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ROBERTO PIGNONI

Density and stability of Morse functions on a stratified space

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Density and Stability of Morse Functions on a Stratified Space (*).

ROBERTO PIGNONI (**)

0. – Introduction.

This article comes out of the preparatory work needed in order to extend to the case of stratified sets (in praticular, of semianalytic sets) the methods of Morse theory over singular spaces. These methods have been outlined by Lazzeri in [7]. The first step towards this generalization of Morse theory consists in giving a qualitative description of a family of generic functions over a stratified set.

Let X be a closed subset of $\mathbb{R}^n$. A function $f: X \to \mathbb{R}$ is said to be k-differentiable if it is the restriction to X of a k-differenziable function over the whole of $\mathbb{R}^n$. By $C^k(X, \mathbb{R})$ we shall mean the space of k-differentiable functions on X, endowed with the Whitney topology. It is a Baire space [6]. With $C_0^k(X, \mathbb{R})$ we shall label the open subspace of proper functions in $C^k(X, \mathbb{R})$.

We shall say that a function f in $C_0^k(X, \mathbb{R})$ is *stable* (*i.e.* topologically stable) when there is an open neighbourhood N of f in $C_0^k(X, \mathbb{R})$ such that $\forall g \in N$ one can find homeomorphisms $h_1: X \to X$ and $h_0: \mathbb{R} \to \mathbb{R}$ which make the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & \mathbb{R} \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} \\ X & \xrightarrow{\;g\;} & \mathbb{R} \end{array}$$

Whenever this happens, it is said that f and g are « topologically equivalent » or have the same « topological type ».

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(**) Istituto Matematico « L. Tonelli », Pisa.
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The problem arises whether we may find a class of functions which are an open dense set in $C_0^k(X, \mathbb{R})$, $3 \leqslant k \leqslant \infty$, and are stable. In this paper we treat the case which occurs when X is a stratified set. We give sufficient conditions on the stratification of X for the existence of a class of functions with the above properties, which can be characterized as the « Morse functions » [1] on the stratification of X. We emphasize (see part 3) that the property of being a Morse function depends strictly on the particular stratification that has been associated to X.

Theorem 1 shows that density of Morse functions is verified on a stratified space with *strongly analytic* strata (an analytic submanifold of $\mathbb{R}^n$ is said to be strongly analytic if it is also a semi-analytic subset of $\mathbb{R}^n$).

Theorem 2 proves the stability of proper Morse functions with distinct critical values on a stratified space, when Whitney's conditions *a* and *b* [13] are fulfilled.

The meaning of the hypothesis in Theorems 1 and 2 is illustrated by supplying counterexamples in both cases, for more general stratifications.

We shall now state a remarkable consequence of these results.

Let X be a closed semi-analytic subset of $\mathbb{R}^n$. Łojasiewicz [8] has shown that in this case we have a stratification of X with strongly analytic strata, satisfying Whitney's conditions *a* and *b*. The stratification introduced by Łojasiewicz is canonically associated to the space, since it is « minimal » among all possible stratifications with the same properties [10].

In this way we can characterize a generic set of stable functions in $C_0^k(X, \mathbb{R})$, $3 \leqslant k \leqslant \infty$, by fixing this stratification of X and taking the family of Morse functions over it which have distinct critical values.

1. – Some results on C-analytic sets.

Let $\mathcal{A}_{\mathbb{R}^n}(\mathcal{H}_{\mathbb{C}^n})$ be the sheaf of germs of analytic (holomorphic) functions in $\mathbb{R}^n(\mathbb{C}^n)$. Let A be a set contained in a domain D of $\mathbb{R}^n(\mathbb{C}^n)$: $\mathcal{A}_D(\mathcal{H}_D)$ shall be the restriction of $\mathcal{A}_{\mathbb{R}^n}(\mathcal{H}_{\mathbb{C}^n})$ to D and $\mathcal{A}_A(\mathcal{H}_A)$ its restriction to A.

