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Theta functions and Barsotti-Tate groups

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Theta Functions and Barsotti-Tate Groups (*)(**).

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Introduction.

In 1970 I. Barsotti published a paper (cfr. [θ]) which developed a theory of theta functions in characteristics 0. The methods used were completely algebraic and for the first time the theta functions were characterized without recours to periods.

The aim of this paper is to construct an analogous theory of theta functions in positive characteristic. Hence it may be useful to illustrate briefly the main ideas of the theory in characteristic 0.

Let A be an abelian variety over a field of characteristic 0, and let R be the completion of the local ring at the identity point of A. Let X be a divisor on A defined by the cocycle $(\varphi_U)_{U\in\mathfrak{U}}$ relative to the affine open covering $\mathfrak{U}$ of A; for every invariant derivation d, the differential $\partial(d\varphi_U/\varphi_U)$ is the exact part on U of a differential of the second kind ω_d. Any such ω_d is determined up to differentials of the first kind on A. Differentials of the second kind are closed, and so there exists an element $\eta_d \in Q(R)$ such that $\partial\eta_d = \omega_d$; here $Q(R)$ denotes the quotient field of R. If the ω_d's are suitably chosen, the map $d \mapsto \eta_d$ is a closed differential ζ of $Q(R)$. Hence the equation $\partial\vartheta/\vartheta = \zeta$ has solutions in $Q(R)$; these solutions are the theta functions of X.

It is clear that this construction cannot be used in positive characteristic. However, in [θ] it is observed that the elements of $Q(R)$ which satisfy the given differential equation are essentially characterized by the « theorem of the cube ». In particular, the theta functions of X are the elements of

(*) Lavoro eseguito nell'ambito del G.N.S.A.G.A. del C.N.R.

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$Q(R)$ which satisfy the equation

$$F = \frac{\big((\iota \otimes \mathbb{P})\mathbb{P}\vartheta\big)(\vartheta \overline{\times} \vartheta \overline{\times} \vartheta)}{(\mathbb{P}\vartheta \overline{\times} 1)\big(sc_{23}(\mathbb{P}\vartheta \overline{\times} 1)\big)(1 \overline{\times} \mathbb{P}\vartheta)}\,, \tag{0.1}$$

where $\mathbb{P}$ is the extension to $Q(R)$ of the coproduct of R, $\overline{\times}$ denotes the completed tensor product, sc_{23} is the automorphism of $R \overline{\times} R \overline{\times} R$ defined by $x \otimes y \otimes z \mapsto x \otimes z \otimes y$, F is an equation of the divisor

$$(p_1 + p_2 + p_3)^* X - (p_1 + p_2)^* X - (p_1 + p_3)^* X - (p_2 + p_3)^* X + \\ + p_1^* X + p_2^* X + p_3^* X\,;$$

the p_i are the obvious projections and p_i^* are the induced maps on the groups of divisors.

This characterization provides the starting point for our paper. We require, as is natural, that the theta functions of a principal divisor coincide with the corresponding rational functions, and that the action of A by translation on the group of thetas corresponds to translation on divisors. Then the thetas of X must be solutions of equation (0.1) in some extension field of the field C of the rational functions on A. Indeed, in section 1 and 2 we show that if the ground field is perfect, equation (0.1) has solutions in a certain completion of the perfect closure of $Q(R)$. The existence of such solutions depends on the fact that, over a perfect base field, every extension of a Barsotti-Tate group by the multiplicative group G_m splits.

The methods used are quite general. In fact they may be applied whenever one has an « analytic » group G whose hyperalgebra contains an order of C, and which has the property that every extension of G by G_m splits. For example, with this method we can also construct l-adic theta functions for every prime $l \neq p$: they are locally constant functions on the Tate space V_l with values in k. Theta functions of this type have already been studied by Mumford [7] and by Barsotti (secret notes).

Those who know the work of Barsotti will clearly see how much the author of the present paper owes to him; but here I wish to thank him particularly for his constant and indispensable aid during the preparation of this paper.

1. – The theta function of a divisor (a special case).

Let k be a perfect field of characteristic $p \neq 0$, and let A be an abelian variety of dimension n over k; we assume that a definite identity point

$e \in A$ has been selected, hence that a definite group law on A has been chosen. $\hat{A}$ will denote the Barsotti-Tate group of A, *i.e.* $\hat{A} = \varinjlim (\ker p^i \iota_A)$ (ι will always denote the identity map). In most of this paper we will use only the local component ${}^{\pi}\hat{A}$ of $\hat{A}$; its hyperalgebra, which will be denoted by R, can be identified with the completion of the local ring $\mathcal{O}_{A,e}$ of e: $R = \hat{\mathcal{O}}_{A,e}$. Hence, if $(x) = (x_1, x_2, \ldots, x_n)$ is a regular set of parameters of $\mathcal{O}_{A,e}$ or of R, we have $R = k[\![x]\!]$. The field $C = k(A)$ of the rational functions on A can be canonically embedded in the quotient field $Q(R)$ of R, and we shall consider it to be so embedded.

Since many copies of A, C and R will be needed, it is convenient to index them; thus, for instance, the completed tensor product $R \overline{\times} R$ (over k) will be denoted by $R_1 \overline{\times} R_2 = k[\![x_1, x_2]\!]$. The single x's, indexed from 1 to n, will never be used again, so that no confusion arises from this notation.

Let X be a divisor on A. If X is linearly equivalent to zero, $X \sim 0$, then $X = (f)$ for some $f \in C \subset Q(R)$. This f, which is uniquely defined by X ud to a nonzero factor in k, is called a *theta element* of X, and will be denoted by ϑ_X. If we assume that none of the polar components of X go through e, ϑ_X can be chosen in R; while if none of the components go through e, ϑ_X can be chosen in such a way that $\vartheta_X \equiv 1 \bmod R^+$ (R^+ is the kernel of the coidentity ε of R).

Next we assume X algebraically equivalent to zero, $X \equiv 0$. If P is a point of A, σ_P will denote the translation by P and σ_P^* is the corresponding map in the group of divisors. We begin by assuming that none of the components of X go through e; this restriction will be eliminated at the end of the following argument.

If p_ι, $i = 1, 2$, stands for the i-th projection of $A \times A$ on A, and p_i^* is the corresponding map between the groups of divisors, it is well known that (cf. [6])

$$Y = (p_1 + p_2)^* X - p_1^* X - p_2^* X \tag{1.1}$$

is linearly equivalent to zero. Therefore Y has a theta element $f = f(x_1, x_2)$ in the previous sense. By the assumption on X, $f \in \mathcal{O}_{A\times A, e\times e} \subset R \overline{\times} R$, and it can be chosen in such a way that

$$f(x_1, 0) = f(0, x_2) = 1 . \tag{1.2}$$

If q_j, $j = 1, 2, 3$, denotes the j-th projection of $A \times A \times A$ on A and p_{ij}, $i = 1, 2$, is the j-th projection of $A \times A \times A$ on the i-th factor of $A \times A$, then $p_l p_{ij} = \delta_{li} q_j$, and we readily obtain

$$(p_{11} + p_{12} + p_{23})^* Y + (p_{11} + p_{22})^* Y = \\ = (p_{11} + p_{22} + p_{23})^* Y + (p_{12} + p_{23})^* Y . \tag{1.3}$$

Therefore, if $\mathbb{P}$ denotes the coproduct of R, and if for short we write $x_1 \dotplus x_2$ instead of $\mathbb{P}x$, then after choosing f as in (1.2), relation (1.3) is equivalent to

$$f(x_1 \dotplus x_2, x_3)f(x_1, x_2) = f(x_1, x_2 \dotplus x_3)f(x_2, x_3)\,. \tag{1.4}$$

If $\hat{G}_m$ denotes the multiplicative formal group over k, and if $k[\![t]\!]$, with $\mathbb{P}t = t\bar{\times}1 + 1\bar{\times}t + t\bar{\times}t$, is its hyperalgebra, then (1.4) shows that the map $t \mapsto f - 1$, from $k[\![t]\!]$ to the completed tensor product $R\bar{\times}R$, corresponds to a factor set of ${}^{\pi}\hat{A}$ on $\hat{G}_m$, and thus to an extension E of ${}^{\pi}\hat{A}$ by $\hat{G}_m$. But it is well known that every such extension splits, i.e. that $E \cong \hat{G}_m \times {}^{\pi}\hat{A}$. Hence there exists an element $\vartheta_X = \vartheta_X(x)$ such that

$$f(x_1, x_2) = \frac{\vartheta_X(x_1 \dotplus x_2)}{\vartheta_X(x_1)\vartheta_X(x_2)}\,. \tag{1.5}$$

This ϑ_X will be called a *theta element* of X.

The quotient of two solutions of (1.5) is a nonzero element $g = g(x) \in R$ such that $g(x_1 \dotplus x_2) = g(x_1)g(x_2)$. Such nonzero elements will be called *multiplicative elements*. The hyperalgebra R has nontrivial multiplicative elements (that is different from 1) if and only if it has a block of slope 1, namely if and only if A has p-division points. Therefore, if there are no such points, there exists a unique ϑ_X which satisfies (1.5). It is uniquely determined by X and by the condition $\vartheta_X \equiv 1 \bmod R^+$. In general X determines ϑ_X up to a nonzero factor in k and a multiplicative factor in R. We shall say for short that X determines ϑ_X up to linear exponentials. The group of the linear exponentials will be denoted by (l.e.). The reason for this terminology will be seen further on.

We can now remove the restriction initially imposed on the support of X. Indeed, let D_0 be the group of principal divisors, and D_1 be the group of divisors algebraically equivalent to zero whose support does not contain e. Then, the preceding argument provides the homomorphisms

$$D_0 \to Q(R)^*/k^* \qquad \text{and} \qquad D_1 \to R^*/(\text{l.e.})$$

(if S is a ring S^* denotes the multiplicative group of units in S). Since $D_a = D_0 + D_1$ coincides with the group of all divisors algebraically equivalent to zero, any $Y \in D_a$ can be written as $Y = Y_0 + Y_1$, with $Y_i \in D_i$; we shall then define ϑ_Y as

$$\vartheta_Y = \vartheta_{Y_0}\vartheta_{Y_1}\,.$$

This is a good definition: indeed, if $Y = Y'_0 + Y'_1$ with $Y'_i \in D_i$, the homomorphisms described above imply that

$$\vartheta_{Y_0}/\vartheta_{Y'_0} = \vartheta_{Y_0 - Y'_0} = \vartheta_{Y'_1 - Y_1} = \vartheta_{Y'_1}/\vartheta_{Y_1};$$

hence $\vartheta_{Y_0}\vartheta_{Y_1} = \vartheta_{Y'_0}\vartheta_{Y'_1}$, (these equalities are really congruences mod (l.c.)).

If K is a k-algebra and G, B are a group over k and its hyperalgebra, we will denote by G_K and B_K the group over K and its hyperalgebra (obtained by scalar extension).

In the next section the following lemma will be needed:

(1.6) LEMMA. *Let K be a perfect extension field of k, S an integrally closed order of K containing k, G a local B-T group over k and E a (commutative) extension of G_S by $(\hat{G}_m)_S$; then $E \cong (\hat{G}_m)_S \times G_S$.*

PROOF. Let R be the hyperalgebra of G. The hypothesis says that E corresponds to a (symmetric) factor set $f \in R_S \overline{\times} R_S$. We must show that teh equation

$$f(x_1, x_2) = \frac{g(x_1 \dot{+} x_2)}{g(x_1)g(x_2)} \tag{1.7}$$

in the unknown g has solutions in R_S.

We shall give the proof only in the case of a single variable, that is when $\dim G = 1$: in general the same reasoning applies.

Since $K = Q(S)$ is perfect, we know by the preceding remarks that (1.7) has solutions $g(x) = \sum_{i \geq 0} a_i x^i$ in R_K. We shall show next that each a_i is a zero of a monic polynomial with coefficients in S. Hence, S being integrally closed, each $a_i \in S$.

Let $f(x_1, x_2) = \sum_{i,j \geq 0} b_{ij} x_1^i x_2^j$, with $b_{ij} \in S$; since f is a unit of $R_S \overline{\times} R_S$ we may suppose that $b_{00} = 1$; f being a (symmetric) factor set, $b_{0i} = 0$ and $b_{ij} = b_{ji}$ for every $i, j > 0$. For $\mathrm{P}x = x_1 \dot{+} x_2$ we will have an expression of the type $x_1 \dot{+} x_2 = x_1 + x_2 + \sum_{i,j>0} \alpha_{ij} x_1^i x_2^j$ with $\alpha_{ij} \in k$. With these notations (1.7) can be rewritten in the following form

$$\sum_{r,s \geq 0} \Big(\sum_{\substack{i+l=r \\ j+m=s}} b_{ij} a_l a_m \Big) x_1^r x_2^s = \sum_{t \geq 0} a_t \Big(x_1 + x_2 + \sum_{u,v>0} \alpha_{uv} x_1^u x_2^v \Big)^t . \tag{1.8}$$

By (1.7) and the hypotheses on f we have $a_0 = 1$; therefore we will use an inductive argument: we will assume that the a_j with $j \leqslant p^i - 1$ belong to S, and we shall prove that also the a_j with $j \leqslant p^{i+1} - 1$ belong to S. To this end, consider the relations $R(p^i, j)$, with $i \geqslant 1$ and $1 \leqslant j \leqslant p^i(p-1)$, obtained by equating the coefficients of the monomials $x_1^{p^i} x_2^j$ in the two sides of (1.8):

$$R(p^i, j) \colon a_{p^i} a_j + \sum_{\substack{1 \leqslant m \leqslant p^i \\ 1 \leqslant l \leqslant j}} b_{ml} a_{p^i - m} a_{j-l} = \binom{p^i + j}{j} a_{p^i + j} + L_{p^i, j} ,$$

where, by the inductive hypothesis, $L_{p^i,j}$ is a linear polynomial with coefficients in S, in the arguments a_l with $p^i \leqslant l \leqslant p^i + j - 1$.

