ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

MARVIN ZEMAN

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 7, nº 2 (1980), p. 257-285

<http://www.numdam.org/item?id=ASNSP_1980_4_7_2_257_0>

On the Uniqueness of the Cauchy Problem for Partial Differential Operators with Multiple Characteristics (*).

MARVIN ZEMAN

Introduction.

We are concerned in this paper with the study of uniqueness in the Cauchy problem for partial differential equations whose real characteristics have multiplicity more than one or whose non-real characteristics have multiplicity more than two. The case where the real characteristics, if any, are simple and the non-real characteristics, if any, are at most double has been studied by, among others, A. P. Calderón [1], [2], L. Hörmander [6], S. Mizohata [11] and R. N. Pederson [16]. In our case there have been results by P. M. Goorjian [5], W. Matsumoto [10], M. Sussman [18], K. Watanabe [20], K. Watanabe and C. Zuily [21] and M. Zeman [22], [23]. The basic difference between the two cases is that in our case some condition has always been imposed on the lower order terms of the equation. These conditions take into account the counterexamples of P. Cohen [4], A. Plís [17] and Hörmander [8].

In Zeman [22] we assumed that the subprincipal symbol vanishes to a certain order on the characteristic set. In this paper we will show that a condition complementary to the above is also sufficient, namely that the subprincipal symbol does *not* vanish at all on the characteristic set. This extends to differential equations having characteristics of arbitrary constant multiplicity the result presented by Matsumoto [10] who dealt only with characteristics all having the same multiplicity.

(*) Work partially supported by NSF Grant No. MCS78-01448.
Pervenuto alla Redazione il 13 Marzo 1978 ed in forma definitiva il 10 Luglio 1979.

As in all of the papers listed above, the proof will involve a Carleman estimate, a weighted L_2 inequality analogous to an L_1 inequality introduced to the study of uniqueness of the Cauchy problem by T. Carleman [3].

§ 1. – First, recall the problem.

Let $P(x, t, \partial_x, \partial_t) = P_m(x, t, \partial_x, \partial_t) + P_{m-1}(x, t, \partial_x, \partial_t) + \ldots$ be a linear partial differential operator of order m and the P_i homogeneous of order i in $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$ and $t \in \boldsymbol{R}^1$. Let $P_m(x, t, \xi, \tau)$ be the principal symbol of P where $\xi \in \boldsymbol{R}^n$ and $\tau \in \boldsymbol{R}^1$.

Assume the hyperplane $t = 0$ is non-characteristic at the origin with respect to P, i.e., $P_m(0, 0, 0, 1) \neq 0$. The Cauchy problem is to find a solution v of $Pv = f$ in a neighborhood of the origin with given (say homogeneous) Cauchy data on the plane $t = 0$: $\partial_t^j v|_{t=0} = 0$, $j = 0, \ldots, m-1$.

We shall make use of the familiar multi-index notation. See, for instance, Hörmander [7]. $S_\xi^{n-1} = \{\xi\colon |\xi| = 1\}$ is the unit sphere for $\xi \in \boldsymbol{R}^n$. L_x^γ denotes the class of homogeneous pseudo-differential operators of order γ in the x-variables and S_x^γ is its corresponding symbol space. See J. J. Kohn and L. Nirenberg [9] for more details.

$L_{x,t}^\gamma$ is the class of operators differential in t and pseudodifferential in x, of order $\gamma = \alpha + \beta$ in (x, t), where α is the order of the operator in t and $\beta \geqslant 0$ is the order of the operator in x. $S_{x,t}^\gamma$ is its symbol space. $L_x^{\gamma,m}$ is the class of pseudo-differential coperators of order γ in the x-variables whose symbol space $S_x^{\gamma,m}$ consists of functions $a(x, t, \xi)$ of the form $a_0(x, t, \xi)|\xi|^\gamma + a_1(x, t, \xi)|\xi|^{\gamma-1/m} + a_2(x, t, \xi)|\xi|^{\gamma-2/m} + \ldots$ where $a_j(x, t, \xi) \in S_x^0$.

(u, v) is the L_2 scalar product of u and v; $\|u\|$ is the corresponding L_2 norm of u. $|||u|||^2 = \int_0^T \|u\|^2 \exp[k(t-T)^2]dt$ where $\|\cdot\|$ is the L_2 norm in the x-variables. $\boldsymbol{H}_m$ is the Hilbert space with norm given by $\|u\|_m^2 = \int (1 + |\xi|^2)^m |\hat{u}(\xi)|^2 d\xi$ where $\hat{u}$ is the Fourier transform of u. $|||u|||_s^2 = \sum_{r=0}^{(s)} \int_0^T \|D_t^r u\|_{s-r}^2 \exp[k(t-T)^2]dt$, where $\|\cdot\|_s$ is the H_s norm in the x-variables and (s) is the smallest integer greater than or equal to s.

$[A, B] = AB - BA$. The letters l.h.s. and r.h.s. will stand for « left-hand side » and « right-hand side » respectively. Finally, C will denote any constant and may vary from line to line.

Since $t = 0$ is non-characteristic at the origin with respect to P we may assume that the coefficient of D_t^m in P_m is 1. It is convenient to make a local transformation of variable so that the surface $t = 0$ is transformed to a convex surface $s\colon t = \alpha \sum_{j=1}^n x_j^2$ where $\alpha > 0$ is constant. The conditions that we impose on the operator P will remain invariant under this change of variable.

§ 2. – We consider the following type of operator

$$P(x, t, \partial_x, \partial_t) = P_m(x, t, \partial_x, \partial_t) + P_{m-1}(x, t, \partial_x, \partial_t) + R_{m-2}(x, t, \partial_x, \partial_t),$$

where we assume the coefficients of P_m and P_{m-1} are real (see the remark in § 6) and, for simplicity, C^∞. While P_m and P_{m-1} are homogeneous in x and t, R_{m-2} need not be.

The underlying assumption throughout this paper is that the multiplicity of the characteristics is constant: if τ_1 and τ_2 are distinct zeros of $P_m(x, t, \xi, \tau) = 0$ on $|\xi| = 1$, then $|\tau_1 - \tau_2| \geqslant \varepsilon$, where ε is a fixed positive number independent of x, t and ξ. Hence we deal with operators whose principal symbol $P_m(x, t, \xi, \tau)$ can be written in the form

$$P_m(x, t, \xi, \tau) = \prod_{i=1}^{p} [\tau - \lambda_i(x, t, \xi)]^{r_i}, \qquad \sum_{i=1}^{p} r_i = m,$$

where $\lambda_i(x, t, \xi)$ are the characteristic roots of P.

Since P_m has real coefficients the characteristic roots are either real or non-real, i.e. either $\operatorname{Im} \lambda_i(x, t, \xi) \equiv 0$ or $|\operatorname{Im} \lambda_i(x, t, \xi)| \geqslant \varepsilon$ for $(x, t, \xi) \in \Omega \times S_\xi^{n-1}$, where $\Omega = \{(x, t) \colon |x| \leqslant \tilde{r}, 0 \leqslant t \leqslant T\}$, for some $\tilde{r}$ and T.

We are now ready to state the main results. Assume the following condition on the lower order terms:

(A) $$P'_{m-1}(0, 0, \xi, \tau)|_{\tau = \lambda_j(0,0,\xi)} \neq 0 \qquad \text{for all } \xi \in S_\xi^{n-1}, \text{ if } r_j \geqslant 2,$$

where $P'_{m-1}(x, t, \xi, \tau)$ is the subprincipal symbol of the operator P defined by

$$P'_{m-1}(x, t, \xi, \tau) = P_{m-1} + \frac{i}{2}\left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\frac{\partial}{\partial \xi_j} P_m + \frac{\partial}{\partial t}\frac{\partial}{\partial \tau} P_m\right).$$

It is a standard fact that $P'_{m-1}(x, t, \xi, \tau)$ is invariantly defined on the characteristic set $\sum = \{(x, t, \xi, \tau) \colon \tau = \lambda_j(x, t, \xi), r_j \geqslant 2\}$.

We then have the following Carleman estimate.

THEOREM 1. *Suppose $t = 0$ is non-characteristic at the origin with respect to P (which is described above). Suppose also that condition* (A) *is satisfied. Then there exists a constant C independent of u such that for $\tilde{r}$, T and k^{-1} suf-*

ficiently small, the following estimate holds:

$$k|||u|||^2_{m-2+1/r} \leqslant C|||Pu|||^2 \tag{2.1}$$

for all $u \in C_0^\infty(\Omega)$, *where* $\Omega = \{(x, t) \colon |x| \leqslant \tilde{r}, 0 \leqslant t \leqslant T\}$ *and* $r = \max_{1 \leqslant i \leqslant p} r_i$.

Theorem 1 is the basic step in the proof of

THEOREM 2. *Suppose the conditions of theorem 1 are satisfied. Then there is a neighborhood* Ω' *of the origin containing* Ω *such that if* $u \in H^{\text{loc}}_{(m)}(\Omega')$ *satisfies* $Pu = 0$ *and* $u = 0$ *in* $\{(x, t) \colon (x, t) \in \Omega', t < 0\}$, *then* $u \equiv 0$ *in* Ω.

PROOF. The proof of uniqueness in the Cauchy problem via a Carleman estimate is standard. See for instance L. Nirenberg [14].

REMARK 1. Although the assumption that the coefficients of

$$P_m(x, t, \partial_x, \partial_t) + P_{m-1}(x, t, \partial_x, \partial_t)$$

are real is not necessary, the theorem is *not* true without *any* assumptions. P. Cohen has presented the following example (see L. Hörmander [7], section 8.9.2) which shows that some conditions on the coefficients of $P_m + P_{m-1}$ are necessary for uniqueness in the Cauchy problem to hold: there is non-uniqueness for the Cauchy problem associated with the operator

$$P = \partial_t^r u + a(x, t)\,\partial_x u\,, \qquad r \text{ an integer } \geqslant 1\,,$$

for some $a(x, t) \in C^\infty(\boldsymbol{R}^2)$.

REMARK 2. For technical reasons, we assumed that condition (A) holds for all λ_i such that $r_i \geqslant 2$. However, if some of the non-real roots are at most double, a weakened version of condition (A) can be shown to be sufficient, namely:

$$P'_{m-1}(0, 0, \xi, \tau)|_{\tau=\lambda_i(0,0,\xi)} \neq 0 \qquad \text{for all } \xi \in S_\xi^{n-1} \tag{A$'$}$$

only for those λ_i satisfying

(*a*) $r_i \geqslant 2$, if λ_i is real,

(*b*) $r_i \geqslant 3$, if λ_i is non-real.

We shall prove this result in a future paper.

EXAMPLE. Let $P(x, t, \partial_x, \partial_t) = \partial_t^2 + a(x, t)\partial_x + b(x, t)\partial_t + c(x, t)$ where a and b are real and $(x, t) \in \boldsymbol{R}^2$, $|x| \leqslant \tilde{r}$, and $0 \leqslant t \leqslant T$, for $\tilde{r}$ and T sufficiently small. In contrast to P. Cohen's non-uniqueness example (see Remark 2), we have uniqueness in the Cauchy problem if $a(0, 0) \neq 0$. It is worth noting that if $a(x, t) \equiv 0$, then uniqueness was already proved in this case. See Zeman [22], Theorem 2.