We shall employ the usual notation $(X, \mathcal{O}_X)$ for an analytic space. We recall that a *smooth* point of dim k of a real analytic space is one for which there is a nbd. isomorphic to the local model $(U, \mathcal{A}_U)$ with U open in $\mathbb{R}^k$; while a *regular* point of dim k for an analytic set $A \subset D$ will be a point in which A is locally a k-dimensional analytic submanifold of D.

Let D be a domain in $\mathbb{R}^n$. As in [3], we shall call *C-analytic* a real analytic set $A \subset D$, whenever the following (equivalent) conditions hold:

1) there is a complex analytic subset B of a complex nbd. of D in $\mathbb{C}^n$, such that $B \cap D = A$.

2) A is the locus of zeros of a coherent sheaf of ideals in $\mathcal{A}_D$.

3) A is the locus of zeros of a finite set $\{g\} = (g_1, \dots, g_s)$ of real analytic functions defined on all D.

Let (B, Ω) be a couple made of an open complex nbd. Ω of D in $\mathbf{C}^n$ and a complex analytic set B in Ω; then one defines a germ $\{Y\}$ over D, which is the (global) germ of complex analytic set induced by B over D. We say that (B, Ω) is a representative of the germ $\{Y\}$. It has been shown [3] that if A is C-analytic, there is *one smallest germ* $\{\hat{Y}\}$ of complex analytic set over D, such that, for any representative (A^*, D^*) of this germ, one has $A^* \cap D = A$.

By *complexification* of A we shall mean one of the representative couples of the germ $\{\hat{Y}\}$, indicated by (A^*, D^*).

Take a complexification (A^*, D^*) of the C-analytic set A. Let I^* be the sheaf of ideals, in $\mathcal{H}_{D^*}$, given in each $p \in D^*$ by the germs of functions vanishing over the germ of A^* in p. I^* is a coherent sheaf over D^*. Let $I^*_{|D}$ be the restriction to D of I^*. Take germs $(\boldsymbol{f}_1, \dots, \boldsymbol{f}_r)$ which generate $I^*_{|D}$ at $a \in A$.

$$(1) \qquad (\mathrm{Re}\boldsymbol{f}_{1|\mathbf{R}^n}, \dots, \mathrm{Re}\boldsymbol{f}_{r|\mathbf{R}^n}, \mathrm{Im}\boldsymbol{f}_{1|\mathbf{R}^n}, \dots, \mathrm{Im}\boldsymbol{f}_{r|\mathbf{R}^n})$$

are germs of real analytic functions in a whose holomorphic extensions are still vanishing on A^* in a complex nbd. of a (for the minimality of the germ induced by A^* over A).

PROPOSITION 1. *$I^*_{|D}$ is the tensor product with C (over $\mathbf{R}$) of a real sheaf of ideals $\hat{I}$ in $\mathcal{A}_D$. Moreover, $\hat{I}$ is the largest coherent sheaf of ideals in $\mathcal{A}_D$ that defines A as its set of zeros.*

PROOF. It is not hard to show that if $\hat{I}_a$ is the ideal generated by the (1), then $\bigcup_{a \in A} \hat{I}_a$ is the restriction to A of a coherent sheaf of ideals $\hat{I}$ in $\mathcal{A}_D$, for which $I^*_{|D} = \hat{I} \otimes_{\mathbf{R}} C$.

Furthermore, one observes that if there was a coherent sheaf of ideals in $\mathcal{A}_D$ null over A and greater than $\hat{I}$, one could easily contradict the minimality of the germ $\{\hat{Y}\}$ induced by A^* ([4], prop. 15). □

A is defined as set of zeros of a family of analytic functions in D: $\{g\} = (g_1, \dots, g_s)$.