If $1 \leqslant j < p^i(p-1)$, then $p \nmid \binom{p^i + j}{j}$. Therefore, for these j's we can obtain the $a_{p^i + j}$ from $R(p^i, j)$ and replace them in the $R(p^i, s)$, where $j < s \leqslant p^i(p-1)$. In this way for each j, $1 \leqslant j \leqslant p^i(p-1)$, we obtain an expression of the form

$$\binom{p^i + j}{j} a_{p^i + j} = P_{p^i, j}(a_{p^i}) , \tag{1.9}$$

where, in wiew of the inductive hypothesis, $P_{p^i,j}(X)$ is a polynomial with coefficients in S. If $(r-1)\, p^i \leqslant j < rp^i$, the degree of $P_{p^i,j}$ is equal to r; in particular $P_{p^i,(r-1)p^i}$ is monic. Since p divides $\binom{p^{i+1}}{p^i(p-1)}$, a_{p^i} is a solution of the equation $P_{p^i, p^i(p-1)}(X) = 0$, and therefore it is in S. Now, from (1.9) we deduce that each a_l with $0 \leqslant l < p^{i+1}$ is in S, Q.E.D.

2. – The theta function of a divisor (general case).

In this section we shall need the perfect closures of R and C: we shall denote them by $\mathfrak{R}^0$ and C^∞ respectively. As R is local, $\mathfrak{R}^0$ is the direct limit of the sequence

$$R \xrightarrow{p^t} R \xrightarrow{p^t} \dots ,$$

and it is endowed with the limit topology. With this topology $\mathfrak{R}^0$ is not complete; its completion will be denoted by $\mathfrak{R}$. The group corresponding to $\mathfrak{R}$ will be denoted by ${}^{\pi}\hat{\mathcal{A}}$: it is the inverse limit of the sequence

$${}^{\pi}\hat{A} \xleftarrow{p^t} {}^{\pi}\hat{A} \xleftarrow{p^t} \dots .$$

As usual, C^∞ will be identified with its image in the field $Q(\mathfrak{R})$.

Let (y) be a non-homogeneous general point of A such that the n-tuple of its first n coordinates, which will be denoted by (x), is a regular set of parameters of $\mathcal{O}_{A,e}$. Identify C with the second factor of $C \otimes C = C_1 \otimes C_2$, and let P be the point of $A^{(1)}$ (extension of A over C_1^∞) where (y_2) assumes values (y_1). Let X be a divisor on A; at first we assume that none of the components of X go through e, and we identify X with its extension to $A_1^{(1)}$. With this identification, $Z = \sigma_P^* X - X$ is a divisor algebraically equivalent to zero on $A_1^{(1)}$, and by the previous paragraph it has a theta element in $C_1^\infty[\![x_2]\!]$.

First, we shall show that this theta element may be chosen in $\mathfrak{R}_1^0[\![x_2]\!]$, and then we shall symmetrize it in such a way as to obtain a factor set in $\mathfrak{R} \overline{\times} \mathfrak{R}$. This factor set will be associated to an extension of ${}^{\pi}\hat{\mathcal{A}}$ by $\hat{G}_m$.

As in paragraph 1, and with the same meanings for the symbols, consider the divisor

$$(2.1) \qquad Y = (p_1 + p_2 + p_3)^* X - (p_1 + p_2)^* X - \\ - (p_1 + p_3)^* X - (p_2 + p_3)^* X + p_1^* X + p_2^* X + p_3^* X$$

on $A \times A \times A$. It is well known, cf. for instance [6], that $Y \sim 0$; hence Y has a theta element $F = F(x_1, x_2, x_3) \in Q(C \otimes C \otimes C)$.

Since none of the components of X go through e, $F \in R \overline{\times} R \overline{\times} R$. As F is symmetric, and $F(0, x_2, x_3)$ is a theta element of the zero divisor, we may assume that

$$(2.2) \qquad F(0, x_2, x_3) = F(x_1, 0, x_3) = F(x_1, x_2, 0) = 1 .$$

If we use the same symbol Y for the extension of Y to $A^{(1)} \times A^{(1)} \times A^{(1)}$, and if P is the point already used in the definition of Z, we have

$$[(p_1 + p_2)^* X - p_2^* X] \cap (P \times A^{(1)} \times A^{(1)}) = p_2^*(\sigma_P^* X - X)$$

$$[(p_1 + p_3)^* X - p_3^* X] \cap (P \times A^{(1)} \times A^{(1)}) = p_3^*(\sigma_P^* X - X)$$

$$[(p_1 + p_2 + p_3)^* X - (p_2 + p_3)^* X] \cap (P \times A^{1)} \times A^{(1)}) = (p_2 + p_3)^*(\sigma_P^* X - X) ,$$

where $\cap$ is the intersection-product; hence

$$(2.3) \qquad Y \cap (P \times A^{(1)} \times A^{(1)}) = (p_2 + p_3)^* Z - p_2^* Z - p_3^* Z .$$

From (2.3) and from section 1 it follows immediately that F is a factor set of $(R_2)_{R_1} \otimes (R_3)_{R_1}$, which supplies an extension of $({}^{\pi}\hat{A})_{R_1}$ by $(\hat{G}_m)_{R_1}$. Now if we observe that $\mathfrak{R}^0$ is integrally closed, and that $Q(\mathfrak{R}^0)$ is a perfect field, we see that, by (1.6), there exists a $\varphi(x_1, x_2) \in \mathfrak{R}_1^0[\![x_2]\!]$ such that

$$F(x_1, x_2, x_3) = \frac{\varphi(x_1, x_2 \dotplus x_3)}{\varphi(x_1, x_2)\varphi(x_1, x_3)}. \tag{2.4}$$

By (2.3) it is clear that each solution φ of (2.4) is a theta element of the copy of Z on $A_2^{(1)}$. If we compute (2.4) first in $(x_1, 0, x_3)$, next in $(0, x_2, x_3)$, and remember assumption (2.2) on F, we conclude that $\varphi(x_1, 0) = 1$ and that $\varphi(0, x_2)$ is a multiplicative element of R_2. Therefore there exists a (unique) solution φ of (2.4) such that

$$\varphi(0, x_2) = \varphi(x_1, 0) = 1. \tag{2.5}$$

The field C will now be identified with the 3-rd factor of $C \otimes C \otimes C$, and P_1, P_2 will denote the points of $A_3^{(1,2)}$ (extension of A over $Q(C_1^\infty \otimes C_2^\infty)$) where (y_3) assumes the values (y_1) and (y_2) respectively.

In a way similar to that just used for $Y \cap (P \times A^{(1)} \times A^{(1)})$, we obtain

$$\begin{aligned} Y \cap (P_1 \times P_2 \times A_3^{(1,2)}) &= \sigma^*_{P_1+P_2} X - \sigma^*_{P_1} X - \sigma^*_{P_2} X + X = \\ &= (\sigma^*_{P_1+P_2} X - X) - (\sigma^*_{P_1} X - X) - (\sigma^*_{P_2} X - X). \end{aligned} \tag{2.6}$$

From (2.6) follows that

$$F(x_1, x_2, x_3) \qquad \text{and} \qquad \frac{\varphi(x_1 \dotplus x_2, x_3)}{\varphi(x_1, x_3)\varphi(x_2, x_3)}$$

are theta elements of the same divisor on $A_3^{(1,2)}$; hence we have:

$$F(x_1, x_2, x_3)\delta(x_1, x_2)\,\mu(x_3) = \frac{\varphi(x_1 \dotplus x_2, x_3)}{\varphi(x_1, x_3)\varphi(x_2, x_3)}, \tag{2.7}$$

where δ is a nonzero element of $Q(C_1^\infty \otimes C_2^\infty)$ and μ is a multiplicative element of $R_3^{(1,2)}$ (the extension of R_3 over $Q(C_1^\infty \otimes C_2^\infty)$). We remember that F and φ are normalized as in (2.2) and (2.5) respectively, and we compute (2.7) first at $(x_1, x_2, 0)$ and then at $(0, 0, x_3)$. The result is $\delta = \mu = 1$, so that

$$F(x_1, x_2, x_3) = \frac{\varphi(x_1 \dotplus x_2, x_3)}{\varphi(x_1, x_3)\varphi(x_2, x_3)}. \tag{2.8}$$

Now, since F is symmetric and ${}^\pi\hat{A}$ is commutative, (2.8) can be rewritten in the form

$$F(x_1, x_2, x_3) = \frac{\varphi(x_2 \dotplus x_3, x_1)}{\varphi(x_2, x_1)\varphi(x_3, x_1)}. \tag{2.9}$$

By comparing (2.9) with (2.4) we see that

$$\chi(x_1, x_2) = \frac{\varphi(x_2, x_1)}{\varphi(x_1, x_2)} \tag{2.10}$$

is a bi-multiplicative element of $\mathfrak{R}\overline{\times}\mathfrak{R}$: that is, χ is a nonzero element such that

$$\frac{\chi(x_1 \dotplus x_2, x_3)}{\chi(x_1, x_3)\chi(x_2, x_3)} = \frac{\chi(x_1, x_2 \dotplus x_3)}{\chi(x_1, x_2)\chi(x_1, x_3)} = 1 .$$

Since there is a one-to-one map between the set of these elements and the set of bi-homomorphisms ${}^{\pi}\hat{\mathcal{A}} \times {}^{\pi}\hat{\mathcal{A}} \to \hat{G}_m$, there exist bi-multiplicative elements in $\mathfrak{R}\overline{\times}\mathfrak{R}$ if and only if $\mathfrak{R}$ has a block of slope $\alpha < 1$. This follows from the theorem of symmetry (cf. chap. 7 of [MA]): in fact, from this theorem we know that $\mathfrak{R}$ has a block B of slope α, with $0 < \alpha < 1$, if and only if it has a block (of slope $1 - \alpha$) isomorphic to the dual $\tilde{B}$ of B. From this fact, if A is ordinary, i.e. $\mathfrak{R}$ has only the block of slope 1, we may conclude that $\chi = 1$, so that φ is symmetric and belongs to $R\overline{\times}R$; in the other cases, i.e. when $\mathfrak{R}$ possesses blocks of slope different from 1, χ supplies informations about the class of X under algebraic equivalence; in particular, if in $\mathfrak{R}$ the block of slope 1 is missing, the map $X \mapsto \chi$ gives an isomorphic image of the Severi group of A in the group of bi-multiplicative elements of $\mathfrak{R}\overline{\times}\mathfrak{R}$ (cf. § 8).

In all cases, $\chi(x_1, x_2) \equiv 1$, so that $\chi(x_1, x_2)^{\frac{1}{2}}$ exists in $\mathfrak{R}\overline{\times}\mathfrak{R}$ and supplies a

$$\psi(x_1, x_2) = \varphi(x_1, x_2)\,\chi(x_1, x_2)^{\frac{1}{2}} \tag{2.11}$$

which is a symmetric element of $\mathfrak{R}\overline{\times}\mathfrak{R}$. The element ψ satisfies both (2.4) and (2.8) (replace φ by ψ), hence it satisfies the equation

$$\psi(x_1, x_2 \dotplus x_3)\,\psi(x_2, x_3) = \psi(x_1 \dotplus x_2, x_3)\,\psi(x_1, x_2) .$$

The element ψ is thus a factor set, associated to an extension of ${}^{\pi}\hat{\mathcal{A}}$ by $\hat{G}_m$. But it is well known (cf. chap. 4 of [MA]) that such extensions split; consequently there exist a $\vartheta_X \in \mathfrak{R}$ such that

$$\psi(x_1, x_2) = \frac{\vartheta_X(x_1 \dotplus x_2)}{\vartheta_X(x_1)\vartheta_X(x_2)} . \tag{2.12}$$

The direct connection between F and ϑ_X appears after comparing (2.12) with (2.4) or (2.9): it is

$$F(x_1, x_2, x_3) = \frac{\vartheta_X(x_1 \dotplus x_2 \dotplus x_3)\vartheta_X(x_1)\vartheta_X(x_2)\vartheta_X(x_3)}{\vartheta_X(x_1 \dotplus x_2)\vartheta_X(x_1 \dotplus x_3)\vartheta_X(x_2 \dotplus x_3)}; \tag{2.13}$$

ϑ_X will be called a *theta element* of X.

This relation, perfectly analogous to the corresponding one in [θ], will be the main tool in the characterization of abstract theta function (cf. § 7).

From the preceding arguments it is clear that ψ is determined by F up to symmetric bi-multiplicative elements of $\mathfrak{R}\overline{\times}\mathfrak{R}$. But every such element $\omega = \omega(x_1, x_2)$ is a factor set, so that it has a decomposition analogous to that of ψ:

$$\omega = \frac{q(x_1 \dotplus x_2)}{q(x_1)q(x_2)}.$$

Such elements q, or more precisely the product of such q's by nonzero elements of k, will be called *trivial theta functions*, or *quadratic exponentials* (q.e. for short). They are characterized by the relation

$$\frac{q(x_1 \dotplus x_2 \dotplus x_3)q(x_1)q(x_2)q(x_3)}{q(x_1 \dotplus x_2)q(x_1 \dotplus x_3)q(x_2 \dotplus x_3)} = \text{constant}.$$

The reason for the first name is thus clear, while the reason for the second one will appear later on. Finally observe that (q.e.) $\supseteq$ (l.e.), and that ϑ_X is determined by X up to (q.e.).