§ 3. – The basic idea underlying the proof of Theorem 1 is estimating $|||Pu|||$ by replacing the operator P with a product of *distinct* first order factors. However, the factorization of $P(x, t, \xi, \tau)$ will be valid only for ξ in $\{\xi\colon |\xi| \geqslant R$, for some fixed $R\}$. In order to make the proof work in general, we shall introduce a simplified partition of unity in $\boldsymbol{R}_\xi^n$.

Since P is restricted to functions whose supports lie in Ω, in Ω the value of Pu is unchanged if the coefficients of P are multiplied by a C^∞ non-negative cut-off function having compact support and identically equal to one in some neighborhood of Ω. Hence we may assume the coefficients of P have compact support.

Now choose θ_1 so that $\theta_1(s) \in C^\infty(\boldsymbol{R}_+^1)$, $0 \leqslant \theta_1(s) \leqslant 1$, $\theta_1(s) = 0$ for $s \geqslant R + 1$ and $\theta_1(s) = 1$ for $s \leqslant R$. Let $\theta_2(s) = 1 - \theta_1(s)$. Choose another C^∞ non-negative cut-off function of x having compact support, $\varphi(x)$, which is identically equal to 1 in some neighborhood of $\{x\colon |x| \leqslant \tilde{r}\}$. Now form the functions $\psi_1(x, \xi) = \varphi(x)\theta_1(|\xi|)$ and $\psi_2(x, \xi) = \varphi(x)\theta_2(|\xi|)$. The operators $\psi_1(x, D_x)$ and $\psi_2(x, D_x)$ are properly supported and belong to L_x^0, and for $(x, t) \in \Omega$, $u(x, t) = \psi_1(x, D_x)u + \psi_2(x, D_x)u$.

Let $\tilde{P}(x, t, \partial_x, \partial_t) = P_m(x, t, \partial_x, \partial_t) + P_{m-1}(x, t, \partial_x, \partial_t)$. First, we shall estimate $|||\tilde{P}u|||$ in the two cases: (a) for $u_1 = \psi_1(x, D_x)u$ where $\operatorname{supp} \psi_1(x, \xi) \subset \subset \{\xi\colon |\xi| \leqslant R + 1\}$, (b) for $u_2 = \psi_2(x, D_x)u$ where $\operatorname{supp} \psi_2(x, \xi) \subset \{\xi\colon |\xi| \geqslant R\}$.

In section 7 we shall provide the proof in the general case. We need the following two lemmas.

LEMMA 3.1. *Let s, s' be two real numbers such that $s' < s$, $-n/2 \leqslant s$. Then to every $\varepsilon > 0$ we can choose T and $\tilde{r}$ so that $|||u|||_{s'} \leqslant \varepsilon |||u|||_s$ for $u \in H_s(\Omega)$ where $\Omega = \{(x, t)\colon |x| \leqslant \tilde{r}, 0 \leqslant t \leqslant T\}$.*

PROOF. See F. Trèves (theorem 0.41 in [19]).

LEMMA 3.2. *Let R belong to $L_{x,t}^\gamma$, $\gamma \leqslant m - 2 + 1/r$, and let S be a partial differential operator of order $m - 2$ with bounded measurable coefficients. If estimate (2.1) is true for P it will still be true if P is replaced by $P + R + S$.*

PROOF. Standard. The details are left to the reader.

We are now ready to estimate $|||\tilde{P}u|||$ for case (a).

PROPOSITION 3.3. *Let ψ_1 be defined as above. Let $\tilde{P}(x, t, \partial_x, \partial_t) = P_m(x, t, \partial_x, \partial_t) + P_{m-1}(x, t, \partial_x, \partial_t)$. Suppose $t = 0$ is non-characteristic at the origin with respect to $\tilde{P}$. Then there exists a constant C independent of u such that for $\tilde{r}$, T, k^{-1} sufficiently small, $k|||\psi_1 u|||^2_{m-1} \leqslant C|||\tilde{P}\psi_1 u|||^2$, for $u \in C_0^\infty(\Omega)$, where $\Omega = \{(x, t)\colon |x| \leqslant \tilde{r},\ 0 \leqslant t \leqslant T\}$.*

PROOF. Since $\psi_1(x, \xi)$ has compact support, then $\|b_j(x, t, \partial_x)\psi_1 u\| \leqslant C(R + 2)^j \|\psi_1 u\|$ for $(x, t) \in \Omega$, for any $b_j(x, t, \partial_x) \in L_j^x$.

This allows us to perturb the coefficients of $\tilde{P}$. If $P_m(x, t, \partial_x, \partial_t) = \partial_t^m + \sum_{j=1}^m a_j(x, t, \partial_x)\partial_t^{m-j}$, then

$$P_m = \partial_t^m + \sum_{j=1}^m \tilde{a}_j(x, t, \partial_x)\partial_t^{m-j} + \sum_{j=1}^m (a_j - \tilde{a}_j)\partial_t^{m-j} = \hat{P}_m(x, t, \partial_x, \partial_t) + \sum_{j=1}^m b_j(x, t, \partial_x)\partial_t^{m-j}, \tag{3.1}$$

where $b_j = a_j - \tilde{a}_j$.

We choose $\tilde{a}_j$ to be real of order j so that $\hat{P}_m(x, t, \partial_x, \partial_t)$ has simple characteristics. We now apply the following result of Calderón [1], which dealt with real operators having simple characteristics:

$$C|||\hat{P}_m u_1|||^2 \geqslant k|||u_1|||^2_{m-1}, \quad \text{where } u_1 = \psi_1 u,\ u \in C_0^\infty(\Omega). \tag{3.2}$$

(A proof of the estimate in the form we have it can be found in L. Nirenberg [14].) (3.1) implies that

$$|||P u_1|||^2 \geqslant |||\hat{P}_m u_1|||^2 - C\sum_{j=1}^m (R + 2)^j |||\partial_t^{m-j} u_1|||^2 .$$

Hence by (3.2) we have

$$C_1|||P u_1|||^2 \geqslant k|||u_1|||^2_{m-1} - C_2 \sum_{j=1}^m |||\partial_t^{m-j} u_1|||^2 . \tag{3.3}$$

Choosing k large enough so that $k > C_2$ and invoking lemma 3.2 we can absorb the second term of the r.h.s. of (3.3) into the first term and get

$$C|||P_m u_1|||^2 \geqslant k|||u_1|||^2_{m-1} . \tag{3.4}$$

By a variation of lemma 3.2 we can replace $P_m(x, t, \partial_x, \partial_t)$ in (3.4) by $P_m(x, t, \partial_x, \partial_t) + P_{m-1}(x, t, \partial_x, \partial_t)$ without affecting the estimate. Hence,

$$C|||\tilde{P} u_1|||^2 \geqslant k|||u_1|||^2_{m-1} .$$

§ 4. – We shall now consider the estimate of $|||\tilde{P}u|||$ for case (*b*). The proof in this case will follow after several preliminary steps. The first step entails replacing $P_m(x, t, \partial_x, \partial_t) + P_{m-1}(x, t, \partial_x, \partial_t)$ with a product of first order factors, modulo terms of order $\leqslant m - 1 - 1/r$. We shall use a method similar to that presented by S. Mizohata and Y. Ohya [12], [13] while studying the well-posedness of Cauchy problem for partial differential equations with multiple characteristics.

We let $\partial_j = D_t - \lambda_j(x, t, D_x)$, $1 \leqslant j \leqslant p$, where

$$\lambda_j(x, t, D_x)u(x, t) = \left(\frac{1}{2\pi}\right)^{n/2}\int \exp[ix\cdot\xi]\,\lambda_j(x, t, \xi)\,\hat{u}(\xi, t)\,d\xi\,.$$

$\partial_0 = I$, the identity.

$\Pi_m = \partial_1^{r_1}\partial_2^{r_2} \dots \partial_p^{r_p}$, where we assume without loss of generality that $r_1 \geqslant r_2 \geqslant \dots \geqslant r_p$.

Define Π_{m-1} by

$$i^{m-1}\Pi_{m-1} = i^m\big(P_m(x, t, D_x, D_t) - \Pi_m\big) + i^{m-1}P_{m-1}\,,$$

where $i = \sqrt{-1}$. In association with λ_j and ∂_j we define $\tilde{\lambda}_j$ and Δ_j, $0 \leqslant j \leqslant m$, as follows: let $\tilde{m}_0 = 0$, $\tilde{m}_1 = r_p$, $\tilde{m}_2 = r_p + r_{p-1}, \dots, \tilde{m}_j = r_p + r_{p-1} + \dots + r_{p-j+1}, \dots, \tilde{m}_p = r_p + \dots + r_1 = m$.

Let $\Delta_0 = \partial_0$. For $\tilde{m}_k + 1 \leqslant j \leqslant m_{k+1}$, where $0 \leqslant k \leqslant p-1$, $\tilde{\lambda}_j(x, t, D_x) = \lambda_{p-k}(x, t, D_x)$, and $\Delta_j = D_t - \tilde{\lambda}_j(x, t, D_x) = \partial_{p-k}$. (i.e. $\Delta_1 = \Delta_2 = \dots = \Delta_{r_p} = \partial_p$, $\Delta_{r_p+1} = \dots = \Delta_{r_p+r_{p-1}} = \partial_{p-1}, \dots, \Delta_{r_p+\dots+r_2+1} = \dots = \Delta_{m-1} = \Delta_m = \partial_1$.)

The ∂_j (and their counterparts Δ_j) are in a sense directional derivatives, and can be used as derivatives, as displayed in the following lemmas.

LEMMA 4.1. (*a*) *For all* $j \geqslant 0$ *there exist* $a_i(x, t, D_x) \in L_x^i$ *such that* $\Delta_j\Delta_{j-1}\dots\Delta_0 = \sum_{i=0}^{j} a_i(x, t, D_x)D_t^{j-i} + T_1$, *where* T_1 *represents the lower order terms, i.e.* $T_1 = \sum_{i=0}^{j-1} c_i(x, t, D_x)D_t^{j-i-1}$, *order* $c_i \leqslant i$.

(*b*) *Conversely, there exist* $b_i(x, t, D_x) \in L_x^i$ *such that*

$$D_t^i = \sum_{i=0}^{j} b_i(x, t, D_x)\Delta_{j-i}\Delta_{j-i-1}\dots\Delta_0 + T_2\,.$$

where

$$T_2 = \sum_{i=0}^{j-1} d_i(x, t, D_x)\Delta_{j-i-1}\Delta_{j-i-2}\dots\Delta_0\,, \qquad \textit{order } d_i \leqslant i\,.$$

(We use the convention that $\Delta_k \equiv I$ for $k \leqslant 0$.)

PROOF. Proof is by induction on j. Details are left to the reader.

COROLLARY 4.2. *Every operator belonging to* $L^k_{x,t}$, *k a non-negative integer can be written as* $\sum_{i=0}^{k} c_i(x, t, D_x)\Delta_{k-1}\Delta_{k-i-1}\dots\Delta_0+T$, *where* $c_i \in L^i_x$ *and* $T\in L^{k-1}_{x,t}$.

COROLLARY 4.3. *There exist* $a_{m-i}\in L^{m-i}_x$ *such that*

$$\Pi_{m-1} = a_{m-1}(x, t, D_x) + a_{m-2}(x, t, D_x)\Delta_1 + \dots + \\ + a_{m-i}(x, t, D_x)\Delta_{i-1}\dots\Delta_0 + \dots + a_0\Delta_{m-1}\dots\Delta_0 + T,$$

where $T\in L^{m-2}_{x,t}$.

