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EDGAR LEE STOUT

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Cauchy-Stieltjes Integrals on Strongly Pseudoconvex Domains (¹).

EDGAR LEE STOUT (*)

Introduction.

A very attractive chapter of classical analysis is that devoted to the study of integrals of Cauchy-Stieltjes type. Given a measure μ on the unit circle in the complex plane or, more generally, on a curve γ, the smoothness properties of μ and γ are shown to be related to those of the holomorphic function F_μ defined by

$$F_\mu(z) = \int_\gamma (\zeta - z)^{-1}\, d\mu(\zeta)\, .$$

This theory is quite well developed and may be found in the books [1] and [7].

Recently it has become feasible to begin an analogous theory in the higher dimensional case. The papers [2], [4], [6] and [11] contain contributions in this direction. In particular, Nagel [6] has studied integrals of the form

$$\int_\Gamma f(w)\big(1 - \langle z, w\rangle\big)^{-N}\, d\mu(w)$$

where μ is a measure concentrated on the smooth curve Γ in the boundary of the unit ball B_N in $\boldsymbol{C}^N$, Γ and μ suitably restricted. In this paper we

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(*) Department of Mathematics, University of Washington, Seattle.
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extend some of Nagel's results. In the first place, we work on strongly pseudoconvex domains in C^N, rather than the ball. Secondly, where Nagel worked with measures concentrated on smooth curves, we are able to treat measures on submanifolds of the boundary of arbitrary dimension.

There is the question of the analogue on a general strongly pseudoconvex domain of the kernel $(1 - \langle z, w\rangle)^{-N}$ in the integral above. Two candidates come to mind. One natural candidate is the kernel of Henkin and Ramírez. The other is the Szegö kernel. We begin by dealing with the Henkin-Ramírez kernel. Given the analysis of this case, we are then able to treat the Szegö kernel by using the analysis given recently by Kerzman and Stein [5].

I would like to acknowledge here some useful discussions I have had with Nagel concerning the results of this paper. In particular, he suggested the idea of studying the Szegö kernel in this context.

1. – Preliminaries.

We fix attention on a strongly pseudoconvex domain D in C^N with $\mathcal{C}^\infty$ boundary. Thus, D is a bounded domain in C^N, and there exists a real-valued $\mathcal{C}^\infty$ function Q on a neighborhood Ω of $\bar{D}$ which is strictly plurisubharmonic and which satisfies

$$D = \{z \in \Omega : Q(z) < 0\}$$

and $dQ \neq 0$ on ∂D. It will become evident in the discussion below that for much of what we do less stringent regularity conditions on ∂D would suffice.

Recall the integral kernel construct. by Ramírez [8]. (Compare this construction with the similar kernel constructed by Henkin [4] as well as with the constructions given by Øvrelid [12] and Fornaess [3].) According to Ramírez, there exists a neighborhood $\mathcal{V}$ of ∂D, a neighborhood $\mathcal{U}$ of $\bar{D}$, and a $\mathcal{C}^\infty$ function $\Phi : \mathcal{U} \times \mathcal{V} \to C$ with the properties that for fixed $\zeta \in \mathcal{V}$, $\Phi(\cdot, \zeta) \in \mathcal{O}(\mathcal{U})$, and $\operatorname{Re} \Phi(z, \zeta) > 0$ for $z \in \bar{D}$, $\zeta \in \partial D$ and $z \neq \zeta$. In addition, there is a decomposition of Φ: There exist $\mathcal{C}^\infty$ functions $g_j : \mathcal{U} \times \mathcal{V} \to C$, $j = 1, 2, \ldots, N$, each holomorphic in the first variable, such that

$$\Phi(z, \zeta) = \sum_{j=1}^{N} (z_j - \zeta_j) g_j(z, \zeta) .$$

According to the theory of Cauchy-Fantappiè forms, there is a constant c_N such that if $f \in \mathcal{O}(\bar{D})$, then for each $z \in D$,

$$f(z) = c_N \int_{\partial D} f(\zeta) \Phi(z, \zeta)^{-N} \Delta_z ,$$

Δ_k a smooth form that depends holomorphically on z.

It becomes natural, therefore, to consider integrals of the type

$$F_\mu^{(s)}(z) = \int_{\partial D} \Phi(z, \zeta)^{-s} \, d\mu(\zeta) ,$$

$s > 0$, with μ a finite measure on ∂D. It is known from [11] that $F_\mu^{(s)} \in H^p(D)$ provided $0 < s < N$ and $p \in (0, N/s)$.

In this paper, as in [6], attention is focused on measures concentrated on certain smooth submanifolds of ∂D, the submanifolds transverse to the holomorphic tangent spaces of ∂D in the following sense. Given a point $p \in \partial D$, let $T_p^{\mathbf{C}}(\partial D)$ denote the maximal complex subspace of $T_p(\partial D)$, $T_p(\partial D)$ the tangent space to ∂D at p. Thus, $\dim_{\mathbf{C}} T_p^{\mathbf{C}}(\partial D) = N - 1$. We shall say that *a submanifold M of ∂D is transverse to the holomorphic tangent space of ∂D at $p \in M$ if $T_p(\partial D) = T_p(M) + T_p^{\mathbf{C}}(\partial D)$*. As $T_p^{\mathbf{C}}(\partial D)$ has codimension one in $T_p(\partial D)$, the condition is equivalent to the condition that $T_p^{\mathbf{C}}(\partial D)$ not contain $T_p(M)$. Notice that ∂D itself has this property at each of its points.

2. – The case of the Henkin-Ramírez kernel.

With the preceding notions in mind, we formulate the following result [2].