What has been said until now concerns divisors whose components do not go through e. But every divisor X is a sum of a divisor Y algebraically equivalent to zero and of a divisor Z whose components do not go through e; thus we will define ϑ_X by means of $\vartheta_X = \vartheta_Y\vartheta_Z$. In this way ϑ_X is well defined, as we may verify with the method used in section 1.

The results of these first two sections may be summarized in the following

(2.14) THEOREM. *To every divisor X on A one can associate an element ϑ_X of the quotient field $Q(\mathfrak{R})$ of $\mathfrak{R}$, unique up to* (q.e.), *with the following properties*: $\vartheta_{X+Y} = \vartheta_X\vartheta_Y$; $\vartheta_X \in \mathfrak{R}$ *if and only if none of the polar components of X go through e*; $X = 0$ *if and only if* $\vartheta_X = 1$; $X \sim 0$ *if and only if* $\vartheta_X \in C$; $\vartheta_X \in Q(R)$ *if* $X \equiv 0$ (*in reality, the previous equalities are congruences mod* (q.e.)).

Observe that, by the first property, the second is equivalent to the following: let X be an effective divisor; then ϑ_X is a unit of $\mathfrak{R}$ if and only if none of the components of X go through e. This assertion follows immediately from (2.13). Observe furthermore that if A is ordinary, then for any X the φ of (2.4) may be chosen in R. Consequently, the converse to the last statement of our theorem does not hold.

Theorem (2.14), and more generally all the arguments used until now, are also true when A is an extension of an abelian variety by a multiplicative group, rather than an abelian variety. Such varieties will appear later on, because it is not possible to distinguish them from abelian varieties by the local components of their B-T groups.

Functions similar to our F have already been studied by Mumford [8]: in his terminology F determines an element of Bi-ext $(A \times A, G_m)$. Our F corresponds to a symmetric bi-extension, and the F's corresponding to such bi-extensions are characterized by the following properties:

$$(2.15)\qquad \begin{cases} F(x_1, x_2, x_3) = F(x_{s_1}, x_{s_2}, x_{s_3})\,, & \text{for every permutation } s \in \mathfrak{S}_3\,; \\ F(x_1 + x_2, x_3, x_4)F(x_1, x_2, x_4) = F(x_1, x_2 + x_3, x_4)F(x_2, x_3, x_4)\,. \end{cases}$$

3. – Generalities on bivectors.

The (Witt) bivectors are the main tool which we will use in analysing the previous results; in particular they allow us to exhibit the analogies between this and the case of characteristic 0.

In this and in the following section we will limit ourselves to brief descriptions and to an inventory of the formulas which will be used later on. For precise definitions and statements we refer to [MA].

In the following discussion B will denote the hyperalgebra of a B-T group (not necessarily local): such hyperalgebras will be called *hyperdomains* (cf. chap. 3 of [MA]). Just as starting from R we have defined $\mathfrak{R}^0$ and $\mathfrak{R}$, so starting from B we may define $\mathfrak{B}^0$ and $\mathfrak{B}$: i.e. $\mathfrak{B}^0$ is the direct limit of the sequence $B \xrightarrow{p\iota} B \xrightarrow{p\iota} \ldots$; it is endowed with the limit topology, and $\mathfrak{B}$ is the completion of $\mathfrak{B}^0$. The hyperalgebras of the type of $\mathfrak{B}^0$ and $\mathfrak{B}$ will be called *bidomains*. For an intrinsic characterization of bidomains cf. chap. 4 of [MA]. Whenever we speak of the canonical embedding of B in $\mathfrak{B}_0$, we are referring to the natural map of the first term of the sequence $B \xrightarrow{p\iota} B \xrightarrow{p\iota} \ldots$ in $\mathfrak{B}^0$.

With vect and cov we shall denote the (group) functors of Witt vectors and covectors. For cov we shall use the definition given at the end of § 5 of [MA]: this definition, unlike other ones given more recently (cf. [4]), makes it possible to define operations as d_* (cf. § 4), even if cov is defined on a category of algebras which, in addition to the local algebras, contains the étale ones. A (Witt) *bivector* with components in the k-algebra S is a sequence $x = (x_i)_{i \in \mathbb{Z}}$ of elements of S such that for every $m \in \mathbb{Z}$ the subsequence $x_{(m)} = (\ldots, x_{m-1}, x_m)$ is a covector. The bivector x will be denoted with $(\ldots, x_{-1}; x_0, x_1, \ldots)$. The bivectors which have the $x_{(m)} = 0$, for some $m \in \mathbb{Z}$, will be called *special.* The special bivectors of the type $(\ldots 0; \xi, 0, 0, \ldots)$ will be denoted by $\{\xi\}$.

For k-algebras of the type of $\mathcal{B}$ (in particular for bidomains), we define biv $\mathcal{B}$: it is the set of bivectors x with components in $\mathcal{B}$, such that, for some, and hence for every $m \in \mathbb{Z}$, $x_{(m)} \in \operatorname{cov} \mathcal{B}$. On biv $\mathcal{B}$ we define a sum in the following way: for x, $y \in \operatorname{biv} \mathcal{B}$, we have $(x+y)_i = (x_{(m)} + y_{(m)})_i$, if $m \geqslant i$. Endowed with this sum biv $\mathcal{B}$ is a group; the subset of its elements such that $x_{(-1)} = 0$ is a subgroup isomorphic to vect $\mathcal{B}$, and biv $\mathcal{B}$/vect $\mathcal{B}$ = cov $\mathcal{B}$.

The topology of $\mathcal{B}$ allows us to define a topology on biv $\mathcal{B}$ (cf. § 9 of [MA]); with this topology biv $\mathcal{B}$ is a topological group. Now $\mathbf{Q}$ vect $\mathcal{B}$, which in addition to the sum has the product inherited from vect $\mathcal{B}$, becomes a dense topological ring in biv $\mathcal{B}$. As a consequence the completion of biv $\mathcal{B}$, which will be denoted by Biv $\mathcal{B}$, has a natural structure of topological ring.

As usual, on biv $\mathcal{B}$ we define the operators π (Frobenius) and t (Verschiebung) in the following way:

$$(\pi x)_i = x_i^p \qquad \text{and} \qquad (tx)_i = x_{i-1}, \qquad\qquad \text{for } x \in \operatorname{biv} \mathcal{B}\,.$$

The operators just defined are continuous, so that it is possible to extend them to Biv $\mathcal{B}$; we have

$$\pi(tx) = t(\pi x) = px, \qquad\qquad \text{for } x \in \operatorname{Biv} \mathcal{B}\,.$$

An element $x \in \operatorname{cov} \mathcal{B}$ will be called *canonical* if $\mathrm{P}x = x \bar{\times} 1 + 1 \bar{\times} x$ (here and later on, if f is a k-algebra homomorphism we shall write f instead of cov f or biv f, etc ...). The set of canonical elements of cov B has the structure of a finite, free K-module, where $K = \operatorname{vect} k$. It will be denoted by $\mathcal{C}B$ and will be called the *canonical K-module* of B. The restriction to $\mathcal{C}B$ of the operator t of cov $\mathcal{B}$ coincides with t_B, the Verschiebung of B. Hence, for $x \in \mathcal{C}B$ we have

$$\pi(tx) = t(\pi x) = (p\iota_B)x\,.$$

In a completely similar manner we define the *canonical bivectors.* The set of the canonical bivectors has the structure of a K'-module, where $K' = \operatorname{biv} k$. It will be denoted by $\mathcal{C}'\mathcal{B}$ and will be called the *canonical K'-module* of $\mathcal{B}$. With regard to t, for $\mathcal{C}'\mathcal{B}$ we have the same situation as for $\mathcal{C}B$: i.e. the restriction of t to $\mathcal{C}'\mathcal{B}$ coincides with $t_{\mathcal{B}}$. Hence, if a canonical bivector has a component in $\mathcal{B}^0$, it has all its components in $\mathcal{B}^0$. The set of such bivectors will be denoted by $\mathcal{C}'\mathcal{B}^0$. It is a sub-$K$-module of $\mathcal{C}'\mathcal{B}$ and it is related to $\mathcal{C}B$ in the following way: the canonical embedding of B in $\mathcal{B}^0$ produces an embedding (K-linear map) τ of $\mathcal{C}B$ in $\mathcal{C}'\mathcal{B}$. This map is characterized by the property $(\tau x)_{(-1)} = x$. The image $\tau(\mathcal{C}B)$ of τ

will be denoted by $\mathcal{C}'B$, and it is related to $\mathcal{C}'\mathfrak{B}^0$ by $\mathcal{C}'\mathfrak{B}^0 = \mathcal{C}'B \otimes_K K'$.

Later on we shall need the following facts about the logarithmic series (for proofs we refer to the last sections of chap. 2 of [MA]).

Suppose (for simplicity) B to be local, denote by $\mathfrak{B}^+$ (this notation will be maintained later on for all other hyperalgebras) the kernel of the coidentity of $\mathfrak{B}$, and let ξ be an element of $\mathfrak{B}$, $\xi \equiv 1 \bmod \mathfrak{B}^+$. Then the series

$$-\sum_{n\geqslant 1} n^{-1}(1-\{\xi\})^n$$

converges in biv $\mathfrak{B}$, and its limit will be denoted by $\log\{\xi\}$.

If η is another element of $\mathfrak{B}$ subject to the same conditions as ξ, so that $\log\{\eta\}$ exists, we have:

$$\log\{\xi\} + \log\{\eta\} = \log\{\xi\eta\}\,.$$

The elements $y \in \operatorname{biv}\mathfrak{B}$ which are of the form $\log\{\xi\}$ for some $\xi \in \mathfrak{B}$, are characterized by the property $ty = y$. For these we have $y_i = L(\xi)$, where $L(\xi)$ is the Artin-Hasse logarithm: i.e. L is defined by the relation $E(L(\xi)) = \xi$, where $E(t) = \prod_{p\nmid n}(1-t^n)^{-\mu(n)/n}$, and μ is the Möbius function.

Now, if we consult (5.51) of [MA], the reason for the term quadratic exponential will be clear.

4. – The Barsotti operators.

Let G be a B-T group and $\tilde{G}$ its dual, and let R and $\tilde{R}$ be their hyperalgebras; then, if D denotes the hyperalgebra $\mathrm{Hom}_{\mathrm{cont.}}(R, k)$ (k is endowed with the discrete topology), $\tilde{\mathfrak{R}}$ is obtainable not only as the completion of the direct limit $\tilde{R} \xrightarrow{p\iota} \tilde{R} \xrightarrow{p\iota} \dots$, but also as the inverse limit of the sequence $D \xleftarrow{p\iota} D \xleftarrow{p\iota} \dots$. Therefore, if we observe that to every element $d \in D$ we may associate the invariant endomorphism $(d\otimes\iota)\mathbb{P}_R$, we see that the elements of $\tilde{\mathfrak{R}}^0$ act (by means of a similar formula) on $\mathfrak{R}$. More precisely, if we identify R with the image in $\mathfrak{R}^0$ of the first term of the sequence $R \xrightarrow{p\iota} R \xrightarrow{p\iota} \dots$, so that $\mathfrak{R}^0 = \bigcup_{i=0}^{\infty}(p\iota)^{-i}R$, the elements of $\tilde{\mathfrak{R}}^0$ (as operators) are exactly the continuous and invariant k-endomorphisms of $\mathfrak{R}^0$ which are $(p\iota)^{-i}R$ linear for some $r \in \mathbb{Z}$ (cf. § 38 of [MA]).

Using the representation of $\tilde{\mathfrak{R}}^0$ in $\mathrm{End}_k\,\mathfrak{R}$ just described, for every $d \in \mathcal{C}'\mathfrak{R}^0$ we may define a continuous and invariant K-endomorphism d_* of $\mathrm{Biv}\,\mathfrak{R}$. The map d_* will be called the *Barsotti operator* associated to d.

Now we shall give a quick schetch of the construction of d_*. A complete treatment is given in chap. 5 of [MA].

Let $d = (d_0, \ldots, d_r)$ be an ordered $(r+1)$-tuple of indeterminates over $\mathbb{Q}$, and consider the $\mathbb{Q}$-algebra of polynomials $L = \mathbb{Q}[d_0, d_1, \ldots, d_r]$. Interpret d as Witt vector whose i-th ghost component is $d^{(i)} = \sum_{j=0}^{i} p^j d_j^{p^{i-j}}$, for $0 \leqslant i \leqslant r$, and endow L with the hyperalgebra structure defined by $\mathbb{P}d_i = (d \otimes 1 + 1 \otimes d)_i$. Then, consider another $(r+1)$-tuple $(x_0, x_1, \ldots, x_r)$ of indeterminates and let $\Lambda = \mathbb{Q}[\ldots, d^\nu x_j, \ldots]$ be the symmetric $\mathbb{Q}$-algebra $\mathbb{S}(L \otimes M)$; here M is the free $\mathbb{Q}$-module on the set $\{x_0, \ldots, x_r\}$, $\nu = (\nu_0, \ldots, \nu_r) \in \mathbb{N}^r$, and d^ν stands for $d_0^{\nu_0} \ldots d_r^{\nu_r}$.

The elements $d^0 x_j$ of Λ will be identified with x_j, so that x may be thought as an element of $\operatorname{vect}_r \Lambda$.