DEFINITION 4.4. For $1\leqslant j\leqslant p$,

$$\begin{aligned} L_j(x, t, \xi) &= \Pi^0_{m-1}(x, t, \xi, \lambda_j(x, t, \xi)) \\ &= a_{m-1}(x, t, \xi) + a_{m-2}(\lambda_j(x, t, \xi) - \tilde{\lambda}_1(x, t, \xi)) + \dots \\ &+ a_{m-\tilde{m}_j-1}(\lambda_j - \tilde{\lambda}_{\tilde{m}_j})\dots(\lambda_j - \tilde{\lambda}_1), \end{aligned}$$

where $\Pi^0_{m-1}(x, t, \xi, \tau)$ is the principal symbol of $\Pi_{m-1}(x, t, D_x, D_t)$.

LEMMA 4.5. $L_j(x, t, \xi) = P'_{m-1}(x, t, \xi, \tau)|_{\tau=\lambda_j(x,t,\xi)}$.

PROOF. See Y. Ohya [15], section 3.

COROLLARY 4.6. *Suppose condition* (A) *is satisfied. For* $r_j\geqslant 2$,

$$|L_j(x, t, \xi)|\geqslant \sigma_0|\xi|^{m-1}, \quad \sigma_0>0, \tag{4.1}$$

for $(x, t, \xi)\in U\times(\boldsymbol{R}^n_\xi\setminus\{0\})$, *where U is some neighborhood of the origin.*

Before we replace $\tilde{P}$ with a product of first order factors, modulo an operator belonging to $L^{m-1-1/r}_{x,t}$, we shall introduce the module V over L^0_x, which is associated with the operator $\Pi_m = \partial_1^{r_1}\dots\partial_p^{r_p}$.

V is generated by monomial operators which are formed as follows: we first describe the operators which generate $V^{(1)}$. They are the operators $a_{ij}(\Pi_m/\partial_i\partial_j)$, where a_{ij} is an arbitrary properly supported operator in $L^{1-1/r^{ij}, r^{ij}}_x$, $r^{ij} = \min(r_i, r_j)$. i may equal j. $\Pi_m/(\partial_i\partial_j)$ is the operator formed by omitting ∂_i and ∂_j from the product in Π_m. Denote this collection of generating operators as $V'^{(1)}$.

$V^{(2)}$ is formed a bit differently from $V^{(1)}$. An operator v_2 in $V'^{(2)}$, the collection of operators generating the module $V^{(2)}$, is of the form $v_2 = b_{i,2}(v_1/\partial_i)$, where v_1 is some operator in $V'^{(1)}$ and properly supported $b_{i.2} \in L_x^{1-1/r_i, r_i}$, $1 \leqslant i \leqslant p$. $V^{(3)}$ is formed the same way as $V^{(2)}$. $v_s \in V'^{(3)}$ is of the form $v_3 = b_{i,3}(v_2/\partial_i)$, where $v_2 \in V'^{(2)}$ and properly supported $b_{i,3} \in L_x^{1-1/r_i, r_i}$. We go on in this manner to form the module $V^{(4)}$, $V^{(5)}$, ... and their corresponding generating sets $V'^{(4)}$, $V'^{(5)}$, ... Finally, let $V' = \bigcup_k V'^{(k)}$. V is the module generated by the operators in V'.

Modulo terms belonging to $L_{x,t}^{m-1-1/r}$, we shall now replace

$$\tilde{P} = P_m(x, t, \partial_x, \partial_t) + P_{m-1}(x, t, \partial_x, \partial_t)$$

with a product of first order factors.

PROPOSITION 4.7. *Let $u_2 = \psi_2(x, D_x)u$, where* supp $\psi_2(x, \xi) \subset \{\xi : |\xi| \geqslant R\}$ *and $u \in C_0^\infty(\Omega)$, then under condition* (A), *$\tilde{P}u_2 = \tilde{\Pi}u_2 + Tu_2 + Ru_2$, where $T \in L_{x,t}^{m-1-1/r}$ is a member of the module V, $R \in L_{x,t}^{m-2}$,*

$$\tilde{\Pi} = \partial_1^{(1)}\partial_1^{(2)} \dots \partial_1^{(r_1)} \dots \partial_p^{(r_p)},$$

where

$$\partial_i^{(j)} = i\big(D_t - \lambda_i^j(x, t, D_x)\big), \quad 1 \leqslant i \leqslant p, \ 1 \leqslant j \leqslant r_i,$$

$$\lambda_i^j(x, t, \xi) = \lambda_i(x, t, \xi) + \sum_{k=1}^{\infty} \nu_{i,k}^j(x, t, \xi)|\xi|^{1-k/r_i}, \tag{4.2}$$

where $\nu_{i,k}^j \in S_x^0$, and where in particular,

$$\nu_{i,1}^j(x, t, \xi) - \nu_{i,1}^k(x, t, \xi) \neq 0 \quad \textit{for } (x, t, \xi) \in \Omega \times S_\xi^{n-1} \textit{ if } r_i > 1 \textit{ and } j \neq k. \tag{4.3}$$

PROOF.

$$\tilde{P}(x, t, \partial_x, \partial_t) = i^m \Pi_m + i^{m-1}\Pi_{m-1} =$$
$$= i^m \partial_1^{r_1}\partial_2^{r_2} \dots \partial_p^{r_p} + i^{m-1}[a_{m-1}(x, t, D_x) + a_{m-2}(x, t, D_x)\Delta_1 + \dots].$$

Hence, we are seeking the roots of

$$i^m \prod_{j=1}^{p} (\tau - \lambda_j)^{r_j} + i^{m-1} \sum_{j=1}^{m} a_{m-j}(x, t, \xi)(\tau - \tilde{\lambda}_{j-1}) \dots (\tau - \tilde{\lambda}_0) = 0 \tag{4.4}$$

where $a_{m-j}(x, t, \xi) \in S_x^{m-j}$.

Since $a_{m-j}(x, t, \xi)$ is homogeneous of order $m - j$,

$$a_{m-j}(x, t, \xi) = a_{m-j}\left(x, t, \frac{\xi}{|\xi|}\right)|\xi|^{m-j}.$$

Let $a_{m-j}(x, t, \xi/|\xi|) = a'_{m-j}(x, t, \xi)$. Then (4.4) becomes

$$(4.5) \quad i^m \prod_{j=1}^{p} [(\tau - \lambda_j(x, t, \xi)]^{r_j} + i^{m-1} \sum_{j=1}^{m} a'_{m-j}(x, t, \xi) |\xi|^{m-j} (\tau - \tilde{\lambda}_{j-1}) \dots (\tau - \tilde{\lambda}_0) = 0 .$$

Let $\tau = \lambda_i + \nu$ and solve (4.5) for ν. Then

$$(4.6) \quad \nu^{r_i} \prod_{\substack{k=1 \\ k \neq i}}^{p} (\lambda_i - \lambda_k + \nu)^{r_k} = +i \sum_{j=1}^{m} a'_{m-j}(x, t, \xi) |\xi|^{m-j} (\lambda_i - \tilde{\lambda}_{j-1} + \nu) \dots (\lambda_i - \tilde{\lambda}_0 + \nu) .$$

Now multiply out both sides of (4.6):

$$\text{l.h.s.} = \nu^{r_i} \prod_{\substack{k=1 \\ k \neq i}}^{p} (\lambda_i - \lambda_k)^{r_k} + b_{m-r_i-1}(x, t, \xi)\nu + b_{m-r_i-2}(x, t, \xi)\nu^2 + \dots + \nu^{m-r_i} ;$$

$$\text{r.h.s.} = + \sqrt{-1} L_i(x, t, \xi) + c_{m-2}(x, t, \xi)\nu + c_{m-3}(x, t, \xi)\nu^2 + \dots + c_0 \nu^{m-1}$$

where $b_j,\ c_j \in S^j_x$.

Let $\nu' = \nu/|\xi|$ and $\xi' = \xi/|\xi|$. Then

$$(\nu')^{r_i} \left\{ \prod_{k \neq i} [\lambda_i(x, t, \xi') - \lambda_k(x, t, \xi')]^{r_k} + b_{m-r_i-1}(x, t, \xi')\nu' + \dots + (\nu')^{m-r_i} \right\} =$$
$$= \frac{1}{|\xi|} \left\{ \sqrt{-1} L_i(x, t, \xi') + c_{m-2}(x, t, \xi')\nu' + \dots + c_0(x, t, \xi')(\nu')^{m-1} \right\}.$$

Hence,

$$(\nu')^{r_i} \left\{ \prod_{k \neq i} [\lambda'_i(x, t, \xi) - \lambda'_k(x, t, \xi)]^{r_k} + b'_{m-r_i-1}(x, t, \xi)\nu' + \dots + (\nu')^{m-r_i} \right\} =$$
$$= \frac{1}{|\xi|} \left\{ \sqrt{-1} L'_i(x, t, \xi) + c'_{m-2}(x, t, \xi)\nu' + \dots + c'_0(x, t, \xi)(\nu')^{m-1} \right\},$$

where $\lambda'(x, t, \xi) = \lambda(x, t, \xi)$, $b'(x, t, \xi) = b(x, t, \xi')$, etc.

Now let

$$\Psi(\nu'; x, t, \xi') = \frac{\sqrt{-1} L'_i + \sum_{j=1}^{m-1} c'_{m-j-1}(\nu')^j}{\prod_{k \neq i} [\lambda'_i - \lambda'_k]^{r_k} + \sum_{j=1}^{m-r_i} b'_{m-r_i-j}(\nu')^j} .$$

If $r_i \geqslant 2$ and condition (A) holds, then for $|\nu'| \leqslant \varepsilon_0$ for a small enough ε_0, we can bound the numerator away from zero since $|L'_i| \geqslant \sigma_0$ in Ω. Similarly for $|\nu'| \leqslant \varepsilon_1$ for a small enough ε_1, we can bound the denominator away from zero since $\prod_{k\neq i}(\lambda_i'-\lambda_k')^{r_k}$ does not vanish at $(x, t, \xi)\in \boldsymbol{R}^{n+1}\times S_\xi^{n-1}$. Hence $[\Psi(\nu'; x, t, \xi')]^{1/r_i}\in C^\infty$ for $|\nu'|\leqslant\varepsilon$ where $\varepsilon=\min(\varepsilon_0, \varepsilon_1)$. If $r_i=1$, then $\Psi\in C^\infty$ for $|\nu'|\leqslant\varepsilon$ even if $L_i'(0, 0, \xi)=0$. Since $(\nu')^{r_i}=(1/|\xi|)\Psi(\nu'; x, t, \xi')$, then

$$\nu'=w^j\frac{1}{|\xi|^{1/r_i}}[\Psi(\nu'; x, t, \xi')]^{1/r_i}, \qquad 1\leqslant j\leqslant r_i,$$

where w is the primitive r_i-th root of unity. Using Lagrange's formula, we have

$$\nu'=\sum_{k=1}^{\infty}\nu_{i,k}(x, t, \xi')(w^j|\xi|^{-1/r_i})^k, \quad 1\leqslant j\leqslant r_i,$$

where

$$\nu_{i,k}(x, t, \xi')=\frac{1}{2\pi k\sqrt{-1}}\int_{|\zeta|=\eta}\frac{[\Psi(\zeta; x, t, \xi')]^{k/r_i}}{\zeta^k}d\zeta .$$

Since for $r_i\geqslant 2$, $\Psi(\zeta; x, t, \xi')$ is not zero if $\zeta>0$ is sufficiently small we have $\nu_{i,k}(x, t, \xi')\in C^\infty$. In particular

$$\begin{aligned}\nu_{i,1}(x, t, \xi')&=\frac{1}{2\pi\sqrt{-1}}\int_{|\zeta|=\eta}\frac{[\Psi(\zeta; x, t, \xi')]^{1/r_i}}{\zeta}d\zeta\\ &=\Psi(0; x, t, \xi')^{1/r_i}=\\ &=\left[\frac{\sqrt{-1}\,L_i'}{\prod_{k\neq i}(\lambda_i'-\lambda_k')^{r_k}}\right]^{1/r_i}.\end{aligned}$$

Finally, we put $\nu_{i,k}^j=w^{(j-1)k}\nu_{i,k}$.