THEOREM I. *Let $M \subset \partial D$ be a locally closed submanifold of class $\mathcal{C}^k$, $k \geq 2$, dimension m, $1 \leq m \leq 2N - 1$, that, at each of its points, is transverse to the holomorphic tangent space of ∂D, and let ψ be a compactly supported function of class $\mathcal{C}^{k-1}$ on M. If μ_m denotes the m-dimensional Hausdorff measure*

[2] Mme Anne-Marie Chollet has informed me that she has obtained the case $m = 1$ of Theorem I.

on C^N and if $F \in \mathcal{O}(D)$ is defined by

$$F(z) = \int_M \Phi(z,\zeta)^{-\sigma}\psi(\zeta)\, d\mu_m(\zeta),$$

$\sigma = s + i\tau$ with $0 < s$, then the derivatives of F of order α, $k - s - 1 < |\alpha| < N + k - s - 1$ belong to $H^p(D)$ for $p \in \big(0, N/(s + |\alpha| - k + 1)\big)$.

The condition that $k - s - 1 < |\alpha|$ guarantees that $s + |\alpha| - k + 1 > 0$, so the range of p, the interval $\big(0, N/(s + |\alpha| - k + 1)\big)$, is nonempty. Notice too that if $k \geqslant s + 1$, then $|\alpha|$ can vary within an interval of length N. On the other hand, if s is large compared with N and k, the condition $|\alpha| < N + k - s - 1$ cannot be satisfied. Finally, for $|\alpha| < N + k - s - 1$ we have $s + |\alpha| - k + 1 < N$, so in particular the derivatives in question belong to $H^1(D)$.

The hypothesis that M be a locally closed submanifold means that M is a closed submanifold of an open subset of ∂D.

PROOF. We execute the proof in two steps. First we deal with the case of curves, *i.e.*, the case that $m = \dim M = 1$, paying some attention to the dependence of the estimates on the differential properties of the curve. This analysis follows the general line of [6], the main point being repeated integration by parts. Once we have the curve case, we are able to deal with the general case by a fibering process.

As above, we denote by Q a $\mathcal{C}^\infty$ strongly plurisubharmonic characterizing function for the domain D. Let P be the associated Levi polynomial $P\colon C^N \times \Omega \to C$ given by

$$P(z,\zeta) = \sum_{j=1}^{N}(z_j - \zeta_j)\frac{\partial Q}{\partial \zeta_j}(\zeta) + \frac{1}{2}\sum_{j,k=1}^{N}(z_j - \zeta_j)(z_k - \zeta_k)\frac{\partial^2 Q}{\partial \zeta_j\, \partial \zeta_k}(\zeta)\,.$$

There is a constant d_1 such that for a certain smooth function H defined on

$$\{(z,\zeta) \in C^N \times \partial D\colon |z - \zeta| < d_1\}\,, \qquad \Phi(z,\zeta) = P(z,\zeta)H(z,\zeta) \tag{1}$$

iwht $H(\cdot,\zeta)$ holomorphic on $\{\zeta\colon |z - \zeta| < d_1\}$ and with

$$c_1 < |H(z,\zeta)| < c_1^{-1} \tag{2}$$

for some constant $c_1 > 0$.

Consider now a locally closed, connected curve Γ in ∂D that is of class $\mathcal{C}^k$ and is transverse to the holomorphic tangent spaces of ∂D. Assume Γ to have length not more than $\frac{1}{4} d_1$. Choose a parameterization

$$\gamma = (\gamma_1, \dots, \gamma_N) \colon (0, 1) \to \Gamma$$

of class $\mathcal{C}^k$, γ' nonvanishing. The $\mathcal{C}^{k-1}$ function ψ is compactly supported in Γ, so $\psi \circ \gamma$ is compactly supported in $(0, 1)$. Set

$$F(z) = \int_\Gamma \Phi(z, \zeta)^{-\sigma} \psi(\zeta)\, d\mu_1(\zeta) = \int_0^1 \Phi(z, \gamma(t))^{-\sigma} \psi(\gamma(t))\, |\gamma'(t)|\, dt\,. \tag{3}$$

By hypothesis the curve Γ is transverse to the holomorphic tangent spaces of ∂D, so for each t, $\gamma'(t) \notin T^{\mathbf{C}}_{\gamma(t)}(\partial D)$. As $D = \{Q < 0\}$, given $w \in \partial D$, the space $T^{\mathbf{C}}_w(\partial D)$ can be identified with the complex subspace

$$\left\{ z \in \mathbf{C}^N \colon \sum_{j=1}^N \frac{\partial Q}{\partial w_j}(w)\, z_j = 0 \right\}$$

of $\mathbf{C}^N$, so $\gamma'(t) \notin T^{\mathbf{C}}_{\gamma(t)}(\partial D)$ is equivalent to

$$\sum_{j=1}^N \frac{\partial Q}{\partial w_j}(\gamma(t))\, \gamma_j'(t) \neq 0\,.$$

Fix a constant $c_2 > 0$ so that for all t in the support of $\psi \circ \gamma$,

$$c_2 < \left| \sum_{j=1}^N \frac{\partial Q}{\partial w_j}(\gamma(t))\, \gamma_j'(t) \right| < c_2^{-1}. \tag{4}$$

For a multiindex $\alpha = (\alpha_1, \dots, \alpha_N)$, let D_α denote the associated differential operator $\partial^{\alpha_1 + \dots + \alpha_N} / (\partial z_1^{\alpha_1} \dots \partial z_N^{\alpha_N})$, and consider $D_\alpha F$. We have

$$D_\alpha F(z) = \int_0^1 D_\alpha \Phi(z, \gamma(t))^{-\sigma} \psi(\gamma(t))\, |\psi'(t)|\, dt\,.$$

$$= \int_0^1 W_\alpha(z, \gamma(t))\, \Phi(z, \gamma(t))^{-|\alpha| - \sigma} \psi(\gamma(t))\, |\gamma'(t)|\, dt\,.$$

If we set

$$\mathcal{U}(\Gamma, \delta) = \{ z \in D \colon \operatorname{dist}(z, \Gamma) < \delta \}\,, \tag{5}$$

we have the estimate

$$(6)\quad \sup\{|D_\alpha F(z)|: z\in D\setminus \mathcal{U}(\Gamma, \tfrac14 d_1)\} \leqslant C_{\sigma,\alpha}(d_1)\sup_\Gamma |\psi| \int_0^1 |\gamma'(t)|\,dt$$

$$= C_{\sigma,\alpha}(d_1)\sup_\Gamma |\psi| \text{ (length } \Gamma)$$

with $C_{\sigma,\alpha}(d_1)$ a constant which depends on Φ, s, α and d_1 but not on Γ.