Let $\{g\}\mathcal{A}_A$ be a sheaf of ideals in $\mathcal{A}_A$, every ideal being generated by the germs induced by $(g_1, \dots, g_s)$ in the points of A. $\hat{I}(A) = \bigcup_{a \in A} \hat{I}_a$ is the restriction of $\hat{I}$ to A.

We turn our attention to the analytic subspace $(A, \mathcal{A}_A/\hat{I}(A))$ of the space $(A, \mathcal{A}_A/\{g\}\mathcal{A}_A)$.

We will say that $(A, \mathcal{A}_A/\hat{I}(A))$ is the *space associated to* A; $\hat{I}(A)$ shall be called the *sheaf of ideals of the space.*

Let $V_r A^*$ be the set of regular r-dimensional points of A^*; $V_r A^* \cap D = V_r A^* \cap A$ is a real analytic submanifold of D, as follows from:

LEMMA 1. *Let $\{f_i\}$ be real analytic functions in a nbd. of a point $b \in \mathbb{R}^n$, such that the equations $f_i(z) = 0$ define, in a complex nbd. of b, a complex analytic submanifold M of dim r: then $M \cap \mathbb{R}^n$ is a real r-dimensional analytic submanifold of dim r in a nbd. of b.*

PROOF. See [4]. □

From Lemma 1 we see that, if $a \in A^* \cap D$ is a regular point of A^*, $\hat{I}_a$ is the *ideal of the set* A, *i.e.* the ideal of all germs of analytic functions vanishing on the germ of A at a (for the present and some following remarks use [12], Prop. 1 and Prop. 4, Chap. V).

We have thus shown that, if a is a regular point of dim r of A^*, $a \in A = A^* \cap D$, then a is also a smooth point of dim r for the space $(A, \mathcal{A}_A/\hat{I}(A))$.

LEMMA 2. *The complexification of a germ of real analytic manifold in a point is a germ of complex analytic manifold.*

PROOF. Immediate from the definition of complexification of a germ of real analytic set in a point (see [12]). □

From Lemma 1 and Lemma 2 and the minimality of the (global) germ $\{\hat{Y}\}$ induced by A^* over A one has

PROPOSITION 2. $\dim A = \dim A^*$.

Let $\dim A = r$, and let $V_r A$ be the set of regular points of dim r of A. Let now $S^* = A^* - V_r A^*$. S^* is a complex analytic set of $\dim < r$ [3].

Set $S = S^* \cap D$: S is a C-analytic set of $\dim < r$, containing the singularities of $(A, \mathcal{A}_A/\hat{I}(A))$. $A = V_r A \cup S$, but it may happen that $V_r A \cap S \neq \emptyset$ [3].

It follows from Lemma 2 that $V_r A \cap S$ is exactly the set of regular points of dim r in A for which the germ of A^* is not a complexification. One verifies that the germ of A^* in $a \in V_r A \cap S$ is not irriducible and also that $V_r A \cap S$ has empty interior in $V_r A$.

The points of $V_r A \cap S$ cannot be smooth for the space $(A, \mathcal{A}_A/\hat{I}(A))$ since a set of germs $(f_1, \ldots, f_m)$ that generate the ideal $\hat{I}_a$ of the space in a point $a \in V_r A \cap S$, generate (with their holomorphic extensions) the ideal I^*_a of the set A^* in a, but are not sufficient to generate the ideal of the set A in a (Lemma 2). This shows

PROPOSITION 3. *The smooth points of maximal dimension of* $(A, \mathcal{A}_A/\hat{I}(A))$ *are precisely the regular points of* A^* *of the same dimension which lie over* A.