Hence

$$\lambda_i^j=\lambda_i+\sum_{k=1}^{\infty}\nu|\xi|^{(1-k)/r_i} \quad 1\leqslant j\leqslant r_i .$$

If $r_i>1$ and $k=1$ we see that $\nu_{i,1}^j\neq\nu_{i,1}^l$ since $w^{j-1}\neq w^{l-1}$ for $j\neq l$.

LEMMA 4.8. *Let $a(x, t, D_x)$ be a properly supported operator belonging to $L_x^{0,k}$, i.e.*

$$a(x, t, \xi)=a_0(x, t, \xi)+a_1(x, t, \xi)|\xi|^{-1/k}+a_2(x, t, \xi)|\xi|^{-2/k}+\dots$$

where $a_j(x, t, \xi)\in S_x^0$.

Then there exists constants C and R independent of u such that

$$\|a(x,t,D_x)\psi_2 u\| \leqslant C\|\psi_2 u\| \quad \textit{for } u \in C_0^\infty(\Omega)$$

such that supp $\psi_2(x,\xi) \subset \{\xi : |\xi| \geqslant R\}$. $\psi_2(x, D_x)$ *is chosen to be properly supported.*

PROOF. If we let $\zeta = |\xi|^{1/k}$ then this is just the standard result about the continuity of classical pseudo-differential operators where a is given as an asymptotic sum in ζ.

§ 5. – In sections 5 to 7, we shall assume that u_2 is of the form $u_2 = \psi_2(x, D_x)u$ where supp $\psi_2(x,\xi) \subset \{\xi : |\xi| \geqslant R\}$. In order to simplify the notation we'll drop the subscript of u_2 and refer to it as u.

Before we proceed with the proof, we will present some technical lemmas, which we will need when we manipulate the first order factors of $\tilde{\Pi}$.

LEMMA 5.1. *Let $\partial_i^{(j)}$ be the operator whose symbol is given by*

$$i(\tau - \lambda_i^j(x,t,\xi)) = i\Big(\tau - \lambda_i(x,t,\xi) - \sum_{k=1}^{\infty} \nu_{i,k}^i(x,t,\xi)|\xi|^{1-k/r_i}\Big),$$

where $\lambda_i(x,t,\xi) \in S_x^1$ and $\nu_{i,k}^j(x,t,\xi) \in S_x^0$.

Then

(*a*) *for any $a(x,t,D_x) \in L_x^{-1/r_i, r_i}$, $b(x,t,D_x) \in L_x^{1-1/r_i, r_i}$, there exist $c(x,t,D_x)$, $d(x,t,D_x) \in L_x^{0,r_i}$ such that*

$$\begin{aligned}(5.1)\quad & c(x,t,D_x)\,\partial_i^{(k)}u + d(x,t,D_x)\,\partial_i^{(l)}u = \\ &\qquad = a(x,t,D_x)D_t u + b(x,t,D_x)u + M(x,t,D_x)u,\end{aligned}$$

where $M \in L_x^{-\infty}$, the class of operators of order $-\infty$, and $k \neq l$.

(*b*) *For any $a' \in L_x^{0,r_i}$ and $b' \in L_x^{1-1/r_i, r_i}$, there exist $c', d' \in L_x^{0,r_i}$ such that*

$$\begin{aligned}& c'(x,t,D_x)\,\partial_i^{(k)}u + d'(x,t,D_x)\,\partial_i^{(l)}u = \\ &\qquad = a'(x,t,D_x)\,\partial_t u + b'(x,t,D_x)u + M'(x,t,D_x)u,\end{aligned}$$

where $M' \in L_x^{-\infty}$, and $k \neq l$.

(c) *For any* $\tilde{a} \in L_x^{0,r_{ij}}$, $\tilde{b} \in L_x^{1,r_{ij}}$, *there exist* $\tilde{c}$, $\tilde{d} \in L_x^{0,r_{ij}}$, *where* $r_{ij} =$ *the greatest common multiple of* r_i *and* r_j, *such that*

$$(5.2) \qquad \tilde{c}(x,t,D_x)\,\partial_i^{(k)}u + \tilde{d}(x,t,D_x)\,\partial_j^{(l)}u = \\ = \tilde{a}(x,t,D_x)D_t u + \tilde{b}(x,t,D_x)u + \tilde{M}(x,t,D_x)u$$

where $\tilde{M} \in L_x^{-\infty}$, *and* $i \neq j$.

All of the above operators are chosen to be properly supported.

Proof. Proof of (*a*): To solve equation (5.1) we have to find $c(x,t,D_x)$ and $d(x,t,D_x)$ such that $c(x,t,D_x) + d(x,t,D_x) = a(x,t,D_x)$ and $c(x,t,D_x)\cdot \cdot\lambda_i^k(x,t,D_x) + d(x,t,D_x)\lambda_i^l(x,t,D_x) = -\,b(x,t,D_x) + M(x,t,D_x)$.

Hence $c(x,t,D_x) = a(x,t,D_x) - d(x,t,D_x)$, where $d(x,t,D_x)$ satisfies the equation

$$(5.3) \qquad d(x,t,D_x)[\lambda_i^l(x,t,D_x) - \lambda_i^k(x,t,D_x)] = \\ = -\,b(x,t,D_x) - a(x,t,D_x)\lambda_i^k(x,t,D_x) + M(x,t,D_x) = \\ = h(x,t,D_x) + M(x,t,D_x)\,,$$

where $h(x,t,D_x) \in L_x^{1-1/r_i,r_i}$ is known. The symbol of the l.h.s. of (5.3) is

$$\sum_\alpha \frac{1}{\alpha!}\,\partial_\xi^\alpha\, d(x,t,\xi)[D_x^\alpha(\lambda_i^l(x,t,\xi) - \lambda_i^k(x,t,\xi))]\,.$$

So solving for $d(x,t,D_x)$ reduces to find $d(x,t,\xi)$ such that

$$(5.4) \qquad \sum_\alpha \frac{1}{\alpha!}\,\partial_\xi^\alpha d(x,t,\xi)[D_x^\alpha(\lambda_i^l(x,t,\xi) - \lambda_i^k(x,t,\xi))] \sim h(x,t,\xi)$$

(where «$\sim$» denotes equality modulo $S_x^{-\infty}$).

We are able to do this since $\nu_{i,1}^l - \nu_{i,1}^k \neq 0$ for $(x,t,\xi) \in \Omega \times S_\xi^{n-1}$ if $r_i > 1$ and $l \neq k$. The proof proceeds in the standard fashion. We leave the details to the reader.

Proof of (*b*). The proof is basically the same as the proof of (*a*). We leave the details to the reader.

Proof of (*c*). The proof of (*c*) runs basically in the same manner as that of (*a*). The reason we get a stronger result lies primarily with the fact that $\partial_i^{(k)}$ differs from $\partial_j^{(l)}$ at the first order since $\lambda_i(x,t,\xi) - \lambda_j(x,t,\xi) \neq 0$ for $(x,t,\xi) \in \Omega \times \mathbf{R}^n \setminus \{0\}$ as opposed to what we have in part (*a*) where $\partial_i^{(k)}$ differs from $\partial_i^{(l)}$ at order $1 - 1/r_i$.

COROLLARY 5.2:

(a) $[\partial_i^{(k)}, \partial_i^{(l)}] = a(x, t, D_x)\,\partial_i^{(k)} + b(x, t, D_x)\,\partial_i^{(l)} + M_1$, *for some* $a, b \in L_x^{0,r_i}$ *and* $M_1 \in L_x^{-\infty}$.

(b) $[\partial_i^{(k)}, \partial_j^{(l)}] = c(x, t, D_x)\,\partial_i^{(k)} + \leqslant(x, t, D_x)\,\partial_j^{(l)} + M_2$ *for some* $c, d \in L_x^{0,r_{ij}}$ *and some* $M_2 \in L_x^{-\infty}$.

PROOF. *(a)* $[\partial_i^{(k)}, \partial_i^{(l)}] = g(x, t, D_x)$ for some $g \in L_x^{1-1/r_i, r_i}$. Hence by lemma 5.1*(a)*, there exist $a, b \in L_x^{0,r_i}$ such that

$$a(x, t, D_x)\,\partial_i^{(k)} + b(x, t, D_x)\,\partial_i^{(l)} + M_1 = g(x, t, D_x)$$

for some $M_1 \in L_x^{-\infty}$.

(b) A similar proof to that of *(a)* works for *(b)*.

The next lemma will be used to control the lower order terms that arise when we permute the factors of $\tilde{\Pi}$.

LEMMA 5.3. *Let*

$$Q_q(x, t, D_x, D_t) = \partial_1^{(1)}\partial_1^{(2)} \dots \partial_1^{(s_1)} \dots \partial_w^{(1)} \dots \partial_w^{(s_w)}:$$

have order q. *Let* $\hat{Q}_q(x, t, D_x, D_t)$ *be obtained from* Q_q *by an arbitrary permutation of the factors* $\partial_i^{(j)}$. *Then* $h(x, t, D_x, D_t) = Q_q(x, t, D_x, D_t) - \hat{Q}_q(x, t, D_x, D_t)$ *is an operator belonging, modulo terms of order* $\leqslant q-2$, *to the module* $S_{(q)}$ *over* $L_x^{0,\beta}$ *(where* $\beta =$ *the greatest common multiple of* r_i, $1 \leqslant i \leqslant w$*), generated by the «monomial» operators* $Q_q/\partial_i^{(j)}$ *formed by omitting one factor at a time from* Q_q.

PROOF. It suffices to carry out the proof for the special permutation

$$s = \partial_1^{(1)} \dots \partial_i^{(k-1)}\partial_i^{(k)}\partial_j^{(l)}\partial_j^{(l+1)} \dots \partial_w^{(s_w)},$$
$$\bar{s} = \partial_1^{(1)} \dots \partial_i^{(k-1)}\partial_j^{(l)}\,\partial_i^{(k)}\partial_i^{(l+1)} \dots \partial_w^{(s_w)},$$

where i can be (but not necessarily) equal to j.