As the curve Γ has length no more than $\frac14 d_1$, it follows that for $\zeta\in\Gamma$, $z\in\mathcal{U}(\Gamma, \frac14 d_1)$, we have $|\zeta - z| < \frac12 d_1$, so for $z\in\mathcal{U}(\Gamma, \frac14 d_1)$, we can write

$$(7)\qquad D_\alpha F(z) = \sum_{\varkappa+\lambda=\alpha} q(\varkappa, \lambda)\int_0^1 \frac{u_\varkappa(z,\gamma(t))\, v_\lambda(z,\gamma(t))\,\psi(\gamma(t))\,|\gamma'(t)|}{P(z,\gamma(t))^{\sigma+|\varkappa|} H(z,\gamma(t))^{\sigma+|\lambda|}}\,dt$$

where the summation is over multiindices $\varkappa$ and λ with nonnegative entries, where the $q(\varkappa, \lambda)$ are certain constants and the functions $u_\varkappa$ and v_λ arise from differentiating the quotient $1/PH$ and depend on P and H respectively. For a fixed t, $u_\varkappa$ is a polynomial in z and v_λ is holomorphic in $\mathcal{U}(\Gamma, \frac34 d)$. As α is fixed, λ in (7) is determined by $\varkappa$. Set

$$(8)\qquad K_\varkappa(z, t) = u_\varkappa v_\lambda H^{-(\sigma+|\lambda|)}$$

so that (7) is

$$(9)\qquad D_\alpha F(z) = \sum_{\varkappa+\lambda=\alpha} q(\varkappa, \lambda)\int_0^1 P(z,\gamma(t))^{-\sigma-|\varkappa|} K_\varkappa(z,t)\,\psi(\gamma(t))\,|\gamma'(t)|\,dt\,.$$

We integrate this by parts.

To this end, notice that

$$\frac{d}{dt}\left[P(z,\gamma(t))^{-(\sigma+|\varkappa|)+1}\right] = -(\sigma+|\varkappa|-1)\,P(z,\gamma(t))^{-\sigma+|\varkappa|}\frac{d}{dt}P(z,\gamma(t))\,.$$

If we write Q_{jk} for $\partial^2 Q/(\partial\zeta_j\partial\zeta_k)$ and use similar notation for other derivatives so that, *e.g.*, Q_j denotes $\partial Q/\partial\zeta_j$, we have

$$P(z,\gamma(t)) = \sum_{j=1}^N (z_j - \gamma_j(t))\,Q_j(\gamma(t)) + \tfrac12 \sum_{j,k=1}^N (z_j-\gamma_j(t))(z_k-\gamma_k(t))\,Q_{jk}(\gamma(t))\,,$$

whence

$$\frac{d}{dt}P(z,\gamma(t)) = -\sum_{j=1}^{N}\gamma_j' Q_j - \frac{1}{2}\sum_{j,k=1}^{N}\big(\gamma_j'(z_k-\gamma_k)+\gamma_k'(z_j-\gamma_j)\big)Q_{jk} \tag{10}$$

$$+\sum_{j=1}^{N}\Big[(z_j-\gamma_j)\sum_{k=1}^{N}Q_{jk}\gamma_j' + Q_{j\bar k}\bar\gamma_k'\Big]$$

$$+\frac{1}{2}\sum_{j,k=1}^{N}\Big[(z_j-\gamma_j)(z_k-\gamma_k)\sum_{r=1}^{N}Q_{jkr}\gamma_r' + Q_{jk\bar r}\bar\gamma_r'\Big] = -\sum_{j=1}^{N}\gamma_j'(t)Q_j(\gamma(t)) + R(z,t)$$

where the remainder term $R(z, t)$ is, for given t, a quadratic polynomial in z that satisfies $R(\gamma(t), t) = 0$. Thus, from (4) it follows that there is a constant d_2 such that

$$\left|\frac{d}{dt}P(z,\gamma(t))\right| > \frac{1}{2}c_2 \tag{11}$$

if dist $(z, \gamma(t)) < d_2$. The constant d_2 depends only on Q and on the magnitude of the first order derivatives of γ. We may take $d_2 < d_1$.

Assume now that $|\gamma'| \leqslant C_1$. Divide $[0, 1]$ into equal intervals $J_1, \ldots, J_L$, $4C_1/d_2 \leqslant L < 4C_1/d_2 + 1$, disjoint except for their endpoints. For a given j, let $\tilde J_j$ be an open interval twice as long as J_j and centered on the center of J_j. Let $\{\eta_j\}_{j=1}^{L}$ be a partition of unity of class $\mathcal{C}^\infty$ on $[0, 1]$ subordinate to $\{\tilde J_j\}_{j=1}^{L}$. The functions η_j can be chosen so that their $\mathcal{C}^k$ norms are bounded by a constant $C(L)$ that depends only on L and hence only on the $\mathcal{C}^1$ norm of γ.

