{Im} z \leqslant 0 \tag{1.42}$$

where c is a constant. The function $\hat{v}(z) = -\theta_+(z) + 1$ is holomorphic and its restriction to $\mathbb{R}$ is in $L^2(\mathbb{R})$. The Paley-Wiener theorem [9] now implies that $v(x) = 0$ almost everywhere for $x > a$. ∎

The argument used above is somewhat easier than that in [7], since we assume $v \in L^2$.

It remains to consider the case in which λ approaches λ_+ or λ_- (notation as in Lemma 1.2). To handle this situation we need to recall a lemma of Krein [7].

LEMMA 1.6 (Lemma 4.1, p. 190 in [7]). *Suppose that $g \in L^1[a_-, a_+]$, where $a_- \leqslant 0$ and $a_+ \geqslant 0$ and c is a complex number. For complex numbers z define $\Psi(z)$ by*

$$\Psi(z) = c + \int_{a_-}^{a_+} e^{izt} g(t)\,dt\,. \tag{1.43}$$

Assume that $\alpha \in C$ is a zero of $\Psi(z)$. Then

$$\Psi(z)(z-\alpha)^{-1} = \int_{a_-}^{a_+} e^{izt} g_\alpha(t)\,dt$$

where $g_\alpha(t)$ is absolutely continuous on $(a_-, 0]$ and $[0, a_+)$ separately, $ig'_\alpha(t) - \alpha g_\alpha(t) = g(t)$ on $(a_-, 0]$ and $[0, a_+)$ separately, $g_\alpha(a_+) = g_\alpha(a_-) = 0$ and $g_\alpha(0^+) - g_\alpha(0^-) = -ic$. In fact one has

$$g_\alpha(t) = \begin{cases} -ie^{-i\alpha t}\displaystyle\int_{a_-}^{t} e^{i\alpha s} g(s)\,ds\,, & a_- \leqslant t < 0 \\ ie^{-i\alpha t}\displaystyle\int_{t}^{a_+} e^{i\alpha s} g(s)\,dx\,, & 0 < t \leqslant a_+\,. \end{cases} \tag{1.44}$$

With the aid of Lemmas 1.4, 1.5 and 1.6 it is now not hard to establish a lemma which will cover the case $\lambda \to \lambda_+$ or $\lambda \to \lambda_-$.

LEMMA 1.7. *Suppose that $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is an even, real-valued function such that $g(x) = 0$ almost everywhere for $|x| > a$ and such that*

$$1 - \hat{g}(\xi) = 1 - \int_{-a}^{a} g(x) e^{i\xi x}\,dx \geqslant 0\,, \quad -\infty < \xi < \infty\,. \tag{1.45}$$

Assume that $\{g_n\} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is a sequence of even, real-valued functions such that $g_n(x) = 0$ almost everywhere for $|x| > a$, $\|g_n - g\|_{L^1} \to 0$ and

$\|g_n - g\|_{L^2} \to 0$ as $n \to \infty$, and

$$1 - \hat{g}_n(\xi) > 0\,, \quad -\infty < \xi < \infty\,. \tag{1.46}$$

Then there exists $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ suchlthat v is real-valued, $v(x) = 0$ almost everywhere for $x < 0$ or $x > a$ and

$$1 - \hat{g}(\xi) = \big(1 - \hat{v}(\xi)\big)\big(1 - \hat{v}(-\xi)\big)\,, \quad -\infty < \xi < \infty\,, \tag{1.47}$$

$$1 - \int_0^a v(x)e^{isx}dx \neq 0\,, \quad s \text{ complex, } \operatorname{Im}(s) > 0\,. \tag{1.48}$$

If $v_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is the unique real-valued function such that $v_n(x) = 0$ almost everywhere for $x < 0$ and such that v_n satisfies equations (1.47) and (1.48) when v_n is substituted for v and g_n for g, then $\|v_n - v\|_{L^2} \to 0$ and $\|v_n - v\|_{L^1} \to 0$ as $n \to \infty$.

PROOF. For z a complex number define $\varphi(z)$ and $\varphi_n(z)$ by

$$\begin{aligned} \varphi(z) &= 1 - \int_{-a}^{a} g(x)e^{izx}dx \\ \varphi_n(z) &= 1 - \int_{-a}^{a} g_n(x)e^{izx}dx\,. \end{aligned} \tag{1.49}$$

It is well known (and not hard to prove) that in any strip G, $G = \{z: c < \operatorname{Im}(z) < d\}$, there is an integer N such that $\varphi(z)$ and $\varphi_n(z)$ have at most N zeros in the strip. Since $\lim_{|z|\to\infty} \varphi_n(z) = 1$ uniformly in n for $z \in \bar{G}$, it follows from Rouche's theorem that if $\varphi(z) \neq 0$ for $\operatorname{Im} z = c$ or $\operatorname{Im} z = d$, then for n large enough $\varphi_n(z)$ and $\varphi(z)$ have the same number of zeros (counting multiplicities) in G. Notice also that if z is a zero of φ (respectively, φ_n) of multiplicity k, then $\bar{z}$, $-\bar{z}$ and $-z$ are also zeros of φ (respectively φ_n) of multiplicity k.

In the situation of Lemma 1.7, we can assume φ has real roots (otherwise Lemma 1.4 gives the result). Inequality (1.45) shows that each of these roots must be of even multiplicity. Select $\varepsilon > 0$ such that $\varphi(z)$ has only real roots $r_1, r_2, \ldots, r_m, -r_1, -r_2, \ldots, -r_m$ and possibly $r_0 = 0$ in the strip $|\operatorname{Im} z| \leqslant \varepsilon$. Let $2k_j$ denote the multiplicity of the zero r_j, $1 \leqslant j \leqslant m$, and $2k_0$ the multiplicity of 0 if 0 is a root. By the remarks above and by further applications of Rouche's theorem one can see that given any $\delta > 0$ there is an integer $N(\delta)$ such that for $n \geqslant N(\delta)$, $\varphi_n(z)$ has precisely $2k_j$ roots in the ball of radius δ about r_j or $-r_j$ (we can assume $B_\delta(r_j)$ and $B_\delta(-r_j)$ are disjoint), $2k_0$ roots in the ball of radius δ about 0 (if 0 is a root of φ) and no other zeros in the strip $|\mathrm{Im}\, z| \leqslant \varepsilon$.

At this point it is convenient to assume that either $\varphi(0) = 0$ and φ has no other real roots (which we shall call case 1) or $\varphi(r) = \varphi(-r) = 0$ for some real $r \neq 0$ and φ has no other real roots (which we shall call case 2). The proof in the general case is essentially the same, but notation becomes cumbersome. Let $2k$ denote the multiplicity of the root 0 (in case 1) or of r (in case 2). Select $\delta > 0$ with $\delta < \varepsilon$ and $\delta < r$ such that for $n \geqslant N(\delta)$ $\varphi_n(z)$ has precisely $2k$ solutions in $B_\delta(0)$ and no other solutions in the strip $|\mathrm{Im}\, z| \leqslant \varepsilon$ (in case 1) or $\varphi_n(z)$ has precisely $2k$ solutions in $B_\delta(r)$ and $B_\delta(-r)$ and no other solutions satisfying $|\mathrm{Im}\, z| \leqslant \varepsilon$ (case 2). Let $z_1^{(n)}, \ldots, z_k^{(n)}$ denote the k roots of $\varphi_n(z)$ in $B_\delta(0)$ (case 1) or in $B_\delta(r)$ (case 2) with positive imaginary part; there must be k such roots because $\varphi_n(z) = \varphi_n(\bar{z})$ and φ_n has no real roots.

In case 1 notice that if $z \in \{z_j^{(n)} : 1 \leqslant j \leqslant k\} = S_n$, then $-\bar{z} \in S_n$. We now define new functions $\psi(s)$ and $\psi_n(s)$ for s a complex number $(i = \sqrt{-1})$:

$$(1.50)\quad \begin{cases} \psi(s) = \left[\left(\dfrac{s+i}{s}\right)\left(\dfrac{s-i}{s}\right)\right]^k \varphi(s) & \text{in case 1} \\ \psi(s) = \left[\left(\dfrac{s+i}{s-r}\right)\left(\dfrac{s-i}{s-r}\right)\left(\dfrac{s+i}{s+r}\right)\left(\dfrac{s-i}{s+r}\right)\right]^k \varphi(s) & \text{in case 2}\,. \end{cases}$$

Define $\psi_n(s)$ by the formulas

$$(1.51)\quad \begin{cases} \psi_n(s) = \left[\prod_{z \in S_n} \left(\dfrac{s+i}{s-z}\right)\left(\dfrac{s-i}{s-z}\right)\right] \varphi_n(s) & \text{in case 1} \\ \psi_n(s) = \left[\prod_{z \in S_n} \left(\dfrac{s+i}{s-z}\right)\left(\dfrac{s-i}{s-z}\right)\left(\dfrac{s+i}{s+z}\right)\left(\dfrac{s-i}{s+z}\right)\right] \varphi_n(s) & \text{in case 2}\,. \end{cases}$$

In either case one can easily check that $\psi_n(s)$ and $\psi(s)$ are even, real-valued and strictly positive for $-\infty < s < \infty$ (ψ is nonnegative and one removes the places where φ is zero).

If we are in case 2 one can write

$$(1.52)\quad \begin{cases} \left(\dfrac{s-i}{s+r}\right)\varphi(s) = \left[1 - \dfrac{r+i}{s+r}\right]\varphi(s) \\ \left(\dfrac{s-i}{s+z}\right)\varphi_n(s) = \left[1 - \dfrac{r+i}{s+z}\right]\varphi(s)\,. \end{cases}$$

If one applies the explicit formulas in Lemma 1.6 and uses the fact that $\bar{z} \in S_n \to r$ as $n \to \infty$, one can see that for real s

$$(1.53)\qquad \begin{cases} \left(\dfrac{s-i}{s+r}\right)\varphi(s) = 1 - \displaystyle\int_{-a}^{a} h(x)\, e^{isx} ds \\[2ex] \left(\dfrac{s-i}{s+r}\right)\varphi_n(s) = 1 - \displaystyle\int_{-a}^{a} h_n(x)\, e^{isx} ds \end{cases}$$

where h and h_n belong to $L^1[-a, a] \cap L^2[-a, a]$ and $\|h_n - h\|_{L^2} \to 0$ and $\|h_n - h\|_{L^1} \to 0$ as $n \to \infty$.

If one repeats this argument $4k$ times (in case 2) or $2k$ times (in case 1) one eventually finds that

$$(1.54)\qquad \begin{aligned} \psi_n(s) &= 1 - \int_{-a}^{a} f_n(x)\, e^{ixs} dx \\ \psi(s) &= 1 - \int_{-a}^{a} f(x)\, e^{ixs} dx \end{aligned}$$

when f_n and f belong to $L^1[-a, a] \cap L^2[-a, a]$ and $\max\left(\|f_n - f\|_{L^2}, \|f_n - f\|_{L^1}\right) \to 0$ as $n \to \infty$.

Since ψ_n and ψ are even and real-valued, it follows that f_n and f are even and real-valued. We have arranged that ψ_n and ψ are positive for real s, so Lemma 1.4 applies. Thus there exist functions w_n and w in $L^2[0, a]$ so $\|w_n - w\|_{L^2} \to 0$ and such that (if w_n and w are extended to be 0 outside $[0, a]$)

$$(1.55)\qquad \begin{aligned} \psi_n(s) &= \left(1 - \hat{w}_n(s)\right)\left(1 - \hat{w}_n(-s)\right) \\ \psi(s) &= \left(1 - \hat{w}(s)\right)\left(1 - \hat{w}(-s)\right) \end{aligned}$$

and

$$(1.56)\qquad \begin{aligned} 1 - \hat{w}_n(s) &\neq 0 \quad \text{for complex } s,\ \operatorname{Im}(s) \geqslant 0 \\ 1 - \hat{w}(s) &= 0 \quad \text{for complex } s,\ \operatorname{Im}(s) \geqslant 0 . \end{aligned}$$

Let Z_+, Z and Z_- be as defined at the beginning of this section. It is an elementary fact that if $\operatorname{Im}(\alpha) > 0$, then $(s - \alpha)^{-1} \in Z_+$ and if $\operatorname{Im}(\alpha) < 0$, then $(s - \alpha)^{-1} \in Z_-$ (see [7], p. 173). Assume for definiteness that we are in case 2. Then our previous work shows that if

$$(1.57)\qquad \begin{cases} Q_n(s)=\left[\prod_{z\in S_n}\left(\dfrac{s+z}{s-i}\right)\left(\dfrac{s-\bar{z}}{s-i}\right)\right](1-\hat{w}_n(s)) \\ Q(s)\ =\left[\prod_{j=1}^{k}\left(\dfrac{s+r}{s-i}\right)\left(\dfrac{s-r}{s-i}\right)\right](1-\hat{w}(s)) \end{cases}$$

then Q and Q_n are elements of Z_+ and

$$(1.58)\qquad \begin{aligned} \varphi_n(s) &= Q_n(s)Q_n(-s) \\ \varphi(s) &= Q(s)Q(-s)\,. \end{aligned}$$

The formula 1.57 shows that $\|Q_n - Q\|_Z \to 0$ as $n \to \infty$ and that $Q_n(s) \neq 0$ for $\operatorname{Im} s > 0$. It follows by the uniqueness result in Lemma 1.4 that

$$Q_n(\xi) = 1 - \hat{v}_n(\xi)$$

and the remarks above show that

$$Q(\xi) = 1 - \hat{v}(\xi)$$

where v is as in the statement of the lemma and $\|v_n - v\|_Y \to 0$ as $n \to \infty$. ■

We can now establish the basic results we shall need about equation (1.1).

THEOREM 1.2. *Suppose that $f(x) \in L^2[0, 1]$ is real-valued and not identically zero and extend $f(x)$ to be even and zero almost everywhere for $|x| > 1$. Define numbers λ_+ and λ_- by*

$$\lambda_+ = \sup\left\{\lambda > 0 : 1 - \lambda\int_{-1}^{1} f(x)e^{i\xi x}dx > 0 \text{ for all real } \xi\right\}$$

$$\lambda_- = \inf\left\{\lambda < 0 : 1 - \lambda\int_{-1}^{1} f(x)e^{i\xi x}dx > 0 \text{ for all real } \xi\right\}.$$

Then for $\lambda_- \leqslant \lambda \leqslant \lambda_+$ there is a solution $u = u_\lambda \in L^2[0, 1]$ of

$$(1.59)\qquad u(x) = f(x) + \lambda\int_{x}^{1} u(y)u(y-x)dy$$

such that

(1.60) $$1-\lambda\hat{u}_\lambda(s)=1-\lambda\int_0^1 u_\lambda(x)\,e^{isx}dx\neq 0\,,\qquad s \text{ complex},\ \operatorname{Im}(s)>0\,.$$

The map $\lambda\to u_\lambda$ *is continuous in the norm topology on* $L^2[0,1]$ *for* $\lambda_-\leqslant\lambda\leqslant\lambda_+$, *and for* $\lambda_-<\lambda<\lambda_+$ *there is one and only one real-valued* $u_\lambda\in L^1[0,1]$ *which satisfies* (1.59) *and* (1.60). *For* $\lambda>\lambda_+$ *or* $\lambda<\lambda_-$, (1.59) *has no real-valued solution in* $L^1[0,1]$.

PROOF. By Lemma 1.1, solving (1.59) is equivalent for $\lambda\neq 0$ to finding a real-valued $v_\lambda(x)\in L^2(\mathbb{R})$, $v_\lambda(x)=0$ for $x\notin[0,1]$, such that

(1.61) $$1-\lambda\hat{f}(\xi)=\big(1-\hat{v}_\lambda(\xi)\big)\big(1-\hat{v}_\lambda(-\xi)\big)\,.$$

In fact our previous lemmas show that for $\lambda_-\leqslant\lambda\leqslant\lambda_+$ there is such a solution v_λ of (1.61) which also satisfies

$$1-\hat{v}_\lambda(s)\neq 0\,,\qquad s \text{ complex, } \operatorname{Im}(s)>0$$

that this solution is unique for $\lambda_-<\lambda<\lambda_+$, and that $\lambda\to v_\lambda$ is continuous. It follows that $u_\lambda=\lambda^{-1}v_\lambda$ satisfies the conditions of Theorem 1.2 and is continuous except possibly at $\lambda=0$.

To complete the proof we have to show that $u_\lambda\to f|[0,1]$ in the $L^2[0,1]$ norm. To do this define a map $\Phi\colon L^2[0,1]\times\mathbb{R}\to L^2[0,1]$ by

(1.62) $$\big(\Phi(u,\lambda)\big)(x)=u(x)-\lambda\int_x^1 u(y)\,u(y-x)\,dy-f(x)\,.$$

It is not hard to see that the map Φ is continuously Fréchet differentiable and that the Fréchet derivative with respect to the u variable at (u,λ) is the linear operator L given by

(1.63) $$(Lh)(x)=h(x)-\lambda\int_x^1 u(y)\,h(y-x)\,dy-\lambda\int_x^1 u(y-x)\,h(y)\,dy\,.$$

(Note that $\|u\|_{L^1}\leqslant\|u\|_{L^2}$ on $[0,1]$, so L is a bounded linear operator.) If $\lambda=0$, this linear operator is just the identity map, so the implicit function theorem for Banach spaces implies that there is $\varepsilon>0$ and a C^1 map

$\lambda \to w_\lambda \in [0,1]$ for $|\lambda| < \varepsilon$ such that $w_0 = f$ and

$$\Phi(w_\lambda, \lambda) = 0 . \tag{1.63'}$$

If λ is so small that $\|\lambda w_\lambda\|_{L^2} < 1$, then because $\int_0^1 |\lambda w_\lambda(x)| dx \leqslant \|\lambda w_\lambda\|_{L^2}$ it is not hard to see that

$$1 - \lambda \int_0^1 w_\lambda(x) e^{isx} dx \neq 0 , \quad s \text{ complex, } \operatorname{Im}(s) \geqslant 0 . \tag{1.64}$$

The uniqueness of solutions satisfying (1.63′) and (1.64) implies that $u_\lambda = w_\lambda$ for λ small enough, so $\lambda \to u_\lambda$ is continuous (and indeed C^1) near $\lambda = 0$.

We are actually interested in (1.1) when $f(x)$ is continuous or has at most a finite number of jump discontinuities, but as we shall now show, this case can be easily analyzed with the aid of Theorem 1.2.

LEMMA 1.8. *Suppose that $f(x)$ is a bounded, measurable function for $0 \leqslant x \leqslant 1$ and that $u \in L^2[0,1]$ satisfies equation (1.1) for $x \notin E$, where E has zero measure. Then for $x \notin E$ one has*

$$|u(x)| \leqslant M + |\lambda| \|u\|_{L^2}^2 , \qquad |f(x)| \leqslant M . \tag{1.65}$$

There exist a function $\omega(\delta)$ for $\delta > 0$ with $\lim_{\delta \to 0^+} \omega(\delta) = 0$ and a constant B such that for any x_1, x_2 with $x_1 \notin E$ and $x_2 \notin E$ one has

$$|u(x_2) - u(x_1)| \leqslant |f(x_2) - f(x_1)| + |\lambda| B^2 |x_1 - x_2| + |\lambda| B \omega(|x_2 - x_1|) . \tag{1.66}$$

In particular, if $f(x)$ is continuous on $[0,1]$, then $u(x)$ can be taken to be continuous on $[0,1]$.

PROOF. If $|f(x)| \leqslant M$, the Cauchy-Schwartz inequality gives (for $x \notin E$)

$$\begin{aligned} |u(x)| &\leqslant M + |\lambda| \int_x^1 |u(y)| |u(y-x)| dy \\ &\leqslant M + |\lambda| \Big(\int_x^1 u(y)^2 dy \Big)^{\frac12} \Big(\int_x^1 u(y-x)^2 dy \Big)^{\frac12} \\ &\leqslant M + |\lambda| \Big(\int_0^1 u(y)^2 dy \Big) . \end{aligned} \tag{1.67}$$

If $|u(x)| \leqslant B$ for $x \notin E$, then for any $x_1, x_2 \in [0,1] - E$, $x_1 < x_2$, one has

$$(1.68)\qquad |u(x_2) - u(x_1)| \leqslant |f(x_2) - f(x_1)| + |\lambda| \int_{x_1}^{x_2} |u(y)||u(y - x_1)|\,dy + |\lambda| \int_{x_2}^{1} |u(y)||u(y - x_2) - u(y - x_1)|\,dy$$
$$\leqslant |f(x_2) - f(x_1)| + |\lambda| B^2 |x_1 - x_2| + |\lambda| B \int_{x_2}^{1} |u(y - x_2) - u(y - x_1)|\,dy\,.$$

Extend u to be 0 outside of $[0,1]$, so

$$\int_{x_2}^{1} |u(y - x_2) - u(y - x_1)|\,dy \leqslant \int_{-\infty}^{\infty} |u(y - x_2) - u(y - x_1)|\,dy\,.$$

It is well known that for any function $u \in L^1(\mathbb{R})$ there is a function $\omega(\delta)$ $(\delta > 0)$ with $\lim_{\delta \to 0^+} \omega(\delta) = 0$ such that

$$(1.69)\qquad \int_{-\infty}^{\infty} |u(y - x_2) - u(y - x_1)|\,dy \leqslant \omega(|x_2 - x_1|)\,.$$

Substituting (1.69) in (1.68) completes the proof. ∎

With the aid of Lemma 1.8 we can prove our basic theorem about continuous solutions of (1.1).

THEOREM 1.3. *Suppose that $f(x)$ is a continuous function for $0 \leqslant x \leqslant 1$ and define real numbers $\lambda_+ > 0$ and $\lambda_- < 0$ as in the statement of Theorem* 1.2. *For $\lambda_- \leqslant \lambda \leqslant \lambda_+$ there exists a continuous function $u_\lambda(x)$, $0 \leqslant x \leqslant 1$, such that $u_\lambda(x)$ satisfies equation* (1.1) *and such that*

$$(1.70)\qquad 1 - \lambda \int_0^1 u_\lambda(x) e^{izx}\,dx \neq 0\,, \quad z \text{ complex, } \operatorname{Im}(z) > 0\,.$$

An L^1 real-valued solution of (1.1) *which also satisfies* (1.70) *is unique for $\lambda_- < \lambda < \lambda_+$. The map $\lambda \to u_\lambda$ is a continuous map from $[\lambda_-, \lambda_+]$ to $C[0,1]$, the Banach space of continuous functions on $[0,1]$ in the usual norm. Equation* (1.1) *has no real-valued L^1 solutions for $\lambda \notin [\lambda_-, \lambda_+]$.*

PROOF. By Lemma 1.8 and Theorem 1.2 one can select for each $\lambda \in [\lambda_-, \lambda_+]$ a continuous function $u_\lambda \in C[0, 1]$ which satisfies (1.1) and (1.70).

It remains to show that $\lambda \to u_\lambda$ is continuous as a map into $C[0, 1]$. By using inequality (1.65) and the fact that $\lambda \to u_\lambda$ is continuous as a map into $L^2[0, 1]$ $\left(\text{so } \sup_{J\in\lambda} \|u\|_{L^2} < \infty \text{ for any compact interval } J \subset [\lambda_-, \lambda_+]\right)$ one can see that for any compact interval $J \subset [\lambda_-, \lambda_+]$ there is a constant M such that

$$(1.71) \qquad |u_\lambda(x)| \leq M\,, \qquad 0 \leq x \leq 1,\ \lambda \in J\,.$$

To show continuity, take any $\lambda \in [\lambda_-, \lambda_+]$ and suppose $\lambda_n \to \lambda$ as $n \to \infty$, where $\lambda_n \in [\lambda_-, \lambda_+]$. For notational convenience write $v = u_\lambda$ and $v_n = u_{\lambda_n}$. By the above comments we can assume $|v_n(x)| \leq M$ and $|v(x)| \leq M$ for $0 \leq x \leq 1$, and we know $v_n \to v$ in L^2 norm. By using the defining equation (1.1) one finds

$$(1.72) \qquad |v_n(x) - v(x)| \leq |\lambda_n - \lambda| \int_x^1 |v_n(y) v_n(y-x)| dy + \\ + |\lambda| \int_x^1 |v_n(y) - v(y)| |v_n(y-x)| dy + |\lambda| \int_x^1 |v(y)| |v(y-x) - v_n(y-x)| dy\,.$$

By using equation (1.72) one sees that

$$(1.73) \qquad |v_n(x) - v(x)| \leq |\lambda_n - \lambda| M^2 + 2|\lambda| M \|v_n - v\|_{L^2}$$

and (1.72) implies that v_n approaches v in the $C[0, 1]$ norm. ■

We shall also need to know that if the function $f(x)$ in equation (1.1) is continuously differentiable and $u(x)$ is a continuous solution, then $u(x)$ is continuously differentiable. This result has been established by G. Pimbley [10] for the case $f(x) \equiv 1$, but his proof applies to the more general case.

PROPOSITION 1.2 (See Theorem 6 in [10]). *Suppose that $u \in C[0, 1]$, $f(x)$ is continuously differentiable for $0 \leq x \leq 1$ and u satisfies the equation*

$$u(x) = f(x) + \lambda \int_x^1 u(y) u(y-x) dy\,, \qquad 0 \leq x \leq 1$$

for some constant λ. Then $u(x)$ is continuously differentiable for $0 \leq x \leq 1$ and $u'(x)$ satisfies the equation

$$(1.74) \qquad u'(x) = f'(x) - \lambda u(x) u(0) - \lambda \int_x^1 u(y) u'(y-x) dy\,.$$

REMARK 1.1. If $f \in C^n[0,1]$ one can see from (1.74) that $u \in C^n$. For integration of (1.74) by parts gives

$$u'(x) = f'(x) + \lambda \int_x^1 u'(y)u(y-x)\,dy - \lambda u(1)u(1-x)\,. \tag{1.75}$$

If $f \in C^2[0,1]$, the right hand side of (1.75) is clearly differentiable with derivative

$$f''(x) + \lambda u(1)u'(1-x) - \lambda u'(x)u(0) - \lambda \int_x^1 u'(y)u'(y-x)\,dy \tag{1.76}$$

so $u \in C^2[0,1]$. It follows that if $f \in C^3[0,1]$ the expression (1.76) has a derivative and $u \in C^3[0,1]$. Clearly, this argument can be continued to show $u \in C^n$.

REMARK 1.2. If we write $w(x) = u'(x) \in C[0,1]$ (for the case $f \in C^1[0,1]$), then (1.75) shows that w satisfies the integral equation

$$w(x) = f'(x) - \lambda u(1)u(1-x) - \lambda \int_x^1 u(y-x)w(y)\,dy\,. \tag{1.76$'$}$$

If u is considered a known function, this equation is well-known to have a unique solution w in $C[0,1]$ and one has the estimate

$$\max_{0\leqslant x\leqslant 1} |w(x)| \leqslant \big(\exp(|\lambda| M_1)\big) M_2 \tag{1.77}$$

$$M_1 = \max_{0\leqslant x\leqslant 1} |u(x)|\,, \qquad M_2 = \max_{0\leqslant x\leqslant 1} |f'(x) - \lambda u(1-x)|\,.$$

Using (1.77) one can see that the obvious analogue of Theorem 1.3 also holds if one works in $C^n[0,1]$ (assuming $f \in C^n[0,1]$).

Until now we have only considered solutions of (1.1) which also satisfy (1.27). We shall need to know the general real-valued solution u of (1.1). This can easily be found using the ideas in [7] and the previous theorems.

LEMMA 1.9. *Let $f(x)$ be an even, real-valued, integrable function such that $f(x) = 0$ for $|x| > 1$, and let λ be a nonzero real number. Assume that $u \in L^1[0,1]$ is a real-valued solution of equation* (1.1) *for almost all $x \in [0,1]$. For com-*

plex numbers z define

$$\varphi(z) = 1 - \lambda \int_0^1 u(x) e^{izx} dx$$

$$\psi(z) = 1 - \lambda \int_{-1}^1 f(x) e^{izx} dx$$

define $S = \{\alpha \in C : \varphi(\alpha) = 0$ *and* $\operatorname{Im}(\alpha) > 0\}$ *and define* $k(\alpha)$ *to be the multiplicity of* α *as a zero of* φ. *Then one has* (1) S *is finite,* (2) S *is a subset of the set of zeros of* ψ *and* $k(\alpha)$ *is less than or equal to the multiplicity of* α *as a zero of* ψ *and* (3) *if* $\alpha \in S$, *then* $-\bar{\alpha} \in S$ *and* $k(-\bar{\alpha}) = k(\alpha)$.

PROOF. Since $u(x)$ is assumed to be real-valued one has $u(-\bar{z}) = \overline{u(z)}$, so condition (3) above is true. We assume that $u \in L^1[0, 1]$, and it follows that

$$\lim_{|z| \to \infty, \operatorname{Im} z \geqslant 0} \varphi(z) = 1 . \tag{1.78}$$

Because $\varphi(z)$ is analytic equation (1.78) implies that S is finite. Finally, condition (2) above follows from equation (1.6) in the proof of Lemma 1.1. ∎

Our real interest is in the converse of Lemma 1.9: given any finite subset S of zeros of $\psi(z)$ as above, there is precisely one real-valued solution u of (1.1) such that $\varphi(z)$ has S as its set of zeros with positive imaginary part. More precisely we have the following theorem.

THEOREM 1.4. *Let* $f(x)$ *be an even, real-valued function such that* $f(x) = 0$ *almost everywhere for* $|x| > 1$ *and such that* f *is integrable. Let* λ *be a real number and for complex numbers* z *define* $\psi(z)$ *by*

$$\psi(z) = 1 - \lambda \int_{-1}^1 f(x) e^{izx} dx \tag{1.79}$$

and assume $\psi(\xi)$ *is nonnegative for all real numbers* ξ. *Let* S *be any finite collection of zeros* α *of* $\psi(z)$ *and for each* $\alpha \in S$ *let* $k(\alpha)$ *be a positive integer. Assume that* S *and the integers* $k(\alpha)$ *satisfy the following properties*: (1) *If* $\alpha \in S$, *one has* $\operatorname{Im}(\alpha) > 0$, $-\bar{\alpha} \in S$ *and* $k(\alpha) = k(-\bar{\alpha})$; (2) *The integer* $k(\alpha)$ *is less than or equal to the multiplicity of* α *as a zero of* $\psi(z)$. *Then there is one and only one real-valued function* $u \in L^1[0, 1]$ *such that* u *satisfies equation* (1.1) *for almost all* x *in* $[0, 1]$ *and such that the set of zeros* α *with* $\operatorname{Im}(\alpha) > 0$ *of*

$$\varphi(z) \stackrel{\text{def}}{=} 1 - \lambda \int_0^1 u(x) e^{izx} dx \tag{1.80}$$

equals S and the multiplicity of $\alpha \in S$ as a zero of $\varphi(z)$ is $k(\alpha)$. (If S is empty it is understood that $\varphi(z) \neq 0$ for $\operatorname{Im} z > 0$). If $f \in L^2(\mathbf{R})$, the above solution $u \in L^2[0,1]$ and if $f|[0,1]$ is continuous, u is continuous. There are no real-valued solutions $u \in L^1[0,1]$ of (1.1) except for the ones described above.

PROOF. Let $v \in L^1[0,1]$ denote a solution of (1.1) such that

$$1 - \lambda \int_0^1 v(x) e^{izx} \neq 0 , \qquad \operatorname{Im} z > 0 .$$

Such a solution is insured by Theorem 1.1; Theorems 1.2 and 1.3 show that v can be taken in $L^2[0,1]$ or $C[0,1]$ if f is in $L^2[0,1]$ or $C[0,1]$ respectively. Define a meromorphic function $Q(z)$ by

$$Q(z) = \prod_{\alpha \in S} \left(\frac{z-\alpha}{z+\alpha} \right)^{k(\alpha)} . \tag{1.81}$$

For ξ real, notice that one can write

$$\frac{\xi - \alpha}{\xi + \alpha} = 1 - \frac{2\alpha}{\xi + \alpha} .$$

If $g_\alpha(x)$ is defined by $g_\alpha(x) = 0$ for $x < 0$ and $g_\alpha(x) = e^{i\alpha x}$ for $x > 0$ one has $g_\alpha \in L^2 \cap L^1$ (since $\operatorname{Im} \alpha > 0$) and

$$-\frac{2\alpha}{\xi + \alpha} = 2\alpha i \hat{g}_\alpha . \tag{1.82}$$

It follows that (in our previous notation) $Q(\xi) \in Z_+$ and

$$Q(\xi)\big(1 - \lambda \hat{v}(\xi)\big) = 1 - \lambda \hat{u}(\xi) \tag{1.83}$$

where $u \in L^1$ if $v \in L^1$, $u \in L^1 \cap L^2$ if $v \in L^1 \cap L^2$ and $u(x) = 0$ for almost all $x < 0$. By grouping the factors corresponding to α and $-\bar{\alpha}$ in the formula for $Q(z)$ and using the fact that $k(\alpha) = k(-\bar{\alpha})$ one can see that

$$\begin{cases} \overline{Q(z)} = Q(-\bar{z}) , & z \text{ complex} \\ |Q(\xi)| = 1 , & \xi \text{ real.} \end{cases} \tag{1.84}$$

By using (1.84) one can see that the function u is real-valued and that for real ξ one has

$$\begin{aligned} 1-\lambda\hat{f}(\xi) &= \big(1-\lambda\hat{v}(\xi)\big)\big(1-\lambda\hat{v}(-\xi)\big) \\ &= Q(\xi)\big(1-\lambda\hat{v}(\xi)\big)Q(-\xi)\big(1-\lambda\hat{v}(-\xi)\big) \\ &= \big(1-\lambda\hat{u}(\xi)\big)\big(1-\lambda\hat{u}(-\xi)\big)\,. \end{aligned} \tag{1.85}$$

Now define an analytic function $\varphi(z)$ by

$$\begin{aligned} \varphi(z) &= 1-\lambda\hat{u}(z)\,, && \operatorname{Im} z \geqslant 0 \\ \varphi(z) &= \big(1-\lambda\hat{f}(z)\big)\big(1-\lambda\hat{u}(-z)\big)^{-1}, && \operatorname{Im} z < 0\,. \end{aligned}$$

Notice that $\varphi(z)$ is everywhere analytic, because if $1-\lambda\hat{u}(-z)=0$ for $\operatorname{Im} z<0$, then $-z\in S$ and $-z$ is a zero of $1-\lambda\hat{f}(z)$. Since we have

$$\lim_{|z|\to\infty,\,\operatorname{Im}(z)\geqslant 0} 1-\lambda\hat{u}(z) = 1 \tag{1.86}$$

uniformly in z in the closed upper half plane, we find (since f has support in $[-1,1]$) that there is a constant C such that

$$|\varphi(z)| \leqslant C\exp\big(|\operatorname{Im}(z)|\big)\,, \qquad \operatorname{Im}(z)\leqslant 0\,. \tag{1.87}$$

For nonzero λ it follows that the same estimate holds for $\hat{u}(z)$, where we define $\hat{u}(z)$ by

$$\hat{u}(z) = \lambda^{-1}\big(1-\varphi(z)\big)\,, \qquad z \text{ complex}\,.$$

Since $u\in L^1$ or $L^2\cap L^1$ (depending on the assumptions on f), the Paley-Wiener theorem implies that u has support in $[0,1]$. Now by working backward from (1.85) and using the ideas of Lemma 1.1, one can see that u satisfies equation (1.1). If f is also continuous on $[0,1]$, Lemma 1.8 shows that u can be taken continuous.