Then

$$s - \bar{s} = \partial_1^{(1)} \dots \partial_i^{(k-1)}[\partial_i^{(k)}, \partial_j^{(l)}]\partial_j^{(l+1)} \dots \partial_w^{(s_w)}.$$

By corollary 5.2, $[\partial_i^{(k)}, \partial_j^{(l)}] = a(x, t, D_x)\,\partial_i^{(k)} + b(x, t, D_x)\,\partial_j^{(l)} + M$ for some $a, b \in L_x^{0,\beta}$ (since $\beta \geqslant r_{ij}$) and some $M \in L_x^{-\infty}$. Thus, modulo terms of order $\leqslant q-2$,

$$s-\bar{s}=\partial_1^{(1)}\dots\partial_i^{(k-1)}a\,\partial_i^{(k)}\partial_j^{(l+1)}\dots\partial_w^{(s_w)}+\partial_1^{(1)}\dots\partial_i^{(k-1)}b\,\partial_j^{(l)}\partial_j^{(l+1)}\dots\partial_w^{(s_w)}.$$

Hence

$$\begin{aligned}s-\bar{s}&=a\partial_1^{(1)}\dots\partial_i^{(k-1)}\partial_i^{(k)}\partial_j^{(l+1)}\dots\partial_w^{(s_w)}\\&+b\partial_1^{(1)}\dots\partial_i^{(k-1)}\partial_j^{(l)}\partial_j^{(l+1)}\dots\partial_w^{(s_w)}\\&+[\partial_1^{(1)}\dots\partial_i^{(k-1)},a]\partial_i^{(k)}\partial_j^{(l+1)}\dots\partial_w^{(s_w)}\\&+[\partial_1^{(1)}\dots\partial_i^{(k-1)},b]\partial_j^{(l)}\partial_j^{(l+1)}\dots\partial_w^{(s_w)}.\end{aligned}$$

The proof is complete once we observe that the last two terms are of order $q-2$.

LEMMA 5.4. *Modulo terms in* $L_{x,t}^{m-2}$, $[\tilde{\Pi},\varphi]\in S_{(m)}$ *for any* $\varphi\in L_x^0$, *where* $S_{(m)}$ *is the module associated with the operator* $\tilde{\Pi}$ *(constructed as in lemma* 5.3).

PROOF. The proof proceeds very much like that of lemma 5.3. The details are left to the reader.

§ 6. – We will now state the proposition basic to the estimate of $|||\tilde{P}u|||$ in case (*b*).

PROPOSITION 6.1. *Let* $\partial_i^{(j)}=D_t-\lambda_i^j(x,t,D_x)$, *where* $\lambda_i^j(x,t,\xi)$ *is as in proposition* 4.7. *Then for* T, $\tilde{r}$ *and* k^{-1} *sufficiently small,*

$$|||u|||^2\leqslant\frac{C}{k}|||\partial_i^{(j)}u|||^2\,,\qquad \text{for } u\in C_0^\infty(\Omega)\,,$$

where $\Omega=\{x,t):|x|\leqslant\tilde{r},\ 0\leqslant t\leqslant T\}$, *and where* C *is independent of* T, $\tilde{r}$, k *and* u.

PROOF. The proof is basically the same as that given by Calderón [1] in the case where $\lambda_i^j(x,t,D_x)=\lambda_i(x,t,D_x)$. See also Nirenberg [14]. The modifications needed to take care of the lower order terms belonging to $L_x^{1-1/r_i,r_i}$ can be found in Matsumoto [10].

REMARK. In order to be able to prove Proposition 6.1 some condition on $\lambda_i^j(x,t,D_x)$ is needed. (See Remark 2 following Theorem 2). It is primarily because of this proposition that we assumed that the coefficients of P_m and P_{m-1} are real. Although this assumption is not optimum (see Nirenberg [14], section 6 for some alternative conditions), we put this limitation on P_m and P_{m-1} for the sake of simplicity.

Much of the proof of the estimate of $|||\tilde{P}u|||$ in case (*b*) will be by induction. In order to keep the notation from being even more cumbersome than it already is we shall introduce the following:

Recall that

$$\Pi_m = \partial_1^{r_1}\partial_2^{r_2}\dots\partial_p^{r_p}\,, \qquad \text{where } r_1 \geqslant r_2 \geqslant \dots \geqslant r_p\,.$$

Let $m_0 = 0$, $m_1 = r_1$, $m_2 = r_1 + r_2, \dots, m_i = r_1 + r_2 + \dots + r_i$. Let $Q_\alpha u = \partial_1^{(1)}\dots\partial_1^{(r_1)}\partial_2^{(1)}\dots\partial_i^{(r_i)}\partial_{i+1}^{(1)}\dots\partial_{i+1}^{(l)}u$ be an operator belonging to $L_{x,t}^{\alpha}$, where $\alpha = m_i + l$, $0 \leqslant i \leqslant p-1$, $0 \leqslant l \leqslant r_{i+1}$. (If $i = 0$, $Q_\alpha u = \partial_1^{(1)}\dots\partial_1^{(l)}u$.) As in lemma 5.3, we can associate with this operator the module $S_{(\alpha)}$. Let $S'_{(\alpha)}$ consist of the operators that generate $S_{(\alpha)}$.

As consequences of proposition 6.1, we have the following lemmas which are essential to the estimate of $|||\tilde{P}u|||$ in case (*b*).

LEMMA 6.2. *There is a constant C independent of u such that for $\tilde{r}$, T, k^{-1} sufficiently small,*

$$C|||u|||_{\alpha-2}^2 + C|||Q_\alpha u|||^2 \geqslant k \sum_{s_\alpha \in S'_{(\alpha)}} |||s_\alpha u|||^2 \qquad \text{for } u \in C_0^\infty(\Omega)\,.$$

PROOF. By proposition 6.1, there is a C such that for k large enough

$$C|||\partial_j^{(\gamma)}[\partial_1^{(1)}\dots\partial_j^{(\gamma-1)}\partial_j^{(\gamma+1)}\dots\partial_{i+1}^{(l)}u]^2 \geqslant k|||\partial_1^{(1)}\dots\partial_j^{(\gamma-1)}\partial_j^{(\gamma+1)}\dots\partial_{i+1}^{(l)}u]|||^2$$

for $1 \leqslant j \leqslant i$, $1 \leqslant \gamma \leqslant r_j$, and for $j = i+1$, $1 \leqslant \gamma \leqslant l$. By lemma 5.3,

$$\partial_j^{(\gamma)}\partial_1^{(1)}\dots\partial_j^{(\gamma-1)}\partial_j^{(\gamma+1)}\dots\partial_{i+1}^{(l)}u = Q_\alpha u + h_j^\gamma u + N_j^\gamma u\,,$$

where $h_j^\gamma \in S_{(\alpha)}$ and $N_j^\gamma \in L_{x,t}^{\alpha-2}$.

Hence,

$$\text{(6.1)} \qquad C|||u|||_{\alpha-2}^2 + C|||h_j^\gamma u|||^2 + C|||Q_\alpha u|||^2 \geqslant k|||\partial_1^{(1)}\dots\partial_j^{(\gamma-1)}\partial_j^{(\gamma+1)}\dots\partial_{(i+1)}^{(l)}u|||^2\,.$$

Adding (6.1) over j and γ, we have

$$C|||u|||_{\alpha-2}^2 + C\sum_{j,\gamma}|||h_j^\gamma u|||^2 + C|||Q_\alpha u|||^2 \geqslant$$
$$\geqslant k\sum_{j,\gamma}|||\partial_1^{(1)}\dots\partial_j^{(\gamma-1)}\partial_j^{(\gamma+1)}\dots\partial_{i+1}^{(l)}u|||^2 \geqslant k\sum_{s_\alpha \in S'_{(\alpha)}}|||s_\alpha u|||^2\,.$$

Since $h_j^\gamma \in S_{(\alpha)}$,

$$\sum_{j,\gamma}|||h_j^\gamma u|||^2 \leqslant C\sum_{s_\alpha \in S'_{(\alpha)}}|||s_\alpha u|||^2\,.$$

Hence,

$$C|||u|||^2_{\alpha-2} + C \sum_{s_\alpha \in S'_{(\alpha)}} |||s_\alpha u|||^2 + C|||Q_\alpha u|||^2 \geqslant k \sum_{s_\alpha \in S'_{(\alpha)}} |||s_\alpha u|||^2 . \tag{6.2}$$

For $k > C$, we can absorb the term $C \sum_{s_\alpha \in S'_{(\alpha)}} |||s_\alpha u|||^2$ of the l.h.s. of (6.2) into the r.h.s. and we have:

$$C|||u|||^2_{\alpha-2} + C|||Q_\alpha u|||^2 \geqslant k \sum_{s_\alpha \in S'_{(\alpha)}} |||s_\alpha u|||^2 ,$$

which is what we set out to prove.

LEMMA 6.3. *Let $\alpha = m_i + l$ be defined as above. There exists a constant C independent of u such that for $\tilde{r}$, T, k^{-1} sufficiently small, the following estimates hold*:

$$a)\quad C|||Q_\alpha u|||^2 \geqslant k|||u|||^2_{\alpha-1-(\alpha-1)/r_1}, \qquad \text{if } i = 0,\ 1 \leqslant \alpha \leqslant r_1 , \tag{6.3}$$

$$b)\quad C|||Q_\alpha u|||^2 \geqslant k|||u|||^2_{\alpha-2+1/r}, \qquad \text{if } 1 \leqslant i,\ \ r_1 \leqslant \alpha \leqslant m . \tag{6.4}$$

PROOF: Proof of (a). The proof is by induction on α. If $\alpha = 1$, (6.3) holds by virtue of proposition 6.1. Suppose (6.3) holds for some $\alpha \geqslant 1$; we'll show it also holds for $\alpha + 1$.

$Q_{\alpha+1} = Q_\alpha \partial_1^{(l+1)}$. Hence, by the induction hypothesis,

$$C|||Q_{\alpha+1} u|||^2 \geqslant k|||\partial_1^{(l+1)} u|||^2_{\alpha-1-(\alpha-1)/r_1} . \tag{6.5}$$

By lemma 5.3,

$$\partial_1^{(1)} \dots \partial_1^{(l-1)} \partial_1^{(l+1)} \partial_1^{(l)} u = Q_{\alpha+1} u + h_1^{\alpha+1} u + N_1^{\alpha+1} u ,$$

where $h_1^{\alpha+1} \in S_{(\alpha+1)}$ and $N_1^{\alpha+1} \in L^{\alpha-1}_{x,t}$.