We have

$$\int_0^1 P(z,\gamma(t))^{-\sigma-|\varkappa|}K_\varkappa(z,t)\,\psi(\gamma(t))\,|\gamma'(t)|\,dt \tag{12}$$
$$= \sum_{j=0}^{L}\int_0^1 P(z,\gamma(t))^{-\sigma-|\varkappa|}K_\varkappa(z,t)\,\psi(\gamma(t))\,\eta_j(t)\,|\gamma'(t)|\,dt\,.$$

In estimating these summands, we restrict our attention to z's in $\mathcal{U}(\Gamma, \frac{1}{4}d_1)$ because of the estimate (6). In the sum (12), there are two kinds of terms. First there are those j for which $z \in \mathcal{U}(\Gamma, \frac{1}{4}d_1) - \mathcal{U}(\gamma(\tilde J_j), \frac{1}{4}d_2)$. These terms are bounded by $C_\varkappa \sup_\Gamma |\psi|$ (length $\gamma(\tilde J_j)$) for a certain constant $C_\varkappa$ that depends on $\varkappa$ and D but not on γ, for P is bounded away from zero uniformly on this set.

If $z \in \mathcal{U}(\gamma(\tilde J_j), \frac{1}{4}d_2)$ then as $\gamma(\tilde J_j)$ has length no more than

$$\sup|\gamma'|\frac{2}{L} \leqslant 2C_1\left(\frac{4C_1}{d_2}\right)^{-1} = \frac{1}{2}d_2\,,$$

it follows that for all $\zeta \in \gamma(\tilde{J}_j)$, $\operatorname{dist}(z, \zeta) < \frac{3}{4} d_2$, so the estimate (11) is at our disposal. Assume now that $\varkappa$ satisfies

$$s + |\varkappa| \geqslant N , \tag{13}$$

and write

$$\int_0^1 P(z, \gamma(t))^{-\sigma-|\varkappa|} K_\varkappa(z, t)\, \psi(\gamma(t))\, \eta_j(t)\, |\gamma'(t)|\, dt \\ = \int_0^1 P(z, \gamma(t))^{-\sigma-|\varkappa|+1} \frac{d}{dt}\left\{\frac{K_\varkappa(z, t)\, \psi(\gamma(t))\, \eta_j(t)\, |\gamma'(t)|}{(-\sigma - |\varkappa| + 1)(d/dt)\, P(z, \gamma(t))}\right\} dt \tag{14}$$

by integration by parts. Introduce a sequence of functions, $G_0, G_1, \ldots$ by

$$G_0(z, t) = K_\varkappa(z, t)\, \psi(\gamma(t))\, \eta_j(t)\, |\gamma'(t)|$$

and, for $j = 1, 2, \ldots$,

$$G_j(z, t) = \frac{d}{dt}\left\{G_{j-1}(z, t)\left[(-\sigma - |\varkappa| + 1) \ldots (-\sigma - |\varkappa| + j)\frac{d}{dt} P(z, \gamma(t))\right]^{-1}\right\}.$$

Iterating the partial integration, we find

$$\int_0^1 P(z, \gamma(t))^{-\sigma-|\varkappa|} K_\varkappa(z, t)\, \psi(\gamma(t))\, \eta_j(t)\, |\psi'(t)|\, dt = \int_0^1 P(z, \gamma(t))^{-\sigma-|\varkappa|+1} G_1(z, t)\, dt \\ \vdots \\ = \int_0^1 P(z, \gamma(t))^{-\sigma-|\varkappa|+r} G_r(z, t)\, dt\,. \tag{15}$$

For a given $\sigma, \varkappa$ and k, we terminate this process for one of two reasons. When $s + |\varkappa| - r \in (0, 1]$, we do not integrate parts again. Also, notice that the functions G_j become progressively less differentiable. The function G_0 is of class C^{k-1} in t, so if $s + |\varkappa| - (k-1) > 0$, we take $r = k - 1$ in (15). Thus, in the former case, we reach

$$\int_0^1 P(z, \gamma(t))^{-\sigma-|\varkappa|} K_\varkappa(z, t)\, \psi(\gamma(t))\, \eta_j(t)\, |\psi'(t)|\, dt = \int_{\tilde{J}_j} P(z, \gamma(t))^{-\sigma-|\varkappa|+r} G_r(z, t)\, dt \tag{16}$$

with $s + |\varkappa| - r \in (0, 1)$, and in the latter case, we find that the integral is

$$\int_{\tilde{J}_j} P(z, \gamma(t))^{-\sigma - |\varkappa| + k - 1} G_{k-1}(z, t)\, dt\,. \tag{16'}$$

(Recall that by its construction, G_{k-1} is supported in $\tilde{J}_j$.) By hypothesis, $|\alpha| < N + k - s - 1$, so as $|\varkappa| \leqslant |\alpha|$, it follows that $|\varkappa| + s - k + 1 < N$.

If we recall (9) and (12), we see that for $z \in \mathcal{U}(\Gamma, \frac{1}{4} d_1)$ we have written $D_\alpha F(z)$ as a sum of $|\alpha| L$ terms of the form

$$\int_{\tilde{J}_j} P(z, \gamma(t))^{-(s' + i\tau)} g(z, t)\, dt \tag{17}$$

with $0 < s' \leqslant s + |\alpha| - k + 1$, g a function continuous on $D \times \tilde{J}$, $g(\cdot, t)$ holomorphic on D, and bounded uniformly by a constant that depends only on the C^k norm of γ, the C^{k-1} norm of ψ and the quantity

$$\inf \left\{ \left| \sum_{j=1}^{N} Q_j(\gamma(t))\, \gamma_j'(t) \right| :\ t \in \operatorname{supp} \psi \right\}^{-1}.$$

As $z \in \mathcal{U}(\Gamma, \frac{1}{4} d_1)$, the integral (17) can be rewritten as

$$\int_{J_j} \Phi(z, \gamma(t))^{-(s' + i\tau)} H(z, \gamma(t))^{s' + i\tau} g(z, t)\, dt\,, \tag{18}$$

and the integral (18) lies in $H^p(\mathcal{U}(\Gamma, \frac{1}{4} d_1))$ for $p \in (0, N/(s' + |\alpha| - k + 1))$. (See [11].)