It remains to prove the uniqueness statement of the lemma. Suppose that u_1 and u_2 are real-valued integrable solutions of (1.1) and that

$$\varphi_1(z) \overset{def}{=} 1-\lambda\int_0^1 u_1(x)e^{izx}dx$$

$$\varphi_2(z) \overset{def}{=} 1-\lambda\int_0^1 u_2(x)e^{izx}dx$$

have the same set of zeros S with positive imaginary part. According to Lemma 1.1 we have for all complex numbers z

$$1-\lambda\hat{f}(z)=\varphi_1(z)\varphi_1(-z)=\varphi_2(z)\varphi_2(-z)\,. \tag{1.88}$$

If z is a real number and $\varphi_j(z)=0$, the multiplicity of z as a zero of $\varphi_j(z)$ is the same as the multiplicity of $-z$ as a zero of $\varphi_j(z)$ (since u_j is real-valued). Therefore equation (1.88) shows that the real zeros of φ_j are precisely the real zeros of $1-\lambda f(z)$, and the multiplicity of a real zero ξ of φ_j must equal $(\frac{1}{2})m(\xi)$, where $m(\xi)$ is the (even) multiplicity of ξ as a zero of $1-\lambda\hat{f}(z)$. It follows that φ_1 and φ_2 have the same zeros z with $\operatorname{Im}(z)\geqslant 0$ (and these zeros have the same multiplicity). By using equation (1.88) one sees that the same is true for zeros of $\varphi_j(z)$ with $\operatorname{Im}(z)<0$. Thus if we define $\theta(z)$ by

$$\theta(z)=\frac{\varphi_1(z)}{\varphi_2(z)}=\frac{\varphi_2(-z)}{\varphi_1(-z)} \tag{1.89}$$

$\theta(z)$ is analytic for all z. Furthermore (1.89) shows $\theta(z)$ approaches 1 as $|z|\to\infty$ in the closed upper half plane and the closed lower half plane and hence in the complex plane. Liouville's theorem implies that $\theta(z)=1$ everywhere, so $\varphi_1(z)=\varphi_2(z)$ for all z.

Lemma 1.9 shows that every solution u of (1.1) comes from some set of zeros S of the type described in the theorem. ■

Theorem 1.4 will play a crucial role in the rest of this paper, for example, in determining the number of positive solutions of (1.1). It provides a reasonably explicit description of all real-valued solutions of (1.1).

2. – The zeros of some holomorphic functions.

The results of the previous section show that an understanding of the structure of the solution set of (1.1) depends on knowledge about the zeros of

$$\varphi(z)=1-\lambda\int_{-1}^{1}f(x)\,e^{izx}\,dx\,. \tag{2.1}$$

For example we shall need detailed information about the number and location of zeros of (2.1) in order to determine how many positive solutions (1.1) has when f and λ are positive. The main tool we shall use is simply Rouche's theorem, but for completeness we state the theorem below (in a slightly more general form than is usually given in complex variables courses).

LEMMA 2.1 (Rouche's Theorem). *Let G be a bounded open subset of the complex plane and suppose that Γ, the boundary of G, consists of a finite number of nonintersecting simple closed curves. Let $\varphi_0(z)$ be a function which is continuous on $\bar{G}$, analytic on G and nonvanishing on Γ. If n denotes the number of zeros of $\varphi_0(z)$ in G (counting multiplicities), then*

$$n = \frac{1}{2\pi i}\int_{\Gamma} \frac{\varphi_0'(z)}{\varphi_0(z)}\, dz\,.$$

If $\varphi\colon \bar{G}\times[0,1] \to C$ is a continuous map such that $\varphi_t(z) \overset{\text{def}}{=} \varphi(z,t) \neq 0$ for $(z,t)\in\Gamma\times[0,1]$, and if φ_0 is analytic on G, then $\deg(\varphi_t, G, 0) =$ *the topological degree of φ_t on G is constant and equals the algebraic number of zeros of φ_0 in G. If φ_1 is also analytic on G, φ_0 and φ_1 have the same number of zeros in G (counting multiplicities of zeros). If G is unbounded, the same conclusion remains true if in addition there exists a number R (independent of $t\in[0,1]$) such that any solution of $\varphi(z,t)=0$ in $G\times[0,1]$ satisfies $|z|\leqslant R$.*

Our next lemma gives a simple formula for the number λ_+ (defined in Lemma 1.2) in the case that $f(x)$ is nonnegative.

LEMMA 2.2. *Let $f(x)$ be a nonnegative, even, integrable function such that $f(x)=0$ almost everywhere for $|x|\geqslant 1$ and $f(x)$ is positive on a set of positive measure. If $\lambda_+ = \left(2\int_0^1 f(x)\,dx\right)^{-1}$, one has*

$$\begin{aligned}
&1-\lambda\hat{f}(\xi) > 0\,, && -\infty<\xi<\infty,\ 0\leqslant\lambda<\lambda_+\\
(2.2)\qquad &1-\lambda_+\hat{f}(\xi) > 0\,, && \infty<\xi<\infty,\ \xi\neq 0\\
&1-\lambda_+\hat{f}(0) = 0\,.
\end{aligned}$$

PROOF. Observe that

$$(2.3)\qquad 1-\lambda\hat{f}(\xi) = 1-2\lambda\int_0^1 f(x)\,\mathrm{cox}\,\xi x\,dx\,.$$

If $E=\{x\in[0,1]\colon f(x)>0\}$ and if $\xi\neq 0$, one has

$$(2.4)\qquad f(x)\cos\xi x < f(x)$$

for almost all $x\in E$. Since E is assumed to have positive measure, inequality (2.2) follows from (2.3) and (2.4).

Our next lemma discusses the pure imaginary solutions of (2.1). As we shall see, the pure imaginary solutions of (2.1) play a special role in a discussion of positive solutions of (1.1).

LEMMA 2.3. *Assume that $f(x)$ satisfies the same assumptions as in Lemma 2.2, write* $\lambda_+ = \left(2\int_0^1 f(x)\,dx\right)^{-1}$ *and* $\lambda_- = \inf\{\lambda < 0 : 1 - \lambda\hat{f}(\xi) > 0$ *for all real* $\xi\}$. *Then for* $0 < \lambda < \lambda_+$ *equation* (2.1) *has precisely two* (*counting multiplicities*) *pure imaginary solutions*, $\pm iw(\lambda)$, $w(\lambda) > 0$. *The function* $w(\lambda)$ *is differentiable*, $w'(\lambda) < 0$ *for* $0 < \lambda < \lambda_+$, $\lim_{\lambda\to 0^+} w(\lambda) = +\infty$ *and* $\lim_{\lambda\to\lambda_+} w(\lambda) = 0$. *For* $\lambda < 0$, *equation* (2.1) *has no pure imaginary solutions. If* $\lambda_- < \lambda \leqslant \lambda_+$, (2.1) *has no solutions* z *such that* $0 < |\mathrm{Re}\,(z)| \leqslant \pi$.

PROOF. If $z = iw$ is a pure imaginary solution of (2.1), one obtains

$$\varphi(iw) \stackrel{def}{=} \psi(w, \lambda) = 1 - \lambda\int_0^1 f(x)[e^{wx} + e^{-wx}]dx = 0\,. \tag{2.5}$$

If $\lambda < 0$, our assumptions show that $\psi(w, \lambda) > 1$, so there can be no pure imaginary solutions in this case. If $0 < \lambda < \lambda_+$ we have

$$\psi(0, \lambda) > \psi(0, \lambda_+) = -\infty.$$

Because $\lambda > 0$ and $f(x)$ is assumed positive on a set of positive measure, we find

$$\lim_{w\to+\infty} \psi(w, \lambda) = 0\,.$$

It follows that the equation $\psi(w, \lambda) = 0$ has at least one positive solution w. Because

$$\frac{\partial\psi}{\partial w} = -\lambda\int_0^1 xf(x)[e^{wx} - e^{-wx}]dx \tag{2.6}$$

we have $\partial\psi/\partial w < 0$ for $w > 0$ and the equation $\psi(w, \lambda) = 0$ has exactly one positive solution $w = w(\lambda)$. Furthermore, the implicit function theorem shows that $w(\lambda)$ is a differentiable function of λ for $0 < \lambda < \lambda_+$ and that $w'(\lambda) < 0$. The fact that $\lim_{\lambda\to 0_+} w(\lambda) = +\infty$ follows because $\psi(w, 0) = 0$ has no solutions, and we have $\lim_{\lambda\to\lambda_+} w(\lambda) = 0$ because $\psi(w, \lambda_+) = 0$ has $w = 0$ as its only real root. Of course $-iw(\lambda)$ is also a solution of (2.1) because φ is an even function.

It remains to show that $\varphi(z) = 0$ has no solutions z such that $0 < |\mathrm{Re}\,(z)| \leqslant \pi$. Since $\varphi(-z) = \varphi(z)$, it suffices to prove this for $z = \mu + iv$,

$0 < \mu \leqslant \pi$. A calculation gives

$$(2.7)\qquad \varphi(z) = 1 - \lambda\int_0^1 f(x)\,\text{cox}\,(\mu x)[e^{\nu x} + e^{-\nu x}]\,dx + \lambda i\int_0^1 f(x)[e^{\nu x} - e^{-\nu x}]\sin(\mu x)\,dx\,.$$

Our definition of λ_+ and λ_- shows that equation (2.1) has no nonzero real roots, so we can assume for definiteness that $\nu > 0$. We can also suppose $\lambda \neq 0$. It follows that the imaginary part of $\lambda^{-1}\varphi(z)$ is obtained by integrating a nonnegative function which is positive on a set of positive measure, so

$$(2.8)\qquad \operatorname{Im}\varphi(z) = \lambda\int_0^1 f(x)[e^{\nu x} - e^{-\nu x}]\sin\mu x\,dx > 0\,.$$

Inequality (2.8) completes the proof of Lemma 2.3. ■

We also need a rough estimate on the size of the imaginary part of zeros of (2.1).

LEMMA 2.4. *Let $f(x)$ be an integrable, even function such that $f(x) = 0$ for almost all x with $|x| > 1$. Assume that there exist positive constants c and δ and a nonnegative integer n such that*

$$(2.9)\qquad f(x) \geqslant c(1-x)^n\,, \qquad 1-\delta \leqslant x \leqslant 1$$

and that there exists a constant $M < \infty$ such that

$$(2.10)\qquad M = \sup\{|f(x)|\colon |x| \leqslant 1-\delta\}\,.$$

Let J be a closed interval which does not contain 0 and for a positive number R, define

$$(2.11)\qquad S = \left\{z\colon 1 - \lambda\int_{-1}^1 f(x)e^{izx}\,dx = 0,\ \lambda \in J,\ |\operatorname{Re}(z)| \leqslant R\right\}.$$

Then there exists a number B which depends only on c, δ, n, M, R and $\varepsilon = \inf\{|u|\colon u \in J\}$ such that $|\operatorname{Im}(z)| \leqslant B$ for $z \in S$.

PROOF. Let $z = \mu + i\nu$ be an element of S. We can assume for definiteness that $\mu \geqslant 0$ and $\nu \geqslant 1$. We divide the proof into two cases: (1) $|\sin\mu| \geqslant |\cos\mu|$ and (2) $|\cos\mu| \geqslant |\sin\mu|$. If $\delta_1 = \sqrt{2}(4R)^{-1}$, it is easy to check that in case 1 one has $|\sin\mu x| \geqslant \sqrt{2}/4$ for $1-\delta_1 \leqslant x \leqslant 1$, while in case 2 one has

$|\cos \mu x| \geq \sqrt{2}/4$ for $1-\delta_1 \leq x \leq 1$. Define $\alpha = \min(\delta, \delta_1)$. If case 1 holds one has (for $\varphi(z)$ as in (2.1))

$$(2.12) \qquad |\operatorname{Im} \varphi(z)| = |\lambda| \left| \int_0^1 f(x) \sin \mu x (e^{\nu x} - e^{-\nu x}) dx \right| \geq$$

$$\geq |\lambda| c \left(\frac{\sqrt{2}}{4}\right) \int_{1-\alpha}^1 (1-x)^n (e^{\nu x} - e^{-\nu x}) dx - M|\lambda| \int_0^{1-\alpha} (e^{\nu x} - e^{-\nu x}) dx =$$

$$= |\lambda|\, c \left(\frac{\sqrt{2}}{4}\right) I_1 - M|\lambda| I_2.$$

To estimate I_1 observe that integration by parts gives

$$(2.13) \qquad I_1 \geq \int_{1-\alpha}^1 (1-x)^n e^{\nu x} dx - 1 \geq$$

$$\geq (n!)(\nu^{-n-1} e^{\nu}) - c_n e^{\nu(1-\alpha)} \nu^{-1} - 1$$

where c_n is a constant that depends only on n (for $\nu \geq 1$). Integration gives

$$(2.14) \qquad I_2 \leq e^{\nu(1-\alpha)} - 1 .$$

By using the estimates (2.13) and (2.14) in (2.12), one can see that there exists a number B_1, which can be chosen to depend continuously on M, n, α, and c (and is independent of λ as long as $\lambda \neq 0$), such that if $|\sin \mu| \geq |\cos \mu|$ and $\nu \geq B_1$, then

$$|\operatorname{Im} \varphi(z)| > 0 .$$

Since we assume $z \in S$ we must have $|\nu| \leq B_1$.

Now assume that $|\cos \mu| > |\sin \mu|$. In this case one has

$$(2.15) \qquad |\operatorname{Re} \varphi(z)| \geq |\lambda| c \left(\frac{\sqrt{2}}{4}\right) \int_{1-\alpha}^1 (1-x)^n (e^{\nu x} + e^{-\nu x}) dx -$$

$$- M|\lambda| \int_0^{1-\alpha} (e^{\nu x} + e^{-\nu x}) dx - 1 = |\lambda| c \left(\frac{\sqrt{2}}{4}\right) I_3 - M|\lambda| I_4 - 1 .$$

Just as before, an integration by parts shows

$$(2.16) \qquad I_3 \geq (n!)(\nu^{-n-1} e^{\nu}) - d_n e^{\nu(1-\alpha)} \nu^{-1}$$

where d_n is a constant that depends only on n (for $\nu \geqslant 1$). Integration gives

$$I_4 \leqslant e^{\nu(1-\alpha)} + 1 . \tag{2.17}$$

By using (2.16), (2.17) and (2.15), one can show that there exists a number B_2, which can be selected to depend continuously on n, M, α, c and ε such that if $|\cos\mu| \geqslant |\sin\mu|$ and $\nu \geqslant B_2$, then

$$|\mathrm{Re}\,\varphi(z)| > 0 . \tag{2.18}$$

Inequality (2.18) shows that $|\nu| \geqslant B_2$ for $z \in S$. ■

We are interested in the location of zeros of (2.1) for a general class of functions f, but to obtain this information it suffices to analyze the simplest function f, namely $f(x) = 1$ for $|x| < 1$ and 0 elsewhere. This analysis is carried out in the next lemma.

LEMMA 2.5. *Define* $\lambda_- = -\min\{-\xi(2\sin\xi)^{-1}: \pi < \xi < 2\pi\}$ *and define* $\theta(z) = \theta(z, \lambda)$ *by*

$$\theta(z) = 1 - \lambda\int_{-1}^{1} e^{izx}dx . \tag{2.19}$$

If n is a nonzero integer and λ any real, $\theta(z, \lambda) \neq 0$ for $\mathrm{Re}\,(z) = n\pi$. For each positive integer n define $U_n = \{z: 2n\pi - \pi \leqslant \mathrm{Re}\,(z) \leqslant 2n\pi + \pi\}$, $V_n = \{z: 2n\pi \leqslant \mathrm{Re}\,(z) \leqslant 2n\pi + \pi\}$, and $W_n = \{z: 2n\pi - \pi \leqslant \mathrm{Re}\,(z) \leqslant 2n\pi\}$. If $\lambda \neq 0$ and n is a positive integer, the equation $\theta(z, \lambda) = 0$ has precisely two solutions in U_n (counting multiplicities). If $\lambda_- < \lambda \leqslant \frac{1}{2}$ and $\lambda \neq 0$ there is precisely one solution $z_n \in U_n$ such that $\mathrm{Im}\,(z_n) > 0$, and this solution varies continuously with λ for $0 < \lambda < \frac{1}{2}$ and for $\lambda_- < \lambda < 0$. If $\lambda > 0$, $\theta(z, \lambda) \neq 0$ for any $z \in W_n$, where n is a positive integer, and $\theta(z,\lambda) = 0$ has precisely two solutions z such that $|\mathrm{Re}\,(z)| \leqslant \pi$. If $\lambda < 0$, $\theta(z, \lambda) \neq 0$ for any $z \in V_n$ and $\theta(z, \lambda) \neq 0$ for any z such that $|\mathrm{Re}\,(z)| \leqslant \pi$.

REMARK 2.1. Since $\theta(-z) = \theta(z)$ and $\theta(\bar{z}) = \theta(z)$, Lemma 2.5 gives a complete picture of all zeros of $\theta(z)$.

Although we only consider the range $\lambda_- < \lambda \leqslant \frac{1}{2}$ in discussing dependence of solutions of $\theta(z, \lambda) = 0$ on the parameter λ, it is not hard to prove that the solutions can be chosen to depend continuously on λ for all $\lambda > 0$ and all $\lambda < 0$. Specifically, Lemma 2.5 shows one can restrict attention to solutions $z \in U_n$. It is not hard to prove that there are numbers $\lambda_n > 0$ and $\mu_n < 0$ such that $\theta(z, \lambda) = 0$ has no real solutions in G_n for $\mu_n < \lambda < \lambda_n$ and precisely two real solutions for $\lambda > \lambda_n$ or $\lambda < \mu_n$. For $\mu_n < \lambda < \lambda_n$ the solutions are a conjugate pair and continuity follows by using Rouche's theorem; for $\lambda \geqslant \lambda_n$ or $\lambda \leqslant \mu_n$, a calculus argument suffices. This same observation applies to the more general situation considered in Theorem 2.1 below.

PROOF OF LEMMA 2.5. Integration gives

$$\theta(z, \lambda) = 1 - \lambda(e^{iz} - e^{-iz})(iz)^{-1} . \tag{2.20}$$

It will be convenient to define $g_0(z) = g_0(z, \lambda)$ by

$$g_0(z, \lambda) = z - \lambda(e^z - e^{-z}) . \tag{2.21}$$

Clearly $z = 0$ is always a solution of $g_0(z, \lambda) = 0$, but aside from this $\theta(w, \lambda) = 0$ if and only if $g_0(iw, \lambda) = 0$. Thus, in order to prove that $\theta(z, \lambda) \neq 0$ for $\mathrm{Re}\,(z) = n\pi$, n a nonzero integer, it suffices to show $g_0(z, \lambda) \neq 0$ for $\mathrm{Im}\,(z) = n\pi$. If $z = \mu + i\nu$, a calculation gives

$$g_0(z, \lambda) = \mu - \lambda(e^{\mu} - e^{-\mu}) \cos \nu + i\big(\nu - \lambda(e^{\mu} + e^{-\mu}) \sin \nu\big) \tag{2.22}$$

so

$$\mathrm{Im}\, g_0(z, \lambda) = n\pi , \qquad z = \mu + in\pi$$

and $g_0(z, \lambda) \neq 0$ for $\mathrm{Im}\,(z) = n\pi$. If n is a negative integer and $2n\pi + \pi \geqslant \nu \geqslant 2n\pi$ and $\lambda > 0$, (2.22) shows that $\mathrm{Im}\, g_0(z, \lambda) \leqslant \nu < 0$ and $g_0(z, \lambda) \neq 0$ for $iz \in W_{-n}$. If $\lambda < 0$ and n is a positive integer we also see that $\mathrm{Im}\, g_0(z, \lambda) \leqslant \nu < 0$ for $iz \in V_n$, so that $\theta(z, \lambda) \neq 0$ for $z \in V_n$.

It is a calculus exercise (which we leave to the reader) to show that λ_- (as defined in this lemma) satisfies

$$\lambda_- = \inf \{\lambda < 0 : \theta(\xi, \lambda) > 0 \text{ for all real } \xi\}.$$

Lemma 4.3 and the above calculations show that $\theta(z, \lambda) = 0$ has, for $\lambda_- < \lambda < 0$, no solutions z such that $|\mathrm{Re}\,(z)| \leqslant \pi$ and has, for $0 < \lambda < \frac{1}{2}$, precisely two solutions z such that $|\mathrm{Re}\,(z)| \leqslant 2\pi$.

Let $I = \{z\colon |\mathrm{Re}\,(z)| \leqslant \pi\}$ and $J = \{z\colon |\mathrm{Re}\,(z)| \leqslant 2\pi\}$. If $\lambda_1 \leqslant \lambda_-$ we want to prove that $\theta(z, \lambda_1) = 0$ has no solutions in I, and if $\lambda_2 \geqslant \frac{1}{2}$ we want to prove that $\theta(z, \lambda_2) = 0$ has precisely two solutions in J. In the first case, select λ_3 with $\lambda_- < \lambda_3 < 0$ and consider the homotopy $\theta(z, \lambda)$ for $\lambda_1 \leqslant \lambda \leqslant \lambda_3$. We have already seen that $\theta(z, \lambda) \neq 0$ for $z \in \partial I$, and it is easy to see that there exists a constant R such that any solution z of $\theta(z, \lambda) = 0$ for $z \in I$ and $\lambda_1 \leqslant \lambda \leqslant \lambda_3$ must satisfy $|z| < R$. Rouche's theorem thus implies that

$\theta(z, \lambda_1) = 0$ has the same number of solutions in I as $\theta(z, \lambda_3) = 0$, namely none. If $\lambda_2 \geqslant \frac{1}{2}$, select λ_4 with $0 < \lambda_4 < \frac{1}{2}$ and consider the homotopy $\theta(z, \lambda)$ for $z \in J$ and $\lambda_4 \leqslant \lambda \leqslant \lambda_2$. An argument like that above shows that $\theta(z, \lambda_2) = 0$ has the same number of solutions in J as $\theta(z, \lambda_4) = 0$, namely two.

It remains to discuss the number of solutions of $\theta(z, \lambda) = 0$ in U_n. For n a positive integer define $G_n = \{z\colon 2n\pi - \pi \leqslant \mathrm{Im}\,(z) \leqslant 2n\pi + \pi\}$. Assume λ is nonzero and for $0 \leqslant t \leqslant 1$ consider the homotopy $g_t(z)$ defined by

$$(2.23) \qquad g_t(z) = (1-t)z - \lambda[e^z - e^{-z}] + |\lambda| t[ei + e^{-1}i]\,.$$

If m is a positive integer and $z = im\pi + \mu$ we have

$$(2.24) \qquad \mathrm{Im}\,\big(g_t(z)\big) = (1-t)m\pi + |\lambda| t(e + e^{-1}) > 0$$

so $g_t(z) - 0$ for $z \in \partial G_n$. If $z = \mu + i\nu \in G_n$, we have

$$(2.25) \qquad |g_t(z)| \geqslant |\lambda|(e^{|\mu|} - 1) - (1-t)|z| - |\lambda| t(e + e^{-1})$$

and (2.25) implies that there is a number R such that any solution $z \in G_n$ of $g_t(z) = 0$ satisfies $|z| < R$. Rouche's theorem implies that $g_0(z)$ and $g_1(z)$ have the same number of zeros in G_n. If $g_1(z) = 0$ for some $z = \mu + i\nu$ in G_n, then by taking real and imaginary parts we find

$$(2.26) \qquad (e^{\mu} - e^{-\mu}) \cos \nu = 0\,,$$

$$(2.27) \qquad \lambda(e^{\mu} + e^{-\mu}) \sin \nu = |\lambda|(c + e^{-1})\,.$$

The absolute value of the left hand side of (2.27) is less than that of the right hand side if $|\mu| < 1$, so we must have $|\mu| \geqslant 1$. Since (2.26) can only hold if $\mu = 0$ or $\cos \nu = 0$, we must have $\cos \nu = 0$ and $\nu = 2n\pi + (\pi/2) = \nu_1$ or $\nu = 2n\pi + (3\pi/2) = \nu_2$. If $\lambda > 0$, (2.27) shows that $\nu = \nu_1$ and $\mu = \pm 1$, while if $\lambda < 0$ we find $\nu = \nu_2$ and $\mu = \pm 1$. A calculation shows that (for $\lambda > 0$) $g_1'(z) \neq 0$ for $z = \pm 1 + \nu_1 i$ and that (for $\lambda < 0$) $g_1'(z) \neq 0$ for $z = \pm 1 + \nu_2 i$. It follows that $g_1(z)$ has precisely two zeros in G_n and consequently that $\theta(z, \lambda) = 0$ has precisely two solutions z in U_n (counting multiplicity).

If $\lambda_- < \lambda \leqslant (\frac{1}{2})$ and $\lambda \neq 0$ we know that $\theta(z, \lambda) = 0$ has no nonzero real solutions z, and the two solutions in U_n are a conjugate pair. Thus there is a unique solution $z_n(\lambda)$ in U_n such that $\mathrm{Im}\,\big(z_n(\lambda)\big) > 0$ $\big($for $\lambda_- < \lambda \leqslant \frac{1}{2}$, $\lambda \neq 0\big)$, and since the multiplicity of this solution is one, the implicit function theorem implies that it varies differentiably with λ for $0 < \lambda < \frac{1}{2}$ and $\lambda_- < \lambda < 0$. ∎

We can now prove the first theorem of this section. The following theorem says that the location of zeros of (2.1) is, for a wide class of functions $f(x)$, qualitatively the same as for the simple function $f_0(x) = 1$ for $|x| \leqslant 1$ and zero otherwise.

THEOREM 2.1. *Let $f(x)$ be a nonnegative, even, real-valued function which is positive on a set of positive measure and which vanishes almost everywhere for $|x| > 1$. Assume that $f|[0,1]$ is integrable and monotonic increasing (not necessarily strictly) and define $\lambda_+ = \left(2\int_0^1 f(x)dx\right)^{-1}$ and $\lambda_- = \inf\{\lambda < 0 : 1 - \lambda\hat{f}(\xi) > 0$ for all real $\xi\}$, where $\hat{f}(\xi) = \int_{-1}^{1} f(x)^{i\xi x}x$. For each positive integer n define $U_n = \{z: 2n\pi - \pi \leqslant \mathrm{Re}\,(z) \leqslant 2n\pi + \pi\}$, $V_n = \{z: 2n\pi \leqslant \mathrm{Re}\,(z) \leqslant 2n\pi + \pi\}$ and $W_n = \{z: 2n\pi - \pi \leqslant \mathrm{Re}\,(z) \leqslant 2n\pi\}$. Define $\varphi(z, \lambda)$ by*

$$\varphi(z, \lambda) \overset{def}{=} 1 - \lambda\int_{-1}^{1} f(x)\, e^{izx} dx\,. \tag{2.28}$$

If $\lambda_- \leqslant \lambda \leqslant \lambda_+$ and if $\mathrm{Re}\,(z) = m\pi$ for a nonzero integer m, then $\varphi(z, \lambda) \neq 0$. If $\lambda_- \leqslant \lambda \leqslant \lambda_+$, $\lambda \neq 0$, and n is a positive integer the equation $\varphi(z, \lambda) = 0$ has precisely two solutions (counting multiplicities) in U_n; and if $\lambda_- < \lambda < \lambda_+$ exactly one of these solutions, say $z_n(\lambda)$, has positive imaginary part. The map $\lambda \to z_n(\lambda)$ is continuous on $[\lambda_-, 0)$ and on $(0, \lambda_+]$. If $0 < \lambda < \lambda_+$ the equation $\varphi(z, \lambda) = 0$ has precisely two solutions z such that $|\mathrm{Re}\,(z)| \leqslant \pi$, and if $\lambda_- \leqslant \lambda < 0$ there are no such solutions. If $0 < \lambda < \lambda_+$ and n is a positive integer, $\varphi(z, \lambda) \neq 0$ for $z \in W_n$, and if $\lambda_- \leqslant \lambda < 0$, $\varphi(z, \lambda) \neq 0$ for $z \in V_n$.

PROOF. If $z = \mu + i\nu$ a calculation (using that f is even) gives

$$\varphi(z, \lambda) = 1 - \lambda\int_0^1 f(x)[e^{\nu x} + e^{-\nu x}] \cos \mu x dx + \\ + \lambda i\int_0^1 f(x)[e^{\nu x} - e^{-\nu x}] \sin \mu x dx\,. \tag{2.29}$$

If $\mu = m\pi$ for m a positive integer and if we define a_j by

$$a_j = \int_{(j-1)m^{-1}}^{jm^{-1}} f(x)[e^{\nu x} - e^{-\nu x}] \sin (m\pi x) dx \tag{2.30}$$

then proving $\operatorname{Im} \varphi(z, \lambda) \neq 0$ is equivalent to proving

$$\sum_{j=1}^{m} a_j \neq 0 . \tag{2.31}$$

To prove (2.31) for $\nu \neq 0$, we may as well assume $\nu > 0$. If we define $f_\nu(x) = f(x)[e^{\nu x} - e^{-\nu x}]$, then for $\nu > 0$ we have that $f_\nu(x)$ is nonnegative monotonic increasing, positive on a set of positive measure and strictly monotonic increasing on the same set of positive measure. It follows from these observations that $|a_j| \leqslant |a_{j+1}|$ for $1 \leqslant j \leqslant m$ and that at least one of these inequalities is strict. Furthermore, it is clear that $(-1)^{j-1} a_j \geqslant 0$ for $1 \leqslant j \leqslant m$. From these remarks we see that

$$\begin{aligned} &\sum_{j=1}^{m} a_j = (a_1 + a_2) + (a_3 + a_4) + \ldots < 0 , && m \text{ even} \\ &\sum_{j=1}^{m} a_j = a_1 + (a_2 + a_3) + (a_4 + a_5) + \ldots > 0 , && m \text{ odd} \end{aligned} \tag{2.32}$$

and it follows that $\varphi(m\pi + i\nu, \lambda) \neq 0$ for $\nu \neq 0$.

It remains to prove that $\varphi(m\pi, \lambda) \neq 0$ for m a nonzero integer and $\lambda_- \leqslant \lambda \leqslant \lambda_+$. If $\lambda_- < \lambda \leqslant \lambda_+$ we know that $\varphi(\mu, \lambda) \neq 0$ for any nonzero real number μ, so we only have to consider the case $\lambda = \lambda_-$. We know from Lemma 1.2 that $\varphi(\xi, \lambda_-) \geqslant 0$ for all real ξ; and thus if $\varphi(\xi_0, \lambda_-) = 0$ for some ξ_0, we have

$$\varphi_\xi(\xi_0, \lambda_-) = 0 . \tag{2.33}$$

Equation (2.33) implies that for such a ξ_0 we have

$$\int_0^1 x f(x) \sin \xi_0 x \, dx = 0 . \tag{2.34}$$

The function $xf(x)$ is nonnegative, monotonic increasing and strictly monotonic increasing on an interval, so the same reasoning used before shows that

$$\begin{aligned} &\int_1^0 x f(x) \sin m\pi x \, dx < 0 , && m \text{ even}, \ m \neq 0 \\ &\int_0^1 x f(x) \sin m\pi x \, dx > 0 , && m \text{ odd} . \end{aligned} \tag{2.35}$$

Equations (2.34) and (2.35) show that $\varphi(\xi, \lambda_-) \neq 0$ for $\xi = m\pi$, m a nonzero integer.

We have actually proved above that if $g(x)$ is a function which satisfies the same hypothesis as $f(x)$ and if $1 - \mu\hat{g}(\xi) \geqslant 0$ for all real ξ and some real number μ, then $1 - \mu\hat{g}(z) \neq 0$ for $z = m\pi + iv$, m a nonzero integer. With this in mind, let λ be a fixed, nonzero number such that $\lambda_- \leqslant \lambda \leqslant \lambda_+$ and define $\lambda_1 = \frac{1}{4}\operatorname{sgn}(\lambda)$. For $0 \leqslant t \leqslant 1$ define a function $g_t(x)$ by

$$(2.36)\qquad \begin{aligned} &g_t(x) = (1-t)\lambda f(x) + t\lambda_1, \qquad |x| \leqslant 1 \\ &g_t(x) = 0, \qquad |x| > 1. \end{aligned}$$

The remark above shows that $1 - \hat{g}_t(z) \neq 0$ for $z = m\pi + iv$, m a nonzero integer. Lemma 2.4 implies that given $C > 0$, there exists a constant R such that $1 - \hat{g}_t(z) = 0$ and $|\operatorname{Re}(z)| \leqslant B$ imply $|z| < R$; an examination of the estimates in Lemma 2.4 shows that R can be chosen independent of t for $0 \leqslant t \leqslant 1$. Rouche's theorem implies that for each positive integer n, $1 - \lambda\hat{f}(z) = 0$ has the same number of solutions in U_n, V_n and W_n respectively as does $1 - \hat{g}_1(z)$. According to Lemma 2.5, $1 - \hat{g}_1(z)$ has precisely two solutions in U_n, no solutions in W_n if $\lambda_1 > 0$ and no solutions in V_n if $\lambda_1 < 0$. The same argument shows that $\varphi(z, \lambda) = 0$ has precisely two solutions z such that $|\operatorname{Re}(z)| \leqslant \pi$ if $0 < \lambda \leqslant \lambda_+$ and no such solutions if $\lambda_- \leqslant \lambda < 0$.

If n is a positive integer and $\lambda_- < \lambda \leqslant \lambda_+$ we know that $\varphi(z, \lambda) = 0$ has no real solutions z in U_n and exactly two complex solutions in U_n: It follows that these solutions must be a conjugate pair and exactly one of these solutions, call it $z_n(\lambda)$, satisfies $\operatorname{Re}(z_n(\lambda)) > 0$. If $\lambda = \lambda_-$ and if the equation $\varphi(z, \lambda) = 0$ has a real solution in U_n, we have already seen that this real solution has multiplicity 2, and it follows that the real solution z is the only solution in U_n. In this case we shall let $z_n(\lambda_-)$ denote the unique solution in U_n. If $\varphi(z, \lambda_-) = 0$ has no real solutions in U_n, the same argument as above shows it has exactly one solution $z_n(\lambda_-) = z$ in U_n with $\operatorname{Re}(z) > 0$.

We claim that the map $\lambda \to z_n(\lambda)$ is continuous for $\lambda_- \leqslant \lambda \leqslant \lambda_+$. The easiest argument is by contradiction. Take μ with $\lambda_- \leqslant \mu \leqslant \lambda_+$ and suppose that there exists a sequence $\lambda_j \to \mu$ and $n > 0$ such that $|z_n(\lambda_j) - z_n(\mu)| \geqslant \varepsilon > 0$. According to Lemma 2.4 the sequence $z_n(\lambda_j)$, $j \geqslant 1$ is bounded, so by taking a subsequence we can assume that $z_n(\lambda_j) \to \zeta$ and $\varphi(\zeta, \mu) = 0$. We know that $\zeta \in U_n$ and $\operatorname{Re}(\zeta) \geqslant 0$ and continuity implies $\varphi(\zeta, \mu) = 0$; the remarks above imply that $\zeta = z_n(\mu)$, a contradiction. ■

We shall also need a theorem which treats the zeros of $\varphi(z, \lambda)$ when $f(x)$ is not monotonic increasing on $[0, 1]$. As the following theorem shows, one can, nevertheless, reduce to the case covered by Theorem 2.1.