Now, observe that there exists a unique $\mathbb{Q}$-bilinear map $L \times \Lambda \to \Lambda$, such that for $\mu, \nu \in \mathbb{N}^{r+1}$, $y, z \in \Lambda$, $0 \leqslant j \leqslant r$,

$$(d^\mu, d^\nu x_j) \mapsto d^{\mu+\nu} x_j \quad \text{and} \quad (d^\mu, yz) \mapsto \mu_\Lambda\big(\mathbb{P}d^\mu(y \otimes z)\big).$$

By this action, to the couple (d, x) corresponds the vector

$$W = W(d_0, \ldots, d_r; x_0, \ldots, x_r) \in \operatorname{vect}_r \Lambda,$$

defined by its ghost components in the following way:

$$W^{(i)} = d_i x^{(i)}.$$

What makes this algorithm useful in our case is the fact that

$$W \in \operatorname{vect}_r \mathbb{Z}[\ldots, d^\nu x_j, \ldots].$$

More precisely (cf. § 43 of [MA]), for $0 \leqslant i \leqslant r$ we have

$$W_i = d_i x_i + P_i(d^\nu x_j), \tag{4.1}$$

where $P_i(d^\nu x_j)$ denotes a polynomial with coefficients in $\mathbb{Z}$ in the arguments $d^\nu x_j$, with $j < i$ and $\nu_s = 0$ if $s > i$. Further, if we attribute weight p^r to x_r and d_r, W_i is isobaric of weight p^i separately in the d's and in the x's.

Now we can define the Barsotti operators: let $d \in \mathbb{C}'\widetilde{\mathfrak{R}}^0$, and let $x \in \operatorname{biv} \mathfrak{R}$ be a special bivector with components in $\mathfrak{R}^0$; then if $-n$ is big enough, and $n \leqslant i \leqslant n + r$ for a positive integer r,

$$W_i(d_n, \ldots, d_{n+r}; x_n, \ldots, x_{n+r})$$

does not depend on n and r (cf. (5.8) and (5.9) of [MA]). This fact allows us to define the (special) bivector $d_* x$ by means of

$$(d_* x)_i = W_i(d_n, \ldots, d_{n+r}; x_n, \ldots, x_{n+r}).$$

The map $d_*: x \mapsto d_*x$ is continuous and K'-linear so that it can be extended to the whole Biv $\mathfrak{R}$. The extension will also be denoted by d_*x: this is the Barsotti operator associated to d.

The more important formal properties of these operators are the following:

$$(4.2)\qquad \begin{cases} d_*(x+y) = d_*x + d_*y \\ d_*(ax) = a(d_*x) \\ (td)_*(tx) = t(d_*x) \\ d_*t_{\mathfrak{R}}x = t_{\mathfrak{R}}(\pi d_*x) \\ \mathbb{P}(d_*x) = (\iota \otimes d_*)\,\mathbb{P}x = (d_* \otimes \iota)\,\mathbb{P}x\,; \end{cases}$$

they hold for $x, y \in \operatorname{Biv} \mathfrak{R}$, $a \in K'$, $d \in \mathcal{C}'\tilde{\mathfrak{R}}^0$.

Even though these operators are not derivations, they link the cohomology of $\mathcal{O}_A^*$ with the cohomology of vect $\mathcal{O}_A$: for this reason they are the key to our entire analysis of theta functions.

The notations of § 3 are still in force. Let $\xi \in R$, $\xi \equiv 1 \mod R^+$; then $\log\{\xi\}$ exists, and for every $d \in \mathcal{C}'\tilde{R}$, i.e. such that $d_{-1}R = 0$, $d_* \log\{\xi\} \in$ $\in \operatorname{vect} R$ (cf. (5.40) of [MA]). If $\xi \in C$, in general $\log\{\xi\}$ does not exist, but, as explained in § 6 of [MC], $d_* \log\{\xi\}$ may be defined directly for every $d \in \mathcal{C}'\tilde{R}$: it is in vect C.

Later on we shall need the following result:

(4.3) Lemma. *Let X be an irreducible subvariety of codimension 1 of A; denote by $Q(X/A)$ the local ring of X and by $Q(X/A)^*$ its group of units. Then for an element $\xi \in C$ the following properties are equivalent*:

i) $\xi \in Q(X/A)^*$,

ii) $d_* \log\{\xi\} \in \operatorname{vect} Q(X/A)$, *for every* $d \in \mathcal{C}'\tilde{R}$,

iii) $d'_*(d_* \log\{\xi\}) \in \operatorname{vect} Q(X/A)$, *for all* $d, d' \in \mathcal{C}'\tilde{R}$.

Proof. By the construction of $d_* \log\{\xi\}$ (cf. § 6 of [MC]) it follows that i) implies ii); for instance $(d_* \log\{\xi\})_{(-1)} = 0$ is a direct consequence of (4.1). Moreover, the components of $d'_*(d_* \log\{\xi\})$ are polynomials whose arguments are invariant hyperderivatives of the components of $d_* \log\{\xi\}$. Therefore ii) implies iii). Now we shall prove that i) follows from ii).

The ring $Q(X/A)$ is a discrete valuation ring; therefore if t denotes a regular parameter of $Q(X/A)$, we will have $\xi = t^r u$, where $u \in Q(X/A)^*$ and $r \in \mathbb{Z}$. Then $d_* \log\{\xi\} = r(d_* \log\{t\}) + d_* \log\{u\}$, and by what we have already seen, ii) implies $r(d_* \log\{t\}) \in \operatorname{vect} Q(X/A)$, for every $d \in \mathcal{C}'\tilde{R}$. In particular, if $r \neq 0$, ii) implies $\delta t/t \in Q(X/A)$ for every invariant derivation δ

of A; but this is impossible. Therefore r must be zero, i.e. $\xi = u \in Q(X/A)^*$. To see this impossibility, it is enough to observe that t belongs to a regular set of parameters of a point $P \in X$. In which case it is well known that there exists an invariant derivation δ such that δt is an unit of $\mathcal{O}_{A,P}$ (suppose for istance $P = e$). As a consequence $\delta t \in Q(X/A)^*$. There remains to be seen that iii) implies i). Observe that if all the invariant hyperderivatives of an element $\eta \in C$ are in $Q(X/A)$, then $\eta \in Q(X/A)$. In fact, if $\eta \notin Q(X/A)$, then $\eta = t^{p^s m} u'$, where $u' \in Q(X/A)^*$, m is a negative integer not divisible by p, and s is a nonnegative integer. But it is well known (cf. [10]) that for every invariant derivation δ of A there exists an invariant hyperderivation δ' of A such that $\delta'(t^{p^s m} u') = (\delta' t^{p^s m}) u' + t^{p^s m} \delta' u'$, and that $\delta'(t^{p^s m}) = (\delta t^m)^{p^s} = m^{p^s} t^{(m-1)p^s} (\delta t)^{p^s}$. Therefore, as there exists a δ such that $\delta t \in Q(X/A)^*$, there also exists a δ' such that $\delta'(t^{p^s m} u') \notin Q(X/A)$. Now, let $\xi = t^r u$, where $u \in Q(X/A)^*$. If $r \neq 0$, from iii) it follows that $\delta'(\delta t/t) \in Q(X/A)$, when δ' ranges over a p-base of the invariant hyperderivation of A. Therefore $\delta'(\delta t/t) \in Q(X/A)$ for every invariant derivation δ and every invariant hyperderivation δ': but this, by what has already been observed, is impossible; hence $r = 0$, Q.E.D.

Since $\mathcal{O}_{A,P} = \bigcap_{P \in X} Q(X/A)$, (4.3) clearly holds for any subvariety of A; therefore we have the following

(4.4) COROLLARY. *Let U be an open set of A and ξ an element of C. Then the following properties are equivalent*:

i) $\xi \in \Gamma(U, \mathcal{O}_A^*)$,

ii) $d_* \log \{\xi\} \in \Gamma(U, \operatorname{vect} \mathcal{O}_A)$ *for every* $d \in \mathcal{C}' \tilde{R}$,

iii) $d'_*(d_* \log \{\xi\}) \in \Gamma(U, \operatorname{vect} \mathcal{O}_A)$ *for all* $d,\ d' \in \mathcal{C}' \tilde{R}$.

5. – Cohomological relations between X and ϑ_X.

In this section we begin the analysis of the results of §§ 1 and 2. The notations are always the same.

Consider the exact sequence of sheaves on A

$$0 \to \mathcal{O}_A \to C \to C/\mathcal{O}_A \to 0\,,$$

where C is identified with the constant sheaf. From this sequence we have the following exact cohomology sequence:

$$0 \to H^0(A, \operatorname{vect} C/K) \to H^0(A, \operatorname{vect} C/\operatorname{vect} \mathcal{O}_A) \to H^1(A, \operatorname{vect} \mathcal{O}_A) \to 0\,.$$

The K-modules $H^0(A, \text{vect } C/K)$, $H^0(A, \text{vect } C/\text{vect } \mathcal{O}_A)$ and $H^1(A, \text{vect } \mathcal{O}_A)$ will be denoted by $\mathcal{E}(A)$, $\mathcal{B}(A)$ and $\mathcal{M}(A)$ respectively. The elements of $\mathcal{B}(A)$ will be called *closed hyperclasses* and those of $\mathcal{E}(A)$ *exact hyperclasses*. The exact hyperclass which corresponds to the element $x \in \text{vect } C$ will be denoted by $\text{cl}\, x$. If $\boldsymbol{b}$, $\boldsymbol{b}'$ are elements of $\mathcal{B}(A)$, the relation $\boldsymbol{b} - \boldsymbol{b}' \in \mathcal{E}(A)$ will be written $\boldsymbol{b} \sim \boldsymbol{b}'$.

The Barsotti operators, in analogy to the derivations in the case of characteristic zero, give maps from the group $D(A)$ of divisors on A and from the Severi group $S(A)$, to $\mathcal{B}(A)$ and $\mathcal{M}(A)$ respectively. Such mappings, which are precisely the link between the cohomology of $\mathcal{O}_A^*$ and the cohomology of vect $\mathcal{O}_A$ alluded to above, will now be described (cf. (6.26) and (6.30) of [MA]).

(5.1) *Let X be a divisor on A defined on the finite open covering $\mathcal{U}$ of A by the cocycle $(\varphi_U)_{U\in\mathcal{U}}$, and let $d \in \mathcal{C}'\tilde{R}$. Then the element of $\mathcal{B}(A)$ associated to them is defined by the cocycle $\big(d_* \log\{\varphi_U\}\big)_{U\in\mathcal{U}}$ and will be denoted by $\boldsymbol{b}(d, X)$. If Y is another divisor and if $X \equiv Y$, then $\boldsymbol{b}(d, X) \sim \boldsymbol{b}(d, Y)$ for every $d \in \mathcal{C}'\tilde{R}$; therefore $X \mapsto \boldsymbol{b}(d, X)$ induces a map from $S(A)$ to $\mathcal{M}(A)$.*

The operators d_* also act on hyperclasses: if $\boldsymbol{b} \in \mathcal{B}(A)$ is defined by the cocycle $(\psi_U)_{U\in\mathcal{U}}$, $d_*\boldsymbol{b}$ will denote the hyperclass defined by the cocycle $(d_*\psi_U)_{U\in\mathcal{U}}$.

The connection between a divisor X and its theta elements is described in the following

(5.2) THEOREM. *Let X be a divisor on A, and ϑ_X be a theta element of X. Then for all d, $d' \in \mathcal{C}'\tilde{R}$ we have*

$$\text{cl}\,\big(d'_*(d_* \log\{\vartheta_X\})\big) = d'_*\boldsymbol{b}(d, X)\,.$$

In particular, if $X \equiv 0$ we may choose a ϑ_X such that

$$\text{cl}\,\big(d_* \log\{\vartheta_X\}\big) = \boldsymbol{b}(d, X)\,.$$

PROOF. Given that every divisor on A is linearly equivalent to a divisor whose components do not go through e, and that the theorem is trivially true for principal divisors, we may, in wiew of (2.14), suppose that $\vartheta_X \in \mathcal{R}$, $\vartheta_X \equiv 1 \mod \mathcal{R}^+$. We begin with the case $X \equiv 0$. If we choose $\vartheta_X \in \mathcal{R}$, by (1.5) and by the invariance of the d_*'s (cf. the last formula of (4.2)) we have

$$(5.3) \qquad (\iota\otimes\varepsilon)\big[(\iota\otimes d_*)\log\{f(x_1, x_2)\}\big] = \\ = (\iota\otimes\varepsilon)\big[\mathbf{P}(d_*\log\{\vartheta_X\})\big] - \varepsilon\big(d_*\log\{\vartheta_X\}\big) = d_*\log\{\vartheta_X\} + \text{const.}\,,$$

whence $d_* \log\{\vartheta_X\} \in \text{vect } C$.

After retaining the notations of (5.1), let U, $U' \in \mathfrak{U}$. Assume that $e \in U'$ and set $W_1 = (p_1 + p_2)^{-1}U$, $W_2 = U \times A$, $W_3 = A \times U'$; then $W = \bigcap_{i=1}^{3} W_i$ is an open set of $A \times A$ which contains $U \times e$. If Y is obtained from X as in (1.1), and if we denote the coproduct of C by $\mathbb{P}$ (cf. § 6), then $\mathbb{P}\varphi_U/(\varphi_U \otimes \varphi_{U'})$ is an equation of Y in W. Thus from (1.5) it follows that

$$(5.4) \qquad \mathbb{P}\varphi_U/(\varphi_U \otimes \varphi_{U'}) = f(x_1, x_2)\, \psi_W(x_1, x_2)\,,$$

where ψ_W is regular and invertible in W.

Now, if we apply the operator $(\iota \otimes \varepsilon)(\iota \otimes d_*) \log \{\ \}$ to both sides of (5.4) and compare the resulting expression with (5.3), we obtain

$$(5.5) \qquad d_* \log \{\varphi_U\} = d_* \log \{\vartheta_X\} + (\iota \otimes \varepsilon)[(\iota \otimes d_*) \log \{\psi_W\}] + \text{const.}\,.$$

It follows from (4.3) that $(\iota \otimes d_*) \log \{\psi_W\}$ is regular in W, and so $(\iota \otimes \varepsilon)[(\iota \otimes d_*) \log \{\psi_W\}]$ is regular in U. Finally (5.5) may be rewritten

$$\boldsymbol{b}(d, X) = \text{cl}\left(d_* \log \{\vartheta_X\}\right),$$

as claimed in the statement of the theorem.