By the induction hypothesis we have, as before,

$$C|||\partial_1^{(1)} \dots \partial_1^{(l-1)} \partial_1^{(l+1)} \partial_1^{(l)} u|||^2 \geqslant k|||\partial_1^{(l)} u|||^2_{\alpha-1-(\alpha-1)/r_1} .$$

Hence

$$C|||\alpha|||^2_{\alpha-1} + C|||h_1^{\alpha+1} u|||^2 + C|||Q_{\alpha+1} u|||^2 \geqslant k|||\partial_1^{(l)} u|||^2_{\alpha-1-(\alpha-1)/r_1} . \tag{6.6}$$

Now by lemma 6.2,

$$C|||u|||^2_{\alpha-1} + C|||Q_{\alpha+1} u|||^2 \geqslant k \sum_{s_{\alpha+1} \in S'_{(\alpha+1)}} |||s_{\alpha+1} u|||^2 .$$

Since $h_1^{\alpha+1} \in S_{(\alpha+1)}$ we have

$$|||h_1^{\alpha+1}u|||^2 \geqslant C \sum_{s_{\alpha+1}\in S'_{(\alpha+1)}} |||s_{\alpha+1}u|||^2 .$$

Hence, (6.6) leads to the estimate

$$(6.7) \qquad C|||u|||^2_{\alpha-1} + C \sum_{s_{\alpha+1}\in S'_{(\alpha+1)}} |||s_{\alpha+1}u|||^2 + C|||Q_{\alpha+1}u|||^2 \geqslant$$
$$\geqslant k \sum_{s_{\alpha+1}\in S'_{(\alpha+1)}} |||s_{\alpha+1}u|||^2 + k|||\partial_1^{(l)}u|||^2_{\alpha-1-(\alpha-1)/r_1} .$$

For $k > C$, we can absorb the term $C \sum_{s_{\alpha+1}\in S'_{(\alpha+1)}} |||s_{\alpha+1}u|||^2$ from the l.h.s. of (6.7) into the r.h.s. and get

$$(6.8) \qquad C|||u|||^2_{\alpha-1} + C|||Q_{\alpha+1}u|||^2 \geqslant k|||\partial_1^{(l)}u|||^2_{\alpha-1-(\alpha-1)/r_1} .$$

Adding (6.5) and (6.8),

$$C|||u|||^2_{\alpha-1} + C|||Q^2_{\alpha+1}u|||^2 \geqslant k\left(|||\partial_1^{(l)}u|||^2_{\alpha-1-(\alpha-1)/r_1} + |||\partial_1^{(l+1)}u|||^2_{\alpha-1-(\alpha-1)/r_1}\right) .$$

By lemma 4.8, we then have

$$(6.9) \qquad C|||u|||^2_{\alpha-1} + C|||Q_{\alpha+1}u|||^2 \geqslant k|||a_1\partial_1^{(l)}u + a_2\partial_1^{(l+1)}u|||^2_{\alpha-1-(\alpha-1)/r_1}$$

for any properly supported $a_1, a_2 \in L_x^{0,r_1}$. By lemma 5.1(a), this implies that

$$(6.10) \qquad C|||u|||^2_{\alpha-1} + C|||Q_{\alpha+1}u|||^2 \geqslant$$
$$\geqslant k|||[a(x, t, D_x)D_t + b(x, t, D_x)]u + Mu|||^2_{\alpha-1-(\alpha-1)/r_1}$$

for any properly supported operators $a \in L_x^{-1/r_1,r_1}$, $b \in L_x^{1-1/r_1,r_1}$ and for some $M \in L_x^{-\infty}$.

Hence,

$$(6.11) \qquad C|||u|||^2_{\alpha-1} + C|||Q_{\alpha+1}u|||^2 \geqslant k|||u|||^2_{\alpha-\alpha/r_1} - k|||Mu|||^2_{\alpha-1-(\alpha-1)/r_1} .$$

Since $M \in L_x^{-\infty}$, then by lemma 3.1, for some ε, $0 < \varepsilon < 1$,

$$|||Mu|||_{\alpha-1-(\alpha-1)/r_1} \leqslant \varepsilon|||u|||_{\alpha-\alpha/r_1} .$$

Hence we can absorb the second term of the r.h.s. of (6.11) into the first. Thus we have

$$|||u|||^2_{\alpha-1} + C|||Q_{\alpha+1}u|||^2 \geqslant k|||u|||^2_{\alpha-\alpha/r_1}. \tag{6.12}$$

By another application of lemma 3.1, $|||u|||_{\alpha-1} \leqslant C|||u|||_{\alpha-\alpha/r_1}$. Hence, for $K > C$, we can absorb the term $C|||u|||^2_{\alpha-1}$ of the l.h.s. of (6.12) into the r.h.s. and arrive at the estimate

$$C|||Q_{\alpha+1}u|||^2 \geqslant k|||u|||^2_{\alpha-\alpha/r_1}.$$

This completes the induction proof of (a).

Proof of (b). If we let $\alpha = r_1$ in (6.3), we have

$$C|||Q_{r_1}u|||^2 \geqslant k|||u|||^2_{r_1-2+1/r_1}.$$

Since $r = \max_{1\leqslant i\leqslant p} r_i = r_1$ (because $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_p$), we see that (6.4) holds for $\alpha = r_1$.

Before we proceed with the proof of (b), we note that estimates (6.10) and (6.12) would hold even if we modified the operator Q_α by replacing the terms $\partial_1^{(k)}$ with $\partial_j^{(k)}$, $j \neq 1$. The proof in this case is precisely the same as for Q_α up to line (6.9). We then apply lemma 5.1(c) instead of lemma 5.1(a). Estimates (6.10) and (6.12) follow because lemma 5.1(c) is « stronger » than lemma 5.1(a).

We are now ready to proceed with the proof of (b). The proof is by induction on α. We suppose (6.4) holds for α, where $\alpha \geqslant r_1$; we'll show this implies that (6.4) holds also for $\alpha + 1$.

By the induction hypothesis, since $Q_{\alpha+1} = Q_\alpha \partial_{i+1}^{(l+1)}$,

$$C|||Q_{\alpha+1}u|||^2 \geqslant k|||\partial_{i+1}^{(l+1)}u|||^2_{\alpha-2+1/r}. \tag{6.13}$$

It can also be shown that if (6.4) holds with Q_α, then it also holds with Q_α replaced with the operator $\partial_1^{(1)} \ldots \partial_i^{(r_i-1)} \partial_{i+1}^{(1)} \ldots \partial_{i+1}^{(l+1)}$; since the two operators have the same order and the same multiplicities of characteristics:

$$C|||\partial_1^{(1)} \ldots \partial_i^{(r_i-1)} \partial_{i+1}^{(1)} \ldots \partial_{i+1}^{(l+1)}u|||^2 \geqslant k|||u|||^2_{\alpha-2+1/r}.$$

This implies that

$$C|||\partial_1^{(1)} \ldots \partial_i^{(r_i-1)} \partial_{i+1}^{(1)} \ldots \partial_{i+1}^{(l+1)} \partial_i^{(r_i)}u|||^2 \geqslant k|||\partial_i^{(r_i)}u|||^2_{\alpha-2+1/r}.$$

By lemma 5.3,

$$\partial_1^{(1)} \dots \partial_i^{(r_i-1)} \partial_{i+1}^{(1)} \dots \partial_{i+1}^{(l+1)} \partial_i^{(r_i)} u = Q_{\alpha+1} u + h_i^{r_i} u + N_i^{r_i} u ,$$

where $h_i^{r_i} \in S_{(\alpha+1)}$ and $N_i^{r_i} \in L_{x,t}^{\alpha-1}$.

Hence

$$(6.14) \qquad C|||u|||_{\alpha-1}^2 + C|||h_i^{r_i} u|||^2 + C|||Q_{\alpha+1} u|||^2 \geqslant k|||\partial_i^{(r_i)} u|||_{\alpha-2+1/r}^2 .$$

As in the proof of (a), since

$$C|||u|||_{\alpha+1}^2 + C|||Q_{\alpha-1} u|||^2 \geqslant k \sum_{s_{\alpha+1} \in S'_{(\alpha+1)}} |||s_{\alpha+1} u|||^2$$

by lemma 6.2 and since $h_i^{r_i} \in S_{(\alpha+1)}$, we can remove the term $C|||h_i^{r_i} u|||^2$ from the l.h.s. of (6.14) and get:

$$(6.15) \qquad C|||u|||_{\alpha-1}^2 + C|||Q_{\alpha+1} u|||^2 \geqslant k|||\partial_i^{(r_i)} u|||_{\alpha-2+1/r}^2 .$$

Adding (6.13) and (6.15) we have

$$C|||u|||_{\alpha-1}^2 + C|||Q_{\alpha+1} u|||^2 \geqslant k\left(|||\partial_{i+1}^{(l+1)} u|||_{\alpha-2+1/r}^2 + |||\partial_i^{(r_i)} u|||_{\alpha-2+1/r}^2\right) .$$

By lemma 4.8, we then have

$$(6.16) \qquad C|||u|||_{\alpha-1}^2 + C|||Q_{\alpha+1} u|||^2 \geqslant k\left(|||a_1 \partial_{i+1}^{(l+1)} u + a_2 \partial_i^{(r_i)} u|||_{\alpha-2+1/r}^2\right) ,$$

for any properly supported $a_1, a_2 \in L_x^{0, r_i, t+1}$. We now apply lemma 5.1(c) to (6.16):

$$C|||u|||_{\alpha-1}^2 + C|||Q_{\alpha+1} u|||^2 \geqslant k|||b_1(x, t, D_x) D_t u + b_2(x, t, D_x) u + \tilde{M} u|||_{\alpha-2+1/r}^2 ,$$

for any properly supported operators $b_1 \in L_x^{0, r_i, t+1}$ and $b_2 \in L_x^{1, r_i, t+1}$ and some $\tilde{M} \in L_x^{-\infty}$. Hence

$$(6.17) \qquad C|||u|||_{\alpha-1}^2 + C|||Q_{\alpha+1} u|||^2 \geqslant k|||u|||_{\alpha-1+1/r}^2 - k|||\tilde{M} u|||_{\alpha-2+1/r}^2 .$$

As in the proof of (a) we can absorb the second term of the r.h.s. of (6.17) into the first. Thus,

$$(6.18) \qquad C|||u|||_{\alpha-1}^2 + C|||Q_{\alpha+1} u|||^2 \geqslant k|||u|||_{\alpha-1+1/r}^2 .$$

Applying lemma 3.1 once again, we can absorb the term $C|||u|||_{\alpha-1}^2$ of the l.h.s. of (6.18) into the r.h.s. and achieve the estimate:

$$C|||Q_{\alpha+1}u|||^2 \geqslant k|||u|||_{\alpha-1+1/r}^2 \, .$$

Hence (6.4) is true for all α, $r_1 \leqslant \alpha \leqslant m$.

We will now describe another module W_α over L_x^0, associated with the operator Q_α.

First we'll describe the operator which generate $W_\alpha^{(1)}$. They form the collection $W'^{(1)}_{(\alpha)}$. If we denote $T_\alpha = \partial_1^{r_1} \dots \partial_1^{r_1} \partial_{i+1}^{l+1}$, the members of $W'^{(1)}_\alpha$ are T_α/∂_i, $b_i(T_\alpha/\partial_i^2)$, $D_t(T_\alpha/\partial_i\partial_j)$, and $b_{ij}(T_\alpha/(\partial_i\partial_j))$, $i \neq j$, where b_i is an arbitrary properly supported operator in $L_x^{1-1/r_i, r_i}$ and b_{ij} is an arbitrary properly supported operator in $L_x^{1, r_{ij}}$, where r_{ij} is the greatest common multiple of r_i and r_j.

$W'^{(2)}_\alpha$ is formed in a similar manner by replacing the operator T_α in the term of $W'^{(1)}_\alpha$ with a member of $W'^{(1)}_\alpha$. We go on in this manner to form $W'^{(3)}_\alpha$, $W'^{(4)}_\alpha$, ... Finally $W'_\alpha = \bigcup_k W'^{(k)}_\alpha$, and W_α is the module generated by all the operators in W'_α.

LEMMA 6.4. *There exists a constant C independent of u such that for $\tilde{r}$, T, k^{-1} sufficiently small, the following estimate holds*:

$$C|||Q_\alpha u|||^2 \geqslant k \sum_{w_\alpha \in W'_\alpha} |||w_\alpha u|||^2 \, , \quad \textit{for } u \in C_0^\infty(\Omega) \, . \tag{6.19}$$

PROOF. The proof is by induction. It is a fairly long proof. In an attempt to make it more manageable we will give it in a series of steps. As in lemma 6.3 we will first consider the case $\alpha = l \leqslant r_1$, and then the case $\alpha = m_i + l$, where $1 \leqslant i$.