We have, therefore, established that *for the function F given by* (3), *$D_\alpha F$ lies in $H^p(D)$ for $p \in (0, N/(s + |\alpha| - k + 1))$ if α satisfies $k - s - 1 < |\alpha| < N + k - s - 1$ and, moreover, the $H^p(D)$ norm of $D_\alpha F$ is bounded by a constant that depends only on the C^k norm of γ, the C^{k-1} norm of ψ and the quantity* $\left(\inf \left\{ \left| \sum_{j=1}^{N} Q_j(\gamma(t))\, \gamma_j'(t) \right| : t \in \operatorname{supp} \psi \right\}\right)^{-1}$.

This completes the discussion of the case $m = 1$ of the theorem. We turn now to the case of general M.

Suppose therefore $M \subset \partial D$ is a locally closed m-dimensional submanifold of class C^k that is transverse to the holomorphic tangent spaces of ∂D, and let $\psi \in C^{k-1}(M)$ have compact support. We suppose, as we may, that M consists of a single coordinate patch so that there is a diffeomorphism (of class C^k) $\gamma\colon \mathbf{R}^m \to M$. Choose $R > 0$ so large that the support of $\psi \circ \gamma$ is contained in

$$B_m(0, R) = \{x \in \mathbf{R}^m\colon |x| < R\}\,.$$

Since M is transverse to the holomorphic tangent spaces of ∂D, there is for each $x \in \mathbf{R}^m$ a unit vector $u_x \in \mathbf{R}^m$ such that

$$\sum_{j=1}^{N} Q_j(\gamma(x)) \frac{d}{dt} \gamma_j(x+tu)\Big|_{t=0} \neq 0 .$$

By compactness, there is an $\varepsilon_1 > 0$ such that for each $x \in B_m(0, R)$ there is a unit vector u_x such that

$$\left| \sum_{j=1}^{N} Q_j(\gamma(x)) \frac{d}{dt} \gamma_j(x+tu_x)\Big|_{t=0} \right| \geqslant \varepsilon_1 .$$

By continuity there exists a δ_1 such that if $x \in B_m(0, R)$ and if x' satisfies $|x - x'| < \delta_1$, then

$$\left| \sum_{j=1}^{N} Q_j(\gamma(x')) \frac{d}{dt} \gamma_j(x'+tu_x)\Big|_{t=0} \right| \geqslant \frac{1}{2} \varepsilon_1 .$$

Choose $x_1, \ldots, x_p \in B(0, R)$ such that $B(0, R) \subset \bigcup_{j=1}^{J} B_j$ where

$$B_j = \{x\colon |x - x_j| < \delta_1\} .$$

Let u_j be the u_x associated with x_j. Let $\{\eta_j\}_{j=1}^J$ be a $\mathcal{C}^\infty$ partition of unity on $B_m(0, R)$ subordinate to the cover $\{B_j\}_{j=1}^J$.

We write

$$\begin{aligned}\int_M \Phi^{-s}(z, \zeta)\, \psi(\zeta)\, d\mu_m(\zeta) &= \int_{\mathbf{R}^m} \Phi^{-s}(z, \gamma(x))\, \psi(\gamma(x))\, J_\gamma(x)\, dx \\ &= \sum_{j=1}^{J} \int_{B_j} \Phi^{-s}(z, \gamma(x))\, \psi(\gamma(x))\, \eta_j(x)\, J_\gamma(x)\, dx\end{aligned}$$

where J_γ denotes the appropriate Jacobian. (See, *e.g.*, [10].) The function J_γ is of class $\mathcal{C}^{k-1}$. For each j, let N_j be the orthogonal complement of the line $x_j + tu_j$, $-\infty < t < \infty$. We can write

$$(16) \qquad \int_{B_j} \Phi^{-s}(z, \gamma(x))\, \psi(\gamma(x))\, J_\gamma(x)\, dx$$

$$= \int_{N_j \cap B_j} \left\{ \int_{|t| < \frac{1}{2}\delta_1} \Phi^{-s}(z, \gamma(x + u_j t))\, \psi(\gamma(x + u_j t))\, \eta_j(x + u_j t)\, J_\gamma(x + u_j t)\, dt \right\} d\mu_{m-1}(x) .$$

By construction and the discussion of the one dimensional case above, for fixed $x \in N_j \cap B_j$, the inner integral, *qua* function of z, belongs to the appropriate H^p-space, with H^p norm bounded uniformly in x. Thus, the integral on the left of (16) belongs to $H^p(D)$, and so the theorem is proved.

COROLLARY. *If M and ψ are of class $\mathcal{C}^\infty$, then the function F of the theorem belongs to $A^\infty(D)$.*

Recall that $A^\infty(D)$ is the space of holomorphic functions on D that together with their derivatives of all orders are continuous on D. The special case that $M = \partial D$ of the corollary is contained in work of Elgueta [2]. See also [5].

3. – The Szegö kernel.

We will now study integrals of the form

$$F(z) = \int_M S(z, \zeta)\, \psi(\zeta)\, d\mu_m(\zeta) \tag{17}$$

wherein S denotes the Szegö kernel of the domain D, and M, ψ and μ_m are as in the preceeding section. *Throughout this section we require D to be a strongly pseudoconvex domain in $\boldsymbol{C}^N$ with boundary of class* $\mathcal{C}^\infty$ with strictly plurisubharmonic defining function Q. Our analysis of the integral (17) is based on the ideas involved in our treatment of the corresponding Henkin-Ramírez integrals and on the recent results of Kerzman and Stein [5] concerning the Szegö kernel.