For the general case we will use a similar argument. Let U, $U' \in \mathfrak{U}$, $e \in U'$; put $W_1 = (p_1 + p_2 + p_3)^{-1}U$, $W_2 = (p_1 + p_2)^{-1}U$, $W_3 = (p_1 + p_3)^{-1}U$, $W_4 = (p_2 + p_3)^{-1}U'$, $W_5 = U \times A \times A$, $W_6 = A \times U' \times A$, $W_7 = A \times A \times U'$; then $\bigcap_{i=1}^{7} W_i = W$ is an open set of $A \times A \times A \times$ which contains $U \times e \times e$.

If Y is obtained from X as in (2.1), then

$$(5.6) \qquad \Phi_W = \frac{(\iota \otimes \mathbb{P})(\mathbb{P}\varphi_U)(\varphi_U \otimes \varphi_{U'} \otimes \varphi_{U'})}{(\mathbb{P}\varphi_U \otimes 1)(1 \otimes \mathbb{P}\varphi_{U'})(\iota \otimes sc_{23})(\mathbb{P}\varphi_U \otimes 1)}$$

is an equation of Y in W; therefore we have

$$(5.7) \qquad \Phi_W = F(x_1, x_2, x_3)\, \psi_W(x_1, x_2, x_3)\,,$$

where ψ_W is regular and invertible in W. From (5.7) and in wiew of (2.13), with a computation analogous to that used for (5.5) we obtain:

$$(5.8) \qquad d'_*(d_* \log \{\vartheta_X\}) = d'_*(d_* \log \{\varphi_U\}) + \\ + (\iota \otimes \varepsilon \otimes \varepsilon)[(\iota \otimes d'_* \otimes d_*) \log \{\psi_W\}] + \text{const.}\,.$$

The arguments already used show that the regularity of

$$(\iota\otimes \varepsilon\otimes \varepsilon)\big[(\iota\otimes d'_*\otimes d_*)\log\{\psi_W\}\big]$$

in U descends from the invertibility of ψ_W in W. Therefore (5.8) is equivalent to

$$\mathrm{cl}\,\big[d'_*(d_*\log\{\vartheta_X\})\big] = d'_*\boldsymbol{b}(d, X)\,, \qquad \text{Q.E.D.}$$

6. – Hyperfields.

In the next section we will need a way to recognize whether a function field is the field of rational functions on a group variety. The properties which characterize such fields give rise to a structure called *hyperfield*. Hyperfields were introduced in [θ] and subsequently studied in depth in [5].

Here we limit ourselves to the essential definitions.

(6.1) A field C is called a *k-hyperfield* if it satisfies the following conditions:

i) C is a regular extension of k;

ii) there exists a k-algebra homomorphism $\mathbb{P}\colon C\to Q\big(C\underset{k}{\otimes} C\big)$, called the *coproduct* of C, which is coassociative and cocommutative, i.e. $(\mathbb{P}\otimes\iota)\,\mathbb{P} = (\iota\otimes\mathbb{P})\,\mathbb{P}$ and $\mathrm{sc}\,\mathbb{P}=\mathbb{P}$;

iii) there exists a k-algebra automorphism $\varrho\colon C\to C$, such that, if μ denotes the extension of the product $m\colon C\otimes C\to C$ to the subring $(C\otimes C)_{(\ker m)}$ of $Q(C\otimes C)$, and if we set $\varepsilon=\mu(\iota\otimes\varrho)\,\mathbb{P}$, then

   a) the domain of ε is a local ring R;

   b) $\varepsilon x = \varepsilon\varrho x\in k$, for every $x\in R$;

   c) $\mu(\varepsilon\otimes\iota)\,\mathbb{P}x = x$, for every $x\in C$;

   ϱ is called the *inversion* of C.

A subfield C' of the k-hyperfield C is said to be a *sub-k-hyperfield* if C' with the restrictions of the coproduct and inversion of C is a k-hyperfield.

A k-hyperfield is said to be *finitely generated* if it is a f.g. k-algebra.

The following is a convenient test for recognizing sub-k-hyperfields of a finitely generated hyperfield.

(6.2) LEMMA. *Let C be a finitely generated k-hyperfield and $C'\supset k$ a subfield. If $\mathbb{P}C'\subseteq Q\big(C'\underset{k}{\otimes} C'\big)$, then C' is a sub-k-hyperfield.*

A proof is given in (2.1) of [θ].

Let C be a k-hyperfield; then a k-automorphism σ of the field C is called *invariant* if $(\iota \otimes \sigma)\mathbf{P} = \mathbf{P}\sigma$.

If V is a commutative group variety over k, the field C of the rational functions on V is a finitely generated hyperfield: $\mathbf{P}$ and ϱ are the mappings induced by the law of composition and by the inversion of V respectively; R is the local ring of the identity of V, and ε is the reduction of R modulo its maximal ideal. Furthermore, the map which sends the point P of V to the automorphism of C corresponding to the translation by P, gives a one-to-one correspondence between the points of V (which do not lie on the *degeneration locus*) and the group of invariant k-automorphisms of C.

Conversely, let C be a finitely generated k-hyperfield, $\bar{k}$ the algebraic closure of k, and let $\bar{C} = C \otimes \bar{k}$ be the $\bar{k}$-hyperfield obtained by extending the coproduct and coidentity of C to $\bar{C}$. The set of all invariant automorphisms of $\bar{C}$ is a subgroup G of the group of $\bar{k}$-automorphisms of $\bar{C}$.

One can easily show that G with the Zariski topology (a base for the open sets of which is given by

$$D(x) = \left\{\tau \mid \tau \in G,\ x \in \tau\left(R \underset{k}{\otimes} \bar{k}\right)\right\}$$

as x varies in $\bar{C}$) and with the structure sheaf defined by

$$\mathcal{O}(D(x)) = \bigcap_{\tau \in D(x)} \tau(R \underset{k}{\otimes} \bar{k}),$$

is (the complement of the degeneration locus of) a group variety over $\bar{k}$. This variety is the extension to $\bar{k}$ of a variety over k such that $k(V) = C$.

For details in regard to group varieties and their degeneration loci, the reader may consult [1].

7. – Abstract theta functions.

In the classical case theta functions in n variables are characterized within the field $\mathcal{M}$ of meromorphic functions on $\mathbb{C}^n$ by means of certain functional equations. In this section we shall concern ourselves with the problem of their characterization in characteristic p.

On the basis of the results established in the first two sections, it is clear that in the present case $\mathbb{C}^n$ should be replaced by a group $\mathcal{G}$ which is the inverse limit of a sequence

$$G \xleftarrow{p\iota} G \xleftarrow{p\iota} \ldots,$$

where G is a local B-T group. Hence $\mathcal{M}$ should be replaced by the field of fractions $Q(\mathfrak{R})$ of the hyperalgebra $\mathfrak{R}$ of $\mathcal{G}$.

The results of this section are analogous to those obtained in [θ] for the case of characteristic zero.

An element $\vartheta \in Q(\mathfrak{R})$ is said to be an *element of type theta* if

$$(7.1)\qquad F(x_1, x_2, x_3) = \frac{\vartheta(x_1 + x_2 + x_3)\vartheta(x_1)\vartheta(x_2)\vartheta(x_3)}{\vartheta(x_1 + x_2)\vartheta(x_1 + x_3)\vartheta(x_2 + x_3)} \in Q(R \otimes R \otimes R)\,.$$

We recall that the middle expression of (7.1) means

$$\frac{((\iota \otimes \mathbf{P})\mathbf{P}\vartheta)(\vartheta \bar{\times} \vartheta \bar{\times} \vartheta)}{(\mathbf{P}\vartheta \bar{\times} 1)(sc_{23}(\mathbf{P}\vartheta \bar{\times} 1))(1 \bar{\times} \mathbf{P}\vartheta)}\,.$$

One should note that the use of tensor product rather than completed tensor product renders the condition (7.1) (more) restrictive.

Further on we will need the following results which are analogous to lemmas 3.2 and 3.3 of [θ].

(7.2) Lemma. *Let $t_1, \dots, t_n, t'_1, \dots, t'_m$ be indeterminates over k, $S = k[\![t_1, \dots, t_n]\!]$, $S' = k[\![t'_1, \dots, t'_m]\!]$ and let φ be an element of $S \bar{\times} S'$. Then one has:*

a) the following conditions are equivalent:

- i) $\varphi \in S \otimes S'$
- ii) $\varphi \in Q(S) \otimes Q(S')$
- iii) *the k-vector space $U \subset S$ generated by the elements $(\iota \otimes y'_\nu)\varphi$ is finite dimensional. Here ν ranges over $\mathbf{N}^m$ and y'_ν is the element of* $\operatorname{Hom}(S', k)$ *defined by $y'_\nu t^\mu = \delta_{\mu\nu}$ for every $\mu \in \mathbf{N}^m$.*

b) If the conditions of a) are fulfilled, the dimension of U coincides with the dimension of the subspace V generated by the elements $(y_\nu \otimes \iota)\varphi$, as ν ranges over $\mathbf{N}^n$. Furthermore $\varphi \in U \otimes V$.

c) Under the same hypothesis as in b), the subfield C_U of $Q(S)$ generated by U over k is the minimal subfield C of $Q(S)$ such that $\varphi \in Q(C \otimes S')$. The analogous property also holds for the field C_V generated by V.

d) C_U and C_V are finitely generated even under the weaker assumption that $\varphi \in Q(S \otimes S')$. Furthermore the minimality properties described in c) remain valid, and in addition $\varphi \in Q(C_U \otimes C_V)$.

PROOF. If φ satisfies the hypothesis of *d*), we have

$$\varphi = \frac{\sum_{i=1}^{r} a_i(t)\, a_i'(t')}{\sum_{i=1}^{r} b_i(t)\, b_i'(t')} = \sum_{\nu\in\mathbf{N}^m} A_\nu(t)\, t'^\nu\,, \tag{7.3}$$

where a_i, b_i and A_ν belong to S, while a_i', b_i' belong to S'. Moreover the A_ν's are uniquely determined by φ. If the denominator in (7.3) may be chosen equal to 1 (i.e. if condition i) of *a*) holds), then U is the space generated by the a_i's. In fact, if r is minimal, $(a_1, \ldots, a_r)$ is a base for U. One can handle V in the same way, so that U and V have the same dimension. Hence i) implies iii) and *b*). Clearly $\varphi \in C_U \otimes S'$, and so *a fortiori* $\varphi \in C_U[\![t']\!]$. But any C such that $\varphi \in C\,[\![t']\!]$ must contain the A_ν's. It follows that C_U has the minimality property described in *c*). A completely similar argument shows that the same property also holds for C_V. Since, for an element $\varphi \in S \overline{\times} S'$, i) is clearly equivalent to ii), and iii) implies immediately i), we have proved *a*), *b*) and *c*).

If now the more general assumption *d*) holds, i.e. if φ has the expression (7.3), then we have

$$a_i(t) = \sum_{\nu\in\mathbf{N}^n} a_{i\nu} t^\nu\,, \quad a_i'(t') = \sum_{\nu\in\mathbf{N}^m} a_{i\nu}' t'^\nu\,, \quad b_i(t) = \sum_{\nu\in\mathbf{N}^n} b_{i\nu} t^\nu$$

and $b_i'(t') = \sum_{\nu\in\mathbf{N}^m} b_{i\nu}' t'^\nu$, where $a_{i\nu}$, $a_{i\nu}'$, $b_{i\nu}$, $b_{i\nu}'$ belong to k. Therefore (7.3) is equivalent to the following relations:

$$\sum_{i=1}^{r} a_{i\nu}' a_i(t) = \sum_{i=1}^{r} \Big(\sum_{\lambda+\mu=\nu} A_\lambda(t)\, b_{i\mu}' \Big)\, b_i(t)$$

for every $\nu \in \mathbf{N}^m$. As one can clearly see from these relations, the A_λ's are solutions of linear systems with coefficients in the field generated by the a_i's and b_i's, for $i = 1, 2, \ldots, r$. Thus the field they generate is indeed finitely generated. From the same relations it follows that, whatever the choice of the a_i' and b_i' may be, the a_i's and b_i's are solutions of a linear system with coefficients in C_U, so that they can be chosen in C_U. Since for C_V the argument is the same, $\varphi \in Q(C_U \otimes C_V)$. The minimality properties of C_U and C_V are now clear, Q.E.D.

At this point we can prove the following

(7.4) THEOREM. *If* (7.1) *holds, there exists a minimal subfield* $C = C_\vartheta$ *of* $Q(R)$ *such that* $F \in Q(C \otimes C \otimes C)$. *The field* C *is a finitely generated* k*-hyperfield.*

PROOF. By the remark of p. 256 of [θ], we may assume $\vartheta \in \mathfrak{R}$ and $\vartheta \not\equiv 0 \bmod \mathfrak{R}^+$. In fact, even if this is not the case, $\vartheta'(x) = \vartheta(x \dotplus x_1')$ satisfies the previous conditions, and Barsotti's remark shows that we can get all statements about $\vartheta(x)$ by specializing the corresponding results about $\vartheta'(x)$.

If ϑ satisfies the previous assumption, as $Q(R \otimes R \otimes R) \cap (\mathfrak{R} \overline{\times} \mathfrak{R} \overline{\times} \mathfrak{R}) \subseteq$ $\subseteq R \overline{\times} R \overline{\times} R$, we have $F \in R \overline{\times} R \overline{\times} R$; therefore from part *d*) of (7.2) and from the symmetry of F we deduce that C is the subfield of $Q(R)$ generated over k by the elements

$$(\iota \otimes y_\mu \otimes y_\nu) F ,$$

where μ, ν range over $\mathbb{N}^n$.