If $\alpha = l$, $Q_\alpha u = Q_l u = \partial_1^{(1)} \dots \partial_1^{(l)} u$. We'll show that

$$C|||Q_l u|||^2 \geqslant k \sum_{w_l \in W'_l} |||w_l u|||^2 \, , \quad 2 \leqslant l \leqslant r_1 \, . \tag{6.20}$$

Step 1. If $l = 2$, then applying proposition 6.1, we have:

$$C|||\partial_1^{(1)}\partial_1^{(2)}u|||^2 \geqslant k|||\partial_1^{(2)}u|||^2 \quad \text{and} \quad C|||\partial_1^{(2)}\partial_1^{(2)}u|||^2 \geqslant k|||\partial_1^{(1)}u|||^2 \, .$$

By lemma 5.3,

$$\partial_1^{(2)}\partial_1^{(1)} = \partial_1^{(1)}\partial_1^{(2)} + h_1^2 + N_1^2 \, ,$$

where $h_1^2 \in S_{(2)}$ and $N_1^2 \in L_0^{x,t}$.

This implies that

$$(6.21)\quad C|||u|||^2 + C|||h_1^2 u|||^2 + C|||\partial_1^{(1)}\partial_1^{(2)}u|||^2 \geqslant k\{|||\partial_1^{(1)}u|||^2 + |||\partial_1^{(2)}u|||^2\} .$$

Since

$$C|||u|||^2 + C|||\partial_1^{(1)}\partial_1^{(2)}u|||^2 \geqslant k \sum_{s_\alpha \in S'_{(2)}} |||s_2 u|||^2$$

and since $h_1^2 \in S_{(2)}$, we can remove, as in the proof of lemma 6.3, the term $C|||h_1^2 u|||^2$ from the l.h.s. of (6.21):

$$C|||u|||^2 + C|||Q_2 u|||^2 \geqslant k\{|||\partial_1^{(1)}u|||^2 + |||\partial_1^{(2)}u|||^2\} .$$

After an application of lemma 4.8, we have

$$(6.22)\qquad C|||u|||^2 + C|||Q_2 u|||^2 \geqslant k\{|||[a_1\partial_1^{(1)} + a_2\partial_1^{(2)}]u|||^2\}$$

for any properly supported operators $a_1, a_2 \in L_x^{0,r_1}$. Hence, by lemma 5.1(*b*) for any properly supported operators $b_1 \in L_x^{0,r_1}$ and $d_{1,2} \in L_x^{1-1/r_1, r_1}$ there exist properly supported $a_1, a_2 \in L_x^{0,r_1}$ and $M_1 \in L_x^{-\infty}$ such that

$$a_1\partial_1^{(1)}u + a_2\partial_1^{(2)}u = b_1\partial_1 u + d_{1,2}u + M_1 u .$$

Since the set W_2' consists of the operators $b_1\partial_1$ and $d_{1,2}$, (6.22) yields

$$C|||u|||^2 + C|||Q_2 u|||^2 \geqslant k \sum_{w_2 \in W_2'} |||w_2 u|||^2 - k|||M_1 u|||^2 .$$

By lemma 6.3, $C|||Q_2 u|||^2 \geqslant k|||u|||^2_{1-1/r_1}$. Hence

$$C|||u|||^2 + C|||Q_2 u|||^2 \geqslant k \sum_{w_2 \in W_2'} |||w_2 u|||^2 + k(|||u|||^2_{1-1/r_1} - |||M_1 u|||^2) .$$

Since $M_1 \in L_x^{-\infty}$, $|||M_1 u|||^2 \leqslant \varepsilon |||u|||^2_{1-1/r_1}$ for any $\varepsilon > 0$. We thus arrive at the estimate:

$$(6.23)\qquad C|||u|||^2 + C|||Q_2 u|||^2 \geqslant k \sum_{w_2 \in W_2'} |||w_2 u|||^2 .$$

By another application of lemma 6.3 and since $|||u|||^2 \leqslant \varepsilon |||u|||^2_{1-1/r_1}$ by lemma 3.1, for $k > C$ we can remove the term $C|||u|||^2$ from the l.h.s. of (6.23) and we have:

$$C|||Q_2 u|||^2 \geqslant k \sum_{w_2 \in W_2'} |||w_2 u|||^2 .$$

Step 2. We suppose now that (6.20) holds for any $l \geqslant 2$; we'll show it also holds for $l+1$. Since $Q_{l+1} = Q_l \partial_1^{(l+1)}$, the induction hypothesis gives

$$C|||Q_{l+1}u|||^2 \geqslant k \sum_{w_l \in W_l'} |||w_l(\partial_1^{(l+1)}u)|||^2 . \tag{6.24}$$

As in step 1, by lemma 5.3,

$$\partial_1^{(1)} \dots \partial_1^{(l-1)} \partial_1^{(l+1)} \partial_1^{(l)} = \partial_1^{(1)} \dots \partial_1^{(l+1)} + h_1^{l+1} + N_1^{l+1} ,$$

where $h_1^{l+1} \in S_{(l+1)}$ and $N_1^{l+1} \in L_{x,t}^{l-1}$.

If we examine the operator $\partial_1^{(1)} \dots \partial_1^{(l-1)} \partial_1^{(l+1)}$, we see that estimate (6.20) holds for this operator as well as for Q_l. Hence we can show that

$$C|||u|||_{l-1}^2 + C|||h_1^{l+1}u|||^2 + C|||Q_{l+1}u|||^2 \geqslant k \sum_{w_l \in W_l'} |||w_l(\partial_1^{(l)}u)|||^2 . \tag{6.25}$$

As in step 1, since $h_1^{l+1} \in S_{(l+1)}$, we can apply lemma 6.2 to remove the term $C|||h_1^{l+1}u|||^2$ from estimate (6.25). And applying lemma 6.3 we can remove the term $C|||u|||_{l-1}^2$.
Hence

$$C|||Q_{l+1}u|||^2 \geqslant k \sum_{w_l \in W_l'} w_l(\partial_1^{(l)}u)|||^2 . \tag{6.26}$$

Combining (6.24) and (6.26) and applying lemma 4.8, we arrive at the following estimate

$$C|||Q_{l+1}u|||^2 \geqslant k \sum_{w_l \in W_l'} |||a_1 w_l(\partial_1^{(l)}u) + a_2 w_l(\partial_1^{(l+1)}u)|||^2$$

for any properly supported $a_1, a_2 \in L_x^{0,r_1}$.

Since the operator $[a_i, w_l]\partial_1^{(j)} \in L_{x,t}^{l-1}$, for $i = 1, 2$ and $j = l, l+1$, where $[A, B]$ represents the commutator of A and B, by another application of lemma 6.3 we have

$$C|||Q_{l+1}u|||^2 \geqslant k \sum_{w_l \in W_l'} |||w_l(a_1 \partial_1^{(l)}u + a_2 \partial_1^{(l+1)}u)|||^2 .$$

Applying lemma 5.1(*b*) once again we have

$$C|||Q_{l+1}u|||^2 \geqslant k \sum_{w_l \in W_l'} |||w_l(b_1 \partial_1^{(l)}u + b_{1,l}u + M_{1,l}u)|||^2$$
$$\geqslant k \sum_{w_l \in W_l'} |||w_l[b_1 \partial_1^{(l)} + b_{1,l}]u|||^2 - k \sum |||w_l M_{1,l}u|||^2$$

for any properly supported $b_1 \in L_x^{0,r_1}$ and $b_{1,l} \in {}_x^{1-1/r_1, r_1}$ and some $M_{1,l} \in L_x^{-\infty}$.

It is easy to see that the set W'_{l+1} consists of the terms $w_l(b_1\partial_1^{(l)} + b_{1,l})$ for the appropriate choice of b_1 and $b_{1,l}$ and for $w_l \in W'_l$.

Therefore,

$$C|||Q_{l+1}u|||^2 \geqslant k \sum_{w_{l+1}\in W'_{l+1}} |||w_{l+1}u|||^2 - k \sum_{w_l \in W'_l} |||w_l M_{1,l} u|||^2 . \tag{6.27}$$

Since $|||w_l M_{1,l}u|||^2 \leqslant \varepsilon |||u|||^2_{l-l/r_1}$ for any $\varepsilon > 0$ and since $C|||Q_{l+1}u|||^2 \geqslant k|||u|||^2_{l-l/r}$ by lemma 6.3 we can remove the last term from the r.h.s. of (6.27) and get

$$C|||Q_{l+1}u|||^2 \geqslant k \sum_{w_{l+1}\in W'_{l+1}} |||w_{l+1}u|||^2 . \tag{6.28}$$

Step 3. Letting $l = r_1$ in (6.28) we see that (6.19) holds for $\alpha = m_1 = r_1$. We will now assume that (6.19) is true for $\alpha = m_i + l$, where $i \geqslant 1$ and $l \geqslant 0$; we'll show that (6.19) holds also for $\alpha + 1$. The proof proceeds in a manner similar to the proof of (6.20).

Since $Q_{\alpha+1} = Q_\alpha \partial_{i+1}^{(l+1)}$, by the induction hypothesis

$$C|||Q_{\alpha+1}u|||^2 \geqslant k \sum_{w_\alpha \in W'_\alpha} |||w_\alpha(\partial_{i+1}^{(l+1)}u)|||^2 . \tag{6.29}$$

As before, by lemma 5.3,

$$\partial_1^{(1)} \dots \partial_i^{(r_i-1)} \partial_{i+1}^{(1)} \dots \partial_{i+1}^{(l+1)} \partial^i_{(r_i)} = Q_{\alpha+1} + h_i^{r_i} + N_i^{r_i} , \tag{6.30}$$

where $h_i^{r_i} \in S_{(\alpha+1)}$ and $N_i^{r_i} \in L^{\alpha-1}_{x,t}$.