THEOREM 2.2. *Let $f(x)$ be a nonnegative, even real-valued function which vanishes almost everywhere for $|x|>1$. Assume that $f|[0,1]$ is continuous, that f is not identically zero, and that there exists a number $\nu_0 \geqslant 0$ such that $(e^{\nu_0 x}-e^{-\nu_0 x})f(x)$ is monotonic increasing on $[0,1]$ (these conditions will be satisfied if $f|[0,1]$ is continuously differentiable and $f(x)>0$ for $0 \leqslant x \leqslant 1$). Let λ_+, λ_-, U_n, V_n, W_n and $\varphi(z,\lambda)$ be as defined in Theorem 2.1. Let J be any compact interval of reals which does not contain 0. Then there exists a positive integer N such that for integers $m \geqslant 2N-1$ and $\lambda \in J$, $\varphi(m\pi+i\nu,\lambda)\neq 0$ and for $\xi \geqslant 2N\pi-\pi$, $\varphi(\xi,\lambda)\neq 0$. The equation $\varphi(z,\lambda)=0$ has precisely two solutions in U_n for $n \geqslant N$. If $\lambda>0$ and $\lambda \in J$, the equation $\varphi(z,\lambda)=0$ has precisely $2(2N-1)$ solutions z such that $|\mathrm{Re}\,(z)| \leqslant (2N-1)\pi$ (counting multiplicities); if $\lambda<0$, the equation $\varphi(z,\lambda)=0$ has precisely $2(2N-1)-2$ solutions such that $|\mathrm{Re}\,(z)| \leqslant (2N-1)\pi$.*

PROOF. Define $f_\nu(x)=(e^{\nu x}-e^{-\nu x})f(x)$. It is a calculus exercise to verify that $(e^{\nu x}-e^{-\nu x})(e^{\nu_0 x}-e^{-\nu_0 x})^{-1}$ is a monotonic increasing function for $x>0$ if $\nu \geqslant \nu_0$, so it follows that $f_\nu|[0,1]$ is monotonic increasing for $\nu \geqslant \nu_0$. An examination of the proof of Theorem 2.1 shows that if m is a positive integer, $\nu \neq 0$ and $f_\nu|[0,1]$ is monotonic increasing then

$$\varphi(m\pi+i\nu,\lambda)\neq 0\,. \tag{2.37}$$

It follows that $\varphi(m\pi+i\nu,\lambda)\neq 0$ for $|\nu|>\nu_0$. Since f is integrable, it is known that given $\varepsilon>0$, there exists a constant A such that if $|\mathrm{Im}\,(z)| \leqslant \nu_0$ and $|z| \geqslant A$ one has $|\hat{f}(z)|<\varepsilon$. It follows that there exists a constant B such that $\varphi(z,\lambda)\neq 0$ for $|\mathrm{Im}\,z| \leqslant \nu_0$, $\lambda \in J$ and $|z| \geqslant B$. If we select an integer N_1 such that $(2N_1-1)\pi \geqslant B$, it follows that for any integer $m \geqslant 2N_1-1$ and any real number ν we have $\varphi(m\pi+i\nu,\lambda)\neq 0$ (assuming $\lambda \in J$). By letting λ vary in J we obtain a homotopy, and Lemmas 2.1 and 2.4 imply that for $n \geqslant N_1$, the number of zeros of $\varphi(z,\lambda)=0$ in U_n, V_n or W_n is independent of λ in J. Furthermore, the number of solutions of $\varphi(z,\lambda)=0$ such that $|z| \leqslant (2N-1)\pi$, $N \geqslant N_1$, is also independent of $\lambda \in J$.

Thus to complete the proof take a fixed $\lambda \in J$, define $f_t(x)=(1-t)\cdot f(x)+t$ for $|x| \leqslant 1$ and $f_t(x)=0$ for $|x|>1$ and define a homotopy $h_t(z)$ by

$$h_t(z)=1-\lambda\hat{f}_t(z)\,. \tag{2.38}$$

Since $(e^{\nu_0 x}-e^{-\nu_0 x})f_t(x)$ is monotonic increasing on $[0,1]$, we see that

$$h_t(m\pi+i\nu)\neq 0\,, \quad \nu \geqslant \nu_0\,, \quad m \text{ a positive integer}\,. \tag{2.39}$$

The same argument as before shows that there exists an integer $N \geqslant N_1$ such that $h_t(z) \neq 0$ for $0 \leqslant t \leqslant 1$ if $\operatorname{Re}(z) \geqslant (2N-1)\pi$ and $|\operatorname{Im}(z)| \leqslant \nu_0$. Lemma 2.1 and the estimates of Lemma 2.4 now imply that for $n \geqslant N$, $h_0(z) = 0$ has the same number of solutions in U_n as $h_1(z) = 0$, and the latter equation has precisely two solutions in U_n (by Lemma 2.5). Furthermore, $h_0(z)$ has the same number of zeros z satisfying $|\operatorname{Re}(z)| \leqslant (2N-1)\pi$ as $h_1(z)$, and Lemma 2.5 implies that $h_1(z)$ has $4N-2$ such zeros if $\lambda > 0$ and $4N-4$ such zeros if $\lambda < 0$ (counting multiplicities). Notice that one can also conclude from the above sort of argument that for $n \geqslant N$, $h_0(z)$ has no zeros in W_n if $\lambda > 0$ and no zeros in V_n if $\lambda < 0$.

To complete the proof of Theorem 2.2 it only remains to prove the claim that if $f|[0,1]$ is C^1 and strictly positive, then $f_\nu(x)$ is monotonic increasing on $[0,1]$ for some $\nu \geqslant 0$. A calculation gives

$$f'_\nu(x) = (e^{\nu x} - e^{-\nu x}) f'(x) + \nu(e^{\nu x} + e^{-\nu x}) f(x) \tag{2.40}$$

if $f'(x) \geqslant -M$ for $0 \leqslant x \leqslant 1$ and if $f(x) \geqslant a > 0$ for $0 \leqslant x \leqslant 1$, equation (2.40) implies

$$f'_\nu(x) \geqslant (e^{\nu x} - e^{-\nu x})(\nu a - M), \quad 0 \leqslant x \leqslant 1 \tag{2.41}$$

so $f'_\nu(x) > 0$ for $0 < x \leqslant 1$ if $\nu a > M$. ∎

If $f(x)$ is C^1 on $[0,1]$, the conclusions of Theorem 2.2 follow under less restrictive hypotheses. Specifically, we need not assume $f(x)$ is strictly positive on $[0,1]$, and we have the following theorem, whose proof we only sketch, since it is essentially the same as the proof of Theorem 2.2.

THEOREM 2.3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an even function such that $f(x) = 0$ for $|x| > 1$, $f|[0,1]$ is continuously differentiable and $f(1) \neq 0$. Define $\varphi(z,\lambda) = 1 - \lambda \hat{f}(z)$ and let J be any compact interval of reals which does not contain 0. Then all the conclusions of Theorem 2.2 remain valid.*

PROOF. Integration by parts gives

$$\begin{aligned} iz\varphi(z,\lambda) &= iz - \lambda iz \int_0^1 f(x)[e^{izx} + e^{-izx}]\,dx = \\ &= iz - \lambda f(1)[e^{iz} - e^{-iz}] - \lambda \int_0^1 f'(x)[e^{izx} - e^{-izx}]\,dx . \end{aligned} \tag{2.42}$$

If we write $I = \int_0^1 f'(x)[e^{izx} - e^{-izx}]\,dx$ and if $z = m\pi + i\nu$ for m an integer we obtain

$$\operatorname{Re}\big(iz\varphi(z,\lambda)\big) = -\nu + \lambda(-1)^m f(1)[e^\nu - e^{-\nu}] - \lambda \operatorname{Re}(I)\,. \tag{2.43}$$

Define $M = \max\{|f'(x)|: 0 \leqslant x \leqslant 1\}$, assume $\nu > 0$ and notice that

$$|\operatorname{Re}(I)| \leqslant M\int_0^1 (e^{\nu x} + e^{-\nu x})\,dx \leqslant M\nu^{-1}e^\nu\,. \tag{2.44}$$

Equations (2.43) and (2.44) imply that (assuming $\nu > 0$)

$$|\operatorname{Im}\big(iz\varphi(z,\lambda)\big)| \geqslant |\lambda||f(1)|[e^\nu - 1] - \nu - M\nu^{-1}e^\nu\,. \tag{2.45}$$

Equation (2.45) implies that there is a number $\nu_0 > 0$ such that if $\nu \geqslant \nu_0$, $\varphi(m\pi + i\nu, \lambda) \neq 0$ and consequently $\varphi(m\pi + i\nu, \lambda) \neq 0$ for $|\nu| \geqslant \nu_0$. Since the equation $\varphi(z, \lambda) = 0$ has only finitely many solutions satisfying $\lambda \in J$ and $|\operatorname{Im}(z)| \leqslant \nu_0$, there exists an integer such that $\varphi(z, \lambda) \neq 0$ for $\operatorname{Re}(z) \geqslant 2N - 1$ and $|\operatorname{Im}(z)| \leqslant \nu_0$ and $\varphi(m\pi + i\nu, \lambda) \neq 0$ for any integer $m \geqslant 2N - 1$, any real ν and any $\lambda \in J$.

To complete the proof, we argue more or less as in Theorem 2.2. Define $f_t(x) = (1-t)f(x) + tf(1)$ for $|x| \leqslant 1$ and $f_t(x) = 0$ for $|x| > 1$ and notice that the estimates obtained above can be taken to be uniform in t for $0 \leqslant t \leqslant 1$. In particular N can be selected so $1 - \lambda\hat{f}_t(z) \neq 0$ for $z \in \partial U_n$ when $n \geqslant N$. By using Lemmas 2.1 and 2.4 (recalling that Lemma 2.5 describes the zeros of $1 - \lambda\hat{f}_1(z)$) and arguing as in Theorem 2.2, one can complete the proof. ∎

It will be useful in subsequent sections to have information about the relative sizes of $\operatorname{Im}(z)$ and $\operatorname{Re}(z)$ when $\varphi(z, \lambda) = 0$. The following proposition will be adequate for our purposes.

Proposition 2.1. *Let $f(x)$ be a real-valued, even function such that $f(x) = 0$ for $|x| > 1$, $f|[0,1]$ is continuously differentiable and $f(1) \neq 0$. If λ is a non-zero real number and $c > 1$, then all but finitely many zeros of*

$$1 - \lambda\hat{f}(z) = 0 \tag{2.46}$$

satisfy

$$|\operatorname{Re}(z)| \leqslant c\ln\big(|\operatorname{Im}(z)|\big)\,. \tag{2.47}$$

Proof. If z is a solution of (2.46), $-z$, $\bar{z}$ and $-\bar{z}$ are also solutions, so we can assume $z = \mu + i\nu$ is a solution of (2.46) and $\mu \geqslant 0$ and $\nu \geqslant 0$.

Integration by parts gives

$$(2.48)\qquad \lambda\hat{f}(z) = \lambda\int_0^1 f(x)[e^{izx} + e^{-izx}]dx =$$
$$= \lambda(iz)^{-1}f(1)[e^{iz} - e^{-iz}] - \lambda(iz)^{-1}\int_0^1 f'(x)[e^{izx} - e^{-izx}]dx .$$

If $M = \max\{|f'(x)|: 0 \leqslant x \leqslant 1\}$, (2.48) implies that

$$(2.49)\qquad 0 = |1 - \lambda\hat{f}(z)| \geqslant |\lambda||\hat{f}(z)| - 1 \geqslant$$
$$\geqslant |\lambda||z|^{-1}|f(1)|[e^{\nu} - 1] - |\lambda||z|^{-1}M\int_0^1 (e^{\nu x} - e^{-\nu x})dx - 1 .$$

Inequality (2.49) implies that for any constant $B > 0$ there can only be finitely many solutions of (2.46) which satisfy (1) $|\mathrm{Im}(z)| \leqslant B$ or (2) $|\mathrm{Re}(z)| \leqslant B$. Of course this is true under less restrictive assumptions on f. Integrating in (2.49) gives

$$(2.50)\qquad 0 \geqslant |\lambda||f(1)|[e^{\nu} - 1] - |\lambda|M\nu^{-1}e^{\nu} - |z| .$$

Write $2a = |\lambda||f(1)| > 0$. Except for a finite number of solutions $z = \mu + i\nu$ of (2.46) we have $|\lambda|M\nu^{-1} < a$ and $2a < |z|$, so (2.50) gives

$$(2.51)\qquad 0 \geqslant ae^{\nu} - 2|z| .$$

Suppose c is as in the statement of the lemma and $\nu = d\ln\mu$ for some $d \geqslant c > 1$; we can assume $\mu > e =$ the base of natural logarithms.

Substituting in (2.51) for ν gives

$$(2.52)\qquad 4(\mu^{2-2d}) + 4d^2(\ln\mu)^2(\mu^{-2d}) \geqslant a^2 .$$

If we write $\psi(\mu, d) = 4d^2(\ln\mu)^{2-2d}$, a calculation shows that

$$(2.53)\qquad \begin{cases} \dfrac{\partial\psi}{\partial\mu} < 0, & \mu > e,\ d > 1 \\ \dfrac{\partial\psi}{\partial d} < 0, & \mu > e,\ d > 1 . \end{cases}$$

Thus if $\mu \geqslant \mu_0 > e$ and $d \geqslant c$ the left hand side of (2.52) achieves its minimum at $\mu = \mu_0$ and $d = c$. If μ_0 is chosen so large that

$$(2.54) \qquad 4(\mu_0^{2-2c}) + 4c^2(\ln \mu_0)^2 \mu_0^{-2c} < a^2$$

equation (2.52) cannot hold for $\mu \geqslant \mu_0$. Since there are only a finite number of solutions of (2.46) with $|\mu| \leqslant \mu_0$, we have proved that (2.47) is valid except for a finite number of z. ■

REMARK 2.2. By a somewhat more careful analysis one can prove that the zeros of $\varphi(z) = 1 - \lambda\hat{f}(z)$ asymptotically look like the zeros of $\psi(z) = 1 - \lambda(iz)^{-1}f(1)[e^{iz} - e^{-iz}]$. More precisely, every sufficiently large solution of $\varphi(z) = 0$ is a simple zero and similarly for $\psi(z)$. Furthermore, given $\varepsilon > 0$ with ε small enough there exists a constant M_ε such that if $\psi(z) = 0$ and $|z| > M_\varepsilon$, then there is exactly one number z_1 such that $|z_1 - z| < \varepsilon$ and $\varphi(z_1) = 0$. Also M_ε can be chosen so that if $\varphi(z_1) = 0$ and $|z_1| > M_\varepsilon$, then $|z_1 - z| < \varepsilon$ for some z such that $\psi(z) = 0$ and $|z| > M_\varepsilon$.

REMARK 2.3. If $f(x)$ satisfies the conditions of Proposition 2.1 except that $f(1) = 0$ and if f is $n+1$ times continuously differentiable with $f^{(j)}(1) = 0$ for $0 \leqslant j \leqslant n-1$ and $f^{(n)}(1) \neq 0$, then there is a constant c (which no longer can be taken arbitrarily close to 1) such that inequality (2.47) is valid for all but finitely many solutions of (2.46). The proof is essentially the same as before except that integration by parts must be repeated $n+1$ times to obtain a suitable expression for $\lambda\hat{f}(z)$.

In the final section of this paper we shall be interested in the spectrum of the Fréchet derivative of the operator $F\colon C[0,1] \to C[0,1]$ defined by

$$(Fu)(x) = f(x) + \lambda\int_x^1 u(y)u(y-x)\,dy\,, \qquad \lambda \neq 0\,.$$

The Fréchet derivative of F at u is the linear operator $L\colon C[0,1] \to C[0,1]$ given by

$$(Lh)(x) = \lambda\int_x^1 u(y)h(y-x)\,dy + \lambda\int_x^1 u(y-x)h(y)\,dy\,.$$

We shall prove later that, for many functions $u \in C[0,1]$, the spectrum of L, $\sigma(L)$, is given by

$$(2.55) \qquad \sigma(L) = \{\lambda\hat{u}(z)\colon \hat{u}(z) = \hat{u}(-z),\ z \text{ complex}\} \cup \{0\}$$

where $u(x)$ is defined to equal zero for $x \notin [0, 1]$. However, to prove (2.55) or to obtain more information about $\sigma(L)$, we need sharp information about the location of zeros of $\theta(z) = \hat{u}(z) - \hat{u}(-z)$. That information is provided by the next few lemmas.

LEMMA 2.6. *Suppose that $u\colon \mathbb{R} \to \mathbb{R}$ is a function such that $u|[0, 1]$ is continuously differentiable and $u(x) = 0$ for $x \notin [0, 1]$. If $u(1) \neq 0$, there exists a constant M such that any solution of $\theta(z) = \hat{u}(z) - \hat{u}(-z) = 0$ satisfies $|\mathrm{Im}\,(z)| < M$.*

PROOF. Integration by parts gives

$$(iz)\theta(z) = -\int_0^1 u'(t)(e^{izt} + e^{-izt})dt + u(1)(e^{iz} + e^{-iz}) - 2u(0)\,. \tag{2.56}$$

If $A \geq |u'(t)|$ for $0 \leq t \leq 1$ and if $z = x + iy$ and $y \geq 0$ (2.56) gives

$$|z||\theta(z)| \geq |u(1)|(e^{y} - 1) - 2|u(0)| - Ay^{-1}e^{y}\,. \tag{2.57}$$

Inequality (2.57) implies that there exists a number M (dependent only on $u(1)$, $u(0)$ and A) such that if $y \geq M$ one has $\theta(z) \neq 0$. Since $\theta(-z) = -\theta(z)$, it follows that $\theta(z) \neq 0$ for $|y| > M$. ∎

THEOREM 2.4. *Suppose that $u\colon \mathbb{R} \to \mathbb{R}$ is a function such that $u|[0, 1]$ is continuously differentiable and $u(x) = 0$ for $x \notin [0, 1]$. Assume that $u(1) \neq 0$. Define $\theta(z) = \hat{u}(z) - \hat{u}(-z)$ and for each positive integer n define*

$$A_n = \{z\colon 2n\pi - \pi \leq \mathrm{Re}\,(z) \leq 2n\pi + \pi\}$$

and

$$B_n = \{z\colon 2n\pi \leq \mathrm{Re}\,(z) \leq 2n\pi + 2\pi\}\,.$$

Then there exists an integer N such that if $n \geq N$ and $u(1)u(0) \geq 0$ the equation $\theta(z) = 0$ has precisely two solutions in A_n (counting multiplicity) and $\theta(z) \neq 0$ for $z \in \partial A_n$, while if $n \geq N$ and $u(1)u(0) \leq 0$ the equation $\theta(z)$ has precisely two solutions in B_n and $\theta(z) \neq 0$ for $z \in \partial B_n$. Furthermore, if $u(1)u(0) \geq 0$ the equation $\theta(z) = 0$ has precisely $4N - 3$ solutions (counting multiplicity) such that $|\mathrm{Re}\,(z)| \leq (2N - 1)\pi$, while if $u(1)u(0) \leq 0$ the equation $\theta(z) = 0$ has precisely $4N - 1$ solutions such that $|\mathrm{Re}\,(z)| \leq 2N\pi$. If z is a solution of $\theta(z) = 0$, so is $\bar{z}$, $-\bar{z}$ and $-z$. If $u(x) \geq 0$ for all x, then $z = 0$ is the only pure imaginary solution of $\theta(z) = 0$.

PROOF. Notice that $\theta(-z) = -\theta(z)$ and $\overline{\theta(z)} = -\theta(\bar{z})$, so one of the claims of the theorem is trivial. Also if $\nu > 0$ we have

$$\theta(i\nu) = \int_0^1 u(x)[e^{\nu x} - e^{-\nu x}]\,dx \tag{2.58}$$

and if $u(x) \geqslant 0$ the integrand in (2.58) is nonnegative and strictly positive at $x = 1$, so $\theta(i\nu) > 0$.

To prove the rest of the theorem, observe that integration by parts gives

$$iz\theta(z) \overset{\text{def}}{=} \psi(z) = u(1)(e^{iz} + e^{-iz}) - 2u(0) - \int_0^1 u'(x)(e^{izx} + e^{-izx})\,dx\,. \tag{2.59}$$

For $0 \leqslant t \leqslant 1$ consider the homotopy given by

$$\psi_t(z) = u(1)(e^{iz} + e^{-iz}) - 2u(0) - (1-t)\int_0^1 u'(x)(e^{izx} + e^{-izx})\,dx\,. \tag{2.60}$$

The constant M in Lemma 2.6 can be chosen so that $\psi_t(z) \neq 0$ for $|\mathrm{Im}\,(z)| \geqslant M$ and $0 \leqslant t \leqslant 1$. If $z = \mu + i\nu$ and $|\nu| \leqslant M$, the Riemann-Lebesgue theorem implies that

$$\lim_{\mu \to +\infty} \int_0^1 u'(x)(e^{izx} + e^{-izx})\,dx = 0 \tag{2.61}$$

where the limit is uniform in ν such that $|\nu| \leqslant M$. We now consider two cases: (*a*) $u(1)u(0) \geqslant 0$ and (*b*) $u(1)u(0) \leqslant 0$. If $z = m\pi + i\nu$ for m an integer we have

$$u(1)(e^{iz} + e^{-iz}) - 2u(0) = (-1)^m u(1)(e^{\nu} + e^{-\nu}) - 2u(0)\,. \tag{2.62}$$

Using (2.62) we see that if $z = m\pi + i\nu$, m is odd and $u(1)u(0) \geqslant 0$

$$|u(1)(e^{iz} + e^{-iz}) - 2u(0)| \geqslant |u(1)| \tag{2.63}$$

while inequality (2.63) is also valid if $u(1)u(0) \leqslant 0$ and m is even. It follows from (2.61) and (2.63) that there exists a positive integer N such that if case (*a*) holds and $m \geqslant 2N - 1$ is an odd integer one has

$$\psi_t(m\pi + i\nu) \neq 0\,, \quad 0 \leqslant t \leqslant 1,\ \nu \text{ real}\,. \tag{2.64}$$

Inequality (2.64) is also valid if case (*b*) holds and $m \geqslant 2N$ is an even integer. Rouche's theorem now implies that in case (*a*) $\psi_0(z) = \psi(z)$ and $\psi_1(z)$ have the same number of zeros in A_n (for $n \geqslant N$) and the same number of zeros such that $|\mathrm{Re}\,(z)| \leqslant (2N-1)\pi$; while in case (*b*), $\psi_0(z)$ and $\psi_1(z)$ have the same number of zeros in B_n for $n \geqslant N$ and the same number of zeros z such that $|\mathrm{Re}\, z| \leqslant 2N\pi$. The zeros of $\psi_0(z) = \psi(z)$ are the same as the zeros of $\theta(z)$ except that the multiplicity of $z = 0$ as a zero of $\psi_0(z)$ is one more than its multiplicity as a zero of $\theta(z)$. The zeros of $\psi_1(z)$ are simply the solutions of

$$e^{iz} + e^{-iz} = 2u(0)\big(u(1)\big)^{-1}. \tag{2.65}$$

One can easily check directly that equation (2.65) has precisely two solutions z_0 and z_1 such that $-\pi < \mathrm{Re}\,(z_j) \leqslant \pi$. If $|u(0)| \leqslant |u(1)|$ both these solutions are real, and they are distinct if $|u(0)| < |u(1)|$. If $u(0)u(1) \geqslant 0$ and $|u(0)| > |u(1)|$, the solutions z_j are of the form $\pm iv$, v real; and if $u(0)u(1) \leqslant 0$ and $|u(0)| > |u(1)|$ the solutions z_j are of the form $\pi \pm iv$. The general solution of (2.65) is of the form $z_j + 2m\pi$, m an integer. Using the above information, a simple counting argument completes the proof. ∎

REMARK 2.4. The division of the above proof into case (*a*) and (*b*) is not particularly significant. However, if one wants to describe the location of zeros of $\theta(z)$ in terms of strips like A_n or B_n, some division into subcases is necessary, because $\theta(z)$ may have zeros z such that $\theta(z) = m\pi$, m an integer.

We shall need more information about the solutions of $\theta(z) = 0$ in order to determine the spectrum of L more precisely for certain classes of functions u.

PROPOSITION 2.2. *Assume that* $u\colon \mathbb{R} \to \mathbb{R}$ *satisfies the hypotheses of Theorem* 2.4. *If* $u'(x) \leqslant 0$ *for* $0 \leqslant x \leqslant 1$ *and* $0 < u(1) < u(0)$, *the equation* $\theta(z) = \hat{u}(z) - \hat{u}(-z) = 0$ *has no real solutions* z *with* $z \neq 0$; *and if* $\theta(z) = 0$ *and* $z \neq 0$, *then* $\hat{u}(z) + \hat{u}(-z)$ *is not real. If* $u'(x) \geqslant 0$ *for* $0 \leqslant x \leqslant 1$ *and if* $0 \leqslant u(0) < u(1)$, *all solutions of* $\theta(z) = 0$ *are real and are simple zeros. For each positive integer* n *there is precisely one zero* ζ_n *such that* $2n\pi - \pi < \zeta_n < 2n\pi$ *and one zero* z_n *such that* $2n\pi < z_n < 2n\pi + \pi$; *and* $z = 0$ *is the only solution of* $\theta(z) = 0$ *such that* $|\mathrm{Re}\,(z)| \leqslant \pi$.

PROOF. First assume u is monotonic decreasing and consider $\theta(\xi)$ for some $\xi > 0$. We have

$$\theta(\xi) = 2i\int_0^\infty u(x) \sin \xi x\, dx = 2i \sum_{j=0}^{\infty} a_j \tag{2.66}$$

where $a_j = \int_{(j-1)\pi\xi^{-1}}^{j\pi\xi^{-1}} u(x)\sin\xi x dx$. Just as in the proof of Proposition 1.1, $(-1)^{j+1}a_j \geqslant 0$ and $|a_j| \geqslant |a_{j+1}|$, with at least one of the latter inequalities being strict. It follows that $\Sigma a_j > 0$, so $\theta(\xi) \neq 0$ for ξ real and nonzero. If $\theta(z) = 0$ for $z \neq 0$, Theorem 2.4 implies that z is not pure imaginary, so $z = \mu + i\nu$ with $\mu \neq 0$ and $\nu \neq 0$. Since $\hat{u}(z) = \hat{u}(-z)$, we can, in evaluating $2\hat{u}(z)$, assume that $\nu > 0$. A calculation shows that

$$2\,\mathrm{Im}\,\hat{u}(z) = 2\int_0^1 u(x)e^{-\nu x}\sin\mu x dx\,. \tag{2.67}$$

The function $u_\nu(x) = u(x)e^{-\nu x}$ is monotonic decreasing and strictly positive on $[0, 1]$, so exactly the argument used above shows (since $\mu \neq 0$) that the right-hand side of (2.67) is nonzero.

Next assume that u is monotonic increasing on $[0, 1]$ and consider $\theta(\xi)$ for $\xi = m\pi$, m a positive integer. We have

$$\theta(\xi) = 2i\sum_{j=1}^{m} a_j \tag{2.68}$$

where a_j is defined as above. In this case we know that $(-1)^{j+1}a_j \geqslant 0$ for $1 \leqslant j \leqslant m$ and $|a_j| \leqslant |a_{j+1}|$, $1 \leqslant j \leqslant m-1$, with at least one of the latter inequalities being strict. Just as in the proof of Theorem 2.2, it follows that $\theta(\xi) > 0$ for m even and $\theta(\xi) < 0$ for m odd. The intermediate value theorem implies that for each positive integer n there exists a real number ζ_n with $2n\pi - \pi < \zeta_n < 2n\pi$ such that $\theta(\zeta_n) = 0$ and a real number z_n with $2n\pi < z_n < 2n\pi + \pi$ such that $\theta(z_n) = 0$. Theorem 2.4 implies that there is an integer N such that for each integer $n \geqslant N$, A_n (defined as in Theorem 2.4) contains precisely two zeros of the equation $\theta(z) = 0$. It follows that z_n and ζ_n must be the only zeros of $\theta(z)$ in A_n and they must be simple zeros. Theorem 2.4 also implies that the equation $\theta(z) = 0$ has precisely $4N-3$ solutions such that $|\mathrm{Re}\,(z)| \leqslant (2N-1)\pi$. However the numbers $\pm z_n$, $\pm \zeta_n$ and 0 for $1 \leqslant n \leqslant N-1$ already give at least $4N-3$ solutions (more if these zeros are not simple), so all these zeros must be simple and there can be no other zeros. The above argument shows that every solution of $\theta(z) = 0$ with $|\mathrm{Re}\,(z)| \leqslant (2N-1)\pi$ is real, and we have already proved the same conclusion for every solution with $|\mathrm{Re}\,(z)| \geqslant (2N-1)\pi$. ■

If u satisfies the hypotheses of Theorem 2.4 and $u(1)u(0) \geqslant 0$, we know that for sufficiently large positive integers there exist precisely two complex numbers z (counting multiplicity) such that $\hat{u}(z) = \hat{u}(-z)$ and $2n\pi - \pi \leqslant \mathrm{Re}\,(z) \leqslant 2n\pi + \pi$. Since these solutions are either both real or are complex conjugates, we can label the solutions z_n and z_n' and assume $\mathrm{Im}\, z_n \geqslant 0$ and $\mathrm{Re}\,(z_n) \geqslant \mathrm{Re}\,(z_n')$. Similarly, if $u(1)u(0) \leqslant 0$ and n is large enough there are precisely two complex solutions z such that $2n\pi \leqslant \mathrm{Re}\,(z) \leqslant 2n\pi + 2\pi$. We can label these solutions ζ_n and ζ_n' and assume $\mathrm{Im}\,(\zeta_n) \geqslant 0$ and $\mathrm{Re}\,(\zeta_n) \geqslant \mathrm{Re}\,(\zeta_n')$. We shall need asymptotic formulas for the numbers z_n and ζ_n. The following proposition simply states (in a cumbersome way) that the zeros of $\theta(z)$ asymptotically look like the zeros of $u(1)(e^{iz} + e^{-iz}) - 2u(0)$.

PROPOSITION 2.3. *Assume that* $u\colon \mathbb{R} \to \mathbb{R}$ *satisfies the hypotheses of Theorem* 2.4 *and let* z_n, z_n', ζ_n *and* ζ_n' *be as defined in the preceding paragraph. If* $u(1)u(0) \geqslant 0$ *and if* $|u(0)| \leqslant |u(1)|$ *there exists a number* a *with* $0 < a \leqslant (\pi/2)$ *such that* $\lim_{n\to\infty} (2n\pi + a - z_n) = 0$ *and* $\lim_{n\to\infty} (2n\pi - a - z_n') = 0$. *If* $u(1)u(0) \geqslant 0$ *and if* $|u(0)| > |u(1)|$ *there exists a number* $b > 0$ *such that* $\lim_{n\to\infty} (2n\pi + ib - z_n) = 0$ *and* $\lim_{n\to\infty} (2n\pi - ib - z_n') = 0$. *If* $u(1)\,u(0) \leqslant 0$ *and if* $|u(0)| \leqslant |u(1)|$ *there exists a number* c *with* $0 < c \leqslant \pi/2$ *such that* $\lim_{n\to\infty} (2n\pi + \pi + c - \zeta_n) = 0$ *and* $\lim_{n\to\infty} (2n\pi + \pi - c - \zeta_n') = 0$. *If* $u(1)u(0) \leqslant 0$ *and if* $|u(0)| > |u(1)|$ *there exists* $d > 0$ *such that* $\lim_{n\to\infty} (2n\pi + \pi + id - \zeta_n) = 0$.

PROOF. The proof of Proposition 2.2 is actually implicit in the proof of Theorem 2.4. We know that there is a number M such that every solution of $\theta(z) = 0$ satisfies $|\mathrm{Im}\,(z)| \leqslant M$. Furthermore, if $z = \mu + i\nu$ we have seen that

$$\lim_{\mu\to\infty} (iz)\theta(z) - \psi_1(z) = 0\,, \qquad \psi_1(z) = u(1)(e^{iz} + e^{-iz}) - 2u(0) \tag{2.69}$$

where the limit is uniform in ν such that $|\nu| \leqslant M$. Given $\delta > 0$ it is easy to see that there exists $\varepsilon > 0$ such that if the distance of z to any zero of $\psi_1(z)$ is greater than δ, then $|\psi_1(z)| \geqslant \varepsilon$. It follows that given any $\varepsilon > 0$, for n large enough the zeros z_n and z_n' (if $u(1)u(0) \geqslant 0$) or ζ_n and ζ_n' (if $u(1)u(0) \leqslant 0$) must be within ε of a zero of $\psi_1(z)$. Assume for definiteness that $u(1)u(0) \geqslant 0$. We have already remarked in the proof of Theorem 2.4 that the zeros of $\psi_1(z)$ in A_n are of the form $2n\pi \pm a$, $0 < a \leqslant \pi/2$, if $|u(0)| \leqslant |u(1)|$ and of the form $2n\pi \pm ib$, $b > 0$, if $|u(0)| > |u(1)|$. This completes the proof in the case that $u(1)u(0)$ is nonnegative; and if $u(1)u(0) \leqslant 0$, the argument is similar. ■

3. – Positive solutions of the integral equation.

In this section we shall assume that $f(x)$ is a nonnegative continuous function for $0 \leqslant x \leqslant 1$ and we wish to investigate the number of nonnegative solutions of the equation

$$(3.1)\qquad u(x) = f(x) + \lambda\int_x^1 u(y)u(y-x)\,dy\,, \qquad 0\leqslant x\leqslant 1\,.$$

If $f(x)=1$ for $0\leqslant x\leqslant 1$ this question has been investigated by G. Pimbley [10] and R. Ramalho [12]. For this $f(x)$, Pimbley claimed to prove the existence of at least two distinct positive solutions u_λ and v_λ for $0<\lambda<\frac{1}{2}$ such that $u_\lambda(x)\leqslant v_\lambda(x)$ for $0\leqslant x\leqslant 1$, the map $\lambda\to u_\lambda$ is continuous for $0\leqslant\lambda\leqslant\frac{1}{2}$, $\lambda\to v_\lambda$ is continuous for $0<\lambda\leqslant\frac{1}{2}$ and $u_\lambda=v_\lambda$ for $\lambda=\frac{1}{2}$. Building upon Pimbley's work, Ramalho claimed to prove that in fact for $0<\lambda<\frac{1}{2}$ (with $f(x)\equiv 1$) equation (3.1) has precisely two solutions. Unfortunately, the proof of Theorem 14 in [10] is wrong, and Theorem 14 plays a crucial role in both papers. As a result, Pimbley's paper only proves the existence of one positive solution for $0\leqslant\lambda\leqslant\frac{1}{2}$, while Ramalho's argument only yields the existence of at least two positive solutions.