Here y_μ and y_ν have the same meaning as in (7.2); therefore if

$$F(x_1, x_2, x_3) = \sum_{\mu,\nu \in \mathbb{N}^n} A_{\mu\nu}(x_1) x_2^\mu x_3^\nu ,$$

we have

$$(7.5) \qquad (\iota \otimes y_\mu \otimes y_\nu) F(x_1, x_2, x_3) = A_{\mu\nu}(x_1) .$$

Now we will prove the second statement. From (7.1) it follows that F is a « cocycle of a symmetric bi-extension », i.e. it satisfies (2.15). From this we deduce that

$$(7.6) \qquad F(x_1 \dotplus x_2, x_3, x_4) = \frac{F(x_1, x_2, x_3 \dotplus x_4) F(x_1, x_3, x_4) F(x_2, x_3, x_4)}{F(x_1, x_2, x_3) F(x_1, x_2, x_4)} .$$

As the right-hand side of (7.6) is in $Q(C_1 \otimes C_2)[\![x_3, x_4]\!]$, $A_{\mu\nu}(x_1 \dotplus x_2) \in Q(C_1 \otimes C_2)$ for all $\mu, \nu \in \mathbb{N}^n$. Therefore, if $\mathbb{P}$ denotes the extension of the coproduct of R to $Q(R)$, we have $\mathbb{P}C \subseteq Q(C \otimes C)$. Now, let ϱ be the extension of the inversion of R to $Q(R)$, and let C' be the smallest subfield of $Q(R)$ which contains C and ϱC. As it is clear that C' is a finite regular extension of k, the statement is a consequence of (6.2) if we can prove that C' is a hyperfield. From relations $\varrho^2 = \iota$ and $\mathbb{P}\varrho = (\varrho \otimes \varrho)\mathbb{P}$ (which hold in R) it follows immediately that $\mathbb{P}C' \subseteq Q(C' \otimes C')$ and $\varrho C' = C'$. Finally, the properties of $\mathbb{P}$ and ϱ required in (6.1) are consequence of the analogous properties in R; therefore C' is a hyperfield, Q.E.D.

From (7.4) it follows that to every ϑ of type theta there corresponds a hyperfield C_ϑ, hence (cf. § 6) a group variety V_ϑ such that $C_\vartheta = k(V_\vartheta)$. Two elements ϑ and ϑ' of $Q(\mathfrak{R})$ such that ϑ/ϑ' is a quadratic exponential will be said to be *associated*. It is clear that if ϑ and ϑ' are associated elements of type theta, then $C_\vartheta = C_{\vartheta'}$.

If $\vartheta \in Q(\mathfrak{R})$ is an element of type theta, by *dimension* of ϑ we shall mean the dimension of the minimal subhyperdomain (cf. § 3) $R_\vartheta \subseteq R$ such that the F of (7.1) is in $Q(R_\vartheta \otimes R_\vartheta \otimes R_\vartheta)$. This R_ϑ (by means of the usual limit, cf. § 3) determines a sub-bidomain $\mathfrak{R}_\vartheta \subseteq \mathfrak{R}$: $\mathfrak{R}_\vartheta$ is the minimal sub-bidomain of $\mathfrak{R}$ containing some element associated to ϑ.

Let S be the sub-k-algebra of R generated by the $A_{\mu\nu}(x_1)$'s of (7.5). Since every element of S is regular at the identity point e of V_ϑ, $S \subseteq \mathcal{O}_{V,e}$; therefore $R_\vartheta \subseteq \hat{\mathcal{O}}_{V,e}$, so that $\dim R_\vartheta \leqslant \dim \hat{\mathcal{O}}_{V,e} =$ trascendency degree of C_ϑ over k. In particular, if $\dim \vartheta =$ trasc. deg. of C_ϑ, $S_{(\ker \varepsilon)} = \mathcal{O}_{V,e}$, hence $R_\vartheta = \hat{\mathcal{O}}_{V,e}$. In this case ϑ will be called a *theta element*. A theta element ϑ will be called *nondegenerate* if $\dim \vartheta = \dim R$, that is if $\mathfrak{R} = \mathfrak{R}_\vartheta$. Observe that from $\mathfrak{R} = \mathfrak{R}_\vartheta$ we can only deduce that R_ϑ is isogeneous to R. Finally, let ϑ be a theta element; as $R_\vartheta = \hat{\mathcal{O}}_{V,e}$ is a hyperdomain we conclude that V_ϑ is an extension of an abelian variety by a multiplicative group G_m^r.

In the next section we shall characterize the elements theta whose corresponding variety is an abelian variety.

Now we shall show how it is possible to recover the hyperfield C_ϑ directly without using the F of (7.1). Also in this case, in wiew of the remark at p. 256 of [θ], we may suppose $\vartheta \in \mathfrak{R}$, $\vartheta \equiv 1 \bmod \mathfrak{R}^+$.

The following easy remark will be used later on.

(7.8) *With the notation of* (7.2), *let* $\varphi \in S \overline{\times} S'$, $\varphi = 1 + \sum_{|\nu|>0} A_\nu(t) t'^\nu$ *where* $A_\nu(t) \in S$. *Denote by* $\sum_{|\nu|>0} B_\nu(t) t'^\nu$ *the Artin-Hasse logarithm* $L(\varphi)$; *then the* A_ν*'s and* B_ν*'s generate the same field over* k.

In fact the B_ν's are polynomials in the arguments A_ν's, with coefficients in the prime field $\mathbf{F}_p$. But also the converse statement is true, because if E denotes the Artin-Hasse exponential, one has $E(L(\varphi)) = \varphi$.

Recall that a k-endomorphism δ of R is an *invariant hyperderivation* if $(\iota \otimes \delta)\mathbf{P} = \mathbf{P}\delta$ and $\ker \delta \subseteq k$. An invariant hyperderivation will be called *canonical* if there exists a $d \in \mathbf{C}'\tilde{R}$ and a non-negative integer i, such that the restriction of d_i to R coincides with δ.

It is well known (cf. chap. 4 of [MA]) that there exists a finite subset $\{d^{(1)}, \ldots, d^{(r)}\}$ of $\mathbf{C}'\tilde{R}$ such that the elements $d_j^{(i)}$, where $j \geqslant 0$ and $1 \leqslant i \leqslant r$, form a p-basis of the k-algebra of the invariant hyperderivations of R. For our purposes it is enough to know that every invariant hyperderivation of R has an expression as a polynomial in the $d_j^{(i)}$'s, with coefficients in k.

The following notations will simplify the description of the facts which concern us. If $i \in \mathbf{N}$ and $\nu = (n_1, \ldots, n_r) \in \mathbf{N}^r$, then $|\nu| = \sum n_j$,

$$d_i^\nu = (d_i^{(1)})^{n_1} \ldots (d_i^{(r)})^{n_r} \quad \text{and} \quad d_{i*}^\nu = \left((t^i d^{(1)})_*\right)^{n_1} \ldots \left((t^i d^{(r)})_*\right)^{n_r}$$

(t is the Verschiebung). Let $s \in \mathbb{N}$ and let $N(s)$ be an ordered $(s+1)$-tuple $(\nu_0, \ldots, \nu_s)$ of elements of $\mathbb{N}^r$, where $|\nu_s| \neq 0$ if $s > 0$. Then we set

$$w(N(s)) = \sum_{i=0}^{s} p^i |\nu_i| , \qquad d^{N(s)} = d_0^{\nu_0} \ldots d_s^{\nu_s} \qquad \text{and} \qquad d_*^{N(s)} = (d_{s*}^{\nu_0})(d_{s-1*}^{\nu_1}) \ldots (d_{0*}^{\nu_s}) .$$

On the set $\mathcal{N}$ of the $N(s)$'s we shall define the following partial-order: $N(s) < N'(s')$ if $w(N(s)) < w(N'(s'))$. Observe that $w(N(s))$ is the weight of the monomial $d^{N(s)}$ when the weight p^j is attributed to $d_j^{(i)}$. Finally, if Δ is a monic monomial of the symmetric algebra $\mathrm{S}(\mathrm{C}'\tilde{R})$, Δ_* will denote the operator obtained by replacing each factor $d \in \mathrm{C}'R$ of Δ with the corresponding d_*. For instance each $d_*^{N(s)}$ is a particular Δ_*.

In the following lemma S stands for a k-algebra without zero-divisors. If σ is a k-homomorphism of the hyperdomain R, the same symbol will be used for the S-homomorphism of $R_S = R \mathop{\overline{\times}}_k S$ which extends σ. The same convention will be used for the operators d_*.

(7.9) LEMMA. *Let $x \in R_S$, $x \equiv 1 \bmod (R_S)^+$; let C be the field generated over k by the elements $\varepsilon(\delta L(x))$ of S (ε is the coidentity of R), when δ ranges over the set of the invariant hyperderivations of R. Then C coincides with the field C' generated over k by the components of the vector $\varepsilon(\Delta_* \log\{x\})$, when Δ ranges over the set of the monic monomials of $\mathrm{S}(\mathrm{C}'\tilde{R})$.*

PROOF. Since $\log\{x\} = (\ldots L(x); L(x), \ldots)$ (see § 3) it is clear that $C \supseteq C'$. Conversely, from what we already have observed about invariant hyperderivations δ of R, we can deduce that every such δ is a linear combination with coefficients in k (of a finite number) of $d^{N(s)}$'s. Therefore it is enough to show that $\varepsilon(d^{N(s)}L(x)) \in C'$ for every $N(s) \in \mathcal{N}$. To this end, recall (cf. (4.1)) that if $d \in \mathrm{C}'\tilde{R}$ and $y = (\ldots, y_{-1}; y_0, y_1, \ldots) \in \operatorname{biv} R_S$, then $(d_* y)_{(-1)} = 0$, while for $i \geqslant 0$ we have

$$(d_* y)_i = d_i y_i + P(\ldots, h y_j, \ldots) ,$$

where P is a polynomial in the arguments $h y_j$ with $0 \leqslant j < i$, and h is a monomial in the d_r's with $0 \leqslant r \leqslant i$. Moreover, if we give weight p^j to d_j and y_j, then $(d_* y)_i$ is isobaric of weight p^i separately in the d_j's and y_j's. As a consequence, if in $(d_* y)_i$ there is a monomial of positive degree in d_i, we may conclude that it is of type $z d_i y_j$, where z is a monomial of positive degree in the y_l with $0 \leqslant l \leqslant i-1$. Recall also (cf. § 4 of [MA]) that if $x, y \in R_S$, then

$$d_i(xy) = \mu_{R_S}(\Phi_i(d_i \otimes \iota, \ldots, d_0 \otimes \iota;\ \iota \otimes d_i, \ldots, \iota \otimes d_0)(x \otimes y)) ,$$

where μ_{R_S} is the product map of R_S and Φ_i is an isobaric polynomial of weight p^i if $d_j \otimes \iota$ and $\iota \otimes d_j$ have weight p^j.

In our case, from the facts just recalled it follows that

$$(d_*^{N(s)} \log\{x\})_i = d^{N(s)} L(x) + P_{N(s)}(d^{N'(s')} L(x)),$$

where $P_{N(s)}$ is a polynomial in the arguments $d^{N'(s')}L(x)$, isobaric of weight $w(N(s))$ in the $d^{N'(s')}$. Moreover, if in $P_{N(s)}$ there is a monomial of positive degree in some $d^{N'(s')}L(x)$, with $w(N'(s')) = w(N(s))$, such monomial is of the type $L(x)^r d^{N'(s')}$ for some positive integer r. As a consequence,

$$\varepsilon(d_*^{N(s)} \log\{x\})_i = \varepsilon(d^{N(s)} L(x)) + P'_{N(s)}\big(\varepsilon(d^{N'(s')} L(x))\big),$$

where the $d^{N'(s')}$ effectively appearing in $P'_{N(s)}$ are such that $w(N'(s')) < w(N(s))$. At this point it is clear that $\varepsilon(d^{N(0)}L(x)) \in C'$ and that if $\varepsilon(d^{N'(s')}L(x)) \in C'$ for every $N'(s')$ with $w(N'(s')) < w(N(s))$, then also $\varepsilon(d^{N(s)}L(x)) \in C'$. Hence, by induction, we conclude that $\varepsilon(d^{N(s)}L(x)) \in C'$ for every $N(s)$, Q.E.D.

(7.10) Corollary. *Notations and assumptions being as in* (7.4), *let* $\vartheta \in \mathfrak{R}$, $\vartheta \equiv 1$ *mod* $\mathfrak{R}^+$; *then the field* C_ϑ *is generated by the components of all vectors* $(\iota \otimes \varepsilon \otimes \varepsilon)\big((\iota \otimes \Delta_* \otimes \Delta'_*) \log\{F\}\big)$ *when* Δ *and* Δ' *range over the set of the monic monomials of positive degree of* $\mathrm{S}(\mathrm{C}'\tilde{R})$.