Consider the operator $\partial_1^{(1)} \dots \partial_i^{(r_i-1)} \partial_{i+1}^{(1)} \dots \partial_{i+1}^{(l+1)}$. It differs from Q_α by containing in its product the extra term $\partial_i^{(l+1)}$ but omitting the term $\partial_i^{(r_i)}$. If we examine the set $\tilde{W}'_\alpha$ associated with this operator (constructed the same way as W'_α) and compare it with the set W'_α associated with Q_α, we can show that

$$C \sum_{\tilde{w}_\alpha \in \tilde{W}'_\alpha} |||\tilde{w}_\alpha u|||^2 \geqslant \sum_{w_\alpha \in W'_\alpha} |||w_\alpha u|||^2 .$$

This is because a typical term in $\tilde{W}'_\alpha$ when compared to a typical term in W'_α involves one more factor $aD_t + b$ and one less factor $c\partial_i + d$, where $a, c \in L_x^{0,r_i,i+1}$, $b \in L_x^{1,r_i,i+1}$ and $d \in L_x^{1-1/r_i,r_i}$. Thus we can show that

$$C|||\partial_1^{(1)} \dots \partial_i^{(r_i-1)} \partial_{i+1}^{(1)} \dots \partial_{i+1}^{(l+1)} u|||^2 \geqslant k \sum_{w_\alpha \in W'_\alpha} |||w_\alpha u|||^2 .$$

As in step 2, we thus show, using (6.30), that

$$C\|\|u\|\|_{\alpha-1}^2 + C\|\|h_i^{r_i}u\|\|^2 + C\|\|Q_{\alpha+1}u\|\|^2 \geqslant k \sum_{w_\alpha \in W'_\alpha} \|\|w_\alpha(\partial_i^{(r_i)}u)\|\|^2 . \tag{6.31}$$

As in step 2 we can remove the terms $C\|\|u\|\|_{\alpha-1}^2 + C\|\|h_i^{r_i}u\|\|^2$ from estimate (6.31). Combining the new estimate with (6.29) we have, after another application of lemma 4.8,

$$C\|\|Q_{\alpha+1}u\|\|^2 \geqslant k \sum_{w_\alpha \in W'_\alpha} \|\|w_\alpha[a\partial_i^{(r_i)} + b\partial_{1+1}^{(l+i)}]u\|\|^2$$

for any properly supported $a, b \in L_x^{0,r_{i,i+1}}$. This leads, after an application of lemma 5.1(c), to:

$$C\|\|Q_{\alpha+1}u\|\|^2 \geqslant k \sum_{w_\alpha \in W'_\alpha} \|\|w_\alpha[cD_t + d + M]u\|\|^2$$

for any properly supported operators $c \in L_x^{0,r_{i,i+1}}$ and $d \in L_x^{1,r_{i,i+1}}$ and some $M \in L_x^{-\infty}$. An argument similar to that used in step 2 then implies that

$$C\|\|Q_{\alpha+1}u\|\|^2 \geqslant k \sum_{w_{\alpha+1} \in W'_{\alpha+1}} \|\|w_{\alpha+1}u\|\|^2 .$$

Hence (6.19) holds for all α.

LEMMA 6.5. *Let V be the module defined in section 4. There exists a constant C independent of u such that the following estimate holds*:

$$C\|\|\tilde{\Pi}u\|\|^2 \geqslant k \sum_{v \in V'} \|\|v(u)\|\|^2 \quad \textit{for } u \in C_0^\infty(\Omega) .$$

PROOF. Since $\tilde{\Pi} = Q_m$, lemma 6.4 implies that $C\|\|\tilde{\Pi}u\|\|^2 \geqslant k \sum_{\in_m w W'_m} \|\|w_m u\|\|^2$. After a comparison of the modules W_m and V, it is easy to see that $C \sum_{w_m \in W'_m} \|\|w_m u\|\|^2 \geqslant \sum_{v \in V'} \|\|v(u)\|\|^2$. Therefore the conclusion of the lemma follows.

COROLLARY 6.6. *Let T be the operator defined in proposition 4.7. There exists a C independent of u such that*

$$C\|\|\tilde{\Pi}u\|\|^2 \geqslant k\|\|Tu\|\|^2 \quad \text{for } u \in C_0^\infty(\Omega) .$$

§ 7. – We are now finally ready to estimate $|||\tilde{P}u|||^2$ in case (*b*).

PROPOSITION 7.1. *Let ψ_2 be defined as in section 3. Let $\tilde{P}(x, t, \partial_x, \partial_t) = P_m(x, t, \partial_x, \partial_t) + P_{m-1}(x, t, \partial_x, \partial_t)$. Suppose $t = 0$ is non-characteristic at the origin with respect to $\tilde{P}$. Then there exists a constant C independent of u such that for $\tilde{r}$, T, k^{-1} sufficiently small we have*

$$k|||\psi_2 u|||^2_{m-2+1/r} \leqslant C|||\tilde{P}\psi_2 u|||^2 \quad \textit{for } u \in C_0^\infty(\Omega),$$

where $\Omega = \{(x, t)\colon |x| \leqslant \tilde{r}, 0 \leqslant t \leqslant T\}$.

PROOF. Let $u_2 = \psi_2 u$. By proposition 4.7

$$\tilde{P}u_2 = \tilde{\Pi}, u_2 + Tu_2 + Ru_2, \quad \text{where } T \in V \text{ and } R \in L^{m-2}_{x,t}. \tag{7.1}$$

Since $Q_m = \tilde{\Pi}$, if we let $\alpha = m$ in (6.19) we have the estimate

$$C|||\tilde{\Pi}u_2|||^2 \geqslant k|||u_2|||^2_{m-2+1/r}.$$

By corollary 6.6, $C|||\tilde{\Pi}u_2|||^2 \geqslant k|||Tu_2|||^2$. Hence, $C|||\tilde{\Pi}u_2|||^2 \geqslant k|||u_2|||^2_{m-2+1/r} + k|||Tu_2|||^2$. (7.1) then implies that

$$C|||u_2|||^2_{m-2+1/r} + C|||Tu_2|||^2 + C|||\tilde{P}u_2|||^2 \geqslant k|||u_2|||^2_{m-2+1/r} + k|||Tu_2|||^2. \tag{7.2}$$

For $k > C$ we can absorb the terms $C|||u_2|||^2_{m-2+1/r} + C|||Tu_2|||^2$ of the l.h.s. of (7.2) into the r.h.s. and get $C|||\tilde{P}u_2|||^2 \geqslant k|||u_2|||^2_{m-2+1/r}$.

We are now ready to complete the proof of theorem 1 by showing how the estimate of $|||\tilde{P}u|||$ in the two cases (*a*) and (*b*) leads to the estimate of $|||\tilde{P}u|||$ in general, which in turn leads to the estimate of $|||Pu|||$.

PROOF OF THEOREM 1.

$$\begin{aligned} \|\tilde{P}u\|^2 &= \|\psi_1\tilde{P}u + \psi_2\tilde{P}u\|^2 \\ &= \|\psi_1\tilde{P}u\|^2 + \|\psi_2\tilde{P}u\|^2 + 2\,\mathrm{Re}\,(\psi_1\tilde{P}u, \psi_2\tilde{P}u) \\ &\geqslant \|\psi_1\tilde{P}u\|^2 + \|\psi_1\tilde{P}u\|^2 - 2\|\tilde{P}u\|^2. \end{aligned}$$

Hence,

$$|||\tilde{P}u|||^2 \geqslant \tfrac{1}{3}(|||\psi_1\tilde{P}u|||^2 + |||\psi_2\tilde{P}u|||^2).$$

We shall now estimate $|||\psi_i \tilde{P} u|||^2$, $i = 1, 2$.

$$|||\psi_i \tilde{P} u|||^2 = |||\tilde{P}\psi_i u + [\tilde{P}, \psi_i]u|||^2 \geqslant |||\tilde{P}\psi_i u|||^2 - |||[\tilde{P}, \psi_i]u|||^2 .$$

Since $u = \psi_1 u + \psi_2 u = u_1 + u_2$,

$$|||[\tilde{P}, \psi_i]u|||^2 \leqslant |||[\tilde{P}, \psi_i]u_1|||^2 + |||[\tilde{P}, \psi_i]u_2|||^2 .$$

Therefore,

$$\begin{aligned} |||\tilde{P}u|||^2 \geqslant & \, C|||\tilde{P}u_1|||^2 + C|||\tilde{P}u_2|||^2 \\ & - C|||[\tilde{P}, \psi_1]u_1|||^2 - C|||[\tilde{P}, \psi_1]u_2|||^2 \\ & - C|||[\tilde{P}, \psi_2]u_1|||^2 - C|||[\tilde{P}, \psi_2]u_2|||^2 . \end{aligned}$$

Rearranging the terms of the last inequality,

$$\begin{aligned} |||\tilde{P}u|||^2 \geqslant & \, C|||\tilde{P}u_1|||^2 + C|||\tilde{P}u_2|||^2 \\ & - C|||[\tilde{P}, \psi_1]u_1|||^2 - C|||[\tilde{P}, \psi_2]u_1|||^2 \\ & - C|||[\tilde{P}, \psi_1]u_2|||^2 - C|||[\tilde{P}, \psi_2]u_2|||^2 . \end{aligned} \tag{7.3}$$

We shall now estimate the terms in the last two lines of (7.3).

Since $\tilde{P} \in L^m_{x,t}$ and $\psi_i \in L^0_x$ and since the coefficient of ∂^m_t in $\tilde{P}$ is 1, $[\tilde{P}, \psi_i] \in L^{m-1}_{x,t}$. Hence,

$$|||[\tilde{P}, \psi_1]u_1|||^2 + |||[\tilde{P}, \psi_2]u_1|||^2 \leqslant C|||u_1|||^2_{m-1} .$$

By proposition 4.7,

$$[\tilde{P}, \psi_i]u_2 = [(\tilde{\Pi} + T + R), \psi_i]u_2 ,$$

where $T \in L^{m-1-1/r}_{x,t}$ and $R \in L^{m-2}_{x,t}$. An examination of the operators T and R shows that they are both of order $\leqslant m - 2$ in t. Hence, $[T, \psi_i] \in L^{m-2}_{x,t}$ and $[R, \psi_i] \in L^{m-2}_{x,t}$.

Therefore, (7.3) implies that

$$\begin{aligned} |||\tilde{P}u|||^2 \geqslant & \, C|||\tilde{P}u_1|||^2 + C|||\tilde{P}u_2|||^2 \\ & - C|||u_1|||^2_{m-1} - C|||u_2|||^2_{m-2} \\ & - C|||[\tilde{\Pi}, \psi_1]u_2|||^2 - C|||[\tilde{\Pi}, \psi_2]u_2|||^2 . \end{aligned} \tag{7.4}$$

By lemma 5.3, $[\tilde{\Pi}, \psi_i] \in S_{(m)}$, modulo terms belonging to $L_{x,t}^{m-2}$. (7.4) therefore implies that

$$\|\!|\tilde{P}u|\!\|^2 \geqslant C|\!\|\tilde{P}u_1|\!\|^2 + C|\!\|\tilde{P}u_2|\!\|^2 - C|\!\|u_1|\!\|_{m-1}^2 - C|\!\|u_2|\!\|_{m-2}^2 - C \sum_{s_m \in S'_{(m)}} |\!\|s_m u_2|\!\|^2 . \tag{7.5}$$

Applying propositions 3.1 and 7.1 and lemma 6.2 to (7.5) we have:

$$C|\!\|\tilde{P}u|\!\|^2 \geqslant k|\!\|u_1|\!\|_{m-1}^2 + k|\!\|u_2|\!\|_{m-2+1/r}^2 + k \sum_{s_m \in S'_{(m)}} |\!\|s_m u_2|\!\|^2 - C|\!\|u_1|\!\|_{m-1}^2 - C|\!\|u_2|\!\|_{m-2}^2 - C \sum_{s_m \in S'_{(m)}} |\!\|s_m u_2|\!\|^2 .$$

Applying lemma 3.1, and choosing $k > C$, we get $C|\!\|\tilde{P}u|\!\|^2 \geqslant k|\!\|u_1|\!\|_{m-1}^2 + k|\!\|u_2|\!\|_{m-2+1/r}^2$. After another application of lemma 3.1, this implies that

$$C|\!\|\tilde{P}u|\!\|^2 \geqslant k|\!\|u_1|\!\|_{m-2+1/r}^2 + k|\!\|u_2|\!\|_{m-2+1/r}^2 \geqslant k|\!\|u|\!\|_{m-2+1/r}^2 .$$

Finally, if we appeal to lemma 3.2, we get the desired estimate:

$$C|\!\|Pu|\!\|^2 \geqslant k|\!\|u|\!\|_{m-2+1/r}^2 .$$

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Southern Illinois University
Department of Mathematics
Carbondale, Illinois 62901