We need to recall some of the Kerzman-Stein results. They introduce an *explicit kernel* $E(z, \zeta)$ by

$$E(z, \zeta) = \frac{\Theta(z, \zeta)}{[g\,(z, \zeta)]^N}$$

with $\Theta\colon \bar{D} \times \partial D \to \boldsymbol{C}$ a $\mathcal{C}^\infty$ function whose principal term is a function given explicitly in terms of Q and the function g given by

$$g(z, \zeta) = \sum_{i=1}^{N} g_i(z, \zeta)(\zeta_i - z_i)$$

with g_i of the form

$$g_i(z, \zeta) = \big(1 - \psi(|z - \zeta|)\big)\, \tilde{g}_i(z, \zeta) + \psi\big(|z - \zeta|\big)(\bar{\zeta}_i - \bar{z}_i)\,.$$

Here $\psi\colon \boldsymbol{R} \to [0, 1]$ is a $\mathcal{C}^\infty$ function satisfying $\psi(s) = 0$ if $s \leqslant \frac{1}{2}s_0$, $\psi(0) = 1$ if $s > s_0$, and

$$\tilde{g}_i(z, \zeta) = 2\,\frac{\partial Q}{\partial \zeta_i}(\zeta) + \sum_{i=1}^{N} \frac{\partial^2 Q}{\partial \zeta_i\, \partial \zeta_j}(\zeta)(z_j - \zeta_j)\,.$$

The constant s_0 is chosen so that if

$$\tilde{g}(z, \zeta) = \sum_{i=1}^{N} (\zeta_i - z_i)\,\tilde{g}_i(z, \zeta)\,,$$

then for some $c'' > 0$,

$$\operatorname{Re} \tilde{g}(z, \zeta) \geqslant c'|z - \zeta|^2$$

if $\zeta \in \partial D$, $z \in \bar{D}$ and $|z - \zeta| < 2s_0$. It follows then that for some $c > 0$

(18) $$\operatorname{Re} g(z, \zeta) \geqslant c|z - \zeta|^2$$

if $\zeta \in \partial D$, $z \in \bar{D}$. Also, for $\zeta \in D$ and

(19) $$z \in \Omega_{\zeta, s_0} = \{z \in D\colon |z - \zeta| < \tfrac{1}{2}s_0\}\,,$$

we have

$$g(z, \zeta) = -\,2P(z, \zeta)\,,$$

P the Levi polynomial used in the last section. The functions $g(z, \zeta)$ and $\overline{g(\zeta, z)}$ are much alike in size near the diagonal of $D \times D$. If $z, \zeta \in \partial D$, then

(20) $$|g(z, \zeta) - \overline{g(\zeta, z)}| \leqslant \text{const } |z - \zeta|^3.$$

According to Kerzman and Stein, the Szegö kernel for the domain D is given as follows. For $z, \zeta \in \partial D$, put

(21) $$K(z, \zeta) = E(z, \zeta) - \overline{E(\zeta, z)}\,.$$

For a fixed integer d, for $\zeta \in \partial D$ and $z \in D$,

(22) $$S(z, \zeta) = E(z, \zeta) + \sum_{j=1}^{d} (-1)^j E_0 K^{(j)}(z, \zeta) + R_d(z, \zeta)$$

in which $R_d(z, \zeta) \in \mathcal{C}^\alpha(\bar{\Omega})$ in z for a fixed $\zeta \in \partial D$ with $\alpha = \alpha(d)$, $\alpha(d) \to \infty$ as $d \to \infty$, and the kernels $E_0 K^{(j)}$ are given by

$$E_0 K^{(j)}(z, \zeta) = \int\limits_{t_1 \in \partial D} \cdots \int\limits_{t_j \in \partial D} E(z, t_1)\, K(t_1, t_2) \ldots K(t_{j-1}, t_j)\, K(t_j, \zeta)\, dS(t_1) \ldots dS(t_j)$$

with dS the surface area measure on ∂D, *i.e.*, $dS = d\mu_{2N-1}$.

Thus, to analyze the integral (17) we must consider three kinds of terms:

$$F_{\mathrm{I}}(z) = \int_M E(z, \zeta)\, \psi(\zeta)\, d\mu_m(\zeta)$$

$$F_{\mathrm{II}}(z) = \int_M E_0 K^{(j)}(z, \zeta)\, \psi(\zeta)\, d\mu_m(\zeta)$$

$$F_{\mathrm{III}}(z) = \int_M R_d(z, \zeta)\, \psi(\zeta)\, d\mu_m(\zeta)\,.$$

This analysis yields the following result.

THEOREM II. *Let $M \subset \partial D$ be a locally closed submanifold of class $\mathcal{C}^k$, $k \geqslant 2$, dimension m, $1 \leqslant m \leqslant 2N-1$, that at each of its points is transverse to the holomorphic tangent space of ∂D, and let ψ be a compactly supported $\mathcal{C}^{k-1}$ function on M. If μ_m denotes m-dimensional Hausdorff measure on $\mathbf{C}^N$, and if $F \in \mathcal{O}(D)$ is defined by*

$$F(z) = \int_M S(z, \zeta)\, \psi(\zeta)\, d\mu_m(\zeta)\,,$$

then the derivatives of F of order α, $k - N - 1 < |\alpha| < k - 1$ belong to $H^p(D)$ for $p \in \big(0, N/(N + |\alpha| - k + 1)\big)$.

This result corresponds to the case $\sigma = N$ of Theorem I which is to be expected on the basis of the representation (22) for S.