Proof. By the assumption on ϑ it follows that

$$F(x_1, x_2, x_3) = 1 + \sum_{\mu,\nu} A_{\mu\nu}(x_1) x_2^\mu x_3^\nu, \qquad \text{where } A_{\mu\nu}(x) \in R \text{ and } A_{\mu 0} = A_{0\nu} = 0\,.$$

By (7.2), (7.4) and (7.8) we know that C_ϑ is generated by the elements $(\iota \otimes y_\mu \otimes y_\nu)L(F)$ when μ and ν range over $\mathbb{N}^n - \{0\}$. Therefore if we set $\delta_\mu = (\iota \otimes y_\mu)\mathbb{P}$, we have $(\iota \otimes \varepsilon \otimes \varepsilon)\big((\iota \otimes \delta_\mu \otimes \delta_\nu)L(F)\big) = (\iota \otimes y_\mu \otimes y_\nu)L(F)$, so that C_ϑ is also generated by the elements $(\iota \otimes \varepsilon \otimes \varepsilon)\big((\iota \otimes \delta \otimes \delta')L(F)\big)$ when δ and δ' range over the set of the invariant hyperderivations of R. Observe that if δ (or δ') is the identity map one has $(\iota \otimes \varepsilon \otimes \varepsilon)\big((\iota \otimes \delta \otimes \delta')L(F)\big) = 0$, so that C_ϑ is also generated by the set of elements $\varepsilon_{R\overline{\times}R}(\delta'' L(F))$ when δ'' ranges over the set of invariant hyperderivations of $R\overline{\times}R$ (here again we use the convention for extended maps already employed in (7.9); thus the previous $\varepsilon_{R\overline{\times}R}$ is really the coidentity of $(R\overline{\times}R)_R$). For the same reason the set $(\iota \otimes \varepsilon \otimes \varepsilon)\big((\iota \otimes \Delta_* \otimes \Delta'_*) \log\{F\}\big)$ of the statement coincides with the set $\varepsilon_{R\overline{\times}R}(\Delta''_* \log\{F\})$ with Δ'' ranging over the set of monic monomials of $\mathrm{S}(\mathrm{C}'(\widetilde{R\overline{\times}R}))$. Therefore our statement follows from (7.9), where $R_S = (R\overline{\times}R)_R$ and $x = F$, Q.E.D.

Finally we can prove

(7.11) THEOREM. *Notations and assumptions as in* (7.4). *Let* $\vartheta \in \mathfrak{R}$, $\vartheta \equiv 1 \bmod \mathfrak{R}^+$; *then the field* C_ϑ *is generated by the components of all the vectors* $\Delta_* \log\{\vartheta\}$ *when* Δ *ranges over the set of monomials of degree* $\geqslant 2$ *of* $\mathcal{S}(\mathcal{C}'\tilde{R})$.

PROOF. If apply the operator $(\iota\otimes\varepsilon\otimes\varepsilon)\big((\iota\otimes\Delta_*\otimes\Delta'_*)\log\{\ \}\big)$ to both terms of (7.1), and recall the invariance of the d_*'s (last formula of (4.2)), we have

$$(7.12)\qquad (\iota\otimes\varepsilon\otimes\varepsilon)\big((\iota\otimes\Delta_*\otimes\Delta'_*)\log\{F\}\big) =$$
$$= (\iota\otimes\varepsilon)(\iota\otimes\Delta_*)\big[(\iota\otimes\iota\otimes\varepsilon)(\iota\otimes\iota\otimes\Delta'_*)(\iota\otimes\mathbb{P})\mathbb{P}\log\{\vartheta\}\big] -$$
$$-(\varepsilon\otimes\varepsilon)(\Delta_*\otimes\Delta'_*)\mathbb{P}\log\{\vartheta\} = (\iota\otimes\varepsilon)(\iota\otimes\Delta_*)\mathbb{P}\big(\Delta'_*\log\{\vartheta\}\big) = (\Delta\Delta')_*\log\{\vartheta\}\,.$$

From (7.12) and (7.10) we obtain the desired result.

We shall now invert theorem (2.14).

(7.13) THEOREM. *Let* $\vartheta \in Q(\mathfrak{R})$ *be a nondegenerate theta element; then there exists a unique divisor* X *on the variety* V_ϑ *corresponding to* ϑ *such that every* ϑ_X *is associated to* ϑ.

PROOF. Let $d, d' \in \mathcal{C}'\tilde{R}_\vartheta$; then if we apply the operator $\big((dd')_*\otimes\iota\otimes\iota\big)\log\{\,\}$ to both sides of (7.1) we obtain

$$(7.14)\qquad \big((dd')_*\otimes\iota\otimes\iota\big)\log\{F\} = (\mathbb{P}\otimes\iota)\mathbb{P}\big((dd')_*\log\{\vartheta\}\big) +$$
$$+\big((dd')_*\log\{\vartheta\}\big)\otimes 1\otimes 1 - \mathbb{P}\big((dd')_*\log\{\vartheta\}\big)\otimes 1 - sc_{23}\big(\mathbb{P}\big((dd')_*\log\{\vartheta\}\big)\otimes 1\big)\,.$$

Now, fix a point P of V. Since $(dd')_*\log\{\vartheta\}$ has its components in C (cf. (7.11)) it is possible to choose points Q, R of V in such a way that all components of $(dd')_*\log\{\vartheta\}$ are regular at $P+Q+R$, $P+Q$, $P+R$, and are such that $(V\times P\times Q)$ is not contained in a pole of F. Now if we identify points P of V with the corresponding natural homomorphisms $\mathcal{O}_{V,P}\to k$, in wiew of the previous choices we have

$$\varphi_U = (\iota\otimes Q\otimes R)F(x_1, x_2, x_3) \in C\,,$$

and

$$(\iota\otimes Q\otimes R)\big[(\mathbb{P}\otimes\iota)\mathbb{P}\big((dd')_*\log\{\vartheta\}\big) - \mathbb{P}\big((dd')_*\log\{\vartheta\}\big)\otimes 1 -$$
$$- sc_{23}\big(\mathbb{P}\big((dd')_*\log\{\vartheta\}\big)\otimes 1\big)\big]$$

is regular in an open set U containing P. From (7.14) and what we have just observed it follows that

$$(7.15)\qquad (dd')_* \log\{\varphi_U\} \equiv (dd')^* \log\{\vartheta\} \mod \Gamma(U, \operatorname{vect} \mathcal{O}_V),$$

for all $d, d' \in \mathcal{C}'R$.

Let $\mathfrak{U}$ be a finite open covering of V consisting of (some of the) open sets previously defined. Since from (7.15) it follows that $(dd')_* \log\{\varphi_U/\varphi_{U'}\}$ is regular in $U \cap U'$, using (4.4) we conclude that $\varphi_U/\varphi_{U'} \in \Gamma(U \cap U', \mathcal{O}_V^*)$, and finally that $(\varphi_U)_{U \in \mathfrak{U}}$ is the cocycle of a divisor X.

By (5.2) we know that ϑ_X is a solution of (7.15), hence that $(dd')_* \log\{\vartheta_X\}/\vartheta = \text{const.}$. But this relation, in wiew of (7.14), implies $\big((dd')_* \otimes \iota \otimes \iota\big) \log\{F\} = 0$; therefore using (4.4) we conclude that $F = \text{const.}$, that is $\vartheta/\vartheta_X \in$ (q.e.), Q.E.D.

Now we shall give some tests which allow us to recognize the effective divisors by their theta elements.

(7.16) THEOREM. *Notations are as in* (7.13). *The following conditions are equivalent*:

i) X *is effective*;

ii) $\theta(x_1, x_2) = \dfrac{\vartheta(x_1 \dotplus x_2)\vartheta(x_1 \dot{-} x_2)}{\vartheta(x_1)^2\vartheta(x_2)\vartheta(\dot{-} x_2)} \in Q(R) \otimes Q(R)$;

iii) $\theta(x_1, x_2) \in C \otimes C$;

iv) $\vartheta(x_1 \dotplus x_2)\vartheta(x_1 \dot{-} x_2) \in \mathfrak{R} \otimes \mathfrak{R}$.

PROOF. With the notations of (7.1) we have $\theta(x_1, x_2) = F^{-1}(x_1, x_2, \dot{-} x_2)$; therefore $\theta(x_1, x_2) \in Q(C \otimes C)$, and the divisor of $\theta(x_1, x_2)$ is

$$Y = (p_1 + p_2)^* X + (p_1 - p_2)^* X - 2(p_1^* X) - p_2^*\big(X + (-\iota)^* X\big).$$

Therefore if X is effective the poles of Y are $2(p_1^* X)$ and $p_2^*\big(X + (-\iota)^* X\big)$, and the converse is also true. Thus, if we observe that an element of $C \otimes C$ has only zeroes of the type $Z \times A$ and $A \times T$ if and only if it has the form $f(x_1)g(x_2)$, we may conclude that i) is equivalent to iii).

Now, an argument similar to that used for proving part d) of (7.2) shows that $\big(Q(R) \otimes Q(R)\big) \cap Q(C \otimes C) = C \otimes C$; therefore ii) implies iii).

If X is effective the denominator of θ is in $\mathfrak{R} \otimes \mathfrak{R}$ and its numerator in $\mathfrak{R} \overline{\times} \mathfrak{R}$; but given that i) implies iii), this numerator is actually in $\big(Q(\mathfrak{R}) \otimes Q(\mathfrak{R})\big) \cap (\mathfrak{R} \overline{\times} \mathfrak{R}) = \mathfrak{R} \otimes \mathfrak{R}$; thus iii) implies iv).

Finally, if iv) holds, $\theta(x_1, x_2) \in \big(Q(\mathfrak{R}) \otimes Q(\mathfrak{R})\big) \cap Q(C \otimes C)$. But, as one may easily verify (for instance using hyperderivations), this intersection is contained in $Q(R) \otimes Q(R)$, so that iv) implies ii), Q.E.D.

In the theorem above we have tacitly assumed that V is an abelian variety. If that is not so, we must consider the degeneration locus of V, but the proof does not change.

A theta element which satisfies the conditions of (7.16) will be called *entire*. If $\vartheta \in \mathfrak{R}$ is an entire theta element, we shall denote by $\mathfrak{L}(\vartheta)$ the set of elements $\vartheta' \in \mathfrak{R}$ such that ϑ' is an entire theta element and $\vartheta/\vartheta' \in C_\vartheta$. $\mathfrak{L}(\vartheta)$ will be called the *linear system* of ϑ. By the arguments of this section and by the general theory of varieties, we know that $\mathfrak{L}(\vartheta)$ is a finite-dimensional vector space over k. Let $(\vartheta_0, \ldots, \vartheta_m)$ be a set of generators of $\mathfrak{L}(\vartheta)$ and V the variety with (homogeneous) general point $(\vartheta_0, \ldots, \vartheta_m)$; then, if V is (biregularly) isomorphic to a variety W we shall say that ϑ *gives a representation* of W.

(7.17) THEOREM. *Let ϑ be an entire theta element; then for every integer $r \geqslant 3$, ϑ^r gives a representation of V_ϑ.*

PROOF. By the general theory of group varieties it is known that if the divisor X is effective, and if $r \geqslant 3$, then rX is very ample. If $f_0 = 1, f_1, \ldots, f_m$ are elements of C_ϑ such that $X_i = rX + (f_i)$, for $0 \leqslant i \leqslant m$, is a basis of the linear systems of rX, then the elements $\vartheta_i = \vartheta^r f_i$ give a basis of the linear system of $\mathfrak{L}(\vartheta^r)$.

Now if t is an indeterminate over $k[f_1, \ldots, f_m]$, $(t, tf_1, \ldots, tf_m)$ is a general point of a variety isomorphic to V_ϑ; but $k[t, tf_1, \ldots, tf_m] \cong k[\vartheta_0, \ldots, \vartheta_m]$, Q.E.D.

8. – Riemann form.

The Riemann form of a divisor X is defined in chapter 7 of [MA]: it is a K'-bilinear alternating form on canonical K'-modules which determines the algebraic equivalence class of X (cf. also [9]).

In this section we shall show how this form may be directly constructed from a theta element of X. Since the results we are concerned with depend uniquely on the algebraic equivalence class of X, we shall assume $\vartheta_X \in \mathfrak{R}$, $\vartheta_X \equiv 1 \bmod \mathfrak{R}^+$.

We begin by constructing an alternating K'-bilinear form associated to ϑ_X: first, starting from ϑ_X by means of (2.13) one constructs F, next using (2.4) one finds a $\varphi(x_1, x_2) \in C_1^\infty[\![x_2]\!]$ unique up to multiplicative elements of R_2, and finally (2.10) gives $\chi(x_1, x_2) \in \mathfrak{R} \bar{\times} \mathfrak{R}$. The element χ is bi-multiplicative and alternating. The construction of χ may be obtained directly using the properties (2.15) of F.

As we have already observed in section 2, χ corresponds to a bi-homomorphism ${}^\pi\hat{\mathcal{A}} \times {}^\pi\hat{\mathcal{A}} \to \hat{G}_m$; therefore if we go from groups to the corresponding canonical K'-modules, from χ we obtain an alternating K'-bilinear form on canonical K'-modules

$$E\colon \mathcal{C}'\widetilde{\mathfrak{R}}\times\mathcal{C}'\widetilde{\mathfrak{R}}\to K',$$

or equivalently a homomorphism of canonical K'-modules

$$e\colon \mathcal{C}'\widetilde{\mathfrak{R}}\to\mathcal{C}'\mathfrak{R}.$$

The map E may be explicitly described: first, since $\chi\equiv 1 \bmod (\mathfrak{R}\overline{\times}\mathfrak{R})^+$, $\log\{\chi\}$ exists; next, as χ is bi-multiplicative and alternating, $\log\{\chi\}$ is an antisymmetric tensor of degree 2 in the tensor algebra $\mathcal{T}(\mathcal{C}'\mathfrak{R}\oplus\mathcal{C}'\mathfrak{R})$. Finally, if we observe that the restrictions to $\mathcal{C}'\mathfrak{R}$ of the Barsotti operators are K'-linear forms, and recall that $t\log\{\chi\}=\log\{\chi\}$, we see that the map

$$(d, d')\mapsto (d_*\otimes d'_*)\log\{\chi\}$$

is a K'-bilinear alternating form of $\mathcal{C}'\widetilde{\mathfrak{R}}\times\mathcal{C}'\widetilde{\mathfrak{R}}$ in K': it is precisely E. Therefore,

$$ed=(d_*\otimes\iota)\log\{\chi\}=(d_*\otimes\iota)\log\{\chi^{\frac12}\}-(\iota\otimes d_*)\log\{\chi^{\frac12}\}. \tag{8.1}$$

Observe that e, being a homomorphism of canonical modules, preserves the slope. As a consequence, if $\mathfrak{R}_\pi$ denotes the multiplicative component of $\mathfrak{R}$ (i.e. $\mathfrak{R}_\pi$ is the block of slope 1) we have $e\mathcal{C}'\widetilde{\mathfrak{R}}_\pi=0$. In fact $\widetilde{\mathfrak{R}}_\pi$ has slope 0 and every block of $\mathcal{C}'\mathfrak{R}$ has slope >0. For this reason we shall only be interested in the restriction of e to the radical part $\mathcal{C}'\widetilde{\mathfrak{R}}_r$ ($\mathfrak{R}_r$ denotes the product of the blocks of slopes <1 of $\mathfrak{R}$).