Since Theorem 14 in [10] plays such an important role in [10] and [12], it may be useful to discuss the error in its proof. If $f(x)\equiv 1$ and u is a positive solution of (3.1), Pimbley defines $L\colon X=C[0,1]\to X$ by

$$(3.2)\qquad (Lh)(x) = \lambda\int_x^1 h(y)u(y-x)\,dy + \lambda\int_x^1 h(y-x)u(y)\,dy\,.$$

If $0<\lambda\leqslant\frac{1}{2}$ he claims that every eigenvalue of L other than its spectral radius is strictly less than one in absolute value. He defines

$$Y=\left\{h\in X\colon\int_0^1 h(x)\,dx=0\right\}\qquad\text{and}\qquad \|h\|_1=\int_0^1|h(x)|\,dx,$$

and he observes that $L\colon Y\to Y$ and that to prove the claim about eigenvalues it would suffice to prove

$$(3.3)\qquad \|L|Y\|_1 \overset{\text{def}}{=} \sup\{\|Lh\|_1\colon h\in Y,\ \|h\|_1\leqslant 1\}<1\,.$$

Pimbley then claims to prove inequality (3.3). However, the jump from the inequalities on p. 121 in [10] to inequality (25) on p. 122 is not justified. In Pimbley's notation in his Theorem 14 he shows that

$$w(x)\leqslant\max\big(w_1(x),\,w_2(x)\big)\,,\qquad 0\leqslant x\leqslant 1$$

where

$$w(x) = \left| \int_x^1 u_\lambda^-(y) f(y-x)\, dy + \int_x^1 u_\lambda^-(y-x) f(y)\, dy \right|$$

$$w_j(x) = \int_x^1 u_\lambda^-(y) |f_j(y-x)|\, dy + \int_x^1 u_\lambda^-(y-x) |f_j(y)|\, dy\,, \qquad j = 1, 2\,.$$

Pimbley then claims (on page 122) that

$$\int_0^1 w(x)\, dx \leqslant \max_{j=1,2} \int_0^1 w_j(x)\, dx$$

while in fact all that one can legitimately conclude is that

$$\int_0^1 w(x)\, dx \leqslant \int_0^1 \max\,(w_1(x),\, w_2(x))\, dx$$

a much weaker result. In fact, Nancy Baxter has proved in her dissertation that inequality (3.3) is false for sufficiently small $\lambda > 0$. The proof involves showing that if v_λ is any positive solution of (3.1) for $f(x) \equiv 1$ different from the fundamental solution u_λ, then if $\varphi \in C[0, 1]$ one has

$$\lim_{\lambda \to 0^+} \lambda \int_0^1 v_\lambda(x) \varphi(x)\, dx = 2\varphi(0)\,.$$

In other words, λv_λ approaches twice the delta function in the sense of distributions; generalizations of this fact will be given in a joint paper of this author with M. Mock. In fact it is unclear whether all eigenvalues of L other than its spectral radius are less than one in absolute value. However, it is a corollary of results in Section 5 of this paper that if $f(x) \equiv 1$, for $0 \leqslant x \leqslant 1$, $\lambda > 0$ and u is a positive solution of (3.1), then the spectral radius of L is its only real eigenvalue; and this fact would have been sufficient to justify the arguments in [10] and [12].

Our basic result will prove the existence of precisely two positive solutions of (3.1) for $0 < \lambda < \lambda_+$ and for reasonably general positive functions $f(x)$. Because of the previous comments this result is new even if $f(x) \equiv 1$, and in fact the result seems inaccessible if one just uses the ideas in [10, 12].

Our first proposition shows that the solution u_λ insured by Theorem 1.2 is nonnegative for $0 \leqslant \lambda \leqslant \lambda_+$ if $f(x)$ is nonnegative.

PROPOSITION 3.1. *Suppose that $f \in C[0,1] = X$ is nonnegative and not identically zero and extend f to be an even function such that $f(x) = 0$ for $|x| > 1$. Define $\lambda_+ = \left(2\int_0^1 f(x)\,dx\right)^{-1}$ and define a continuous map $F_\lambda\colon X \to X$ by*

$$(F_\lambda v)(x) = f(x) + \lambda \int_0^1 v(y)v(y-x)\,dy\,, \qquad 0 \leqslant x \leqslant 1\,. \tag{3.4}$$

For $\lambda \leqslant \lambda_+$ and $\lambda \neq 0$ define numbers I_λ^+ and I_λ^- by

$$\begin{cases} I_\lambda^- = \left[1 - \left(1 - 2\lambda\int_0^1 f(x)\,dx\right)^{\frac{1}{2}}\right]\lambda^{-1} \\ I_\lambda^+ = \left[1 + \left(1 - 2\lambda\int_0^1 f(x)\,dx\right)^{\frac{1}{2}}\right]\lambda^{-1} \end{cases} \tag{3.5}$$

and define $I_\lambda^- = \int_0^1 f(x)\,dx$ for $\lambda = 0$. For $0 \leqslant \lambda \leqslant \lambda_+$ there is a unique, nonnegative continuous function $u(x) = u_\lambda(x)$ which satisfies (3.1) for $0 \leqslant x \leqslant 1$ and is such that

$$\int_0^1 u_\lambda(x)\,dx = I_\lambda^-\,. \tag{3.6}$$

The function $u_\lambda(x)$ satisfies

$$1 - \lambda\int_0^1 u_\lambda(x)e^{izx}\,dx \neq 0\,, \qquad \operatorname{Im}(z) > 0,\ 0 < \lambda < \lambda_+ \tag{3.7}$$

and the map $\lambda \to u_\lambda \in X$ is continuous for $0 \leqslant \lambda \leqslant \lambda_+$. If $v(x) = v_\lambda(x)$ is any other nonnegative, continuous solution of (3.1), then $u(x) \leqslant v_\lambda(x)$ for $0 \leqslant x \leqslant 1$. Furthermore, if for fixed λ, $0 < \lambda \leqslant \lambda_+$, one defines $w_0(x) = f(x)$ for $0 \leqslant x \leqslant 1$ and $w_n = F_\lambda(w_{n-1})$ for $n \geqslant 1$, then one has

$$w_n(x) \leqslant w_{n+1}(x) \leqslant u_\lambda(x)\,, \qquad 0 \leqslant x \leqslant 1 \tag{3.8}$$

and $w_n \to u_\lambda$ in the $C[0,1]$ topology.

PROOF. Let $u_\lambda(x)$ be the unique, continuous real-valued solution of (3.1) such that equation (3.7) is satisfied for $0 \leqslant \lambda \leqslant \lambda_+$. The existence and uniqueness of such a u_λ is insured by Theorems 1.3 and 1.4, and these theorems also guarantee that $\lambda \to u_\lambda$ is continuous as a map into X. Taking $\xi = 0$ in equation (1.6) (in Lemma 1.1) gives

$$1 - 2\lambda \int_0^1 f(x)\,dx = \left(1 - \lambda \int_0^1 u(x)\,dx\right)^2 \tag{3.9}$$

and one obtains from (3.9) that

$$\int_0^1 u_\lambda(x)\,dx = I_\lambda^+ \quad \text{or} \quad \int_0^1 u_\lambda(x)\,dx = I_\lambda^- . \tag{3.10}$$

Since $\int_0^1 u_\lambda(x)\,dx$ varies continuously with λ for $0 \leqslant \lambda \leqslant \lambda_+$ and I_λ^+ approaches $+\infty$ as λ approaches 0 we must have $\int_0^1 u_\lambda(x)\,dx = I_\lambda^-$ for λ small, and since $I_\lambda^- < I_\lambda^+$ for $0 < \lambda < \lambda_+$ it follows that $\int_0^1 u_\lambda(x)\,dx = I_\lambda^-$ for $0 \leqslant \lambda \leqslant \lambda_+$.

We next claim that $u_\lambda(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$ and $0 \leqslant \lambda \leqslant \lambda_+$. It suffices to prove this for $0 \leqslant \lambda < \lambda_+$ because we know that $u_\lambda(x) \to u_{\lambda_+}(x)$ as $\lambda \to \lambda_+$. First assume that $f(x)$ is strictly positive for $0 \leqslant x \leqslant 1$ and define λ_1 by

$$\lambda_1 = \sup\{\lambda \geqslant 0 : u_s(x) \geqslant 0 \text{ for } 0 \leqslant s \leqslant \lambda,\ 0 \leqslant x \leqslant 1,\ \lambda \leqslant \lambda_+\} . \tag{3.11}$$

We know that $u_\lambda = f$ for $\lambda = 0$ and $\lambda \to u_\lambda$ is continuous, so $\lambda_1 > 0$. If $\lambda_1 < \lambda_+$, the definition of λ_1 implies that $u_{\lambda_1}(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$ and $u_{\lambda_1}(x) = 0$ for some x with $0 \leqslant x \leqslant 1$. However, if $\lambda \geqslant 0$ and u is a nonnegative solution of (3.1), one can see directly from (3.1) that $u(x) = u_{\lambda_1}(x) \geqslant f(x) > 0$. Thus we have proved that $u_\lambda(x) > 0$ for $0 \leqslant \lambda \leqslant \lambda_+$, $0 \leqslant x \leqslant 1$, (if we assume $f(x)$ is strictly positive).

Next assume only that $f(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$, take $\lambda < \lambda_+$ and define $f_\varepsilon(x) = f(x) + \varepsilon$ for $\varepsilon > 0$. For ε small enough the equation

$$u(x) = f_\varepsilon(x) + \lambda \int_x^1 u(y)u(y-x)\,dy , \qquad 0 \leqslant x \leqslant 1$$

has by our remarks above a positive solution $u = h_\varepsilon$ such that

$$1 - \lambda \int_0^1 h_\varepsilon(x) e^{izx}\,dx \neq 0 , \qquad \operatorname{Im}(z) \geqslant 0 .$$

Lemma 1.7 implies that h_ε approaches u_λ in the $L^1[0,1]$ and $L^2[0,1]$ norms, and Lemma 1.8 shows that there is a constant M independent of $\varepsilon > 0$ such that

$$|h_\varepsilon(x)| \leqslant M\,, \qquad |u_\lambda(x)| \leqslant M\,, \qquad 0 \leqslant x \leqslant 1\,. \tag{3.12}$$

The argument used in the proof of Theorem 1.3 now shows that h_ε approaches u_λ in the sup norm as $\varepsilon \to 0$, so $u_\lambda(x)$ is nonnegative for $0 \leqslant x \leqslant 1$.

For notational convenience introduce a partial ordering on X by $u \leqslant v$ if $u(x) \leqslant v(x)$ for $0 \leqslant x \leqslant 1$. One can see that if $0 \leqslant u \leqslant v$ (where 0 denotes the function identically zero) and if $\lambda \geqslant 0$, one has $f \leqslant F_\lambda(u) \leqslant F_\lambda(v)$. Thus if w_n is defined as in the statement of the proposition and if $\lambda \geqslant 0$, one has $w_1 \geqslant w_0 = f$, and induction implies that $w_{n+1} \geqslant w_n$ for $n \geqslant 1$. If $\lambda \leqslant \lambda_+$ we also have that $w_0 \leqslant u_\lambda$ and generally

$$w_{n+1} = F_\lambda(w_n) \leqslant F_\lambda(u_\lambda) = u_\lambda \tag{3.13}$$

for $n \geqslant 0$. It follows that $w_n(x) \leqslant u_\lambda(x) \leqslant B$ and $w_n(x)$ is a monotonic increasing sequence with $\lim_{n\to\infty} w_n(x) = w(x)$. The Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_0^1 \big(w(x) - w_n(x)\big)\,dx = 0\,. \tag{3.14}$$

We leave as an exercise the fact that F_λ gives a continuous map of $L^1[0,1]$ into itself, so (3.14) implies that

$$w = \lim_{n\to\infty} w_{n+1} = \lim_{n\to\infty} F_\lambda(w_n) = F_\lambda(w) \tag{3.15}$$

where the limits in (3.15) are taken in the $L^1[0,1]$ norm. We know that $w(x) \leqslant B$ for $0 \leqslant x \leqslant 1$, so Lemma 1.8 shows that $w(x)$ can be taken to be continuous on $[0,1]$ and w satisfies (3.1). Our construction shows that $f \leqslant w \leqslant u_\lambda$, and since the integral of w must equal I_λ^+ or I_λ^-, we conclude that

$$\int_0^1 w(x)\,dx = I_\lambda^-\,. \tag{3.16}$$

We find from (3.16) that

$$0 = I_\lambda^- - I_\lambda^- = \int_0^1 \big(u_\lambda(x) - w(x)\big)\,dx\,. \tag{3.17}$$

The integrand in (3.17) is nonnegative, so we must have $u_\lambda = w$. Notice that $w(x) - w_n(x) = h_n(x)$ is a decreasing sequence of continuous functions such that $\lim_{n\to\infty} h_n(x) = 0$ for $0 \leqslant x \leqslant 1$, so Dini's theorem says that the limit is uniform in x, $0 \leqslant x \leqslant 1$, and $\lim_{n\to\infty} w_n = u_\lambda$ in the $C[0, 1]$ norm.

The previous argument actually shows that if $v_\lambda = v$ is any nonnegative, continuous solution of (3.1), then $u_\lambda = w \leqslant v_\lambda$. To see this, note that $w_0 = f \leqslant v_\lambda$, and one obtains by induction that

$$w_{n+1} = F_\lambda(w_n) \leqslant F_\lambda(v_\lambda) = v_\lambda \,. \tag{3.18}$$

Taking the limit as $n \to \infty$ gives $u_\lambda \leqslant v_\lambda$. If v_λ is a nonnegative, continuous solution with integral equal to I_λ^-, then just as before we find

$$0 = \int_0^1 \big(v_\lambda(x) - u_\lambda(x)\big)\,dx \,. \tag{3.19}$$

Since the integrand in (3.19) is nonnegative, we have $u_\lambda = v_\lambda$. ■

COROLLARY 3.1. *Let notation and assumptions be as in Proposition* 3.1. *If* $\lambda = \lambda_+ = \Big(2\int_0^1 f(x)\,dx\Big)^{-1}$, *equation* (3.1) *has precisely one nonnegative, continuous solution.*

PROOF. The proof of Proposition 3.1 showed that any real-valued, integrable solution u of (3.1) must have integral equal to I_λ^+ or I_λ^- and that there is exactly one nonnegative continuous solution of (3.1) for $0 \leqslant \lambda \leqslant \lambda_+$ with integral equal to I_λ^-. Since $I_\lambda^+ = I_\lambda^-$ when $\lambda = \lambda_+$, the proof is completed. ■

REMARK 3.1. The proof that there is at most one nonnegative solution of (3.1) with integral equal to I_λ^- is implicit in Ramalho's paper [12].

Our next proposition shows that under the assumptions of Proposition 3.1 there must always be at least one nonnegative continuous solution v_λ of (3.1) for $0 < \lambda \leqslant \lambda_+$ such that $v_\lambda \neq u_\lambda$ for $0 < \lambda < \lambda_+$ and $\lambda \to v_\lambda$ is continuous.

PROPOSITION 3.2. *Assume that* $f(x)$ *satisfies the same hypotheses as in Proposition* 3.1 *and for* $0 < \lambda < \lambda_+ = \Big(2\int_0^1 f(x)\,dx\Big)^{-1}$ *let* $i\beta_\lambda$ *be the unique pure imaginary zero* z *of* $1 - \lambda\hat{f}(z) = 0$ *such that* $\beta_\lambda > 0$. (*The existence of* β_λ *is insured by Lemma* 2.3). *For* $0 < \lambda < \lambda_+$ *let* v_λ *be the unique, real-valued, continuous solution of* (3.1) *such that* $i\beta_\lambda = z$ *is the only solution of* $1 - \lambda\,\hat{v}_\lambda(z) = 0$ *with positive imaginary part* $\big(v_\lambda(x) = 0$ *for* $x \notin [0, 1]\big)$. (*The existence of* v_λ

follows from Theorem 1.4). *Define* $v_\lambda = u_\lambda$ *for* $\lambda = \lambda_+$. *Then the map* $\lambda \to v_\lambda$ *is continuous as a map of* $(0, \lambda_+]$ *into* $X = C[0, 1]$ *and* v_λ *is a nonnegative function for* $0 < \lambda \leq \lambda_+$.

PROOF. The proof of Theorem 1.4 shows that

$$1 - \lambda\hat{v}_\lambda(z) = (z - i\beta_\lambda)(z + i\beta_\lambda)^{-1}(1 - \lambda\hat{u}_\lambda(z)). \tag{3.20}$$

Equation (3.20) shows that

$$\hat{v}_\lambda(z) = \hat{u}_\lambda(z) + (2i\beta_\lambda)(\lambda^{-1})(z + i\beta_\lambda)^{-1}(1 - \lambda\hat{u}_\lambda(z)) \tag{3.21}$$

so to prove that $\lambda \to v_\lambda$ is continuous for $0 < \lambda < \lambda_+$ it suffices to show that

$$\hat{v}_\lambda(z) - \hat{u}_\lambda(z) = \hat{w}_\lambda(z) \tag{3.22}$$

where $w_\lambda|[0, 1]$ is continuous, $w_\lambda(x) = 0$ for $x \notin [0, 1]$, and $\lambda \to w_\lambda \in X = C[0, 1]$ is continuous for $0 < \lambda < \lambda_+$.

However, Krein's lemma (Lemma 1.6) gives an explicit formula for $w_\lambda(x)$ for $0 \leq x \leq 1$:

$$w_\lambda(x) = 2\beta_\lambda \exp(-\beta_\lambda x)\int_x^1 \exp(\beta_\lambda t)u_\lambda(t)\,dt. \tag{3.23}$$

Since the map $\lambda \to \beta_\lambda$ is continuous for $0 < \lambda < \lambda_+$ and $\beta_\lambda = 0$ for $\lambda = \lambda_+$, one can obtain directly from (3.23) that $\lambda \to w_\lambda \in X$ is continuous for $0 < \lambda < \lambda_+$ and w_λ approaches the zero function as λ approaches λ_+.

It remains to prove that v_λ is nonnegative. First assume that $f(x)$ is strictly positive for $0 \leq x \leq 1$ and define $\lambda_1 = \inf\{\lambda > 0 : v_s \text{ is nonnegative for } \lambda \leq s \leq \lambda_+\}$. We know that $v_\lambda = u_\lambda$ for $\lambda = \lambda_+$ and that $u_\lambda \geq f$ for $0 \leq \lambda \leq \lambda_+$. It follows that v_{λ_+} is strictly positive on $[0, 1]$ and the continuity of $\lambda \to v_\lambda$ implies that v_λ is strictly positive for λ near λ_+. If $\lambda_1 > 0$, the continuity of $\lambda \to v_\lambda$ implies that v_{λ_1} must be nonnegative. Furthermore $v_{\lambda_1}(x) = 0$ for some $x \in [0, 1]$, since otherwise v_λ will be nonnegative for $\lambda_1 - \varepsilon \leq \lambda \leq \lambda_+$. On the other hand, any nonnegative solution v of (3.1) for $\lambda > 0$ satisfies $v(x) \geq f(x)$ for $0 \leq x \leq 1$, so we have a contradiction.

It remains to show that v_λ is nonnegative if f is nonnegative, and it suffices to prove this for $0 < \lambda < \lambda_+$. For $\varepsilon > 0$ define $f_\varepsilon(x) = f(x) + \varepsilon$ for $|x| < 1$, $f_\varepsilon(x) = 0$ for $|x| > 1$, let $i\beta_\lambda^{(\varepsilon)}$ be the unique pure imaginary solution of $1 - \lambda\hat{f}_\varepsilon(z)$ such that $\beta_\lambda^{(\varepsilon)} > 0$ (for $0 < \lambda < \lambda_+^{(\varepsilon)}$), and let $v_\lambda^{(\varepsilon)}$ be the corresponding solution as in the statement of our proposition. We leave it to the reader to show that $\lim_{\varepsilon \to 0} \beta_\lambda^{(\varepsilon)} = \beta_\lambda$ and that $\lim_{\varepsilon \to 0} v_\lambda^{(\varepsilon)} = v_\lambda$ (use Lemmas 1.6 and 1.7). Since $v_\lambda^{(\varepsilon)}$ is strictly positive on $[0, 1]$, for $\varepsilon > 0$ and ε small, it follows that v_λ is nonnegative.

We shall also need a classical result from the theory of analytic maps, of several variables. The following result is due to Jane Cronin-Scanlon [5]; a nice presentation of these ideas can be found in [11]. The reader who is not familar with topological degree may want to ignore Lemma 3.1 below and just accept Lemma 3.2 as a basic fact about analytic maps.

LEMMA 3.1 (*see* [5]). *Let U be a bounded open subset of $\mathbb{C}^n$, complex n dimensional space, and let $h\colon \bar{U} \to \mathbb{C}^n$ be a continuous map such that $h|U$ is an analytic map and $h(z) \neq 0$ for $z \in \partial U$. Then it follows that the equation $h(z) = 0$ has only finitely many solutions in U, $\deg(h, U, 0) \geqslant 0$ and $\deg(h, U, 0) \geqslant 1$ if $h(z) = 0$ for some $z \in U$.*

LEMMA 3.2. *Let G be a bounded open subset of $\mathbb{C}^n$ and let $h\colon \bar{G}\times[a, b] \to \mathbb{C}^n$ be a continuous map such that $h(\cdot, \lambda)\colon G \to \mathbb{C}^n$ is analytic for $a \leqslant \lambda \leqslant b$ and $h(z, \lambda) \neq 0$ for $(z, \lambda) \in (\partial G)\times[a, b]$. Assume that $h(w_0, a) = 0$ for some $w_0 \in G$. Let $S = \{(w, \lambda) \in G\times[a, b]\colon h(w, \lambda) = 0\}$. Then there exists a connected subset $S_0 \subset S$ such that $(w_0, a) \in S_0$ and $(w_1, b) \in S_0$ for some $(w_1, b) \in S$.*

PROOF. Define $A = \{(w_0, a)\}$ and $B = \{(w, b) \in S\}$; A and C are closed, disjoint subsets of the compact metric space S. Assume that there does not exist a connected subset D of S such that $D \cap A$ and $D \cap B$ are nonempty. Theorem 9.3 in Chapter I of [15] implies that there is an open subset Ω of $\bar{G}\times[a, b]$ (open in the relative topology) such that $A \subset \Omega$, $B \cap \bar{\Omega}$ is empty, and $h(w, \lambda) \neq 0$ for $(w, \lambda) \in \bar{\Omega} - \Omega$. For notational convenience define $h_\lambda(z) = h(z, \lambda)$, $\Omega_\lambda = \{z\colon (z, \lambda) \in \Omega\}$. The homotopy property for topological degree implies that

$$\deg(h_a, \Omega_a, 0) = \deg(h_b, \Omega_b, 0)\,. \tag{3.24}$$

On the other hand, $h(w_0, a) = 0$ and $w_0 \in \Omega_a$, so

$$\deg(h_a, \Omega_a, 0) \geqslant 1\,. \tag{3.25}$$

The construction of Ω_b implies $h(w, b) \neq 0$ for $w \in \bar{\Omega}_b$, so

$$\deg(h_b, \Omega_b, 0) = 0\,. \tag{3.26}$$

The above equations give a contradiction, so the initial assumption that there does not exist a set like D above was wrong. ∎

With the aid of Lemma 3.2 we are now in a position to prove our basic theorem about positive solutions of equation (3.1).

THEOREM 3.1. *Assume that* $f\colon[0,1]\to\mathbb{R}$ *is* n *times continuously differentiable* $(n\geqslant 0)$, $f^{(j)}(1)=0$ *for* $0\leqslant j<n$ *and* $(-1)^n f^{(n)}(1)>0$. *Suppose also that* $f(x)\geqslant 0$ *for* $0\leqslant x\leqslant 1$. *Then for* $0<\lambda<\lambda_+=\left(2\int_0^1 f(x)\,dx\right)^{-1}$ *equation* (3.1) *has precisely two positive solutions* $u_\lambda(x)$ *and* $v_\lambda(x)$. *For* $\lambda=\lambda_+$ *equation* (3.1) *has exactly one positive solution. The solutions* u_λ *and* v_λ *satisfy* $f(x)\leqslant u_\lambda(x)\leqslant \leqslant v_\lambda(x)$ *for* $0\leqslant x\leqslant 1$, *the map* $\lambda\to u_\lambda\in C[0,1]$ *is continuous for* $0\leqslant\lambda\leqslant\lambda_+$ *and* $\lambda\to v_\lambda\in C[0,1]$ *is continuous for* $0<\lambda\leqslant\lambda_+$.

PROOF. Assume first that $f|[0,1]$ is continuously differentiable and $f(x)>0$ for $0\leqslant x\leqslant 1$. Let u_λ and v_λ be as in Propositions 3.1 and 3.2, respectively. Suppose that for some λ_0 with $0<\lambda_0<\lambda_+$ equation (3.1) has a nonnegative (hence strictly positive) solution u with $u\neq u_{\lambda_0}$, $u\neq v_{\lambda_0}$. Let $S=\{\alpha\colon \operatorname{Im}(\alpha)>0,\,1-\lambda_0 u(\alpha)=0\}$ and for each $\alpha\in S$ let $k(\alpha)$ denote the multiplicity of α as a zero of $1-\lambda_0\hat{u}(z)$. (As usual we have defined $u(x)=0$ for $x\notin[0,1]$). We know that if $f(x)$ is extended to be even and $f(x)=0$ for $|x|>1$, then

$$1-\lambda_0\hat{f}(z)=\big(1-\lambda_0\hat{u}(z)\big)\big(1-\lambda_0\hat{u}(-z)\big)$$

so by Lemma 2.3 $1-\lambda_0\hat{u}(z)=0$ has at most one pure imaginary solution $i\beta$ with $\beta>0$. Assume for definiteness that $1-\lambda_0\hat{u}(z)$ has such a pure imaginary solution and define $\Sigma=\{\alpha\in S\colon \operatorname{Re}(\alpha)>0\}$. By using the explicit formula given in the proof of Theorem 1.4 we find

$$1-\lambda\hat{u}(z)=\big(1-\lambda\hat{v}_\lambda(z)\big)\left[\prod_{\alpha\in\Sigma}\left(\frac{z-\alpha}{z+\alpha}\right)^{k(\alpha)}\left(\frac{z+\bar{\alpha}}{z-\bar{\alpha}}\right)^{k(\alpha)}\right],\qquad \lambda=\lambda_0. \tag{3.27}$$

If $1-\lambda_0\hat{u}(z)$ has no pure imaginary zeros $i\beta$ with $\beta>0$, v_{λ_0} is replaced by u_{λ_0} in (3.27). Let m denote the number of elements in Σ, counting multiplicities, so that α is counted $k(\alpha)$ times. By Theorem 2.2 there exists an integer N such that $(2N-1)\pi>\operatorname{Re}(\alpha)$ for $\alpha\in\Sigma$ and $1-\lambda\hat{f}(z)\neq 0$ for $\lambda_0\leqslant\lambda\leqslant\lambda_+$ and for $\operatorname{Re}(z)=(2N-1)\pi$. By Lemma 2.4 there exists a constant M such that $\operatorname{Im}(\alpha)<M$ for $\alpha\in\Sigma$ and such that any solution of $1-\lambda\hat{f}(z)=0$ for $\lambda_0\leqslant\lambda\leqslant\lambda_+$ and $\operatorname{Re}(z)\leqslant(2N-1)\pi$ satisfies $\operatorname{Im}(z)<M$. According to Lemma 2.3, $1-\lambda\hat{f}(z)\neq 0$ for $0<\operatorname{Re}(z)\leqslant\pi$ and $0\leqslant\lambda\leqslant\lambda_+$, and Lemma 2.2 implies that there exists $\varepsilon>0$ such that $1-\lambda\hat{f}(z)\neq 0$ for (λ,z) such that $0\leqslant\lambda\leqslant\lambda_+$, $\pi\leqslant\operatorname{Re}(z)\leqslant(2N-1)\pi$ and $|\operatorname{Im}(z)|\leqslant\varepsilon$. With this notation, define a set U by

$$U=\{z\in\mathbf{C}\colon \pi<\operatorname{Re}(z)<(2N-1)\pi,\ \varepsilon<\operatorname{Im}(z)<M\}$$

and define an open subset G of $\mathbf{C}^m$ by

$$G = \{(z_1, \ldots, z_m) \in \mathbf{C}^m : z_j \in U \text{ for } 1 \leqslant j \leqslant m\}.$$

We shall assume that the elements α of Σ have been ordered in some way, say as $(\alpha_1, \ldots, \alpha_m)$ (each element is repeated as many times as its multiplicity $k(\alpha)$). Our construction shows that $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in G$ and that $1 - \lambda\hat{f}(z) \neq 0$ for $z \in \partial U$ and $\lambda_0 \leqslant \lambda \leqslant \lambda_+$. Define a map $h: \bar{G} \times [\lambda_0, \lambda_+] \to \mathbf{C}^m$ by $h(z_1, z_2, \ldots, z_m \lambda) = (w_1, \ldots, w_m)$, where $w_j = 1 - \lambda\hat{f}(z_j)$. According to Lemma 3.2, there exists a connected subset D of $\bar{G} \times [\lambda_0, \lambda_+]$ such that $h(z, \lambda) = 0$ for $(z, \lambda) \in D$, $(\alpha, \lambda_0) = (\alpha_1, \alpha_2, \ldots, \alpha_m, \lambda_0) \in D$ and $(w, \lambda_+) \in D$ for some $w \in G$.

For each element $(z_1, \ldots, z_m) \in D$ define a continuous function $\lambda u = \Phi(z_1, z_2, \ldots, z_m, \lambda) \in C[0, 1]$ by defining $u(x) = 0$ for $x \notin [0, 1]$ and writing

$$(3.28) \qquad 1 - \lambda\hat{u}(z) = (1 - \lambda\hat{v}_\lambda(z)) \left[\prod_{j=1}^{m} \left(\frac{z - z_j}{z + z_j}\right)\left(\frac{z + \bar{z}_j}{z - \bar{z}_j}\right)\right].$$

We have already seen in the first section that this defines a function $u \in C[0, 1]$, and we have shown that $\lambda \to v_\lambda \in C[0, 1]$ is continuous. Since $\operatorname{Im}(z_j) \geqslant \varepsilon > 0$, equation (1.82) in Section 1 enables one to give a formula for λu as the convolution of certain functions with v_λ, and one can see directly from the formula that Φ is a continuous map into $C[0, 1]$.

We claim that $\lambda^{-1}\Phi(z_1, z_2, \ldots, z_m, \lambda)$ gives a nonnegative solution of (3.1) for each $(z, \lambda) \in D$. Define $\mathcal{O}_1 = \{(z, \lambda) \in D : \lambda^{-1}\Phi(z, \lambda)$ is a nonnegative solution of (3.1)$\}$.

We know that $\mathcal{O}_1$ is nonempty and closed (because Φ is continuous); however, if u is a nonnegative solution of (3.1), $u(x) \geqslant f(x) > 0$ for $0 \leqslant x \leqslant 1$, so continuity of Φ also shows that $\mathcal{O}_1$ is open. Since D is connected we conclude that $\mathcal{O}_1 = D$. If $(w, \lambda_+) \in D$, we conclude that $\varphi = \Phi(w, \lambda_+)$ is a nonnegative solution of (3.1) such that $1 - \lambda_+\hat{\varphi}(z)$ has zeros with positive imaginary part. It follows that $\varphi \neq u_{\lambda_+}$, and since u_{λ_+} is the only nonnegative solution of (3.1) for $\lambda = \lambda_+$, we have a contradiction.

It remains to prove that there are precisely two nonnegative solutions under the hypotheses of the theorem. As before, suppose that $0 < \lambda < \lambda_+$, that u is a nonnegative solution of (3.1), and that $u \neq u_\lambda$, $u \neq v_\lambda$. We shall obtain a contradiction. Extend $u(x) = 0$ for $x \notin [0, 1]$ and let S and Σ be as defined before. If α is a zero of $1 - \lambda\hat{u}(z)$ of multiplicity $k(x)$ we know that α is a zero of $1 - \lambda\hat{f}(z)$ of multiplicity $m(\alpha) \geqslant k(\alpha)$ (where $f(\alpha)$

is extended to be even and $f(x)=0$ for $|x|>1$). Let n be as in the statement of the theorem. If $n=0$, let $f_j(x)$ be a sequence of real-valued functions such that $f_j(x)$ approaches $f(x)$ uniformly in x for $0\leqslant x\leqslant 1$, $f_j|[0,1]$ is continuously differentiable, and $f_j(x)>0$ for $0\leqslant x\leqslant 1$; if $n\geqslant 1$, define $f_j(x)=f(x)+j^{-1}(1-x)^{n-1}$ for $0\leqslant x\leqslant 1$. Extend $f_j(x)$ to be even and equal to zero for $|x|>1$. Given $\varepsilon>0$, it is an application of Rouche's theorem to show that there exists $N>0$ such that for $j\geqslant N$ and any $\alpha\in S$, $1-\lambda\hat{f}_j(z)$ has precisely $m(\alpha)$ zeros such that $|z-\alpha|<\varepsilon$ and no zeros z with $|z-\alpha|=\varepsilon$ for some $\alpha\in S$ and $\operatorname{Im}(z)\geqslant 0$. We can assume that ε is so small that ε discs about elements of S are pairwise disjoint and contain no complex numbers with nonpositive imaginary part. Let T_j denote the set of zeros of $1-\lambda\hat{f}_j(z)$ such that $|z-\alpha|<\varepsilon$ for some $\alpha\in S$. Let S_j denote a subset of T_j. For each $\beta\in T_j$ let $m_j(\beta)$ denote the multiplicity of β as a zero of $1-\lambda\hat{f}_j(z)$ and for $\beta\in S_j$ let $k_j(\beta)$ denote a positive integer such that $k_j(\beta)\leqslant m_j(\beta)$. We can assume that if $\beta\in S_j$, $-\bar{\beta}\in S_j$ and $k_j(\beta)=k_j(-\bar{\beta})$. Furthermore we can arrange that

$$\sum_{\beta\in S_j,\,|\beta-\alpha|<\varepsilon} k_j(\beta)=k(\alpha)\,.$$

According to Theorem 1.4 of Section 1 there is one and only one solution u_j of the equation

$$v(x)=f_j(x)+\lambda\int_x^1 v(y)v(y-x)\,dy\,,\qquad 0<x\leqslant 1 \tag{3.29}$$

such that the zeros β of $1-\lambda\hat{u}_j(z)$ with positive imaginary part are precisely the elements of S_j with multiplicity (as a zero of $1-\lambda\hat{u}_j(z)$) $k_j(\beta)$. By using the formula (3.27) and the results of Section 1 one can see that $u_j\to u$ in $C[0,1]$ (we leave the details to the reader).

Now assume that $n=0$, so $f(1)>0$ and $f(x)$ is continuous and nonnegative on $[0,1]$. We claim that $u(x)>0$ for $0\leqslant x\leqslant 1$. We have

$$u(1)=f(1)>0\qquad u(0)=f(0)+\lambda\int_0^1 u(y)^2\,dy \tag{3.30}$$

so there certainly exists $\delta>0$ such that $u(x)\geqslant a>0$ for $1-\delta\leqslant x\leqslant 1$ and $0\leqslant x\leqslant\delta$. We claim that $u(x)>0$ for $0\leqslant x\leqslant 1$. If not, define $x_0=\inf\{x>0: u(y)>0 \text{ for } x\leqslant y\leqslant 1\}$. Our construction implies that $u(x_0)=0$, $u(y)>0$ for $x_0<y\leqslant 1$ and $\delta<x_0<1-\delta$. However, because $u(x)$ is assumed nonnegative, we obtain from (3.1) that

$$(3.31)\qquad \begin{aligned} u(x_0) = 0 &= f(x_0) + \lambda\int_{x_0}^{1} u(y)u(y-x_0)\,dy \\ &\geqslant \lambda\int_{x_0}^{x_0+\delta} u(y)u(y-x_0)\,dy \\ &\geqslant \lambda a\int_{x_0}^{x_0+\delta} u(y)\,dy > 0\,. \end{aligned}$$

Equation (3.31) gives a contradiction, so we conclude that $u(x) > 0$ for $0 \leqslant x \leqslant 1$. Since u_j approaches u uniformly in $x \in [0, 1]$, we conclude that $u_j(x) > 0$ for $0 \leqslant x \leqslant 1$ and for j large. We also know that u_j is not one of the nonnegative solutions of (3.29) insured by Propositions 3.1 and 3.2, because $1 - \lambda\hat{u}_j(z) = 0$ for some z with $\mathrm{Re}\,(z) > 0$ and $\mathrm{Im}\,(z) > 0$. On the other hand, f_j is strictly positive and C^1 on $[0, 1]$, so the first part of the proof shows that (3.29) has precisely two nonnegative solutions in this case. This contradiction (for $n = 0$) shows that the original assumption of a third nonnegative solution u was wrong.