PROOF. We shall not execute the proof in detail; to do so would merely be to repeat much of the proof of Theorem I.

The first point to be made is that as in the preceeding section, it is sufficient to treat the case that M is a curve, say Γ; the case of higher dimensional M reduces to this as before. Thus, we fix a $\mathcal{C}^k$ parameterization $\gamma\colon [0, 1] \to \partial D$ of Γ; $\gamma'(t)$ is transverse to the holomorphic tangent directions of ∂D at $\gamma(t)$.

Also, by using a smooth partition of unity, we can suppose that the diameter of γ is small, say diam $\Gamma < \frac{1}{8} s_0$.

Thus, if

$$\begin{aligned}\Omega_{\Gamma,\delta} &= \bigcup \{\Omega_{\zeta,\delta}\colon \zeta \in \Gamma\} \\ &= \{z \in D\colon \operatorname{dist}(z, \Gamma) < \delta\}\,,\end{aligned}$$

then for $\zeta \in \Gamma$, $z \in \Omega_{\Gamma,\frac{1}{8}s_0}$, $g(z, \zeta) = -2P(z, \zeta)$.

We can now dispatch the integral F_{I}. The function F_{I} is smooth on $\bar{D}\setminus\Gamma$ but not holomorphic on all of D, though for fixed $\zeta \in \Gamma$, $E(\cdot, \zeta)$ is holomorphic on $\Omega_{\Gamma,\frac{1}{8}s_0}$. To determine the behavior of F_{I} and its derivatives, we need only examine the behavior near Γ itself. However, as noted in the last paragraph, for z near Γ, the kernel of the integral defining $F(z)$ has the same singularity as the one we handled in analyzing the Henkin-Ramírez integral. Thus, F_{I} behaves as the theorem asserts.

We shall see that F_{II} and F_{III} behave better.

Consider F_{II}. If we set

$$G(\tau) = \int_M K(\tau, \zeta)\ \psi(\zeta)\ d\mu_1(\zeta)\,,$$

then for $z \in D$,

$$F_{\mathrm{II}}(z) = \int_{t_1 \in \partial D} \cdots \int_{t_j \in \partial D} E(z, t_1)\, K(t_1, t_2) \ldots K(t_{j-1}, t_j)\, G(t_j)\, dS(t_1) \ldots dS(t_j)\,.$$

The $t_1, \ldots, t_j$- and the ζ-integrations can be interchanged because Kerzman and Stein have shown

$$\int_{\partial D} |K(t_1, t_2)|\ dS(t_1) \leqslant C \tag{23}$$

and

$$\int_{\partial D} |K(t_1, t_2)|\, dS(t_2) \leqslant C \tag{23'}$$

for some constant C independent of t_1 and t_2.

The definition shows that $G(\tau)$ is defined for all $\tau \in \partial D\setminus\Gamma$. If we write

$$G(\tau) = G'(\tau) + G''(\tau)$$

with

$$G'(\tau) = \int_\Gamma E(\tau, \zeta)\, \psi(\zeta)\, d\mu_1(\zeta)\,,$$

and

$$G'(\tau) = \int_\Gamma \overline{E(\zeta, \tau)}\, \psi(\zeta)\, d\mu_1(\zeta)\,,$$

then the function G' is defined and smooth on all of $\bar{D}\setminus\Gamma$. Near the diagonal of $\partial D \times \partial D$, the singularity of E is essentially $1/P^N$, so the analysis used to treat the Henkin-Ramírez kernel applies. The derivatives $D_\alpha F_{\mathrm{II}}$ are smooth on $\bar{D}\setminus\Gamma$, and they are in the appropriate H^p class near Γ. Thus, the boundary values of G' are in $L^p(\partial D)$ for $p \in \big(0, N/(N + |\alpha| - k + 1)\big)$ if $|\alpha|$ lies in the range $k - N - 1 < |\alpha| < k - 1$.

To treat G'', we notice that the equation (20) implies that

$$\frac{d}{dt}\overline{g(\gamma(t),\tau)}\bigg|_{\tau=\gamma(t)}=\frac{d}{dt}g(\tau,\gamma(t))\bigg|_{\tau=\gamma(t)}. \tag{24}$$

If now $X_1, \ldots, X_r$ are smooth vector fields on ∂D and we put

$$D_{X_1\ldots X_r}G''=X_1\ldots X_r G'',$$

which is surely defined on $\partial D\setminus\Gamma$, then for $\tau\in\partial D\setminus\Gamma$, we have

$$D_{X_1\ldots X_r}G''(\tau)=\int_0^1\Theta_1(t,\tau)\,\psi(\gamma(t))\overline{[g(\gamma(t),\tau)]}^{-N-r}|\gamma'(t)|\,dt$$

for a suitable smooth function Θ_1. By virtue of (24), we can integrate by parts, just as in the Henkin-Ramírez case, provided we have made Γ short enough. The process will be terminated here just as in the earlier case. If we take $r\leqslant k-1$, we find

$$D_{X_1\ldots X_r}G''(\tau)=\int_0^1\varphi(t,\tau)\overline{[g(\gamma(t),\tau)]}^{-N+r-1}\,dt$$

for a continuous function φ — φ is continuous in τ so long as τ is close to Γ. According to (20) we may write

$$\overline{g(\zeta,\tau)}=g(\tau,\zeta)+u(\tau,\zeta)\,,$$

$|u(\tau,\zeta)|\leqslant\text{const}\,|\tau-\zeta|^3$, so we have then

$$\overline{[g(\zeta,\tau)]^{-1}}=[g(\tau,\zeta)]^{-1}\big(1+u(\tau,\zeta)\,g(\tau,\zeta)^{-1}\big)\,.$$

Since $\operatorname{Re}g(\tau,\zeta)\geqslant\text{const}\,|\tau-\zeta|^2$, we get, again provided Γ is short enough and τ is close enough to Γ,

$$\overline{[g(\zeta,\tau)]}^{-N+r-1}=[g(\tau,\zeta)]^{-N+r-1}q(\tau,\zeta)$$

with q a continuous function—the term q is at least continuous, but it is not clear how smooth it is. In any event, we reach

$$D_{X_1\ldots X_r}G''(\tau)=\int_0^1\frac{\varphi(t,\tau)\,q(\tau,\gamma(t))}{[g(\tau,\gamma(t))]^{N-r+1}}\,dt$$

for τ near Γ. This is an integral of the type we dealt with earlier; the conclusion is that provided $k - N - 1 < r < k - 1$, $D_{X_1 \dots X_r} G''$ has values on ∂D belonging to $L_p(\partial D)$, p in the asserted range.