Now we state the main result of this section:

(8.2) THEOREM. *The restriction of e to $\mathcal{C}'\widetilde{\mathfrak{R}}_r$ coincides with the restriction of the homomorphism φ_X of chapter 6 of* [MA]. *As a consequence, the restriction of E to $\mathcal{C}'\widetilde{\mathfrak{R}}_r\times\mathcal{C}'\widetilde{\mathfrak{R}}_r$ coincides with the restriction of the Riemann form of X (as defined in chapter 7 of* [MA]).

Before proving (8.2) we shall describe the following procedure (cf. § 64 of [MA]).

Let $\boldsymbol{b}$ be a closed hyperclass of A (cf. § 4); let p_i, p_i^*, where $i=1,2$, be the projections of $A\times A$ on A and respectively the corresponding maps between hyperclasses; then (cf. [11] and chap. 6 of [MA])

$$(p_1+p_2)^*\boldsymbol{b}-p_1^*\boldsymbol{b}-p_2^*\boldsymbol{b}=\mathrm{cl}\,(g(x_1, x_2)), \tag{8.3}$$

where $g(x_1, x_2) \in \operatorname{vect} Q(C \otimes C)$ is a factor set of A in the (infinite) Witt group W. Denote by A' the provariety obtained as inverse limit of the sequence $A \xleftarrow{p\iota} A \xleftarrow{p\iota} \dots$, and by C' its field of rational functions. By using the projection of A' on the first term of the sequence, one can see that g corresponds to an extension of A' by W. But (cf. last page of [2]) every such extension splits, so that there exists $z \in \operatorname{vect} C'$ such that

$$g(x_1, x_2) = z(x_1 \dotplus x_2) - z(x_1) - z(x_2): \tag{8.4}$$

this z is the x of (6.12) of [MA].

Observe that $\boldsymbol{b}$ determines g up to an additive constant, while g determines z uniquely: this is so, because there is no canonical vector with components in C'. However if z is allowed to be in biv $\mathfrak{R}$, it is determined only up to elements of $\mathrm{C}'\mathfrak{R}$. In fact since canonical vectors with components in $\mathfrak{R}$ do not exist, if $z' \in \operatorname{biv} \mathfrak{R}$ is a solution of (8.4), it follows that

$$z' = z - \eta\,, \tag{8.5}$$

where $\eta \in \mathrm{C}'\mathfrak{R}$ is determined by the condition $(z + \eta)_{(-1)} = 0$. Therefore the sequence $\big((p\iota)^{-r} p^r z'\big)_{r \in \mathrm{N}}$ converges in biv $\mathfrak{R}$, and

$$\eta = \lim_{r \to \infty} (p\iota)^{-r} p^r z'\,. \tag{8.6}$$

Now we may prove (8.2).

Proof. Apply the procedure just described to the hyperclass $\boldsymbol{b}(d, X)$ (cf. (5.1)) and denote by $z_d = z_d(X)$, η_d and g_d the elements previously denoted by z, η and g respectively. An argument similar to that used in (5.2) shows that $g_d(x_1, x_2)$ may be chosen in such a way that

$$g_d(x_1, x_2) = (\iota \otimes \iota \otimes \varepsilon)\big((\iota \otimes \iota \otimes d_*) \log \{F\}\big)\,; \tag{8.7}$$

therefore using (2.13) we have

$$g_d(x_1, x_2) = \mathrm{P}\big(d_* \log \{\vartheta_X\}\big) - \big(d_* \log \{\vartheta_X\}\big) \otimes 1 - 1 \otimes \big(d_* \log \{\vartheta_X\}\big) + \varepsilon\big(d_* \log \{\vartheta_X\}\big)\,. \tag{8.8}$$

By comparing this with (8.4), and by (8.5), we obtain

$$z_d = \big(d_* \log \{\vartheta_X\} - c_d\big) + \eta_d\,, \tag{8.9}$$

where $c_d = \varepsilon\big(d_* \log \{\vartheta_X\}\big)$.

Observe that $\varepsilon z_d = 0$, and that for a suitable divisor $Y \equiv 0$, (8.9) written for ϑ_{X+Y} gives a z_d regular on every p-division point of A'. In this situation (cf. (6.14) of [MA]), since $d \in \mathrm{C}'\tilde{R}_r$, the sequence $(p^{-s}(p\iota)^s z_d)_{s\in\mathbf{N}}$ converges in biv $\mathfrak{R}_r$, and with the notations of §§ 67, 68 of [MA], we have

$$\varphi_X d = \lim_{s\to\infty} p^{-s}(p\iota)^s z_d .$$

Since this limit, when it exists, depends only on X, without loss of generality we may suppose $Y = 0$. Since η_d is a canonical bivector, $p^{-s}(p\iota)^s\eta_d = \eta_d$, for every $s \in \mathbb{Z}$, hence

$$\varphi_X d = \lim_{s\to\infty} p^{-s}(p\iota)^s\big(d_* \log\{\vartheta_X\} - c_d\big) + \eta_d . \tag{8.10}$$

Now we may compare φ_X with e: by § 2 we know that

$$\varphi(x_1, x_2)\chi^{\frac{1}{2}}(x_1, x_2) = \frac{\vartheta_X(x_1 \dotplus x_2)}{\vartheta_X(x_1)\vartheta_X(x_2)} ; \tag{8.11}$$

then if we set

$$\lambda_d = (\iota\otimes \varepsilon)(\iota\otimes d_*)\log\{\varphi\} \quad \text{and} \quad \xi_d = (\iota\otimes d_*)\log\{\chi^{\frac{1}{2}}\} ,$$

from (8.11) it follows that

$$\lambda_d + \xi_d = d_* \log\{\vartheta_X\} - c_d , \qquad \text{for every } d \in \mathrm{C}'\tilde{R}_r . \tag{8.12}$$

At this point, recalling that $C^\infty \subseteq C'$ (cf. § 2), we have $\lambda_d \in \operatorname{vect} C'$ and $\xi_d \in \mathrm{C}'\mathfrak{R}_r$, so that by comparing (8.12) with (8.9) we conclude that

$$\xi_d = -\eta_d \quad \text{and} \quad \lambda_d = z_d . \tag{8.13}$$

If instead we set

$$\mu_d = (\varepsilon\otimes \iota)(d_*\otimes \iota)\log\{\varphi\} \quad \text{and} \quad \zeta_d = (d_*\otimes \iota)\log\{\chi^{\frac{1}{2}}\} ,$$

by (8.11) it follows that

$$\mu_d + \zeta_d = d_* \log\{\vartheta_X\} - c_d . \tag{8.14}$$

Now however, $\mu_d \in \operatorname{biv} R_r$ and $\zeta_d \in \mathrm{C}'\mathfrak{R}_r$; moreover from (8.14) it follows that every component of μ_d is in R_r^+. Now, we shall show that $\lim_{s\to\infty} p^{-s}(p\iota)^s\mu_d = 0$. If w denotes the valuation of $\mathfrak{R}_r$ normalized in such a way that the group of w-values of R_r is $\mathbb{Z}$, a fundamental set of neighbourhoods of zero in biv $\mathfrak{R}_r$ is given by the sets

$$U_i(\beta) = \{x \in \operatorname{biv} \mathfrak{R}_r,\ p^{-j}w(x_j) \geqslant \beta \text{ when } j \leqslant i\}$$

obtained by varying i in $\mathbb{Z}$ and β in $\mathbb{R}^+$.

In our case, for every integer s there exists an integer h_s such that $t_{R_r}^{\,s} R_r^+ \subset \pi^{h_s} R^+$, and h_s tends to ∞ with s.

Since every component of μ_d has a w-value > 0, it follows that $(p\iota)^s \mu_d \in U_i(p^{-i+s+h_s})$ and thus $p^{-s}(p\iota)^s \mu \in U_{i-s}(p^{-i+s+h_s})$. Now we must show that for every integer $H > 0$ there exists an integer s', such that for every $s \geqslant s'$ it is possible to choose $i \geqslant s + H$ and $p^{-i+s+H} \geqslant H$. But since $h_s \to \infty$ when $s \to \infty$, we may choose $i = s + H$, and s' such that $h_{s'} \geqslant H + \log_p H$. From what we have just shown, recalling that $p^{-s}(p\iota)^s \zeta_d = \zeta_d$ for every integer s, it follows that

$$\zeta_d = \lim_{s\to\infty} p^{-s}(p\iota)^s \big(d_* \log \{\vartheta_X\} - c_d\big)\,. \tag{8.15}$$

Finally, from (8.10), (8.15), the first of (8.13) and (8.1) it follows that

$$\varphi_X d = \zeta_d - \xi_d = ed\,, \qquad \text{Q.E.D.}$$

At this point we have shown how, starting from ϑ_X, we may construct the restriction of φ_X to $\mathrm{C}'\tilde{R}_r$. If A is an abelian variety, in wiew of the results of chap. 7 of [MA], φ_X is completely known when, besides this previous restriction, one knows also its restriction to $\mathrm{C}'\tilde{R}_\pi$: this restriction too may be obtained directly from ϑ_X.

(8.16) COROLLARY. *Let $\vartheta = \vartheta_X$ be a theta element, denote with R_t the hyperalgebra of the étale component of $\hat{A}$; then for every $d \in \mathrm{C}'\tilde{R}_\pi$ there exists $Z \equiv X$ such that the sequence $\big(p^{-s}(p\iota)^s(d_* \log\{\vartheta_Z\} - \varepsilon(d_* \log\{\vartheta_Z\}))\big)_{s\in\mathbf{N}}$ converges in $\mathrm{C}'R_t$. In this case we have*

$$\varphi_X d = \lim_{s\to\infty} p^{-s}(p\iota)^s(d_* \log\{\vartheta_Z\} - \varepsilon(d_* \log\{\vartheta_Z\}))\,.$$

PROOF. We have already observed that there is a divisor Z as in the statement. Now, remember that the map $d \mapsto \eta_d$ (cf. (8.9)) preserves the slope; thus, since the elements d of $\mathrm{C}'\tilde{R}_\pi$ have slope 1, in the present case η_d must be 0. In conclusion, the formula analogous to (8.10) now becomes

$$\varphi_X d = \lim_{s\to\infty} p^{-s}(p\iota)^s\Big(d_* \log\{\vartheta_Z\} - \varepsilon\big(d_* \log\{\vartheta_Z\}\big)\Big)\,, \qquad \text{Q.E.D.}$$

REMARK. *Theorem* (7.11) *and formula* (8.9) *tell us that* C_ϑ *may also be generated by the components of the vectors* $\Delta * z_d$ *obtained when* d *ranges over* $\mathcal{C}'\tilde{R}$ *and* Δ *over the set of monic monomials* $\mathcal{S}(\mathcal{C}'\tilde{R})$ *of positive degree.*

At this point it is possible to recognize the entire theta elements whose corresponding varieties are abelian varieties. The main fact used for this identification is that the abelian varieties are characterized among group varieties without periodic factors by the possession of a self-dual B-T group.

(8.17) THEOREM. *Let* ϑ *be an entire nondegenerate theta element,* $A = V_\vartheta$ *its corresponding variety, and* X *the divisor of* ϑ *on* A. *Denote with* M *the quotient module of the* K*-module generated by the* z_d*'s of* (8.9) (*d ranges over* $\mathcal{C}'\tilde{R}_\pi$ *and* X *over its algebraic equivalence class*) *by its sub-*K*-module generated by the elements which are in vect* C. *Then* A *is an abelian variety if and only if* M *is a free* K*-module of rank* f, *where* $f = \mathit{rank}\,(\ker p\iota_A) = \dim_K \mathcal{C}'\tilde{R}_\pi$.

PROOF. Every element $m \in M$ contains an element z_d for which $\lim_{s\to\infty} p^{-s}(p\iota)^s z_d = \varphi_X d$ exists: such limit depends uniquely on m; thus we have a map $\beta\colon m \mapsto \varphi_X d$ of M in $\mathcal{C}'R_t$. One may easily verify (cf. (6.15) of [MA]) that β is a K-linear injective map. Now if $X \equiv Y$, $z_d(X) \equiv z_d(Y)$ mod vect C, therefore $d \mapsto z_d$ induces a K-linear map α of $\mathcal{C}'\tilde{R}_\pi$ in M, and it is clear that $\beta\alpha = \varphi_X$.

If A is an abelian variety and X is the divisor on A such that $\vartheta_X = \vartheta$, the only point P such that $\sigma_P^* X = X$ is the identity point e. But, since X is effective, this condition implies that X is nondegenerate, that is that the polarization corresponding to X is an isogeny (cf. 10 of [3]). In this case (cf. chap. 7 of [MA]) φ_X is an isomorphism, thus also α is an isomorphism. If, on the contrary, A is not an abelian variety, $\dim_K \mathcal{C}'\tilde{R}_\pi > \dim \mathcal{C}'R^t$ so that $\dim M < \dim \mathcal{C}'\tilde{R}_\pi$, Q.E.D.

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