We have proved Theorem 3.1 for $n = 0$. If we can prove that if the theorem is true for a given $n \geqslant 0$ then it is true for $n + 1$, we will be done by mathematical induction. Thus suppose the theorem true for n and let $f(x)$ be a nonnegative function such that $f \in C^{n+1}[0, 1]$, $f^{(i)}(1) = 0$ for $0 \leqslant i \leqslant n$ and $(-1)^{n+1}f^{(n+1)}(1) > 0$. As before, define $f_j(x) = f(x) + j^{-1}(1 - x)^n$. By inductive hypothesis, if $0 < \lambda < \lambda_+ = \left(2\int_0^1 f(x)\,dx\right)^{-1}$ and j is large enough, equation (3.29) has precisely two nonnegative solutions. If, as before, we suppose that $u \neq u_\lambda$, $u \neq v_\lambda$, is a nonnegative solution of (3.1) and that u_j is defined as before, we know that $u_j \to u$ in the $C[0, 1]$ norm. It then follows with the aid of Remark 1.2 in Section 1 that $\lim_{j\to\infty} u_j^{(k)}(x) = u^{(k)}(x)$ uniformly in $x \in [0, 1]$ for $0 < k \leqslant n + 1$. If we can show that $u_j(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$ and for j large enough, then just as in the case $n = 0$, we will have a contradiction.

Thus it remains to show that $u_j(x)$ is nonnegative for j large. Starting from the formula

$$(3.32)\qquad u'(x) = f'(x) - \lambda u(x)u(0) - \lambda\int_x^1 u(y)u'(y-x)\,dy$$

it is relatively easy to see that $u^{(j)}(1) = 0$ for $0 \leqslant j \leqslant n$ and $(-1)^{n+1}u^{(n+1)}(1) > 0$;

by continuity there exists $\delta > 0$ such that $(-1)^{n+1}u^{(n+1)}(x) > 0$ for $1-\delta \leqslant x \leqslant 1$. Similarly, starting from the equation

$$u_j'(x) = f_j'(x) - \lambda u_j(x)u_j(0) - \lambda\int_x^1 u_j(y)u_j'(y-x)\,dy \tag{3.33}$$

one can show that

$$\begin{aligned} &u_j^{(k)}(1) = 0\,, \quad 0 \leqslant k \leqslant n-1 \\ &(-1)^n u_j^{(n)}(1) \geqslant 0\,. \end{aligned} \tag{3.34}$$

If j is selected so large that $(-1)^{n+1}u_j^{(n+1)}(x) > 0$ for $1-\delta \leqslant x \leqslant 1$ and $j \geqslant J$ and if Taylor's formula with remainder term is used at $x = 1$, one concludes that for $j \geqslant J$

$$u_j(x) > 0\,, \quad 1-\delta \leqslant x \leqslant 1\,. \tag{3.35}$$

Similarly, Taylor's formula implies that $u(x) > 0$ for $1-\delta \leqslant x < 1$. If one uses the same argument used before for $n = 0$ one can conclude that $u(x) > 0$ for $0 \leqslant x \leqslant 1-\delta$. Since $u_j(x)$ approaches $u(x)$ uniformly in $x \in [0, 1-\delta]$, one concludes that for j large enough $u_j(x) > 0$ for $0 \leqslant x \leqslant 1$. ∎

REMARK 3.1. It is important to allow the possibility $f(1) = 0$ in Theorem 3.1. In fact, if the original integral equation which leads one to consider (3.1) comes from a three dimensional problem by assuming radial symmetry [3, 4], then it is *necessary* that $f(1) = 0$.

It is plausible that Theorem 3.1 is true under the weaker assumption that $f(x)$ is nonnegative, continuous and not identically zero on $[0, 1]$, but we have been unable to prove this. We would like to mention, with only an outline of the proof, a result which suggests that something sharper than Theorem 3.1 should be true.

PROPOSITION 3.3. *Suppose that* $f|[0,1]$ *is continuously differentiable,* $f(1) = 0$, $f(0) > 0$ *and* $f'(x) \leqslant 0$ *for* $0 \leqslant x \leqslant 1$. *Then for* $0 < \lambda < \lambda_+ = \left(2\int_0^1 f(x)\,dx\right)^{-1}$, *equation* (3.1) *has precisely two nonnegative solutions.*

Outline of proof. First one proves that, under the assumptions on f, any nonnegative solution u of (3.1) satisfies $u(1) = 0$ and $u'(x) \leqslant 0$ for $0 \leqslant x \leqslant 1$. According to Theorem 1.4 and Lemma 2.3, proving that (3.1) has precisely two positive solutions for $0 < \lambda < \lambda_+$ is equivalent to showing that

$$1-\lambda\int_0^1 u(x)e^{izx}=0 \tag{3.36}$$

has no solutions $z=\alpha+i\beta$ with $\alpha>0$ and $\beta>0$. If (3.36) had such a solution, then taking the imaginary part would give

$$\int_0^\infty v(x)\sin\alpha x\,dx=0\,,\qquad v(x)=u(x)e^{-\beta x}\,,\qquad u(x)=0\qquad\text{for } x>1\,. \tag{3.37}$$

The function $v(x)$ is monotonic decreasing on $[0,\infty)$, continuous and $\lim_{x\to\infty} v(x)<v(0)$. By using an argument like that at the beginning of the proof of Theorem 2.1, one shows that this implies

$$\int_0^\infty v(x)\sin\alpha x\,dx>0\,,\qquad \alpha>0\,. \tag{3.38}$$

Notice that strict inequality may fail in (3.38) if $v(x)$ is not continuous, e.g., if $v(x)\equiv 1$ for $0\leqslant x\leqslant 1$, $v(x)=0$ for $x>1$. ■

We now want to consider the question of uniqueness of positive solutions of (3.1) when $\lambda<0$. The situation here is more complicated than for positive λ; equation (3.1) may not have a positive solution for a given λ with $\lambda_-<\lambda<0$. For a given $f(x)$ our results will give only a crude idea of the range of negative λ for which (3.1) has a positive solution or a unique positive solution.

We begin with some simple lemmas.

LEMMA 3.3. *Let $f(x)$ be a continuous, real-valued function for $0\leqslant x\leqslant 1$ and define $A=\max|f(x)|$ for $0\leqslant x\leqslant 1$. If $B>0$ and if λ is such that*

$$|\lambda|\leqslant B(B+A)^{-2}\qquad\text{and}\qquad |\lambda|<(2B+2A)^{-1} \tag{3.39}$$

then equation (3.1) has one and only one continuous solution $u(x)=u$ such that $\|u-f\|\leqslant B$ (the norm is the sup norm).

PROOF. Define $D=\{u\in C[0,1]\colon \|u-f\|\leqslant B\}$ and define $F_\lambda\colon D\to C[0,1]$ by

$$(F_\lambda u)(x)=f(x)+\lambda\int_x^1 u(y)u(y-x)\,dy\,.$$

If $u, v \in D$, then straightforward estimates give

$$(3.40)\quad |(F_\lambda u)(x) - (F_\lambda v)(x)| \leqslant \leqslant |\lambda| \int_x^1 |u(y)-v(y)||u(y-x)| + |v(y)||u(y-x)-v(y-x)|dy \leqslant \leqslant 2|\lambda|(B+A)\|u-v\| .$$

Equation (3.40) shows that if (3.39) holds, $F_\lambda|D$ is a contraction mapping with Lipschitz constant $k = 2|\lambda|(B+A) < 1$. If $u \in D$, $\|u\| \leqslant B + A$ and one has

$$(3.41)\qquad \|F_\lambda u - f\| \leqslant |\lambda|(B+A)^2 \leqslant B, \qquad u \in D$$

so $F_\lambda(D) \subset D$. It follows that F_λ is a contraction mapping of D into D and therefore has a unique fixed point in D. ∎

Our next lemma is a slight modification of Lemma 3.2 to allow unbounded G.

LEMMA 3.4. *Let G be an open subset of $\mathbb{C}^n$ and let $h: \bar{G} \times [a, b) \to \mathbb{C}^n$ be a continuous map such that $h(\cdot, \lambda): G \to \mathbb{C}^n$ is analytic for $a \leqslant \lambda < b$ and $h(z, \lambda) \neq 0$ for $(z, \lambda) \in \partial G \times [a, b]$. Given any $\varepsilon > 0$ assume that there exists $M_\varepsilon > 0$ such that any solution $(z, \lambda) \in \bar{G} \times [a, b-\varepsilon]$ of $h(z, \lambda) = 0$ satisfies $|z| \leqslant M_\varepsilon$. Assume that $h(w_0, a) = 0$ for some $w_0 \in G$ and write $S = \{(w, \lambda) \in G \times [a, b): h(w, \lambda) = 0\}$. Then there exists a connected subset S_0 of S such that $(w_0, a) \in S_0$ and for every λ with $a \leqslant \lambda < b$ there exists $w_\lambda \in G$ such that $(w_\lambda, \lambda) \in S_0$.*

PROOF. Let $b_n < b$ be a monotonic sequence approaching b and write $a = b_0$. Let $S^{(n)} = \{(w, \lambda) \in G \times [b_n, b_{n+1}]: h(w, \lambda) = 0\}$. By applying Lemma 3.2 and using the fact that each $S^{(n)}$ is bounded and hence can be considered a subset of $G_n \times [b_n, b_{n+1}]$, where G_n is a bounded, open subset of G such that $h(w, \lambda) \neq 0$ for $(w, \lambda) \in \partial G_n \times [b_n, b_{n+1}]$, there exists a connected subset $S_0^{(0)}$ of $S^{(0)}$ such that $(w_0, a) \in S_0^{(0)}$ and $(w_1, b_1) \in S_0^{(0)}$ for some $w_1 \in G$. Applying Lemma 3.2 again, there exists a connected subset $S_0^{(1)}$ of $S^{(1)}$ such that $(w_1, b_1) \in S_0^{(1)}$ and $(w_2, b_2) \in S_0^{(1)}$ for some $w_2 \in G$. Generally, there exists a connected subset $S_0^{(n)}$ of $S^{(n)}$ such that $(w_n, b_n) \in S_0^{(n-1)} \cap S_0^{(n)}$ and $(w_{n+1}, b_{n+1}) \in S_0^{(n)}$ for some $w_{n+1} \in G$. Define $S_0 = \{(w, \lambda): (w, \lambda) \in S_0^{(n)}$ if $b_n \leqslant \lambda \leqslant b_{n+1}, n \geqslant 0\}$. One can check that S_0 is a connected subset of S. We claim that given λ with $a < \lambda < b$, there exists $w \in G$ such that $(w, \lambda) \in S_0$. If not, we could write S_0 as the disjoint union of two nonempty relatively open subsets, namely

$$S_0 = Q_0 \cup R_0$$

(3.42)
$$Q_0 = \{(w, \mu) \in S_0 : a \leqslant \mu < \lambda\}$$

$$R_0 = \{(w, \lambda) \in S_0 : \lambda < \mu < b\}$$

$\big(R_0$ is nonempty because $(w_n, b_n) \in R_0$ for $b_n > \lambda\big)$. Equation (3.42) contradicts the connectedness of S_0. ■

Using Lemmas 3.3 and 3.4 we can obtain a result concerning existence and uniqueness of nonnegative solutions of (3.1) for $\lambda < 0$.

THEOREM 3.2. *Assume that* $f\colon [0, 1] \to \mathbb{R}$ *is a continuously differentiable, strictly positive function. Let* $\lambda_* \leqslant 0$ *be a number such that if* $u(x)$, $0 \leqslant x \leqslant 1$, *is any continuous nonnegative solution of*

$$u(x) = f(x) + \lambda \int_x^1 u(y) u(y - x) dy, \qquad 0 \leqslant x \leqslant 1 \tag{3.43}$$

for some λ *with* $\lambda_* \leqslant \lambda \leqslant 0$, *then in fact* $u(x) > 0$ *for* $0 \leqslant x \leqslant 1$. *Then equation* (3.43) *has exactly one positive solution* u *for each* λ *with* $\lambda_* \leqslant \lambda \leqslant 0$ *and* $\lambda > \lambda_-$ (λ_- *is defined as in Lemma* 1.2, *Section* 1).

REMARK 3.2. Notice that there is no assumption of existence of positive solutions of (3.42) in the definition of λ_*.

As we shall see later, for a given function $f(x)$ one can give a rough estimate of λ_*.

PROOF. As usual, for $\lambda_- \leqslant \lambda < 0$, let $u_\lambda(x)$ denote the unique solution of (3.43) such that $1 - \lambda \hat{u}_\lambda(z) \neq 0$ for $\operatorname{Im}(z) > 0$ $\big($where $u_\lambda(x) = 0$ for $x \notin [0, 1]\big)$. We have already seen in Section 2 that $\lim_{\lambda \to 0} u_\lambda = f$ in $C[0, 1]$, so $u_\lambda(x)$ is strictly positive on $[0, 1]$ for λ small and negative. We claim that $u_\lambda(x) > 0$ for $\lambda_* \leqslant \lambda < 0$ and $0 \leqslant x \leqslant 1$. If not, define λ_1 by

$$\lambda_1 = \inf\{\lambda > \lambda_* : u_s(x) > 0 \text{ for } \lambda \leqslant s \leqslant 0 \text{ and } 0 \leqslant x \leqslant 1\}.$$

The continuity of the map $\lambda \to u_\lambda$ shows that $u_{\lambda_1}(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$, and by assumption $\lambda_1 \geqslant \lambda_*$ and $u_{\lambda_1}(x) = 0$ for some x. However, the definition of λ_* shows that this cannot happen.

It remains to show that there is precisely one positive solution for $\lambda_* \leqslant \lambda \leqslant 0$ and $\lambda > \lambda_-$. The general idea of the proof is the same as that of Theorem 3.1, so we will be sketchy. Suppose that for λ_0 with $\lambda_* \leqslant \lambda_0 < 0$

and $\lambda_- < \lambda_0$ equation (3.43) has a nonnegative solution u with $u \neq u_{\lambda_0}$. Define $u(x) = 0$ for $x \notin [0, 1]$ and as in Theorem 3.1 define $S = \{\alpha: \mathrm{Im}\,(\alpha) > 0, 1 - \lambda_0 \hat{u}(\alpha) = 0\}$ and define $k(\alpha)$ to be the multiplicity of α as a zero of $1 - \lambda_0 \hat{u}(z) = 0$. Since any zero of $1 - \lambda_0 \hat{u}(z) = 0$ is a zero of $1 - \lambda_0 \hat{f}(z) = 0$ (where $f(x)$ is even and $f(x) = 0$ for $|x| > 1$) Lemma 2.3 implies that $|\mathrm{Re}\,(\alpha)| > \pi$ for every element of S. The explicit formula in Theorem 1.4 implies that if $\Sigma = \{\alpha \in S: \mathrm{Re}\,(\alpha) > 0\}$, then

$$(3.44) \qquad 1 - \lambda \hat{u}(z) = \left(1 - \lambda \hat{u}_\lambda(z)\right) \left[\prod_{\alpha \in \Sigma} \left(\frac{z - \alpha}{z + \alpha}\right)^{k(\alpha)} \left(\frac{z + \bar{\alpha}}{z - \bar{\alpha}}\right)^{k(\alpha)} \right], \qquad \lambda = \lambda_0 .$$

Just as in Theorem 3.1, there exists an integer $N > 0$ such that $(2N - 1)\pi > \mathrm{Re}\,(\alpha)$ for $\alpha \in \Sigma$ and $1 - \lambda \hat{f}(z) \neq 0$ for $\lambda_0 \leqslant \lambda < 0$, and $\mathrm{Re}\,(z) = (2N - 1)\pi$. Since we are assuming $\lambda_- < \lambda_0$, there exists $\varepsilon > 0$ such that $1 - \lambda \hat{f}(z) \neq 0$ for $\lambda_0 \leqslant \lambda \leqslant 0$ and for z with $|\mathrm{Im}\,(z)| \leqslant \varepsilon$. Lemma 2.3 implies that $1 - \lambda \hat{f}(z) \neq 0$ for z with $|\mathrm{Re}\,(z)| \leqslant \pi$ and for $\lambda_0 \leqslant \lambda \leqslant 0$; and Lemma 2.4 implies that if $\lambda_* \leqslant \lambda \leqslant -\delta < 0$, there is a constant $A(\delta)$ such that if $1 - \lambda \hat{f}(z) = 0$ and $|\mathrm{Re}\,(z)| \leqslant (2N - 1)\pi$ then $|\mathrm{Im}\,(z)| \leqslant A(\delta)$. We now define an unbounded open set U by

$$U = \{z \in C: \pi < \mathrm{Re}\,(z) < (2N - 1)\pi, \mathrm{Im}\,(z) > \varepsilon\}$$

and note that $1 - \lambda \hat{f}(z) \neq 0$ for $z \in \partial U$ and $\lambda_0 \leqslant \lambda \leqslant 0$.

As in Theorem 3.1, let m denote the number of elements in Σ, counting the multiplicity $k(\alpha)$, and assume that the elements of Σ have been ordered in some way, say as $(\alpha_1, \ldots, \alpha_m)$. Define an unbounded, open subset G of $\mathbb{C}^m$ by

$$G = \{(z_1, \ldots, z_m) \in \mathbb{C}^m: z_j \in U \text{ for } 1 \leqslant j \leqslant m\}$$

and define a map $h: \bar{G} \times [\lambda_0, 0) \to \mathbb{C}^m$ by $h(z_1, \ldots, z_m, \lambda) = (w_1, \ldots, w_m)$, where $w_j = 1 - \lambda \hat{f}(z_j)$. Our construction shows that the hypotheses of Lemma 3.4 are satisfied, so there exists a connected subset D of $G \times [\lambda_0, 0)$ such that $h(z, \lambda) = 0$ for $(z, \lambda) = D$, $(\alpha_1, \ldots, \alpha_m, \lambda_0) \in D$ and if λ is such that $\lambda_0 \leqslant \lambda < 0$, there exists $z \in G$ with $(z, \lambda) \in D$.

For each $(z, \lambda) \in D$, $z = (z_1, z_2, \ldots, z_m)$, define a continuous function $\lambda w = \Phi(z, \lambda) \in C[0, 1]$ by $w(x) = 0$ for $x \notin [0, 1]$ and

$$(3.45) \qquad 1 - \lambda \hat{w}(z) = \left(1 - \lambda \hat{u}_\lambda(z)\right) \left[\prod_{j=1}^{m} \left(\frac{z - z_j}{z + z_j}\right) \left(\frac{z + \bar{z}_j}{z - \bar{z}_j}\right) \right].$$

Essentially the same argument used before still shows that the map $(z, \lambda) \to \lambda^{-1} \Phi(z, \lambda)$ is a continuous map of D into $C[0, 1]$ and that $\lambda^{-1} \Phi(z, \lambda)$ is a nonnegative solution of (3.43) for every $(z, \lambda) \in D$. It follows that for every $\lambda_0 \leqslant \lambda < 0$ there is a nonnegative solution $w_\lambda \neq u_\lambda$ of equation 3.43. However, any nonnegative solution w of (3.43) must satisfy $w(x) \leqslant f(x)$. This shows that there exists a constant $B = \|f\|$ such that $\|w_\lambda\| \leqslant B$ for $\lambda_0 \leqslant \lambda < 0$. However, Lemma 3.3 shows that for λ small enough, equation (3.43) has precisely one solution u with $\|u - f\| \leqslant B$, whereas u_λ and w_λ are both solutions with this property. This contradiction completes the proof. ■

REMARK 3.3. One can also prove a version of Theorem 3.2 for functions $f(x)$ with $f(1) = 0$, e.g., $f(x) = 1 - x$ for $0 \leqslant x \leqslant 1$. However, there is a large gap between what one can prove about nonnegative solutions of (3.43) for such $f(x)$ (for $\lambda < 0$) and what is probably true. For instance, numerical studies and a variety of heuristic arguments suggest that for $f(x) = 1 - x$ equation (3.43) has a positive solution (probably unique) for every $\lambda < 0$. The results we have actually been able to prove for this function are much weaker, and we omit them.

As an example we would like to apply Theorem 3.2 to equation (3.43) for $f(x) \equiv 1$. The result we shall prove was claimed by Ramalho [12], though, as is observed in [2], there is a gap in the proof. A correct proof, different from the one given here, was obtained by N. Baxter in [2]. Numerical studies in [2] suggest that for the $f(x)$ above, equation (3.43) has a unique positive solution for (approximately) $-2.1 < \lambda < 0$. The number λ_- is approximately -2.3 in this case.

COROLLARY 3.2. *The equation*

$$u(x) = 1 + \lambda \int_x^1 u(y - x) u(y)\, dy, \qquad 0 \leqslant x \leqslant 1 \tag{3.46}$$

has precisely one positive, continuous solution $u(x)$ *for* $-\frac{3}{2} \leqslant \lambda < 0$.

PROOF. Suppose that $-\frac{3}{2} \leqslant \lambda < 0$ and that $u(x)$ is a nonnegative continuous solution of (3.46). According to Theorem 3.2 it suffices to show that $u(x) > 0$ for $0 \leqslant x \leqslant 1$. The Cauchy-Schwartz inequality implies

$$\int_x^1 u(y - x) u(y)\, dy \leqslant \int_0^1 u(y)^2\, dy\,. \tag{3.47}$$

Equations (3.46) and (3.47) imply that

$$u(0) = 1 + \lambda \int_0^1 u(y)^2\, dy \leqslant u(x)\,. \tag{3.48}$$

Since $0 \leqslant u(x) < 1$ for $0 \leqslant x \leqslant 1$, (3.48) implies that

$$1 + \lambda\int_0^1 u(y)\,dy < u(x)\,, \qquad 0 \leqslant x \leqslant 1\,. \tag{3.49}$$

As was observed in Proposition 3.1, we have

$$\lambda\int_0^1 u(y)\,dy = 1 \pm \sqrt{1-2\lambda}\,. \tag{3.50}$$

However, Theorem 1.4 shows that if $w(x)$ is any continuous, real-valued solution of equation (3.1) for $\lambda < 0$, one can only have the possibility

$$\lambda\int_0^1 u(y)\,dy = 1 + \sqrt{1-2\int_0^1 f(x)\,dx}$$

if the equation $1-\lambda\hat{f}(z) = 0$ $\big(f(x)$ even, $f(x) = 0$ for $|x| > 1\big)$ has a pure imaginary solution $i\mu$, $\mu > 0$. The results of Section 2 show that $1-\lambda\hat{f}(z)$ has no such pure imaginary solutions if $f(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$. Thus in equation (3.49) one must take the minus sign and using this in (3.49) gives

$$2-\sqrt{1-2\lambda} < u(x)\,, \qquad 0 \leqslant x \leqslant 1\,. \tag{3.51}$$

If $-\frac{3}{2} \leqslant \lambda < 0$, the left hand side of (3.51) is nonnegative. ∎

4. – A formula for $1-\lambda\hat{u}_\lambda(z)$.

In this section we shall give an explicit formula in terms of the zeros of $1-\lambda\hat{f}(z)$ for $1-\lambda\hat{u}_\lambda(z)$, where $u_\lambda(x) = 0$ for $x \notin [0, 1]$,

$$u_\lambda(x) = f(x) + \lambda\int_x^1 u_\lambda(y)u_\lambda(y-x)\,dy\,, \qquad 0 \leqslant x \leqslant 1 \tag{4.1}$$

and

$$1-\lambda\hat{u}_\lambda(z) \neq 0 \qquad \text{for } \operatorname{Im}(z) > 0\,. \tag{4.2}$$

Elementary complex variable theory provides a formula for $1-\lambda\hat{u}(z)$ as an infinite product. We will show that the infinite product can be written in such a way that the only unknown constants are the zeros of $1-\lambda\hat{f}(z)$.

First, suppose only that $f(x)$ is a real-valued, L^1 function such that $f(-x) = f(x)$ for all x and $f(x) = 0$ for $|x| > 1$. Assume that $1 - \lambda\hat{f}(\xi) \geqslant 0$ for all real ξ and let $u(x)$ be the unique, real-valued L^1 solution of (4.1) and (4.2) such that $u(x) = 0$ for $x \notin [0, 1]$. Define $S = \{\beta \in \mathbf{C} - \{0\}: 1 - \lambda\hat{f}(\beta) = 0\}$ and $T = \{\beta \in S: \operatorname{Im}(\beta) \leqslant 0\}$. It follows directly from the facts that $f(x)$ and $u(x)$ have support in $[-1, 1]$ that $1 - \lambda\hat{f}(z)$ and $1 - \lambda\hat{u}(z)$ are entire functions of order less than or equal to one and hence (by Theorem 7 on p. 186 of [1]) of genus less than or equal to one. It follows therefore (remembering [4.2]) that one has the formulas

$$1 - \lambda\hat{f}(z) = Az^{2p}e^{iaz}\prod_{\beta\in S^1}\left(1 - \frac{z}{\beta}\right)\exp\frac{z}{\beta} \tag{4.3}$$

$$1 - \lambda\hat{u}(z) = Bz^{p}e^{ibz}\prod_{\beta\in T}\left(1 - \frac{z}{\beta}\right)\exp\frac{z}{\beta}\,. \tag{4.4}$$

One writes $2p$ for the (even) multiplicity of 0 as a zero of $1 - \lambda\hat{f}(z) = 0$. It is understood in equation (4.3) that the factor $(1 - z/\beta)\exp(z/\beta) = Q(z; \beta)$ is repeated a number of times equal to the multiplicity of β as a zero of $1 - \lambda\hat{f}(z) = 0$; similarly, in (4.4) $Q(z; \beta)$ is repeated a number of times equal to the multiplicity of β as a zero of $1 - \lambda\hat{u}(z) = 0$. Thus, if the multiplicity of β as a solution of $1 - \lambda\hat{f}(\beta) = 0$ is k, the factor $Q(z; \beta)$ is repeated k times in (4.4) if $\operatorname{Im}(\beta) < 0$ and $\frac{1}{2}k$ times if β is real. The infinite products in (4.3) and (4.4) converge absolutely and uniformly on compact subsets of $\mathbf{C}$ (see [1, p. 186]).

Recall that if β is a solution of $1 - \lambda\hat{f}(z) = 0$, so is $\bar{\beta}$, $-\beta$ and $-\bar{\beta}$. Define $S_2 = \{\beta \in S: \beta$ is real and $\beta > 0$ or β is pure imaginary and $\operatorname{Im}(\beta) > 0\}$ and $S_1 = \{\beta \in S: \operatorname{Re}(\beta) > 0$ and $\operatorname{Im}(\beta) > 0\}$. By grouping $Q(z; \beta)$ and $Q(z; -\beta)$ together one obtains from (4.3) that

$$1 - \lambda\hat{f}(z) = Az^{2p}e^{iaz}\left[\prod_{\beta\in S_1}\left(1 - \frac{z^2}{\beta^2}\right)\left(1 - \frac{z^2}{\bar{\beta}^2}\right)\right]\left[\prod_{\beta\in S_2}\left(1 - \frac{z^2}{\beta^2}\right)\right]. \tag{4.5}$$

Because $f(x)$ is even and real-valued, $1 - \lambda\hat{f}(z)$ is even and this implies that $a = 0$ in (4.5). By taking $z = 0$ in (4.5) and assuming that $1 - \lambda\hat{f}(0) \neq 0$ one finds

$$A = 1 - 2\lambda\int_0^1 f(x)\,dx \qquad \text{if } 1 - 2\lambda\int_0^1 f(x)\,dx \neq 0\,. \tag{4.6}$$

If one defines $T_2 = \{\beta \in T: \beta$ is pure imaginary$\}$ and $T_1 = \{\beta \in T: \beta \notin T_2\}$ and if one rearranges the terms in (4.4) purely formally by grouping $Q(z; \beta)$

and $Q(z;-\bar{\beta})$ one obtains

$$(4.7) \qquad 1-\lambda\hat{u}(z) = Bz^p e^{icz}\left[\prod_{\beta\in T_1}\left(1-\frac{z}{\beta}\right)\left(1+\frac{z}{\bar{\beta}}\right)\right]\left[\prod_{\beta\in T_2}\left(1-\frac{z}{\beta}\right)\right]$$

where we define c by

$$(4.8) \qquad c = b - \sum_{\beta\in T_1}(2\operatorname{Im}\beta)|\beta|^{-2} - \sum_{\beta\in T_2}(\operatorname{Im}\beta)|\beta|^{-2}.$$

In order to justify the regrouping of terms in (4.4), to prove that the infinite product in (4.7) converges absolutely and uniformly on compact sets and to make sense of (4.8), one needs to know that c in (4.8) is finite. This follows from Theorem 2 on p. 225 of [8], which implies that

$$(4.9) \qquad \sum_{\beta\in S}|\operatorname{Im}\beta||\beta|^{-2} < \infty.$$

Notice that if one assumes that $f|[0,1]$ is C^1 and $f(1)\neq 0$, Theorem 2.3 and Proposition 2.1 give sharper information than (4.9). In fact Theorem 2.3 and Proposition 2.1 generalize to the case when $f|[0,1]$ is C^{n+1} and $f^{(n)}(1)\neq 0$ and again provide sharper results than (4.9).

The remainder of this section is devoted to showing that, under mild further assumptions on f, one has $c=\frac{1}{2}$ in (4.7) and, if

$$1-2\lambda\int_0^1 f(x)\,dx \neq 0, \qquad B = \sqrt{1-2\lambda\int_0^1 f(x)\,dx}.$$

For simplicity we shall eventually restrict ourselves to the case $f|[0,1]$ is C^1 and $f(1)\neq 0$, but the same formula will hold if $f|[0,1]$ is C^{n+1} and $f^{(n)}(1)\neq 0$.

LEMMA 4.1. *Let $f(x)$ be a real-valued, even integrable function such that $f(x)=0$ for $|x|>1$. Let λ be a real number, such that $1-\lambda\hat{f}(\xi)\geqslant 0$ for all real numbers ξ. If $1-\lambda\hat{f}(0)=0$, let $2p$ denote the multiplicity of 0 as a solution of $1-\lambda\hat{f}(z)=0$. Let $u=u_\lambda$ denote the unique solution of equations* (4.1) *and* (4.2) *and let B and c be constants as in* (4.7). *Then c is a real number. If*

$$1-\lambda\hat{f}(0)\neq 0, \qquad B = \sqrt{1-2\lambda\int_0^1 f(x)\,dx}.$$

In general, if $p > 0$, $(i)^p B > 0$, *where*

$$i = \sqrt{-1}, \qquad \text{and} \qquad B^2 = \left(-2\lambda \int_0^1 x^{2p} f(x)\,dx\right)\left(\frac{1}{(2p)!}\right).$$

PROOF. We have, from Section 1, the basic formula

$$1 - \lambda\hat{f}(z) = \big(1 - \lambda\hat{u}(z)\big)\big(1 - \lambda\hat{u}(-z)\big). \tag{4.10}$$

Furthermore, because $u(x)$ is real-valued we obtain

$$\overline{1 - \lambda\hat{u}(\xi)} = 1 - \lambda\hat{u}(-\xi), \qquad \xi \text{ a real number}. \tag{4.11}$$

By using (4.7) and (4.11) we find

$$\bar{B}\xi^p \exp(-i\bar{c}\xi) = B(-\xi)^p \exp(-ic\xi), \qquad -\infty < \xi < \infty. \tag{4.12}$$

By taking absolute values on both sides of (4.12), one derives that c is real and thus $\bar{B} = (-1)^p B$, so B is real for p even and pure imaginary for p odd.

If $p = 0$, the formula for B follows by setting $z = 0$ in equation (4.7) If $p > 0$, write the Taylor series about $z = 0$ for $1 - \lambda\hat{f}(z)$ and $1 - \lambda\hat{u}(z)$. By assumption, the coefficient of z^{2p} is the first nonzero term in the Taylor series for $1 - \lambda\hat{f}(z)$ and Bz^p is the first nonzero term in the Taylor series for $1 - \lambda\hat{u}(z)$. It follows from (4.10) that

$$(-1)^p B^2 = \big((2p)!\big)^{-1}\left(\frac{d^{2p}}{dz^{2p}}\right)\big(1 - \lambda\hat{f}(z)\big)\Big|_{z=0} = \tag{4.13}$$

$$= \big((2p)!\big)^{-1}\left(-\lambda\int_{-1}^{1}(ix)^{2p} f(x)\,dx\right).$$

Equation (4.13) gives the desired formula for B^2.

It remains to show that $(i)^p B > 0$ (which, together with (4.13), uniquely determines B). We assume that $1 - \lambda\hat{u}(z) \neq 0$ for $\operatorname{Im}(z) > 0$. Taking $z = ir$ for $r > 0$ and recalling that $1 - \lambda\hat{u}(ir)$ is real and $\lim_{r\to\infty} 1 - \lambda\hat{u}(ir) = 1$, we see that we must have

$$1 - \lambda\hat{u}(ir) > 0, \qquad r > 0. \tag{4.14}$$

For (4.14) to hold for $r > 0$ and r small, the corresponding inequality must hold for the first nonzero term in the Taylor series for $1 - \lambda\hat{u}(z)$. Thus we obtain

$$B(ir)^p > 0\,, \qquad r > 0 \tag{4.15}$$

which is the desired result. ■

Our next lemma is a simple exercise in integration by parts.

LEMMA 4.2. *Let $f(x)$ be a real-valued, even, L^1 function such that $f(x) = 0$ for $|x| > 1$. Assume there exist positive constants a and b such that*

$$|f(x)| \geqslant a(1-x)^n\,, \qquad 1-b \leqslant x \leqslant 1\,. \tag{4.16}$$

Then there exists a constant k such that for every real $r \geqslant 1$ one has

$$|1-\lambda\hat{f}(-ir)| \geqslant a|\lambda| n!\, r^{-n-1}e^r - ke^{r(1-b)}\,. \tag{4.17}$$

In particular, if $\lambda \neq 0$ and $\varepsilon > 0$ there exists a constant R_ε such that

$$|1-\lambda\hat{f}(-ir)| \geqslant e^{r(1-\varepsilon)}\,, \qquad r \geqslant R_\varepsilon\,. \tag{4.18}$$

PROOF. It suffices to prove (4.17). For $r \geqslant 1$ one has

$$\begin{aligned} |1-\lambda\hat{f}(-ir)| &= \left|1-\lambda\int_0^1 f(x)e^{rx}dx - \lambda\int_0^1 f(x)e^{-rx}dx\right| \geqslant \\ &\geqslant |\lambda|\left|\int_{1-b}^1 f(x)e^{rx}dx\right| - 1 - |\lambda|\left|\int_0^{1-b} f(x)e^{rx}dx\right| - |\lambda|\left|\int_0^1 f(x)dx\right| \geqslant \\ &\geqslant |\lambda| a\int_{1-b}^1 (1-x)^n e^{rx}dx - |\lambda|\left(e^{r(1-b)}+1\right)\int_0^1 |f(x)|dx\,. \end{aligned} \tag{4.19}$$

Equation (4.17) follows from (4.19) if one can prove that for $r \geqslant 1$ there exists a constant k_1 such that

$$\int_{1-b}^1 (1-x)^n e^{rx}dx = I_n \geqslant n!\, r^{-n-1}e^r - k_1 e^{r(1-b)}\,. \tag{4.20}$$

A simple integration by parts shows that

$$I_n = -b^n r^{-1}\exp(r-rb) + nr^{-1}I_{n-1} \tag{4.21}$$

and by using (4.21) repeatedly, one obtains (4.20). ■

We can now obtain a lower bound on the constant c in equation (4.7).