We now know that both G' and G'' have the kind of boundary behavior we claim for F. Since, as Kerzman and Stein show, the kernel K is a smoothing kernel, a kernel of type 1 in their terminology, and since E is a kernel of type 0, the definition of F_{II} implies that the boundary behavior of F_{II} is even better than that claimed for F.

It remains for us to discuss the integral F_{III}. This follows the same lines as the treatment of F_{II}. We have to recall the form of the remainder term $R_d(z, \zeta)$. By construction it is, except for a term that is of class $\mathcal{C}^\infty$ on $\partial D \times \bar{D}$,

$$R_d(z, \zeta) = \int_{w \in \partial D} S(z, w) \left\{ \int_{\partial D} \dots \int_{\partial D} A(w, t_1) A(t_1, t_2) \dots A(t_d, \zeta) \, dS(t_1) \dots dS(t_d) \right\} dS(w) .$$

Here the kernel A is given by

$$A(\xi, \eta) = \overline{E(\eta, \xi)} - E(\xi, \eta) + \overline{C(\xi, \eta)} - C(\eta, \xi) ,$$

the function C of class $\mathcal{C}^\infty$ on $\bar{D} \times \partial D$, and $S(z, w)$ is the Szegö kernel itself. This leads to

$$\int_\Gamma R_d(z, \zeta) \, d\mu_1(\zeta) = \int_{w \in \partial D} S(z, w) \, G(w) \, dS(w)$$

with

$$G(w) = \int_{\partial D} \dots \int_{\partial D} A(w, t_1) A(t_1, t_2) \dots A(t_{d-1}, t_d) \left\{ \int_\Gamma A(t_d, \zeta) \, d\mu_1(\zeta) \right\} dS(t_1) \dots dS(t_d) .$$

As in our discussion of F_{II}, this function G is smooth on $\bar{D}$, and becomes progressively smoother as k increases. It follows that by making d large, we can make F_{III} as smooth as we wish.

The theorem is proved.

4. – An example.

We have considered integrals of smooth functions over smooth manifolds, and our work has used this smoothness in an essential way, in our repeated partial integration. We give here a simple example to illustrate what can happen when we relax the smoothness conditions.

Let Γ be the circle in ∂B_2, B_2 the open unit ball in $\boldsymbol{C}^2$, given by

$$\gamma(t) = (e^{it}, 0) \qquad -\pi \leqslant t \leqslant \pi .$$

We have $\langle \gamma(t), \gamma'(t) \rangle = -i$, so $\gamma'(t)$ is transverse to $T^{\boldsymbol{C}}_{\gamma(t)}(\partial B_2)$ for each t. Consequently, if $f \in \mathcal{C}^{k-1}(\Gamma)$, and

$$(25) \qquad F(z) = \int_{\Gamma} \frac{f(\zeta)}{(1 - \langle z, \zeta \rangle)^2} \, d\mu_1(\zeta) = \int_0^{2\pi} \frac{f(e^{it})\, dt}{(1 - z_1 e^{-it})^2} ,$$

then $\partial^r F / \partial z^r \in H^p(B_2)$ for $p \in (0, 2)$ if $r = k - 2$.

It is not unreasonable to ask whether, granted mere continuity on f, the function F enjoys any unanticipated smoothness properties brought about by the special geometry of Γ. The answer seems to be that it does not.

Recall that for $g \in \mathcal{C}(\partial B_2)$,

$$\int_{\partial B_2} g(z)\, dS(z) = \int_{|\zeta|<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\zeta, \sqrt{1 - |\zeta|^2}\, e^{i\theta})\, d\theta\, d\lambda(\zeta) ,$$

λ the Lebesgue measure on $\boldsymbol{C}$. It follows that $F \in H^1(B_2)$ if and only if

$$\int_{|z_1|<1} |F(z_1, 0)|\, d\lambda(z_1) < \infty .$$

We have

$$F(z_1, z_2) = \frac{1}{i} \int_{|\zeta|=1} \frac{f(\zeta)\, \zeta\, d\zeta}{(\zeta - z_1)^2} = \frac{1}{i} \frac{d}{dz_1} \int_{|\zeta|=1} \frac{f(\zeta)\, \zeta\, d\zeta}{\zeta - z_1} .$$

If f belongs to the disc algebra, *i.e.*, f is continuous on the closed unit disc, holomorphic on its interior, then we find

$$F(z_1, z_2) = 2\pi[f(z_1) + z_1 f'(z_1)] ,$$

so $F \in H^1(B_2)$ if and only if

$$\int_{|\zeta|<1} |f'(\zeta)|\, d\lambda(\zeta) < \infty .$$

It is known however [9] that there exist functions g in the disc algebra for which

$$V(\theta) = \int_0^1 |g'(re^{i\theta})|\, dr = \infty$$

for almost all values of θ.

Thus, *the integral* (25) *need not belong to* $H_1(B_2)$, *even for* f *the boundary value of function in the disc algebra.*

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