LEMMA 4.3. *Assume that $f(x)$ is an even, real-valued, integrable function such that $f(x) = 0$ for $|x| > 1$. Assume that there exist positive constants a and b and an integer n such that $|f(x)| \geqslant a(1-x)^n$ for $1-b \leqslant x \leqslant 1$. Let λ be a real number such that $1-\lambda\hat{f}(\xi) \geqslant 0$ for $-\infty < \xi < \infty$ and $u(x) = u_\lambda(x)$ be the corresponding real-valued solution of* (4.1) *and* (4.2). *Then the constant c in equations* (4.7) *satisfies $c \geqslant \frac{1}{2}$.*

PROOF. Recall that $T_2 = \{\beta : 1-\lambda\hat{f}(\beta) = 0,\ \beta$ is pure imaginary, and $\operatorname{Im}(\beta) < 0\}$ and $T_1 = \{\beta : 1-\lambda\hat{f}(\beta) = 0, \operatorname{Im}(\beta) \leqslant 0$ and $\beta \notin T_1\}$. Define $k(\beta)$ to be the multiplicity of β as a solution of $1-\lambda\hat{f}(\beta) = 0$ and $2p$ to be the multiplicity of 0 as a solution of $1-\lambda\hat{f}(z) = 0$ (possibly $p = 0$). We have seen that

$$1-\lambda\hat{u}(z) = Bz^p e^{icz}\left[\left(\prod_{\beta\in T_1}\left(1-\frac{z}{\beta}\right)\left(1+\frac{z}{\bar{\beta}}\right)\right)\right]\left[\prod_{\beta\in T_2}\left(1-\frac{z}{\beta}\right)\right] \quad \overset{\text{def}}{=} Bz^p e^{icz}P(z).$$

It is understood that if $\beta \in T_2$, the term $(1-z/\beta)$ is repeated $k(\beta)$ times. If $\beta \in T_1$ and β is real the term $(1-z/\beta)(1+z/\bar{\beta})$ is repeated $\frac{1}{2}k(\beta)$ times; otherwise it is repeated $k(\beta)$ times. The constant B is determined by Lemma 4.1 and c is real. Equation (4.10) implies that if $r > 0$ we have

$$|1-\lambda\hat{f}(-ir)| = |B|^2 r^{2p}|P(ir)P(-ir)|. \tag{4.22}$$

If z is pure imaginary a calculation gives

$$\begin{cases} \left|\left(1-\dfrac{z}{\beta}\right)\left(1+\dfrac{z}{\bar{\beta}}\right)\right| = |\beta|^{-2}|z-\beta|^2 \\ \left|\left(1-\dfrac{z}{\beta}\right)\right| = |\beta|^{-1}|z-\beta| \end{cases} \tag{4.23}$$

and it is clear that, if $\operatorname{Im}(\beta) \leqslant 0$, the right hand side of (4.23) has a value for $z = ir$, $r \geqslant 0$, which is greater than or equal to its value for $z = -ir$. It follows that $|P(ir)| \geqslant |P(-ir)|$ for $r \geqslant 0$ and using this fact we obtain from (4.22) that

$$|1-\lambda\hat{f}(-ir)| \leqslant |B|^2 r^{2p}|P(ir)|^2. \tag{4.24}$$

Equation (4.24) and Lemma 4.2 imply that given any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for $r \geqslant R_\varepsilon$ we have

$$\exp\left(r(1-\varepsilon)\right) \leqslant |1-\lambda\hat{f}(-ir)|. \tag{4.25}$$

Equations (4.24) and (4.25) together give (for $r \geqslant R_\varepsilon$)

$$|B|^{-1} r^{-p} \exp\left(\left(\tfrac{1}{2}\right) r(1-\varepsilon)\right) \leqslant |P(ir)| . \tag{4.26}$$

On the other hand we know that

$$|1-\lambda\hat{u}(ir)| = |B|^{1} r^{p} |P(ir)| e^{-cr} \tag{4.27}$$

and since $|1-\lambda\hat{u}(z)|$ is bounded for $\operatorname{Im}(z) \geqslant 0$, (4.26) and (4.27) together imply that we must have $c \geqslant (\frac{1}{2})(1-\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we have the desired result. ■

It remains to show that $c \leqslant \frac{1}{2}$. The next lemma is the crucial step.

LEMMA 4.4. *Let c_1 and d_1 be positive real numbers and for $n \geqslant N =$ a positive integer, let a_n be a complex number such that $|\operatorname{Re}(a_n)| \geqslant c_1 n$ and $|\operatorname{Im}(a_n)| \leqslant d_1 \log(n)$. Define an entire function $Q(z)$ by*

$$Q(z) = \prod_{n=N}^{\infty} \left(1 - \frac{z}{a_n}\right)\left(1 + \frac{z}{\bar{a}_n}\right). \tag{4.28}$$

Then there exist positive constants k_1 and k_2 and a positive number R such that

$$|Q(-ir)||Q(ir)|^{-1} \geqslant k_1 r^{-k_2}, \quad r \geqslant R . \tag{4.29}$$

PROOF. A calculation shows that if $a_n = \alpha_n + i\beta_n$ and r is real

$$\begin{aligned} |Q(-ir)||Q(ir)|^{-1} &= \prod_{n=N}^{\infty} |a_n - ir|^2 |a_n + ir|^{-2} \\ &= \prod_{n=N}^{\infty} \left(1 - (4\beta_n r)(\alpha_n^2 + (\beta_n + r)^2)^{-1}\right). \end{aligned} \tag{4.30}$$

For convenience we define $\varepsilon_n(r) = \varepsilon_n$ by

$$\varepsilon_n(r) = 4\beta_n r[\alpha_n^2 + \beta_n^2 + 2\beta_n r + r^2]^{-1} . \tag{4.31}$$

Notice that it suffices to prove Lemma 4.4 for a function $Q_1(z)$ defined by replacing the integer N in (4.28) by a larger integer N_1 (because a finite number of terms in the infinite product does not affect $|Q(ir)||Q(ir)|^{-1}$ for large r). Our first claim is that, possibly after increasing N, we can assume

$$|\varepsilon_n(r)| \leqslant \left(\tfrac{1}{2}\right) \tag{4.32}$$

for all $r > 0$ and for $n \geqslant N$. To prove (4.32), observe that $|\varepsilon_n(r)| \leqslant [r|\beta_n|] \cdot [r - 2|\beta_n|]^{-1}$, so (4.32) certainly holds if

$$4|\beta_n| \leqslant (\tfrac{2}{5})\, r\,. \tag{4.33}$$

In particular, because of the estimate on $|\beta_n|$, (4.32) will hold if $4d_1 \log(n) \leqslant (\frac{2}{5})r$. Therefore, we can assume that $10d_1 \log(n) \geqslant r$, and in this case we find

$$\varepsilon_n(r) \leqslant \big(4d_1 \log(n)\big)(10d_1 \log n)(c_1^2 n^2)^{-1}\,. \tag{4.34}$$

The right hand side of (4.34) will be less than $\frac{1}{2}$ for $n \geqslant N$ if N is originally chosen large enough.

In view of (4.32), the Taylor series for $\log(1-x)$ gives

$$\log\big(1 - \varepsilon_n(r)\big) = -\sum_{j=1}^{\infty} j^{-1}\big(\varepsilon_n(r)\big)^j \geqslant \geqslant -|\varepsilon_n(r)| \left(1 + \frac{|\varepsilon_n|}{2}\left(\sum_{k=0}^{\infty} (\tfrac{1}{2})^k\right)\right) \geqslant -\tfrac{3}{2}|\varepsilon_n(r)|\,. \tag{4.35}$$

Using (4.35) gives

$$\prod_{n=N}^{\infty}\big(1 - \varepsilon_n(r)\big) \geqslant \exp\left(-\tfrac{3}{2}\sum_{n=N}^{\infty}|\varepsilon_n(r)|\right). \tag{4.36}$$

It follows that we must estimate the summation on the right hand side of (4.36). First observe that for n large enough and any $r > 0$ one has

$$c_1 n^2 + \beta_n^2 + 2\beta_n r + r^2 \geqslant (\tfrac{1}{2})(c_1 n^2 + r^2)\,. \tag{4.37}$$

If $\beta_n \geqslant 0$, (4.37) is obvious, and if $\beta_n < 0$, (4.37) is equivalent to (writing $\varepsilon = \sqrt{2}/2$)

$$(\varepsilon r - |\beta_n|\varepsilon^{-1})^2 - \beta_n^2 + (\tfrac{1}{2})c_1 n^2 \geqslant 0\,. \tag{4.38}$$

Since $|\beta_n| \leqslant d_1 \log(n)$, (4.38) is true for n large, and by increasing N, we can assume it true for $n \geqslant N$. Using (4.37) we have the estimate

$$\sum_{n=N}^{\infty}|\varepsilon_n(r)| \leqslant 8d_1 r \sum_{n=N}^{\infty}(\log n)(c_1 n^2 + r^2)\,. \tag{4.39}$$

The summation in (4.39) can be divided into two parts, those terms for which $n \leqslant r$ and those for which $n > r$. For the first terms we have

$$\sum_{N \leqslant n \leqslant r} \frac{\log n}{c^2 n^2 + r^2} \leqslant \sum_{N \leqslant n \leqslant r} \frac{\log r}{r^2} \leqslant \frac{\log r}{r}\,. \tag{4.40}$$

For the second group of terms, we note that the function $(\log x)(c_1x^2 + r^2)$ is monotonic decreasing for $x \geqslant r$ if r is large enough $\big(2c_1 \log r > (c_1 + 1)$ insures that r is large enough$\big)$. Thus for r large enough we have

$$(4.41) \qquad \sum_{n>|r|} (\log n)(c_1 n^2 + r^2) < \int_r^\infty (\log x)(c_1 x^2 + r^2)^{-1} dx = I\,.$$

If we make the substitution $x = ru$ in the integral I in (4.41) we obtain

$$(4.42) \qquad I = \left(\frac{1}{r}\right)\int_1^\infty [\log r + \log u][c_1 u^2 + 1]^{-1} du = (c_2 \log r) r^{-1} + c_3 r^{-1}\,.$$

Formulas (4.39), (4.40) and (4.42) give (for r large enough)

$$(4.43) \qquad \sum_{n=N}^{\infty} |\varepsilon_n(r)| \leqslant 8d_1(\log r + c_2 \log r + c_3)$$

and (4.43) and (4.36) give the lemma. ■

We can now give our explicit formula for $1 - \lambda\hat{u}_\lambda(z)$.

THEOREM 4.1. *Let $f(x)$ be an even, continuous, real-valued function such that $f(x) = 0$ for $|x| > 1$. Assume that $f|[0, 1]$ is continuously differentiable and $f(1) \neq 0$. Let λ be a real number such that $1 - \lambda\hat{f}(\xi) \geqslant 0$ for* **all** *real ξ and let $u(x) = u_\lambda(x)$ be the unique continuous, real-valued solution of equations* (4.1) *and* (4.2) $\big(u(x)=0$ *for* $x \notin [0, 1]\big)$. *Define $T = \{\beta \in \mathbf{C} - \{0\} : \operatorname{Im}(\beta) \leqslant 0,\ 1 - \lambda\hat{f}(\beta) = 0\}$, $T_2 = \{\beta \in T : \beta$ is pure imaginary$\}$, $T_1 = \{\beta \in T : \beta \notin T_2\}$ and $k(\beta) =$ multiplicity of $\beta \in T$ as a zero of $1 - \lambda\hat{f}(z) = 0$. Then we have*

$$(4.44) \qquad 1 - \lambda\hat{u}(z) = Bz^p \exp\big((\tfrac{1}{2})iz\big)\left[\prod_{\beta\in T_1}\left(1 - \frac{z}{\beta}\right)\left(1 + \frac{z}{\bar\beta}\right)\right]\left[\prod_{\beta\in T_2}\left(1 - \frac{z}{\beta}\right)\right].$$

The term $(1 - z/\beta)(1 + z/\bar\beta)$ is repeated $k(\beta)$ times for β not real and $\frac{1}{2}k(\beta)$ times for β real; the term $(1 - z/\beta)$ in the second product is repeated $k(\beta)$ times. The multiplicity of 0 as a solution of $1 - \lambda\hat{f}(z) = 0$ is $2p$ (possibly $p = 0$). If $1 - \lambda\hat{f}(0) > 0$, we have $B = \sqrt{1 - \lambda\hat{f}(0)}$; otherwise

$$i^p B > 0 \qquad \textit{and} \qquad B^2 = \left(-2\lambda\int_0^1 x^{2p} f(x)\,dx\right)\left(\frac{1}{(2p)!}\right).$$

REMARK 4.1. Formula (4.44) is also valid if $f|[0,1]$ is C^{n+1} for some $n\geqslant 1$ and $f^{(n)}(1)\neq 0$, but the proof requires establishing analogues of Theorem 2.3 and Proposition 2.1 which we have omitted for reasons of length.

PROOF. Because of the previous lemmas it only remains to show that the constant c in (4.7) satisfies $c\leqslant\frac{1}{2}$. According to Theorem 2.3 there exists a positive integer N such that for every integer $n\geqslant N$ the equation $1-\lambda\hat{f}(z)=0$ has precisely two solutions in $U_n=\{z\colon 2n\pi-\pi\leqslant \operatorname{Re}(z)\leqslant 2n\pi+\pi\}$, and these solutions are not real, do not lie in ∂U_n and are complex conjugates of each other. Thus let a_n denote the unique solution of $1-\lambda\hat{f}(z)=0$ such that $a_n\in U_n$ and $\operatorname{Im}(a_n)<0$. Proposition 2.1 insures that there is a constant $c_1>0$ such that $|\operatorname{Im}(a_n)|\leqslant c_1\log(n)$. Define entire functions $P(z)$ and $Q(x)$ by

$$P(z)=\left[\prod_{\beta\in T_1}\left(1-\frac{z}{\beta}\right)\left(1+\frac{z}{\bar{\beta}}\right)\right]\left[\prod_{\beta\in T_2}\left(1-\frac{z}{\beta}\right)\right]$$
$$Q(z)=\prod_{n=N}^{\infty}\left(1-\frac{z}{a_n}\right)\left(1+\frac{z}{\bar{a}_n}\right).$$

It is easy to see that (for $r>0$)

$$\lim_{r\to+\infty}|P(-ir)P(ir)^{-1}Q(-ir)^{-1}Q(ir)|=1$$

so Lemma 4.4 implies that there exist positive constants k_1 and k_2 and a positive number R such that

$$|P(-ir)P(ir)^{-1}|\geqslant k_1r^{-k_2},\qquad r\geqslant R. \tag{4.45}$$

It follows from (4.10) and (4.45) that for $r\geqslant R$

$$|1-\lambda\hat{f}(ir)|=|B|^2r^{2p}|P(ir)||P(-ir)|\geqslant|B|^2kr^{2p-k_2}|P(ir)|^2. \tag{4.46}$$

Equation (4.46) implies that

$$|P(ir)|\leqslant ar^b\exp\big((\tfrac{1}{2})r\big),\qquad r\geqslant R_1>0 \tag{4.47}$$

where a, b and R_1 are positive constants. We have used here the fact that $|1-\lambda\hat{f}(ir)|$ is dominated by a constant multiple of e^r for r large. Using (4.47) we have

$$|1-\lambda\hat{u}(ir)|=|B|r^{2p}e^{-cr}|P(ir)|\leqslant|B|ar^{2p+b}\exp\big((\tfrac{1}{2})r-cr\big),\qquad r\geqslant R_1. \tag{4.48}$$

The left hand side of (4.48) approaches 1 as r approaches $+\infty$, so the constant c in (4.48) must satisfy $c \leqslant \frac{1}{2}$. This completes the proof. ■

5. – The spectrum of a linear operator.

In the previous sections we have sought solutions $u \in C[0,1]$ of $u = F_\lambda(u)$, where the operator F_λ is defined by

$$(5.1) \qquad (F_\lambda u)(x) = f(x) + \lambda \int_x^1 u(y)u(y-x)\,dy\,, \qquad 0 \leqslant x \leqslant 1$$

and $f(x)$ is a given continuous function. One can easily verify that the Fréchet derivative of F_λ at u is the linear operator $L\colon C[0,1] \to C[0,1]$ defined by

$$(5.2) \qquad (Lh)(x) = \lambda \int_x^1 u(y)h(y-x)\,dy + \lambda \int_x^1 u(y-x)h(y)\,dy$$
$$= \lambda \int_0^{1-x} u(y+x)h(y)\,dy + \lambda \int_x^1 u(y-x)h(y)\,dy\,.$$

It is part of the folklore of the subject that F_λ is *not* compact as a map of $C[0,1]$ into itself (recall that a map is compact if it is continuous and takes bounded sets to sets with compact closure); to see this just observe that the image of $S = \{u_n\colon u_n(x) = \sin n\pi x,\ n \geqslant 1\}$ under F_λ is not equicontinuous. However, Ramalho [12] has observed that F_λ is compact if it is viewed as a map of $C^1[0,1]$ into $C^1[0,1]$ (assuming $f(x)$ is C^1). Furthermore, by using (5.2) one can show that L takes bounded sets in $C[0,1]$ into equicontinuous sets, so L is compact as a map of $C[0,1]$ into itself (see [10] for details). Also, it is not hard to show that the equation (5.2) defines a compact linear map L of $L^2[0,1]$ into $L^2[0,1]$ (assuming only that $u \in L^2[0,1]$). It follows that the spectrum of L as a map of $L^2[0,1]$ into itself consists only of 0 and point spectrum and that if $z = 0$ is an eigenvalue of L, the algebraic multiplicity of $z = \dim\{h \in L^2[0,1]\colon (z-L)^m h = 0$ for some $m \geqslant 1\}$ is finite. A similar statement holds for L as a map of $C[0,1]$ into itself. Note that all Banach spaces here consist of complex valued functions.

We are interested in the spectrum of L as a map of $C[0,1]$ into itself, but it will be convenient to work in $L^2[0,1]$. Our first lemma shows that the spectrum is the same in either Banach space.

LEMMA 5.1. *Assume that $u(x)$ is a continuous function and let $L\colon L^2[0,1] \to \to L^2[0,1]$ be defined by (5.2). If $z \neq 0$ is an eigenvalue of algebraic multiplicity m for L considered as a map of L^2 into itself, it is also an eigenvalue of algebraic multiplicity m for L considered as a map of $C[0,1]$ into itself.*

PROOF. It suffices to show that if $h \in L^2[0,1]$ and $(z-L)^k(h)$ is a continuous function for some integer k, then h is a continuous function. Suppose we have proved this for $k=1$ and we have

$$(z-L)^k h = g \in C[0,1] \tag{5.3}$$

for some $h \in L^2$. Then by applying the result for $k=1$ we find that $(z-L)^{k-1}h$ is a continuous function, and after k steps we find h is a continuous function. Thus we can assume $k=1$ in (5.3). We have

$$h(x) = z^{-1}g(x) + z^{-1}\lambda\int_0^{1-x} u(y+x)h(y)\,dy + z^{-1}\lambda\int_x^1 u(y-x)h(y)\,dy \tag{5.4}$$

so the Cauchy-Schwartz inequality implies $|h(x)| \leqslant B < \infty$ (this is true even if $u \in L^2$ but u is not continuous). If $0 \leqslant x_1 \leqslant x_2 \leqslant 1$, equation (5.4) implies

$$\begin{aligned} |h(x_1)-h(x_2)| \leqslant |z|^{-1}|g(x_1)-g(x_2)| &+ |z^{-1}\lambda|\int_0^{1-x_2} |u(y+x_2)-u(y+x_1)|B\,dy \,+ \\ &+ |z^{-1}\lambda|\int_{1-x_2}^{1-x_1} |u(y+x_1)|B\,dy + |z^{-1}\lambda|\int_{x_1}^{x_2} |u(y-x_1)|B\,dy \,+ \\ &+ |z^{-1}\lambda|\int_{x_2}^{1} |u(y-x_2)-u(y-x_1)|B\,dy\,. \end{aligned} \tag{5.5}$$

The continuity of u shows that given $\varepsilon > 0$, there is a $\delta > 0$ such that the right hand side of (5.5) is less than ε whenever $|x_1 - x_2| < \delta$. This proves that h is continuous. ■

Suppose now that $u, h \in L^2[0,1]$. Extend u and h to be zero outside $[0,1]$, and, as usual, if v is a function, define $\tilde{v}(x) = v(-x)$. Then one can see from (5.2) that

$$h_1(x) = (Lh)(x) = \lambda[(\tilde{u} * h)(x) + (u * \tilde{h})(x)]\,, \qquad 0 \leqslant x \leqslant 1\,. \tag{5.6}$$

The right hand side of (5.6) is defined for all x (since u and h are defined for all x) and is an even function which vanishes for $|x|>1$. It follows that if we define $h_1(x)=(Lh)(x)$ for $0\leqslant x\leqslant 1$ and $h_1(x)=0$ for $x\notin[0,1]$, then we have for all x

$$(5.7)\qquad \begin{aligned} &h_1(x)+\tilde{h}_1(x)=\lambda[(\tilde{u}*h)(x)+(u*\tilde{h})(x)]\,, \qquad -\infty<x<\infty \\ &u(x)\,,\quad h(x)\,,\quad h_1(x)=0 \quad \text{for } x\notin[0,1]\,. \end{aligned}$$

If $u,h\in L^2[0,1]$ and $Lh=\mu h$ for some $\mu\in\mathbf{C}$, (5.7) implies

$$(5.8)\qquad \begin{aligned} &\mu\big(h(x)+\tilde{h}(x)\big)=\lambda[(\tilde{u}*h)(x)+(u*\tilde{h})(x)]\,, \qquad -\infty<x<\infty \\ &u(x)\,,\quad h(x)=0 \quad \text{for } x\notin[0,1]\,. \end{aligned}$$

Taking the Fourier transform and evaluating at $z\in\mathbf{C}$ gives

$$(5.9)\qquad \mu\big(\hat{h}(z)+\hat{h}(-z)\big)=\lambda[\hat{u}(-z)\hat{h}(z)+\hat{u}(z)\hat{h}(-z)]\,.$$

The right hand side of (5.9) can be written in the following form

$$(5.10)\qquad \mu[\hat{h}(z)+\hat{h}(-z)]=\left(\frac{\lambda}{2}\right)[\hat{u}(z)+\hat{u}(-z)][\hat{h}(z)+\hat{h}(-z)]- \\ -\left(\frac{\lambda}{2}\right)[\hat{u}(z)-\hat{u}(-z)][\hat{h}(z)-\hat{h}(-z)].$$

Notice that if $\hat{u}(z)=\hat{u}(-z)$ and $\hat{h}(z)+\hat{h}(-z)\neq 0$ equation (5.10) implies that

$$(5.11)\qquad \mu=\left(\frac{\lambda}{2}\right)[\hat{u}(z)+\hat{u}(-z)]=\lambda\hat{u}(z)\,.$$

In fact (5.11) will hold under slightly less restrictive assumptions. Let $\theta(\zeta)=\hat{u}(\zeta)-\hat{u}(-\zeta)$ and suppose $\theta(z)=0$ and z is a zero of θ of multiplicity m. Suppose that h satisfies (5.10) and $(d/dz)^j\big(\hat{h}(z)+\hat{h}(-z)\big)\neq 0$ for some $j<m$, but that all derivatives of $\hat{h}(\zeta)+\hat{h}(-\zeta)$ of order less than j vanish at $\zeta=z$. Then by differentiating (5.10) j times one can still see that (5.11) holds.

On the basis of the above calculations one might conjecture that the point spectrum of L consists precisely of those points μ given by (5.11) for which $\hat{u}(z)=\hat{u}(-z)$. The remainder of this section is devoted to proving this conjecture. The basic tool we shall use is the following theorem, whose proof we defer to the end of this section.

THEOREM 5.1. *Let $u\colon[0,1]\to\mathbf{R}$ be a real-valued, continuously differentiable function such that $u(1)\neq 0$. Extend $u(x)=0$ for $x\notin[0,1]$, define*

$\theta(z) = \hat{u}(z) - \hat{u}(-z)$ and let $S = \{z \in \mathbf{C}: \theta(z) = 0\}$. For each $z \in S$, let $m(z)$ denote the multiplicity of z as a solution of $\theta(z) = 0$ and let $m_1(z) = m(z)$ for $z \neq 0$ and $m_1(z) = m(z) + 1$ for $z = 0$. Define A by $A = \{x^j e^{zx}: -1 \leqslant x \leqslant 1, \theta(z) = 0$ and $0 \leqslant j < m_1(z)\}$. Then the closed linear span of A in $L^2[-1,1]$ is all of $L^2[-1,1]$. If any element of A is removed to give a set A_1, the closed linear span of A_1 in $L^2[-1,1]$ is not all of $L^2[-1,1]$.

Most of our work will involve proving Theorem 5.1. With it we can easily analyze the spectrum of L. We begin with the following lemma.

LEMMA 5.2. *Let notation and assumptions be as in Theorem* 5.1. *For a fixed $\zeta \in S$, define V to be the orthogonal complement in $L^2[-1,1]$ of the collection of functions $\{x^j e^{izx}: z \in S, 0 \leqslant j < m_1(z), z \neq \pm\zeta\} = B$. Then V is finite dimensional and* $\dim(V) = 2m(\zeta)$ *if $\zeta \neq 0$ or* $\dim(V) = m(\zeta) + 1$ *if $\zeta = 0$. Define $V_e = \{g \in V: g(-x) = g(x), -1 \leqslant x \leqslant 1\}$ and $V_0 = \{g \in V: g(-x) = -g(x), -1 \leqslant x \leqslant 1\}$. Then V is the orthogonal direct sum of V_e and V_0. If $\zeta \neq 0$,* $\dim V_e = \dim V_0 = m(\zeta)$ *and if $\zeta = 0$, $m(\zeta)$ is odd and* $\dim(V_e) = (\frac{1}{2})(m(\zeta) + 1)$.

PROOF. Define M to be the closed linear span of B. Basic functional analysis implies that $\dim V = \operatorname{codim}(M)$, and $\operatorname{codim}(M) = \dim(F)$ where F is any finite dimensional subspace of $L^2[-1,1]$ such that $F \cap M = \{0\}$ and $F + M = L^2[-1,1]$. If we take $F =$ linear span of $\{x^j \exp(\pm i\zeta x): 0 \leqslant j < m_1(\zeta)\}$, it is clear that the dimension of F is $2m(\zeta)$ if $\zeta \neq 0$ and $m_1(\zeta)$ if $\zeta = 0$ (since the functions $x^j \exp(\pm i\zeta x)$ are linearly independent). Theorem 5.1 implies that $F + M = L^2[-1,1]$; and if $F \cap M$ contained a nonzero element, it would follow that $L^2[-1,1]$ would be the closed linear span of some proper subset of A, which would contradict Theorem 5.1.

If $h(x) = x^j e^{izx}$, $z \in S$, $0 \leqslant j < m(z)$, $z \neq \pm\zeta$, so $h \in M$, notice that $g(x) = h(-x)$ is also an element of M (because $-z \in S$ if $z \in S$). Since elements of M are limits of finite linear combinations of such functions, it follows that if $h \in M$, $g \in M$, where $g(x) = h(-x)$.

If $f \in V$, so f is orthogonal to every element of M, and if $f_1(x) = f(-x)$, then $f_1 \in V$. To see this, take an arbitrary $h \in V$ and observe that

$$\int_{-1}^{1} f_1(x) h(x)\, dx = \int_{-1}^{1} f(x) h(-x)\, dx = 0$$

since $g(x) = h(-x)$ is an element of M. It follows that if $f \in V$, then $f_1(x) =$ odd part of $f = \frac{1}{2}\big(f(x) - f(-x)\big)$ is an element of V and $f_2(x) =$ $=$ even part of $f = \frac{1}{2}\big(f(x) + f(-x)\big)$ is an element of V. This shows that $V = V_e + V_0$, and since any even function in $L^2[-1,1]$ is orthogonal to any odd function, V_e is orthogonal to V_0.

It remains to prove the assertion about $\dim(V_e)$. Define $F =$ linear span $\{x^j \exp(\pm i\zeta x): 0 \leqslant j < m_1(\zeta)\}$. It is clear that if $f \in F$, then $g \in F$, where $g(x) = f(-x)$, and thus the even and odd parts of f are also elements of F. Define F_e to be the set of even functions in F and F_0 the set of odd functions in F. If $\zeta \neq 0$, one can see that $\dim(F_e) = \dim(F_0)$. If $\zeta = 0$, observe that $\theta^{(j)}(0) = 0$ for any even integer j ($\theta(z)$ as in Theorem 5.1), so the multiplicity of 0 as a solution of $\theta(0) = 0$ must be odd. Since $m(\zeta)$ is odd for $\zeta = 0$ one also sees that $\dim(F_e) = \frac{1}{2}(m(\zeta) + 1)$ for $\zeta = 0$.

Our claim is that $\dim(V_e) = \dim(F_e)$. Write $n = \dim(F_e)$, $\dim(V_e) = p$ and $\dim(V_0) = q$. We shall suppose that $p < n$ and obtain a contradiction. Let $u_1, u_2, \ldots, u_p$ be an orthonormal basis of V_e, $v_1, v_2, \ldots, v_q$ be an orthonormal basis of V_0 and $r_1, \ldots, r_n$ be an orthonormal basis of F_e. If $(\cdot,\cdot)$ denotes the inner product in $L^2[-1, 1]$ and if we recall that odd functions and even functions are orthogonal, we obtain (for suitable elements m_j of M)

$$
\begin{aligned}
r_j &= \sum_{i=1}^{p} a_{ij} u_i + \sum_{i=1}^{q} b_{ij} v_i + m_j, \qquad 1 < j < n \\
a_{ij} &= (r_j, u_i), \qquad b_{ij} = (r_j, v_i) = 0, \qquad m_j \in M. \qquad (5.12)
\end{aligned}
$$

Since we are assuming that $p < n$, there are constants $d_1, d_2, \ldots, d_n$, not all zero, such that

$$
(a_{ij}) \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (5.13)
$$

where matrix multiplication is indicated in (5.13). Equation (5.13) implies that

$$
\sum_{j=1}^{n} d_j r_j = \sum_{j=1}^{n} d_j m_j \in M. \qquad (5.14)
$$

Since the left hand side of (5.14) is a nonzero element of F, we have a contradiction, and we conclude that $\dim(V_e) \geqslant \dim(F_e)$. An exactly analogous proof shows that $\dim(V_0) \geqslant \dim(F_0)$. However, we know that

$$
\dim(V_e) + \dim(V_0) = \dim(F) = \dim(F_e) + \dim(F_0) \qquad (5.15)
$$

so we conclude that $\dim(V_e) = \dim(F_e)$ and $\dim(V_0) = \dim(F_0)$. ■

With these preliminaries we can establish our main theorem.

THEOREM 5.2. *Suppose that* $u\colon [0,1] \to \mathbb{R}$ *is a continuously differentiable function such that* $u(1) \neq 0$ *and for* $\lambda \in \mathbb{R}$ *define a linear operator* $L\colon C[0,1] \to \to C[0,1]$ *by equation* (5.2). *Extend* $u(x)$ *by* $u(x) = 0$ *for* $x \notin [0,1]$, *define* $\theta(z) = \hat{u}(z) - \hat{u}(-z)$; *and if* $\theta(z) = 0$, *define* $m(z)$ *to be the multiplicity of* z *as a solution of* $\theta(z) = 0$. *Then* L *is a compact map and the point spectrum of* L *consists of those numbers* μ *such that* $\mu = \lambda\hat{u}(z)$ *and* $\theta(z) = 0$. *If* $\mu \neq 0$ *and* μ *is in the point spectrum of* L *and if we write* $S(\mu) = \{z : \theta(z) = 0$ *and* $\lambda\hat{u}(z) = \mu\}$ *and* $T(\mu) = \{z \in S(\mu)\colon \mathrm{Re}\,(z) > 0$ *or* $\mathrm{Re}\,(z) = 0$ *and* $\mathrm{Im}\,(z) \geqslant 0\}$, *then the algebraic multiplicity of* μ *as an eigenvalue of* L *is finite and equals* $\sum_{z \in T(\mu)} n(z)$, *where* $n(z) = m(z)$ *if* $z \neq 0$ *and* $n(z) = (\frac{1}{2})\big(m(z) + 1\big)$ *if* $z = 0$. *If* $u(x) \geqslant 0$ *for* $0 \leqslant x \leqslant 1$, *then* $\lambda\hat{u}(0)$ *is an eigenvalue of* L *of algebraic multiplicity* 1 *and* $|\lambda|\hat{u}(0)$ *is the spectral radius of* L.

PROOF. The fact that L is compact is proved in [10], and Lemma 5.1 shows that we can view L as a map of $L^2[0,1]$, into itself. If h is any element of $L^2[0,1]$, define h on all of $\mathbb{R}$ by $h(x) = 0$ for $x \notin [0,1]$, define $\tilde{h}(x) = h(-x)$ and let $\hat{h}(z)$ denote the Fourier transform for $z \in \mathbb{C}$. If $\theta(\zeta) = 0$, define W_ζ by

$$W_\zeta = \Big\{h \in L^2[0,1]\colon \left(\frac{d^j}{dz^j}\right)\big(\hat{h}(z) + \hat{h}(-z)\big) = 0 \quad \text{for } z \text{ with } \theta(z) = 0,\ z \neq \pm\zeta, \text{ and } 0 \leqslant j < m_1(z)\Big\},$$

where $m_1(z)$ is as defined in Theorem 5.1. If $h \in L^2[0,1]$, define $Eh = g \in \in L^2[-1,1]$ by $g(x) = h(x)$ for $0 \leqslant x \leqslant 1$ and $g(-x) = g(x)$. It is easy to see that $E(W_\zeta)$ is just the set of even functions in $L^2[-1,1]$ which are orthogonal to the set of functions

$$B = \{x^j e^{izx}, -1 \leqslant x \leqslant 1 : z \neq \pm\zeta,\ \theta(z) = 0,\ 0 \leqslant j < m_1(z)\}\,.$$

Conversely, any such even function gives rise to an element of W_ζ by restriction to $[0,1]$. Thus Lemma 5.2 implies that $\dim(W_\zeta) = n(\zeta)$.

We next claim that $L(W_\zeta) \subset W_\zeta$. To see this, take $h \in W_\zeta$, define $h_1 = Lh$ and define both functions to be 0 outside $[0,1]$. Taking the Fourier transform of (5.7) for complex z gives

$$\hat{h}_1(z) + \hat{h}_1(-z) = \left(\frac{\lambda}{2}\right)[\hat{u}(z) + \hat{u}(-z)][\hat{h}(z) + \hat{h}(-z)] - \left(\frac{\lambda}{2}\right)\theta(z)[\hat{h}(z) - \hat{h}(-z)]\,. \tag{5.16}$$

One can see from (5.16) that if

$$\theta(z_0) = 0 \qquad \text{and} \qquad \left(\frac{d}{dz}\right)^j (\hat{h}(z) + \hat{h}(-z))\Big|_{z=z_0} = 0 \qquad \text{for } 0 \leqslant j < m(z_0)\,,$$

then

$$\left(\frac{d}{dz}\right)^j (\hat{h}_1(z) + \hat{h}_1(-z))\Big|_{z=z_0} = 0 \qquad \text{for } 0 \leqslant j < m(z_0)\,.$$

If $\zeta \neq 0$ and $z_0 = 0$ in (5.16), we also have to show that

$$\left(\frac{d}{dz}\right)^j (\hat{h}_1(z) + \hat{h}_1(-z))\Big|_{z=0} = 0\,, \qquad j = m(0)\,.$$

However $m(0)$ is odd and $\hat{h}_1(z) + \hat{h}_1(-z)$ is an even function of z, so the equality holds automatically.

If $h \in W_\zeta$ and $h \neq 0$, we must have

$$\left(\frac{d}{dz}\right)(\hat{h}(z) + \hat{h}(-z))\Big|_{z=\zeta} \neq 0$$

for some j with $0 \leqslant j < m(\zeta)$. Otherwise, Theorem 5.1 implies that $Eh = 0$. Notice that the above inequality also holds for $\zeta = 0$. Using this fact and equation (5.10) we conclude that if $Lh = \mu h$ for some $h \in W_\zeta$ then $\mu = \lambda\hat{u}(\zeta)$. Since W_ζ is a complex vector space of dimension $m_1(\zeta)$, we conclude that $\lambda\hat{u}(\zeta)$ is the only eigenvalue of $L|W_\zeta$ and that it has algebraic multiplicity $m_1(\zeta)$ as an eigenvalue of $L|W_\zeta$.

We have already seen that if $h \in L^2[0, 1]$ is an eigenvector of L with eigenvalue μ and if $(d/dz)^j(\hat{h}(z) + \hat{h}(-z)) \neq 0$ for some z with $\theta(z) = 0$ and $0 \leqslant j < m(z)$, then $\mu = \lambda\hat{u}(z)$. On the other hand, if $(d/dz)^j(\hat{h}(z) + \hat{h}(-z)) = 0$ for all pairs (j, z) with $\theta(z) = 0$ and $0 \leqslant j < m(z)$, then Eh is orthogonal to $\{x^j e^{izx}, -1 \leqslant x \leqslant 1 : \theta(z) = 0,\ 0 \leqslant j < m_1(z)\}$ and Theorem 5.1 implies $Eh = 0$. Thus we have found all eigenvalues.

It remains to prove the statement about the algebraic multiplicity of a nonzero eigenvalue μ of L. Since L is compact, the multiplicity of μ is finite; and standard functional analysis implies that there exist a finite dimensional subspace F of $L^2[0, 1]$, $F = \{h \in L^2[0,1] : (\mu - L)^m h = 0$ for some $m \geqslant 0\}$, and a closed subspace $G \subset L^2[0, 1]$ such that $F \cap G = \{0\}$, $F + G = L^2[0, 1]$, $L(F) \subset F$, $L(G) \subset G$, the spectrum of $L|F$ equals $\{\mu\}$ and the spectrum of $L|G$ does not contain μ. The latter conditions imply that if $\theta(\zeta) = 0$ and $\mu = \lambda\hat{u}(\zeta)$, then $W_\zeta \subset F$ (because we have already seen that the spectrum of $L|W$ is $\{\mu\}$). Notice that if $\theta(\zeta_1) = \theta(\zeta_2) = 0$ and $\zeta_1 \neq \pm\zeta_2$ then $W_{\zeta_1} \cap W_{\zeta_2} = \{0\}$ (because their images under E are orthogonal in

$L^2[-1, 1]$). Since $-\zeta \notin T(\mu)$ if $\zeta \in T(\mu)$ and $\zeta \neq 0$, we conclude that $T(\mu)$ has only a finite number of elements, say $\zeta_1, \zeta_2, \ldots, \zeta_k$ (otherwise, $\dim F = \infty$). If we define W by

$$W = W_{\zeta_1} + W_{\zeta_2} + \ldots + W_{\zeta_k}$$

we have seen that this is a direct sum, so $\dim(W) = \sum_{\zeta \in T(\mu)} m_1(\zeta)$, and $W \subset F$.

To complete the proof it suffices to show that $W = F$. Suppose not, so there exists $h \in L^2[0, 1]$ such that $(\mu - L)^m(h) = 0$ for some integer m, but $(d/dz)^j(\hat{h}(z) + \hat{h}(-z)) \neq 0$ for $z = \zeta_p$ and $0 \leqslant j < m(\zeta_p)$, $1 \leqslant p \leqslant k$.

According to Theorem 5.1, there must exist z_0 with $\theta(z_0) = 0$ such that $z_0 \neq \zeta_p$ for $1 \leqslant p \leqslant k$, and a smallest integer $j < m(z_0)$ such that

$$\left(\frac{d}{dz}\right)^j (\hat{h}(z) + \hat{h}(-z))\Big|_{z=z_0} \neq 0 . \tag{5.17}$$

As usual, $h(x)$ in (5.17) has been extended to be 0 for $x \notin [0, 1]$. Define $h_1 = (\mu - L)(h)$ and define $h_1(x) = 0$ for $x \notin [0, 1]$. We claim that

$$\left(\frac{d}{dz}\right)^j (\hat{h}_1(z) + \hat{h}_1(-z))\Big|_{z=z_0} \neq 0 \tag{5.18}$$

where j is as in (5.17). In order to prove (5.18), observe that (writing $\tilde{u}(x) = u(-x)$, etc.)

$$h_1(x) + h_1(-x) = \mu h(x) + \mu h(-x) - \lambda(u * \tilde{h})(x) - \lambda(\tilde{u} * h)(x) . \tag{5.19}$$

Taking Fourier transforms gives

$$\begin{aligned} \hat{h}_1(z) + \hat{h}_1(-z) &= \mu(\hat{h}(z) + \hat{h}(-z)) - \\ &- \left(\frac{\lambda}{2}\right)[\hat{u}(z) + \hat{u}(-z)][\hat{h}(z) + \hat{h}(-z)] + \left(\frac{\lambda}{2}\right)[\hat{u}(z) - \hat{u}(-z)][\hat{h}(z) - \hat{h}(-z)] . \end{aligned} \tag{5.20}$$

Taking the j-th derivative of both sides of (5.20), evaluating at $z = z_0$, and using the minimal nature of $j < m(z_0)$, one obtains

$$\left(\frac{d}{dz}\right)^j (\hat{h}_1(z) + \hat{h}_1(-z))\Big|_{z=z_0} = [\mu - \lambda\hat{u}(z_0)]\left(\frac{d}{dz}\right)^j (\hat{h}(z) + \hat{h}(-z))\Big|_{z=z_0} \neq 0 . \tag{5.21}$$

Here we used the fact that $\mu \neq \lambda\hat{u}(z_0)$. It is also clear that

$$\left(\frac{d}{dz}\right)^p (\hat{h}_1(z) + \hat{h}_1(-z))\Big|_{z=z_0} = 0 \qquad \text{for } p < j$$

(a vacuous condition if $j = 0$).

The above argument can be applied to $h_2 = (\mu - L)h_1 = (\mu - L)^2 h$ and proves that

$$(5.22)\qquad \left(\frac{d}{dz}\right)^j (\hat{h}_2(z) + \hat{h}_2(-z))\Big|_{z=z_0} = 0$$

but that any lower order of differentiation in (5.22) gives zero. By induction we prove (the argument is always the same) that if $h_p = (\mu - L)^p h = (\mu - L)h_{p-1}$, then

$$(5.23)\qquad \left(\frac{d}{dz}\right)^j (\hat{h}_p(z) + \hat{h}_p(-z))\Big|_{z=z_0} \neq 0$$

where j is as in (5.17). Since $h_m = (\mu - L)^m h = 0$, equation (5.23) gives a contradiction and the original assumption that $F \neq W$ is wrong. It follows that $\dim(F)$, which is the algebraic multiplicity of μ as an eigenvalue of L, equals $\dim(W)$.

It only remains to prove that if $u(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$, then $\mu = \lambda\hat{u}(0)$ has absolute value equal to the spectral radius and that the algebraic multiplicity of μ in this case is 1. To see this, suppose μ_1 is another eigenvalue of L. By our previous comments, $\mu_1 = \lambda\hat{u}(z)$, where $\hat{u}(z) = \hat{u}(-z)$ and we can assume $z = \alpha + i\beta$ with $\beta \geqslant 0$. Since u is nonnegative and not identically zero we have

$$(5.24)\qquad \begin{aligned} |\mu| &= |\lambda|\int_0^1 u(x)\,dx \\ |\mu_1| &= |\lambda|\left|\int_0^1 u(x)\,e^{-\beta x}e^{i\alpha x}dx\right| . \end{aligned}$$

If $\beta > 0$, we clearly have $|\mu_1| < |\mu|$ just by taking the absolute value inside the integral sign. If $\beta = 0$, we can write $\mu_1 = (\lambda/2)\big(u(z) + \hat{u}(-z)\big)$ and we have

$$(5.25)\qquad |\mu_1| = |\lambda|\left|\int_0^1 u(x)\cos\alpha x\,dx\right| .$$

Since $\alpha + i\beta \neq 0$ (otherwise $\mu_1 = \mu$), we have $\alpha \neq 0$ in (5.25), and taking the absolute value under the integral again implies that $|\mu_1| < |\mu|$.

Notice that the above argument actually shows $|\lambda|\hat{u}(0) > |\lambda||\hat{u}(z)|$ for any $z \neq 0$ such that $\theta(z) = 0$. It follows that the multiplicity of μ as an eigenvalue of L is just $(\frac{1}{2})(m(0) + 1)$, where $m(0)$ is the multiplicity of 0 as a solution of $\theta(z) = 0$. However, we have $\theta'(0) = i\int_0^1 xu(x)\,dx \neq 0$, so $m(0) = 1$, and the proof is complete. ■

REMARK 5.1. The results in Theorem 5.2 concerning the spectral radius of L can also be obtained by a functional analytic argument using refinements of the classical Krein-Rutman theorem, and in fact such a result is claimed in [10]. Some care, however, is necessary, since no iterate of L maps the interior of the cone K of nonnegative functions in $C[0,1]$ into itself $\big((Lh)(1) = 0$ for every $h\big)$.

We shall now present several simple corollaries of Theorem 5.2. Our first corollary could be used in [12] to replace the use of Theorem 14 from [10] and then would provide a correct proof of the main result in [12].

COROLLARY 5.1. *Assume that* $u\colon [0,1] \to \mathbb{R}$ *is a* C^1 *real-valued function such that* $u'(x) \leqslant 0$ *for* $0 \leqslant x \leqslant 1$ *and* $u(0) > u(1) > 0$. *Then* $\lambda \int_0^1 u(x)\,dx$ *is the only real eigenvalue of the linear operator* L *defined by equation* (5.2). *If* $f\colon [0,1] \to \mathbb{R}$ *is a* C^1 *real-valued map such that* $f'(x) \leqslant 0$ *for* $0 \leqslant x \leqslant 1$ *and if* $u(x)$ *is a continuous, positive solution of* (5.1) *for some* $\lambda > 0$, *then* $u'(x) \leqslant 0$ *for* $0 \leqslant x \leqslant 1$ *and* $u(0) > u(1) > 0$.

PROOF. Theorem 5.2 implies that the eigenvalues of L are the numbers $\lambda \hat{u}(z)$ such that $\hat{u}(z) = \hat{u}(-z)$. Proposition 2.2 implies that $\lambda \hat{u}(0)$ is the only eigenvalue which is real.

If u is a continuous, positive solution of (5.1) and f is C^1, we know that u is C^1 and

$$u'(x) = f'(x) + \lambda \int_x^1 u'(y) u(y-x)\,dy - \lambda u(1) u(1-x)\,. \tag{5.26}$$

Equation (5.26) shows that $u'(1) < 0$, so define $x_0 < 1$ to be $\inf \{x\colon 0 \leqslant x < 1,\ u'(y) < 0 \text{ for } x \leqslant y \leqslant 1\}$. If $x_0 > 0$ we obtain

$$u'(x_0) \leqslant f'(x_0) - \lambda u(1) u(1-x_0) < 0 \tag{5.27}$$

where we have used the fact that $u'(y) < 0$ for $x_0 < y < 1$ and $u(y-x) \geqslant 0$ for $x_0 \leqslant y \leqslant 1$. But if $u'(x_0) < 0$, we have contradicted the choice of x_0, so we must have $x_0 = 0$. ■

The argument used to prove the second part of Corollary 5.1 is the same used to prove Theorem 7 in [10].

COROLLARY 5.2. *If* $u\colon [0,1] \to \mathbb{R}$ *is* C^1, $u'(x) \geqslant 0$ *for* $0 \leqslant x \leqslant 1$ *and* $0 \leqslant u(0) < u(1)$, *then every solution of* $\hat{u}(z) = \hat{u}(-z)$ *is real* $\big(u(x) = 0$ *for* $x \notin [0,1]\big)$ *and every eigenvalue of the linear operator* L *in* (5.2) *is real.*

If $f\colon [0,1] \to \mathbb{R}$ *is a* C^1 *function such that* $f'(x) \geqslant 0$ *and* $f(x) > 0$ *for* $0 \leqslant x \leqslant 1$ *and if* u_λ *denotes the unique, real-valued solution of* (5.1) *such that* $1 - \lambda \hat{u}_\lambda(z) \neq 0$

for $\mathrm{Im}(z) > 0$ (*assuming* (5.1) *has a solution*), *assume that* $\lambda_0 < 0$ *is such that* $u_\lambda(x) > 0$ *for* $\lambda_0 \leqslant \lambda \leqslant 0$ *and* $0 \leqslant x \leqslant 1$. *Then one has* $u'_\lambda(x) > 0$ *for* $\lambda_0 \leqslant \lambda < 0$ *and* $0 \leqslant x \leqslant 1$, *and every eigenvalue of the operator* L *associated with* u_λ *is real for* $\lambda_0 \leqslant \lambda < 0$.

PROOF. The first part of the corollary follows immediately from Theorem 5.2 and Proposition 2.2.

To prove the second part of the corollary we first have to show that $u'_\lambda(x) > 0$ for $0 \leqslant x \leqslant 1$ and for $|\lambda|$ small. We proceed by approximating equation (5.1) by a more suitable equation and then using a somewhat clumsy limiting argument. Observe that there exists $\varepsilon > 0$ such that for $0 < \delta \leqslant 1$ and $-\varepsilon \leqslant \lambda < 0$ the equation

$$u(x) = f(x) + \delta x + \lambda \int_x^1 u(y) u(y - x)\, dy \tag{5.28}$$

has a solution $u_{\lambda,\delta} = u$ which depends continuously on (λ, δ) and is such that $1 - \hat{u}_{\lambda,\delta}(z) \neq 0$ for $\mathrm{Im}(z) > 0$ and $u_{\lambda,\delta}(x) > 0$ for $0 \leqslant x \leqslant 1, -\varepsilon \leqslant \lambda \leqslant 0$, $0 < \delta \leqslant 1$. We leave the proof to the reader. Results of the first section show that $\lim_{\delta \to 0^+} u_{\lambda,\delta} = u_\lambda$ and $\lim_{\delta \to 0^+} u'_{\lambda,\delta} = u'_\lambda$, so to prove that $u'_\lambda(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$ and $-\varepsilon \leqslant \lambda < 0$, it suffices to prove the same for $u_{\lambda,\delta}$. Since $u_{0,\delta}(x) = f(x) + \delta x$ and $f'(x) \geqslant \delta > 0$ for $0 \leqslant x \leqslant 1$, there exists a number $\eta > 0$ (depending on δ) such that $u'_{\lambda,\delta}(x) > 0$ for $-\eta \leqslant \lambda \leqslant 0$ and $0 \leqslant x \leqslant 1$. Define $\lambda_1 = \inf\{-\eta : 0 < \eta \leqslant \varepsilon,\ u'_{\lambda,\delta}(x) > 0$ for $0 \leqslant x \leqslant 1$ and for $-\eta \leqslant \lambda \leqslant 0\}$. If $-\lambda_1 < \varepsilon$, we must have $u'_{\lambda_1,\delta}(x) = 0$ for some x with $0 \leqslant x \leqslant 1$; by continuity in λ we have $u'_{\lambda_1,\delta}(x) \geqslant 0$ for all x with $0 \leqslant x \leqslant 1$. For convenience write $u(x) = u'_{\lambda_1,\delta}(x)$. If x is such that $u'(x) = 0$ we have

$$u'(x) = f'(x) + \delta + \lambda_1 \int_x^1 u'(y) u(y - x)\, dy - \lambda_1 u(1) u(1 - x)\,. \tag{5.29}$$

Since u is monotonic increasing we have

$$\begin{aligned} \lambda_1 \int_x^1 u'(y) u(y - x)\, dy - \lambda_1 u(1) u(1 - x) \geqslant \\ \geqslant -\lambda_1 \Big[u(1) u(1 - x) - u(1 - x) \int_x^1 u'(y)\, dy \Big] = |\lambda_1| u(x) u(1 - x) > 0\,. \end{aligned} \tag{5.30}$$

Combining (5.29) and (5.30) we find that $u'(x) > 0$, and this contradiction proves that we must have $\lambda_1 = -\varepsilon$. If we let δ approach zero, we now find that $u'_\lambda(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$ and for $-\varepsilon \leqslant \lambda < 0$.

The argument used above actually shows that if $\lambda < 0$, $g(x)$ is C^1 and $g'(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$, and v is a strictly positive solution of

$$v(x) = g(x) + \lambda \int_x^1 v(y) v(y - x) dy\,, \qquad 0 \leqslant x \leqslant 1$$

such that $v'(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$, then in fact $v'(x) > 0$ for $0 \leqslant x \leqslant 1$. Using this fact we see that $u'_\lambda(x) > 0$ for $0 \leqslant x \leqslant 1$ and $-\varepsilon \leqslant \lambda < 0$. Furthermore, if $\lambda_2 = \inf \{\lambda < 0 : \lambda \geqslant \lambda_0, u'_s(x) > 0 \text{ for } \lambda \leqslant s < 0,\ 0 \leqslant x \leqslant 1\}$, we know by a continuity argument and the definition of λ_0 that $u_{\lambda_2}(x) > 0$ and $u'_{\lambda_2}(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$. By the above remark we must have $u'_{\lambda_2}(x) > 0$ for $0 \leqslant x \leqslant 1$, so continuity implies that u_λ and u'_λ are strictly positive for λ near λ_2, and we must have $\lambda_2 = \lambda_0$. ∎

Remark 5.2. Notice that it was necessary to prove the existence of $\varepsilon > 0$ such that $u'_\lambda(x) > 0$ for $-\varepsilon \leqslant \lambda < 0$ and $0 \leqslant x \leqslant 1$ in order to define λ_2: If $f'(x) > 0$ for $0 \leqslant x \leqslant 1$, the existence of such an ε is trivial and the proof of Corollary 5.2 becomes easier.

Corollary 5.3. *Suppose that $f\colon [0, 1] \to \mathbb{R}$ is a real-valued, continuously differentiable function such that $f(1) \neq 0$ (or f can be as in Remark 5.1) and that $u\colon [0, 1] \to \mathbb{R}$ is a continuous, real-valued solution of*

$$u(x) = f(x) + \lambda \int_x^1 u(y) u(y - x) dy \tag{5.31}$$

for some real $\lambda \neq 0$. Extend $f(x)$ to be even and such that $f(x) = 0$ for $|x| > 1$ and assume that the equation $1 - \lambda \hat{f}(z) = 0$ (z complex) has only simple zeros. Then 1 is not in the spectrum of the linear operator L defined by equation (5.2) (for u as in (5.31)).

Proof. If we define $u(x) = 0$ for $x \notin [0, 1]$, we know that

$$\big(1 - \lambda \hat{f}(z)\big) = \big(1 - \lambda \hat{u}(z)\big)\big(1 - \lambda \hat{u}(-z)\big)\,. \tag{5.32}$$

If 1 were in the spectrum of L, Theorem 5.2 implies that $\lambda \hat{u}(z_0) = 1$ for some z_0 such that $\hat{u}(z_0) = \hat{u}(-z_0)$. But then equation (5.32) implies that z_0 gives a zero of multiplicity at least 2 of $1 - \lambda \hat{f}(z) = 0$, a contradiction. ∎

Remark 5.3. If 1 is not in the spectrum of L, the implicit function theorem provides a solution of

$$u(x) = f(x) + \lambda_1 \int_x^1 u(y) u(y - x) dy$$

for values of λ near λ_1. If $f\colon [0,1] \to \mathbb{R}$ satisfies $f'(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$, $f(0) \geqslant 0$ and $f(1) \geqslant 0$, Theorem 2.1 implies that $1 - \lambda \hat{f}(z) = 0$ has only simple zeros for $\lambda_- < \lambda < \lambda_+$ (λ_- and λ_+ defined as in Lemma 1.2). In this case, it follows from Corollary 5.3 that if $u(x)$ satisfies (5.31) for some $\lambda = \lambda_0 \neq 0$, one can analytically continue this solution on the whole interval $[\lambda_-, 0)$ if $\lambda_0 < 0$ or the whole interval $(0, \lambda_+]$ if $\lambda_0 > 0$.

(The fact that the continued solution approaches a solution as $\lambda \to \lambda_+$ or $\lambda \to \lambda_-$ requires some further argument).

We now turn to the problem of proving Theorem 5.1. Our approach is a generalization of ideas used by Paley and Wiener [9, Chapter 6], and the reader may want to compare the arguments there.

LEMMA 5.3. *Let $u\colon [0,1] \to \mathbb{R}$ be a real-valued, C^1 function such that $u(1) \neq 0$. Extend $u(x) = 0$ for $x \notin [0,1]$, define $\theta(z) = \hat{u}(z) - \hat{u}(-z)$, define $S = \{z \colon z\theta(z) = 0\}$ and define $m_1(\zeta)$ to be the multiplicity of $\zeta \in S$ as a zero of $z\theta(z) = 0$. Then the closed linear span of $A = \{x^j e^{izx} \colon -1 \leqslant x \leqslant 1,\ z \in S,\ 0 \leqslant j < m_1(z)\}$ is all of $L^2[-1,1]$.*

PROOF. We apply a result in Appendix III of [8] on page 418. Define $\varphi(z) = z\theta(z)$, which is clearly an entire function of exponential type. According to the result in [8], Lemma 5.3 is true if

$$\overline{\lim_{|v| \to \infty}} |\varphi(iv)| e^{-|v|} > 0 \tag{5.33}$$

where v in equation (5.33) is real and we have modified equation (11) on page 418 of [8] to account for the fact that we are working on the interval $[-1,1]$ instead of $[-\pi, \pi]$. To prove (5.33), we may as well assume $v > 0$ ($|\varphi(-iv)| = |\varphi(iv)|$), and integration by parts gives

$$|\varphi(iv)| e^{-v} = \left| u(1)(1 - e^{-2v}) - \int_0^1 u'(x) e^{v(x-1)}\, dx - \int_0^1 u'(x) e^{-v(x+1)}\, dx \right|. \tag{5.34}$$

If $|u'(x)| \leqslant M$ for $0 \leqslant x \leqslant 1$, each of the integral terms is dominated by $\int_0^1 M e^{v(x-1)} dx \leqslant M v^{-1}$, and it follows that the integral terms approach zero as $v \to \infty$ and $\lim_{|v| \to \infty} |\varphi(iv)| e^{-|v|} = |u(1)| > 0$. ∎

REMARK 5.4. We originally proved Lemma 5.3 without knowledge of the result in [8]. The method of proof was to associate an entire function $F(z)$ to the set S ($F(z)$ is defined below) and to generalize results of Paley and Wiener [9, Chapter 6] by proving that A has closed linear span all of $L^2[-1,1]$ if and only if $F|\mathbb{R}$ is not an element of $L^2(\mathbb{R})$. The novelty was that the zeros of F are no longer necessarily real and distinct as they are in [9]. One can show that $F|\mathbb{R} \notin L^2(\mathbb{R})$ by variants of arguments we shall use later to show that $\left(1/(|z|+1)\right)F|\mathbb{R} \in L^2(\mathbb{R})$. Our original argument is more elementary than Levin's, but for reasons of length we have omitted it.

It is convenient at this point to define the function $F(z)$ mentioned above. Let u and S be as in Lemma 5.3.

DEFINITION 5.1. $F(z) = z^{p+1} \prod_{\zeta \in S-\{0\}} (1-z/\zeta)\exp(z/\zeta)$, where $p =$ the multiplicity of 0 as a solution of $\theta(z) = \hat{u}(z) - \hat{u}(-z) = 0$ and each term $\left(1-(z/\zeta)\right)\exp(z/\zeta)$ is repeated a number of times equal to the multiplicity of ζ as a zero of $\theta(z) = 0$.

Theorem 2.4 implies that there exists an integer $m \geqslant 1$ such that for each integer $k \geqslant 0$ the equation $\theta(z) = 0$ has precisely two zeros α_k and α_k' in the strip $\Gamma_k = \{z: m\pi + 2k\pi \leqslant \operatorname{Re}(z) \leqslant m\pi + 2k\pi + 2\pi\}$ (counting multiplicities) and none on the boundary of Γ_k; α_k and α_k' are either both real or are complex conjugates. Furthermore, $\theta(z)$ has precisely $2m-1$ zeros ζ such that $\operatorname{Re}(\zeta) < m\pi$ ($2m-1 =$ the number of integers j such that $|j| < m$), and if ζ is such a zero, so are $-\zeta$, $-\bar{\zeta}$ and $\bar{\zeta}$. It follows that one can write $F(z)$ in the form

$$F(z) = P(z) \prod_{k=0}^{\infty} \left(1 - \left(\frac{z}{\alpha_k}\right)^2\right)\left(1 - \left(\frac{z}{\alpha_k'}\right)^2\right) \tag{5.35}$$

$P(z) =$ a polynomial of degree $2m$ with real coefficients.

If we assume that $u(1)u(0) \geqslant 0$ and if we use the notation from the end of Section 2, formula 5.35 becomes (writing $m = 2N-1$)

$$F(z) = P(z) \prod_{n=N}^{\infty} \left(1 - \left(\frac{z}{z_n}\right)^2\right)\left(1 - \left(\frac{z}{z_n'}\right)^2\right). \tag{5.36}$$

A similar formula holds if $u(1)u(0) \leqslant 0$. If $u(1)u(0) \geqslant 0$, Proposition 2.3 implies that $\lim_{n\to\infty} 2n\pi + a = z_n = 0$ and $\lim_{n\to\infty} 2n\pi - a - z_n' = 0$, where either $0 \leqslant a \leqslant \pi/2$ or a is pure imaginary. We shall have to determine whether $z^{-1}F(z)|\mathbb{R} \in L^2(\mathbb{R})$ and to do this is makes sense to study the limiting case $z_n = 2n\pi + a$. Then we get

$$\left\{\begin{aligned} G_1(z) &= Q(z) \prod_{n=N}^{\infty} \left(1 - \left(\frac{z}{2n\pi + a}\right)^2\right)\left(1 - \left(\frac{z}{2n\pi - a}\right)^2\right) \\ Q(z) &= \prod_{n=1}^{N-1} \left(1 - \left(\frac{z}{2n\pi + a}\right)^2\right)\left(1 - \left(\frac{z}{2n\pi - a}\right)^2\right). \end{aligned}\right. \tag{5.37}$$

Since the degree of $Q(z)$ is $4N-4$ and the degree of $(1/z)P(z)$ is $2(2N-1)-1=4N-3$, a natural first question is whether $zG_1(z)\in L^2(\mathbb{R})$ since if this is the case, it at least becomes plausible that $z^{-1}F(z)\in L^2(\mathbb{R})$. A similar analysis shows that if $u(1)u(0)\leqslant 0$ it is important to analyze

$$G_2(z)=\prod_{n=0}^{\infty}\left(1-\left(\frac{z}{2n\pi+\pi+a}\right)^2\right)\left(1-\left(\frac{z}{2n\pi+\pi-a}\right)^2\right) \tag{5.38}$$

where $0\leqslant a\leqslant(\pi/2)$ or a is pure imaginary and see if $\big(1/(|z|+1)\big)G_2(z)$ is an element of $L^2(\mathbb{R})$. Our next lemma gives a precise formula for $G_1(z)$ and $G_2(z)$ and shows that $zG_1(z)$ and $\big(|z|+1\big)^{-1}G_2(z)$ are elements of $L^2(\mathbb{R})$.

LEMMA 5.4. *If a is a complex number and $a\neq 2n\pi$, n an integer, then*

$$\sin\left(\frac{z+a}{2}\right)\sin\left(\frac{a-z}{2}\right)=\left(\sin\left(\frac{a}{2}\right)\right)^2\left(1-\left(\frac{z}{a}\right)^2\right)G_1(z) \tag{5.39}$$

where $G_1(z)$ is given as in equation (5.37). *If $a=0$,*

$$\sin\left(\frac{z}{2}\right)\sin\left(\frac{-z}{2}\right)=-\left(\frac{1}{4}\right)z^2G_1(z)\,. \tag{5.40}$$

If $a\neq 2n\pi+\pi$, n an integer, then

$$\cos\left(\frac{z+a}{2}\right)\cos\left(\frac{z-a}{2}\right)=\left(\cos\left(\frac{a}{2}\right)\right)^2G_2(z) \tag{5.41}$$

where $G_2(z)$ is as in (5.38).

PROOF. Define $f(z)=\sin\big((z+a)/2\big)\sin\big((a-z)/2\big)$, so $f(z)$ is an even, entire function of exponential type. Basic complex variables implies [1, p. 186] that if $f(0)\neq 0$

$$f(z)=c\exp(\gamma z)\prod_{\alpha\in T}\left(1-\left(\frac{z}{\alpha}\right)\right)\exp(\alpha z) \tag{5.42}$$

where T is the collection of zeros of $f(z)$ and factors are repeated according to multiplicity. Since f is even, $\gamma=0$, and grouping the terms corresponding to α and $-\alpha$ in T gives $\big($assuming $f(0)\neq 0\big)$

$$f(z)=f(0)\prod_{\alpha\in T_1}\left(1-\left(\frac{z}{\alpha}\right)^2\right) \tag{5.43}$$

where T_1 is a subset of zeros of T such that if $\alpha \in T_1$, $-\alpha \notin T_1$ and if $\alpha \in T$, $-\alpha$ or α is in T_1. If a is not an integral multiple of 2π, T comprises $+a$, $-a$ and $\{2n\pi \pm a, -2n\pi \pm a: n \text{ a positive integer}\}$, and using this fact, equation (5.39) follows from (5.43).

If $a = 0$, $f(z)$ has a double zero at $z = 0$ and $\lim_{z\to 0} z^{-2}f(z) = -\frac{1}{4}$. Equation (5.40) now follows by the same sort of reasoning used above.

To prove equation (5.41) (assuming $a \neq 2m\pi + \pi$ for an integer m) it suffices to observe that $g(z) = \cos\big((z+a)/2\big)\cos\big((z-a)/2\big)$ is an even, entire function of exponential type and that its zeros are the same as those of $G_2(z)$. ∎

Before proceeding further let us recall some elementary calculus. We know that

$$\sum_{n=1}^{m}\left(\frac{1}{n}\right) - \log(m) = \gamma + \varepsilon_m \tag{5.44}$$

where $\varepsilon_m \to 0$ as $m \to \infty$, γ is Euler's constant and log denotes natural logarithm. Similarly, if m is a positive integer and $w > m - 1$ we have

$$\begin{aligned}
\sum_{n=1}^{m-1}\frac{1}{w-n} &= \sum_{n=1}^{m-1}\int_{n-1}^{n}(w-x)^{-1}dx + \int_{n-1}^{n}(n-x)(w-x)^{-1}(w-n)^{-1}dx \leqslant \\
&\leqslant \log w - \log(w-m+1) + \sum_{n=1}^{m-1}\int_{n-1}^{n}(w-x)^{-1}(w-n)^{-1}dx \leqslant \\
&\leqslant \log w - \log(w-m+1) + \big(1+(w-m+1)^{-1}\big)\int_{0}^{m-1}(w-x)^{-1}dx \leqslant \\
&\leqslant \log w - \log(w-m+1) + \big((w-m+1)^{-1} - w^{-1}\big)\big(1+(w-m+1)^{-1}\big).
\end{aligned} \tag{5.45}$$

We shall use the above elementary estimates in proving that $z^{-1}F(z) \in L^2(\mathbb{R})$.

LEMMA 5.5. *Let $u\colon [0,1] \to \mathbb{R}$ be a real-valued, continuously differentiable function such that $u(1) \neq 0$ and let $F(z)$ be as defined in Definition* 5.1. *Then it follows that $z^{-1}F(z) \in L^2(\mathbb{R})$.*

PROOF. We shall restrict attention to the case $u(1)u(0) \geqslant 0$, since the proof when $u(1)u(0) \leqslant 0$ is exactly analogous. Clearly we can also assume that $u(1) > 0$. We know from Section 2 that there exists an integer $N \geqslant 1$ such that the equation $\theta(z) = \hat{u}(z) - \hat{u}(-z) = 0$ has precisely two zeros z_n and z_n' in each strip $2n\pi - \pi \leqslant \mathrm{Re}\,(z) \leqslant 2n\pi + \pi$ and no zeros on the boundary of the strip. According to our previous remarks we have

$$F(z) = P(z)\prod_{n=N}^{\infty}\left(1-\left(\frac{z}{z_n}\right)^2\right)\left(1-\left(\frac{z}{z_n'}\right)^2\right)$$

where $P(z)$ is a polynomial of degree $2(2N-1)$. Proposition 2.3 implies that there is a complex number a such that $\lim_{n\to\infty} z_n - 2n\pi = a$ and $\lim_{n\to\infty} z_n' - 2n\pi = -a$. Thus, given any $\varepsilon > 0$, we can choose an integer $N = N(\varepsilon)$ such that $|z_n - 2n\pi - a| < \varepsilon$ and $|z_n' - 2n\pi + a| < \varepsilon$ for $n \geqslant N$. We shall specify ε more precisely later. We adopt the convention that $z_n = 2n\pi + a$ and $z_n' = 2n\pi - a$ for $1 \leqslant n < N$. With this convention, one can see that $z^{-1}F(z) \in L^2(\mathbb{R})$ if and only if

$$H(z) \stackrel{\text{def}}{=} z\prod_{n=1}^{\infty}\left(1-\left(\frac{z}{z_n}\right)^2\right)\left(1-\left(\frac{z}{z_n'}\right)^2\right) \in L^2(\mathbb{R}) \tag{5.46}$$

because the ratio of $H(z)$ in (5.46) to $z^{-1}F(z)$ approaches one as $|z| \to \infty$. We shall try to prove that $H(z) \in L^2(\mathbb{R})$ by comparing it with $G_1(z)$ in Lemma 5.4 and proving that $|H(z)| \leqslant c|z|^{-p}$ for $|z| \geqslant 1$, where c is some constant and $p > \frac{1}{2}$. Since we are assuming $u(1) > 0$ and $u(0) \geqslant 0$, the proof divides naturally into two cases.

Case 1. Assume $u(1) > u(0) \geqslant 0$. Proposition 2.3 implies in this case that $0 < a \leqslant (\pi/2)$ and that z_n and z_n' are real for n large enough. Thus by selecting N large enough we can assume $2n\pi + a - \varepsilon < z_n < 2n\pi + a + \varepsilon$ for $n \geqslant N$ and hence, by the convention that $z_n = 2n\pi + a$ for $1 \leqslant n < N$, for all n. Similarly, we can assume $2n\pi - a - \varepsilon < z_n' < 2n\pi - a + \varepsilon$ for $n \geqslant N$ and define $z_n' = 2n\pi - a$ for $1 \leqslant n < N$. Of course, z_n and z_n' are only zeros of $\theta(z)$ for $n \geqslant N$. We shall specify ε later.

It suffices to estimate $H(z)$ for $z \geqslant \pi$ since H is odd and entire, so assume $2m\pi - \pi \leqslant z \leqslant 2m\pi + \pi$, where m is a positive integer. As a first step define polynomials $P_1(z)$ and $P_2(z)$ by

$$P_1(z) = \prod_{n=1}^{m-1}\left(1-\left(\frac{z}{z_n}\right)^2\right) \qquad P_2(z) = \prod_{n=1}^{m-1}\left(1-\left(\frac{z}{2n\pi + a}\right)^2\right). \tag{5.47}$$

If we assume $\varepsilon < \pi/2$, our construction insures that $(z/z_n)^2 > 1$, so we find

$$|P_1(z)| = \prod_{n=1}^{m-1}\left(\left(\frac{z}{z_n}\right)^2 - 1\right) \leqslant \prod_{n=1}^{m-1}\left(\left(\frac{z}{2n\pi + a - \varepsilon}\right)^2 - 1\right) \tag{5.48}$$

If we use (5.47) and (5.48) and do some algebraic manipulation we find

$$(5.49)\quad \left|\frac{P_1(z)}{P_2(z)}\right| \leqslant \prod_{n=1}^{m-1}\left[1+\frac{\varepsilon}{2n\pi+a-\varepsilon}\right]^2\left[1+\frac{\varepsilon}{z-2n\pi-a}\right]\left[1-\frac{\varepsilon}{z+2n\pi+a}\right].$$

Standard estimates using the power series for $\log(1+w)$ when $|w|<1$ imply

$$(5.50)\quad \begin{cases} \log\left(1+\dfrac{\varepsilon}{2n\pi+a-\varepsilon}\right)^2 \leqslant \dfrac{2\varepsilon}{2n\pi+a-\varepsilon} \\ \log\left(1+\dfrac{\varepsilon}{z-2n\pi-a}\right) \leqslant \dfrac{\varepsilon}{z-2n\pi-a} \\ \log\left(1-\dfrac{\varepsilon}{z+2n\pi+a}\right) \leqslant -u+\dfrac{1}{2}u^2, \qquad u=\left(\dfrac{\varepsilon}{z+2n\pi+a}\right). \end{cases}$$

Using (5.50) we find that

$$(5.51)\quad \left|\frac{P_1(z)}{P_2(z)}\right| \leqslant \left[\exp\left(\sum_{n=1}^{m-1}\frac{2\varepsilon}{2n\pi+a-\varepsilon}+\frac{\varepsilon}{z-2n\pi-a}-\frac{\varepsilon}{z+2n\pi+a}\right)\right]R(z)$$

$$R(z)=\exp\left(\frac{1}{2}\varepsilon^2\sum_{n=1}^{m-1}\left(\frac{1}{z+2n\pi+a}\right)^2\right).$$

It is clear that $R(z)$ is bounded by a constant c_1 independent of m, z, $\varepsilon<\pi/2$ and $a>0$.

In order to estimate the various summations in (5.51), first recall that one can easily obtain from (5.44) that

$$(5.52)\quad \sum_{n=0}^{p}\frac{1}{2n+1}=\left(\frac{1}{2}\right)\log p+\log 2+\frac{1}{2}\gamma+\eta_p$$

where γ is Euler's constant and $\lim_{p\to\infty}\eta_p=0$. Using (5.52) and the assumption $\varepsilon<\pi/2$ one finds that

$$(5.53)\quad \sum_{n=1}^{m-1}\frac{2\varepsilon}{2n\pi+a-\varepsilon}\leqslant 2\varepsilon\sum_{n=1}^{m-1}\frac{1}{2n\pi-\pi}\leqslant\frac{\varepsilon}{\pi}\log m+c_2$$

where c_2 is a constant independent of $\varepsilon<\pi/2$, m, z and $a\geqslant 0$. Similarly, using (5.45) one finds that

$$(5.54)\quad \varepsilon\sum_{n=1}^{m-1}\frac{1}{z-2n\pi a}\leqslant\varepsilon\sum_{n=1}^{m-1}\left(\frac{1}{2\pi}\right)\left(m-\frac{a}{2\pi}-\frac{1}{2}-n\right)^{-1}\leqslant$$

$$\leqslant\frac{\varepsilon}{2\pi}\log w+c_3\leqslant\frac{\varepsilon}{2\pi}\log m+c_3$$

where $w = m - (a/2\pi) - (\frac{1}{2})$ and c_3 is a constant independent of $\varepsilon < \pi/2$, z, m and $a \leqslant \pi/2$. Using (5.53) and (5.54) and the assumption that $2m\pi - \pi \leqslant z$ we find that

$$(5.55) \qquad \sum_{n=1}^{m-1}\left(\frac{2\varepsilon}{2n\pi + a - \varepsilon} + \frac{\varepsilon}{z - 2n\pi - a} - \frac{\varepsilon}{z + 2n\pi + a}\right) \leqslant \left(\frac{3\varepsilon}{2\pi}\right)\log(z) + c_4$$

where c_4 is a constant independent of $\varepsilon < \pi/2$, $a \leqslant \pi/2$, z and m. Equations (5.51) and (5.55) imply that

$$(5.56) \qquad \left|\frac{P_1(z)}{P_2(z)}\right| \leqslant c_5|z|^{\delta_1}, \qquad \delta_1 = \frac{3\varepsilon}{2\pi}.$$

The next step in the proof is to estimate $\prod_{n=m+1}^{\infty}(1 - (z/z_n)^2)$. Since we are assuming $z \leqslant 2m\pi + \pi$ and $\varepsilon < \pi/2$, one can see that $z/z_n < 1$ and one has

$$(5.57) \qquad \prod_{n=m+1}^{\infty}\left(1 - \left(\frac{z}{z_n}\right)^2\right) < \prod_{n=m+1}^{\infty}\left(1 - \left(\frac{z}{2n\pi + a + \varepsilon}\right)^2\right) \stackrel{\text{def}}{=} Q_1(z).$$

If we define

$$Q_2(z) = \prod_{n=m+1}^{\infty}\left(1 - \left(\frac{z}{2n\pi + a}\right)^2\right),$$

a calculation shows

$$(5.58) \qquad \left|\frac{Q_1(z)}{Q_2(z)}\right| \leqslant$$
$$\leqslant \prod_{n=m+1}^{\infty}\left(1 - \frac{\varepsilon}{2n\pi + a + \varepsilon}\right)^2\left(1 + \frac{\varepsilon}{2n\pi + a - z}\right)\left(1 + \frac{\varepsilon}{2n\pi + a + z}\right).$$

Just as before one finds that

$$(5.59) \qquad \begin{cases} \log\left(1 - \dfrac{\varepsilon}{2n\pi + a + \varepsilon}\right)^2 \leqslant -\left(\dfrac{2\varepsilon}{2n\pi + a + \varepsilon}\right) + \left(\dfrac{\varepsilon}{2n\pi + a + \varepsilon}\right)^2 \\[2ex] \log\left(1 + \dfrac{\varepsilon}{2n\pi + a - z}\right) \leqslant \dfrac{\varepsilon}{2n\pi + a - z} \\[2ex] \log\left(1 + \dfrac{\varepsilon}{2n\pi + a + z}\right) \leqslant \dfrac{\varepsilon}{2n\pi + a + z}. \end{cases}$$

Just as before we find that

$$(5.60)\qquad \left|\frac{Q_1(z)}{Q_2(z)}\right| \leqslant d_1 \exp\left(\varepsilon \sum_{n=m+1}^{\infty}\left(\left[-\frac{2}{2n\pi+a+\varepsilon}+\frac{2}{2n\pi}\right]+ + \left[\frac{1}{2n\pi+a-z}-\frac{1}{2n\pi}\right]+\left[\frac{1}{2n\pi+a+z}-\frac{1}{2n\pi}\right]\right)\right)$$

where d_1 is a constant independent of $\varepsilon < (\pi/2)$, $a < \pi/2$, z and m. In obtaining (5.60) we have added and subtracted $1/n\pi$ from the terms in the infinite series and regrouped terms. If we recall that $a > 0$, we can see that there is a constant d_2, independent of z, m, etc., such that

$$(5.61)\qquad \left|\frac{Q_1(z)}{Q_2(z)}\right| \leqslant d_2 \exp\left(\varepsilon \sum_{n=m+1}^{\infty} \frac{1}{2n\pi+a-z}-\frac{1}{2n\pi}\right).$$

Since $a > 0$ and $z \leqslant 2m\pi + \pi$ we have

$$(5.62)\qquad \sum_{n=m+1}^{\infty}\left(\frac{1}{2n\pi+a-z}-\frac{1}{2n\pi}\right) \leqslant \sum_{n=m+1}^{\infty}\left(\frac{1}{2n\pi-(2m+1)\pi}-\frac{1}{2n\pi}\right)=$$

$$= \lim_{N\to\infty}\left(\frac{1}{\pi}\sum_{n=1}^{N-m-1}\frac{1}{2n-1}-\frac{1}{2\pi}\sum_{n=m+1}^{N}\frac{1}{n}\right)=$$

$$=\left(\frac{1}{\pi}\right)\lim_{N\to\infty}\left[\frac{1}{2}\log(N-m-1)-\frac{1}{2}\log(N)+\frac{1}{2}\log(m)+c+\varrho_N\right]$$

where $c = \log 2 + \frac{1}{2}\gamma + \frac{1}{2}\varepsilon_m$, $\gamma =$ Euler's constant, ε_m is an in (5.44) and $\lim_{N\to\infty} \varrho_N = 0$. Equation (5.62) implies that

$$(5.63)\qquad \sum_{n=m+1}^{\infty}\left(\frac{1}{2n\pi+a-z}-\frac{1}{2n\pi}\right) \leqslant \left(\frac{1}{2\pi}\right)\log(m)+d_3.$$

Using (5.61) and (5.63) we find that

$$(5.64)\qquad \left|\frac{Q_1(z)}{Q_2(z)}\right| \leqslant d_4|z|^{\delta_2}, \qquad \delta_2=\frac{\varepsilon}{2\pi}$$

where d_4 is independent of z, m, etc. (as long as z, m, ε and a satisfy the usual constraints). Combining (5.56) and (5.64) we find that

$$(5.65)\qquad \left|\prod_{\substack{n=1\\ n\neq m}}^{\infty}\left(1-\left(\frac{z}{z_n}\right)^2\right)\right| \leqslant c_6 \prod_{\substack{n=1\\ n\neq m}}^{\infty}\left|\left(1-\left(\frac{z}{2n\pi+a}\right)^2\right)\right| |z|^{\delta}, \qquad \delta=\frac{2\varepsilon}{\pi}.$$

Essentially the same analysis as that above yields

$$\left|\prod_{\substack{n=1\\n\neq m}}^{\infty}\left(1-\left(\frac{z}{z_n'}\right)^2\right)\right| \leqslant c_7\left|\prod_{\substack{n=1\\n\neq m}}^{\infty}\left(1-\left(\frac{z}{2n\pi-a}\right)^2\right)\right||z|^\delta, \qquad \delta=\frac{2\varepsilon}{\pi}. \tag{5.66}$$

If we combine (5.65) and (5.66) and use the formula from Lemma 5.4 we find (setting $\delta = 2\varepsilon/\pi$)

$$\begin{cases} \left|\prod_{n=1}^{\infty}\left(1-\left(\frac{z}{z_n}\right)^2\right)\left(1-\left(\frac{z}{z_n'}\right)^2\right)\right| \leqslant \\ \qquad \leqslant c_8\theta_m(z)|z|^{2\delta}\left[\left(\frac{z}{a}\right)^2-1\right]^{-1}\left|\sin\left(\frac{z+a}{2}\right)\sin\left(\frac{a-z}{2}\right)\right| \\ \theta_m(z) \stackrel{\text{def}}{=} \left|\left(1-\left(\frac{z}{z_m}\right)^2\right)\left(1-\left(\frac{z}{z_m'}\right)^2\right)\cdot\right. \\ \qquad \left.\cdot\left(1-\left(\frac{z}{2m\pi+a}\right)^2\right)^{-1}\left(1-\left(\frac{z}{2m\pi-a}\right)^2\right)^{-1}\right|. \end{cases} \tag{5.67}$$

In equation (5.67) c_8 denotes a constant independent of z, m, etc. and $(2m-1)\pi \leqslant z \leqslant (2m+1)\pi$. If we use the fact that $(2m-1)\pi \leqslant z \leqslant (2m+1)\pi$, so that $|z-z_m| \leqslant \pi$ and $|z-z_m'| \leqslant \pi$ a computation and some obvious estimates yield

$$\begin{aligned} &\left|\theta_m(z)\sin\left(\frac{z+a}{2}\right)\sin\left(\frac{z-a}{2}\right)\right| \leqslant \\ &\qquad \leqslant c_9\left|\sin\left(\frac{z+a}{2}\right)\sin\left(\frac{z-a}{2}\right)(2m\pi+a-z)^{-1}(2m\pi-a-z)^{-1}\right| \end{aligned} \tag{5.68}$$

where c_9 is independent of z, m, a and ε. If we write $z = 2m\pi + w$, with $|w| \leqslant \pi$, we find that the right hand side of (5.68) equals

$$\left|\sin\left(\frac{w+a}{2}\right)\sin\left(\frac{w-a}{2}\right)(a-w)^{-1}(a+w)^{-1}\right| \stackrel{\text{def}}{=} \psi(w).$$

Since $\psi(w)$ extends to a continuous function on $[-\pi, \pi]$, we conclude that there exists a constant c_{10}, independent of z and m such that

$$\left|\theta_m(z)\sin\left(\frac{z+a}{2}\right)\sin\left(\frac{z-a}{2}\right)\right| \leqslant c_{10}. \tag{5.69}$$

Using this estimate in (5.67) we obtain (for an appropriate constant c_{11}

and for $|z| \geqslant \pi$)

$$(5.70) \qquad \left| z \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{z_n}\right)^2\right)\left(1 - \left(\frac{z}{z_n'}\right)^2\right) \right| \leqslant c_{11} |z|^{1+2\delta} \left[\left(\frac{z}{a}\right)^2 - 1\right]^{-1}, \qquad \delta = \frac{2\varepsilon}{\pi}.$$

It follows that if $\varepsilon < \pi/8$, the right hand side of (5.70) and hence $z^{-1}F(z)$ are elements of $L^2(\mathbb{R})$. Notice that we have actually proved that if z_n and z: are (eventually) real numbers with

$$\limsup_{n\to\infty} |z_n - 2n\pi - a| < \frac{\pi}{8} \text{ and } \limsup_{n\to\infty} |z_n' - 2n\pi - a| < \frac{\pi}{8} \text{ and if } 0 < a \leqslant \frac{\pi}{2},$$

then the right hand side of (5.70) is in $L^2(\mathbb{R})$. The condition on a is not strongly used; for example, if $a = 0$, one can prove the same result by using equation (5.40) in Lemma 5.4.

Case 2. Assume $0 < u(1) \leqslant u(0)$. The proof in this case is similar to that in Case 1, although some details are different because z_n and z_n' need not be real. We know that $\lim_{n\to\infty} (z_n - 2n\pi - ib) = 0$ for some $b \geqslant 0$, $\lim_{n\to\infty} (z_n' - 2n\pi + ib) = 0$ and $\bar{z}_n = z_n'$ if z_n or z_n' is not real. Just as in case 1, for purposes of proving $zH(z) \in L^2(\mathbb{R})$, we can assume $|\mathrm{Re}\,(z_n) - 2n\pi| < \varepsilon$ and $|\mathrm{Im}\,(z_n) - b| < \varepsilon$ for all $n \geqslant 1$ and similarly for z_n. Furthermore, if $b > 0$ we can assume that none of the z_n or z_n' is real. The number $\varepsilon > 0$ will be specified later but will always be assumed less than $\pi/2$.

As before, assume that $(2m-1)\pi \leqslant z \leqslant (2m+1)\pi$. We leave as a calculus exercise that if $m \geqslant m_0$, where m_0 depends only on b, then for $n \geqslant m+1$

$$(5.71) \qquad \left| \left(1 - \left(\frac{z}{z_n}\right)^2\right)\left(1 - \left(\frac{z}{z_n'}\right)^2\right) \right| \leqslant$$
$$\leqslant \left| \left(1 - \left(\frac{z}{2n\pi + i(b+\theta)}\right)^2\right)\left(1 - \left(\frac{z}{2n\pi - i(b+\theta)}\right)^2\right) \right|$$
$$n \geqslant m+1, \; i\theta = \varepsilon(1+i).$$

Since we only have to estimate $H(z)$ for z large, we can assume that $m \geqslant m_0$.

We proceed as in case 1. We define

$$P_1(z) = \prod_{n=1}^{m-1} \left(1 - \left(\frac{z}{z_n}\right)^2\right) \qquad \text{and} \qquad P_2(z) = \prod_{n=1}^{m-1} \left(1 - \left(\frac{z}{2n\pi + ib}\right)^2\right)$$

and seek to estimate $|P_1(z)P_2(z)^{-1}|$ (assuming $(2m-1)\pi \leqslant z \leqslant (2m+1)\pi$).

If we define $\varepsilon_n = 2n\pi + ib - z_n$, so $|\varepsilon_n| \leqslant \sqrt{2}\varepsilon$, a calculation yields

$$(5.72)\quad |P_1(z)P_2(z)^{-1}| = \prod_{n=1}^{m-1} \left|1 + \frac{\varepsilon_n}{z_n}\right|^2 \left|1 - \frac{\varepsilon_n}{2n + ib - z}\right| \left|1 + \frac{\varepsilon_n}{2n + ib + z}\right| \leqslant$$

$$\leqslant \prod_{n=1}^{m-1} \left(1 + \frac{\sqrt{2}\varepsilon}{|z_n|}\right)^2 \left(1 + \frac{\sqrt{2}\varepsilon}{|2n\pi + ib - z|}\right)\left(1 + \frac{\sqrt{2}\varepsilon}{2n\pi + ib + z}\right).$$

Taking the logarithm of (5.72) and making some simple estimates gives

$$(5.73)\quad \log|P_1(z)P_2(z)^{-1}| \leqslant \sqrt{2}\varepsilon \sum_{n=1}^{m-1}\left(\frac{2}{2n\pi - \varepsilon} + \frac{1}{z - 2n\pi} + \frac{1}{2n\pi + z}\right) \leqslant$$

$$\leqslant \sqrt{2}\varepsilon \sum_{n=1}^{m-1}\left(\frac{2}{2n\pi - \pi} + \frac{1}{2m\pi - \pi - 2n\pi} + \frac{1}{2n\pi + (2m-1)\pi}\right) \leqslant$$

$$\leqslant (\sqrt{2}\varepsilon)\left(\frac{3}{2\pi}\right)\log m + k_1$$

where k_1 is a constant independent of z and m. To obtain (5.73) we have used equation (5.52). We obtain from (5.73) that

$$(5.74)\quad \left|\prod_{n=1}^{m-1}\left(1 - \left(\frac{z}{z_n}\right)^2\right)\right| \leqslant k_2|z|^{\alpha_1}\left|\prod_{n=1}^{m-1}\left(1 - \left(\frac{z}{2n\pi + ib}\right)^2\right)\right|, \quad \alpha_1 = \left(\frac{3\sqrt{2}}{2\pi}\right)\varepsilon.$$

A similar formula holds if z_n in (5.74) is replaced by z_n', so we obtain

$$(5.75)\quad \left|\prod_{n=1}^{m-1}\left(1 - \left(\frac{z}{z_n}\right)^2\right)\left(1 - \left(\frac{z}{z_n'}\right)^2\right)\right| \leqslant$$

$$\leqslant k_3|z|^{\alpha_2}\left|\prod_{n=1}^{m-1}\left(1 - \left(\frac{z}{2n\pi + ib}\right)^2\right)\left(1 - \left(\frac{z}{2n\pi - ib}\right)^2\right)\right|, \quad \alpha_2 \overset{def}{=} \left(\frac{3\sqrt{2}}{\pi}\right)\varepsilon.$$

The constant k_3 in (5.75) is independent of z and m.

It remains to estimate

$$R_1(z) = \prod_{n=m+1}^{\infty}\left(1 - \left(\frac{z}{z_n}\right)^2\right)\left(1 - \left(\frac{z}{z_n'}\right)^2\right)$$

by comparing it with

$$R_2(z) = \prod_{n=m+1}^{\infty}\left(1 - \left(\frac{z}{2n\pi + ib}\right)^2\right)\left(1 - \left(\frac{z}{2n\pi - ib}\right)^2\right).$$

If we use equation (5.71) and simplify we find

$$(5.76)\quad \begin{cases} |R_1(z)R_2(z)^{-1}| \leqslant |S_1(z)S_2(z)| \\ S_1(z) \overset{def}{=} \prod_{n=m+1}^{\infty}\left(1-\dfrac{i\theta}{2n\pi+ib_1}\right)^2\left(1+\dfrac{i\theta}{2n\pi+ib-z}\right)\left(1+\dfrac{i\theta}{2n\pi+ib+z}\right) \\ S_2(z) \overset{def}{=} \prod_{n=m+1}^{\infty}\left(1-\dfrac{i\theta}{2n\pi-ib_1}\right)^2\left(1+\dfrac{i\theta}{2n\pi-ib-z}\right)\left(1-\dfrac{i\theta}{2n\pi-ib+z}\right) \\ i\theta=\varepsilon(1+i),\ b_1=b+\theta\,. \end{cases}$$

To estimate S_1 and S_2 it is natural to take logarithms and use the Taylor series for $\log(1+w)$ when $|w|<1$. If we do this for $S_1(z)$ we find

$$(5.77)\quad \begin{cases} S_1(z)=T_1(z)T_2(z)T_3(z)T_4(z) \\ T_1(z) \overset{def}{=} \exp\left(\sum_{n=m+1}^{\infty} -\dfrac{2i\theta}{2n\pi+ib_1}+\dfrac{i\theta}{2n\pi+ib-z}+\dfrac{i\theta}{2n\pi+ib+z}\right) \\ T_2(z)=\exp\left(\sum_{n=m+1}^{\infty} +\dfrac{2i\theta}{2n\pi+ib_1}+2\log\left(1-\dfrac{i\theta}{2n\pi+ib_1}\right)\right) \\ T_3(z)=\exp\left(\sum_{n=m+1}^{\infty}\log\left(1+\dfrac{i\theta}{2n\pi+ib-z}\right)-\dfrac{i\theta}{2n\pi+ib-z}\right) \\ T_4(z)=\exp\left(\sum_{n=m+1}^{\infty}\log\left(1+\dfrac{i\theta}{2n\pi+ib+z}\right)-\dfrac{i\theta}{2n\pi+ib+z}\right). \end{cases}$$

Each of the functions $T_j(z)$, $2\leqslant j\leqslant 4$, has absolute value dominated by a constant independent of z and m. For example, if we write $\mu=i\theta/(2n\pi+ib-z)$, so $|\mu|<\sqrt{2}/2$ (since $\varepsilon<\pi/2$ and $|2n\pi+ib-z|\geqslant\pi$) we have

$$(5.78)\quad |\log(1+\mu)-\mu| = \left|\sum_{j=2}^{\infty}\frac{(-1)^j\mu^j}{j}\right| \leqslant \frac{1}{2}\sum_{j=2}^{\infty}|\mu|^j=\left(\frac{1}{2}\right)\frac{|\mu|^2}{1-|\mu|}\leqslant \leqslant(2-\sqrt{2})^{-1}|\mu|^2\,.$$

Using (5.78) we see that

$$(5.79)\quad \sum_{n=m+1}^{\infty}\left|\log\left(1+\frac{i\theta}{2n\pi+ib-z}\right)-\frac{i\theta}{2n\pi+ib+z}\right|\leqslant \leqslant(2-\sqrt{2})^{-1}\sum_{n=m+1}^{\infty}\frac{\sqrt{2}}{(2n\pi-z)^2+b^2}\leqslant k_4$$

where k_4 is a constant independent of z and m. A similar argument applies to $T_2(z)$ and $T_4(z)$, so we conclude that there is a constant k_5, independent of z and m, such that

$$|S_1(z)| \leqslant k_5 |T_1(z)|\,. \tag{5.80}$$

To estimate $T_1(z)$ we proceed as in case 1. We have

$$T_1(z) = \exp\left(i\theta \sum_{n=m+1}^{\infty}\left(\frac{-2}{2n\pi + ib_1} + \frac{2}{2n\pi}\right) + \right.$$
$$\left. + \left(\frac{1}{2n\pi + ib - z} - \frac{1}{2n\pi}\right) + \left(\frac{1}{2n\pi + ib + z} - \frac{1}{2n\pi}\right)\right).$$

A simple calculation shows that

$$\exp\left(i\theta\left(\sum_{n=m+1}^{\infty}\frac{-2}{2n\pi + ib_1} + \frac{2}{2n\pi}\right)\right) \leqslant \exp\left(|\theta||b_1| \sum_{n=m+1}^{\infty}\frac{1}{n\pi(2n\pi + \varepsilon)}\right) \leqslant k_6\,, \tag{5.81}$$

where k_6 is a constant independent of m, z and $\varepsilon < \pi/2$. A similar calculation shows that

$$\left|\exp\left(i\theta \sum_{n=m+1}^{\infty}\left(\frac{1}{2n\pi + ib + z} - \frac{1}{2n\pi}\right)\right)\right| = \tag{5.82}$$
$$= \left|\exp\left(i\theta(ib + z)\sum_{n=m+1}^{\infty}\frac{1}{2n\pi(2n\pi + ib - z)}\right)\right| \leqslant$$
$$\leqslant \exp\left((b + z)|\theta| \sum_{n=m+1}^{\infty}\frac{1}{2n\pi(2n\pi + z)}\right) \leqslant$$
$$\leqslant \exp\left(|\theta|(bz^{-1} + 1)\lim_{N\to\infty}\sum_{n=m+1}^{N}\left(\frac{1}{2n\pi} - \frac{1}{2n\pi + 2m\pi + 2\pi}\right)\right) \leqslant k_7\,,$$

where k_7 is a constant independent of $z \geqslant 1$ and m. We have used equation (5.44) to show that

$$\lim_{N\to\infty}\sum_{n=m+1}^{N}\left(\frac{1}{n} - \frac{1}{n + m + 1}\right)$$

is dominated by a constant independent of m.

To estimate the remaining term in $T_1(z)$ observe that

$$\left|\exp\left(i\theta \sum_{n=m+1}^{N}\frac{1}{2n\pi + ib - z} - \frac{1}{2n\pi}\right)\right| \leqslant \tag{5.83}$$
$$\leqslant \exp\left(\sqrt{2}\varepsilon(z + b)\sum_{n=m+1}^{\infty}\frac{1}{(2n\pi - z)(2n\pi)}\right) \leqslant$$
$$\leqslant \exp\left(\sqrt{2}\varepsilon(1 + bz^{-1})\sum_{n=m+1}^{\infty}\frac{1}{2n\pi - z} - \frac{1}{2n\pi}\right) \leqslant (\text{since } z \leqslant (2m + 1)\pi) \leqslant$$
$$\leqslant \exp\left(\sqrt{2}\varepsilon(1 + bz^{-1})\lim_{N\to\infty}\sum_{n=m+1}^{N}\frac{1}{2n\pi - 2m\pi - \pi} - \frac{1}{2n\pi}\right).$$

By using (5.52) and (5.44) we see that

$$\lim_{N\to\infty}\sum_{n=m+1}^{N}\left(\frac{1}{2n\pi-2m\pi-\pi}-\frac{1}{2n\pi}\right)\leqslant\frac{1}{2\pi}\log m+k_8\leqslant\frac{1}{2\pi}\log z+k_8, \tag{5.84}$$

where k_8 is a constant independent of m and z. If the estimate (5.84) is used in (5.83) one obtains

$$\left|\exp\left(i\theta\sum_{n=m+1}^{\infty}\left(\frac{1}{2n\pi+ib-z}-\frac{1}{2n\pi}\right)\right)\right|\leqslant k^9|z|^{\alpha_3}, \qquad \alpha_3=\frac{\sqrt{2}\varepsilon}{2\pi}. \tag{5.85}$$

If we combine (5.81), (5.82) and (5.85) we obtain an estimate for $T_1(z)$, and equation (5.80) then implies

$$|S_1(z)|\leqslant k_{10}|z|^{\alpha_3}, \qquad \alpha_3=\frac{\sqrt{2}\varepsilon}{2\pi}. \tag{5.86}$$

An analogous argument proves that $|S_2(z)|\leqslant k_{11}|z|^{\alpha_3}$, so we obtain from (5.76) that

$$\left|\prod_{n=m+1}^{\infty}\left(1-\left(\frac{z}{z_n}\right)^2\right)\left(1-\left(\frac{z}{z_n'}\right)^2\right)\right|\leqslant \tag{5.87}$$
$$\leqslant k_{12}|z|^{2\alpha_3}\left|\prod_{n=m+1}^{\infty}\left(1-\left(\frac{z}{2n\pi+ib}\right)^2\right)\left(1-\left(\frac{z}{2n\pi-ib}\right)^2\right)\right|.$$

Combining inequalities (5.75) and (5.87) yields

$$\left|\prod_{\substack{n=1\\ n\neq m}}^{\infty}\left(1-\left(\frac{z}{z_n}\right)^2\right)\left(1-\left(\frac{z}{z_n'}\right)^2\right)\right|\leqslant \tag{5.88}$$
$$\leqslant k_{13}|z|^{\alpha}\left|\prod_{\substack{n=1\\ n=m}}^{\infty}\left(1-\left(\frac{z}{2n\pi+ib}\right)^2\right)\left(1-\left(\frac{z}{2n\pi-ib}\right)^2\right)\right|, \qquad \alpha=\left(\frac{4\sqrt{2}\varepsilon}{\pi}\right).$$

The constant k_{13} in (5.88) is independent of z, m and $\varepsilon<\pi/2$.

The remainder of the proof closely parallels case 1. Define $\theta_m(z)$ by

$$\theta_m(z)=\left|\left(1-\left(\frac{z}{z_m}\right)^2\right)\left(1-\left(\frac{z}{z_m'}\right)^2\right)\left(1-\left(\frac{z}{2m\pi+ib}\right)^2\right)^{-1}\left(1-\left(\frac{z}{2m\pi-ib}\right)^2\right)^{-1}\right|.$$

If we recall that $(2m-1)\pi\leqslant z\leqslant(2m+1)\pi$, a calculation shows that

$$|\theta_m(z)|\leqslant k_{14}|(2m\pi-ib-z)^{-1}(2m\pi+ib-z)^{-1}|$$

where k_{14} is independent of z and m. If $b > 0$ and if we use (5.88) and (5.39) (with $a = ib$) and the above estimate we find

$$(5.89)\quad \left| z\prod_{n=1}^{\infty}\left(1-\left(\frac{z}{z_n}\right)^2\right)\left(1-\left(\frac{z}{z_n'}\right)^2\right)\right| \leqslant$$

$$\leqslant k|z|^{1+\alpha}\left(1+\left(\frac{z^2}{b^2}\right)^{-1}\theta_m(z)\left|\sin\left(\frac{z+ib}{2}\right)\sin\left(\frac{ib-z}{2}\right)\right|\right) \leqslant$$

$$\leqslant k'|z|^{1+\alpha}\left(1+\frac{z^2}{b^2}\right)^{-1}\left|\sin\left(\frac{w+ib}{2}\right)\sin\left(\frac{ib-w}{2}\right)(ib+w)^{-1}(ib-w)^{-1}\right|$$

$$\alpha = \frac{4\sqrt{2}\varepsilon}{\pi} \quad \text{and} \quad w = z - 2m\pi\,.$$

The constants k and k' are independent of z and m. An examination of (5.89) shows that there is a different constant k, independent of z and m, such that

$$\left| z\prod_{n=1}^{\infty}\left(1-\left(\frac{z}{z_n}\right)^2\right)\left(1-\left(\frac{z}{z_n'}\right)^2\right)\right| \leqslant k|z|^{\alpha-1}$$

for $|z|$ large enough. Thus to insure $z^{-1}F(z) \in L^2(\mathbb{R})$ in this case it suffices to assume $\alpha < \frac{1}{2}$.

The above proof has to be modified slightly in the case $b = 0$, since then one must use (5.40) instead of (5.39). We leave the details to the reader. ■

We need one more lemma in order to prove theorem 5.1.

LEMMA 5.6. *Let $u\colon [0, 1] \to \mathbb{R}$ be a real-valued, continuously differentiable function such that $u(1) \neq 0$ and let $F(z)$ be as defined in Definition* 5.1. *Then given any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$(5.90)\quad |z^{-1}F(z)| \leqslant C_\varepsilon \exp\big((1+\varepsilon)|\operatorname{Im} z|\big)\,, \quad z \text{ complex}\,.$$

PROOF. By relabelling the zeros of $F(z)$ one can write

$$(5.91)\quad z^{-1}F(z) = z\prod_{n=1}^{\infty}\left(1-\left(\frac{z}{w_n}\right)^2\right),$$

where $|w_n - n\pi| \leqslant L$ and L is a constant independent of n. Notice that the zeros of $F(z)$ are now regarded as indexed by all positive integers instead of just the even integers. It follows from equation (5.91) that

$$(5.92)\quad |z^{-1}F(z)| \leqslant |z|\prod_{n=1}^{\infty}\left(1+\frac{|z|^2}{|w_n|^2}\right).$$

However, the same calculation used on the bottom of page 86 and the top of page 87 in [9] shows that given $\varepsilon > 0$, there is a constant C_ε such that the right hand side of (5.91) is dominated by $C_\varepsilon \exp\big((1+\varepsilon)|z|\big)$. In particular, we have

$$|y^{-1}F(iy)\exp\big(i(1+\varepsilon)z\big)| \leqslant C_\varepsilon, \qquad y \geqslant 0\,. \tag{5.93}$$

The argument in Lemma 5.5 actually showed that $z^{-1}F(z)$ approaches zero as z approaches $\pm\infty$ through the reals, so by increasing C_ε we can assume

$$|z^{-1}F(z)\exp\big(i(1+\varepsilon)z\big)| \leqslant C_\varepsilon, \qquad z \text{ real}\,. \tag{5.94}$$

Since we have already shown (equation (5.92)) that $z^{-1}F(z)$ is of exponential type, the Phragmén-Lindelöf theorem implies that

$$|z^{-1}F(z)\exp(1+\varepsilon)z| \leqslant C_\varepsilon, \qquad \operatorname{Im}(z) \geqslant 0 \tag{5.95}$$

which is the desired inequality when $\operatorname{Im}(z) \geqslant 0$. Since $z^{-1}F(z)$ is odd, we obtain (5.90) for general z from the case $\operatorname{Im}(z) \geqslant 0$. ■

We are finally ready to prove Theorem 5.1.

PROOF OF THEOREM 5.1. We have already seen (Lemma 5.3) that the closed linear span of A is $L^2[-1,1]$. If A_1 denotes A with one element, say $x^j e^{\zeta x}$, removed, we have to show the closed linear span of A_1 is not all of $L^2[-1,1]$. Define $\Phi(z)$ by

$$\Phi(z) = \left(1 - \frac{z}{\zeta}\right)^{-1} F(z)\,.$$

Clearly $\Phi(z)$ is an entire function and $\lim_{|z|\to\infty} \Phi(z)\big(z^{-1}F(z)\big)^{-1} = 1$, so Lemma 5.5 implies that $\Phi|\mathbb{R}$ is in $L^2(\mathbb{R})$ and Lemma 5.6 implies that $|\Phi(z)| \leqslant C_\varepsilon \exp\big((1+\varepsilon)|\operatorname{Im}(z)|\big)$ for every $\varepsilon > 0$. The Paley-Wiener theorem implies that $\Phi(z) = \hat{f}(z)$, where $f \in L^2(\mathbb{R})$ and $f(x)$ has support in the interval $[-1,1]$. But this means $f \in L^2[-1,1]$ and f is orthogonal to every element of A_1, so the closed linear span of A_1 is not all of $L^2[-1,1]$. ■

REMARK 5.5. In all of our previous work we have assumed that $u\colon [0,1] \to \mathbb{R}$ is C^1 and $u(1) \neq 0$. Analogues of Theorem 5.1 and 5.2 hold if $u\colon [0,1] \to \mathbb{R}$ is C^k, $k \geqslant 1$, $u^{(j)}(1) = 0$ for $0 \leqslant j < k-1$ and $u^{(k-1)}(1) \neq 0$; however, the proofs are not completely mechanical extensions of the previous work, and for reasons of length we have restricted ourselves to the case $u(1) \neq 0$. In any event, if $u\colon [0,1] \to \mathbb{R}$ is only continuous and

L: $C[0,1] \to C[0,1]$ is defined by equation (5.2), a simple limiting argument (using Theorem 5.2) shows that every complex number $\mu = \lambda\hat{u}(z)$, where $\hat{u}(z) = \hat{u}(-z)$, is in the spectrum of L. The difficulty is to determine whether there are other eigenvalues and to find the algebraic multiplicity of μ.

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