An immediate consequence of these considerations is the following assertion of existence of a smooth C-analytic filtration for A:

PROPOSITION 4. *Let* A *be a* C*-analytic set with the associated analytic space* $(A, \mathcal{A}_A/\hat{I}(A))$. *There exists a sequence of* C*-analytic subsets of* A, $\{A^i\}_{0\leqslant i\leqslant r}$ *with the following properties*:

1) $A^0 = A$ *and* $A^{i+1} \subset A^i$ *for all* $i \geqslant 0$;

2) $\dim A^i > \dim A^{i+1}$;

3) $A^i - A^{i+1}$ *is an open subspace of* $(A^i, \mathcal{A}_{A^i}/\hat{I}(A^i))$ *consisting of all smooth points of maximal dimension.*

2. – Limit planes of real strongly-analytic submanifolds.

If Σ_r is a differentiable r-dimensional submanifold of $\mathbb{R}^n$, we define $\tau(\Sigma_r)$: $\{(p, T) \in \mathbb{R}^n \times G^{n,r},\ p \in \Sigma_r,\ T$ is the element of $G^{n,r}$ corresponding to the tangent plane to Σ_r in $p\}$.

In the following pages, we shall be primarily concerned with the closure $\overline{\tau(\Sigma_r)}$ of this set in $\mathbb{R}^n \times G^{n,r}$.

An element $H \in G^{n,r}$ is called a *limit plane* of Σ_r in $p \in \partial\Sigma_r$ if $(p, H) \in \overline{\tau(\Sigma_r)}$. When Σ_r, Σ_s are differentiable submanifolds of $\mathbb{R}^n$ of dimension $s < r$, with $\Sigma_s \subset \partial\Sigma_r$ (we shall then write: $\Sigma_s < \Sigma_r$), set $\tau(\Sigma_r, \Sigma_s) = \{\Sigma_s \times G^{n,r} \cap \overline{\tau(\Sigma_r)}\}$.

DEFINITION. A subset E of an analytic manifold M is *semianalytic* if every point of M has a nbd. W such that

$$E \cap W = \bigcup_{i=1}^{t} \left\{ \bigcap_{j=1}^{s} [x \in W : h_{ij}(x) > 0] \cap [x \in W : \eta_{ij}(x) = 0] \right\}$$

with h_{ij}, η_{ij} analytic in W.

For the properties of semianalytic sets that shall be exploited hereafter the reader is referred to [8].

Now, let Σ_r be a strongly-analytic submanifold of $\mathbb{R}^n$ of $\dim r$. Take a point p in the closure of Σ_r. It is immediate that there exists a nbd. U

of p in which

$$\Sigma_U = \Sigma_r \cap U = \bigcup_{i=1}^{m} \left\{ \left[\bigcap_{j=1}^{k} (x \in U : g_{ij}(x) > 0) \right] \cap \left[\bigcap_{j=1}^{k} (x \in U : f_{ij}(x) = 0) \right] \right\} = \bigcup_{i=1}^{m} [A_i \cap V_i]$$

where f_{ij} and g_{ij} are analytic in U and so the A_i are C-analytic sets and the V_i open sets in U defined by analytic inequalities. Set $A_U = \bigcup_{i=1}^{m} A_i$: A_U is a C-analytic set containing Σ_U.

LEMMA 3. *Each A_i can be chosen so that* $\dim A_i = \dim (A_i \cap V_i)$.

PROOF. It follows from the existence of a C-analytic filtration for A_i (Prop. 4). □

$\dim A_U$ is equal to $\max\{\dim A_i\}$. From now on we suppose all the sets A_i are taken as in Lemma 3.

LEMMA 4. $\dim A_U = r$.

PROOF. Let $p \in \Sigma_U$. The germ of A_U in p contains the germ of Σ_U at p, and the latest is the germ of an r-dimensional analytic manifold. From this easily follows that $\dim A_U \geqslant r$.

But $\dim A_U \leqslant r$, too. Since otherwise $\dim A_{i_0} \cap V_{i_0} > r$ for some i_0 and then the germ of Σ_U at some point $q \in A_{i_0} \cap V_{i_0}$ would contain a germ of analytic manifold of $\dim > r$. □

To A_U is associated the space $(A_U, \mathcal{A}_{A_U}/\hat{I}(A_U))$. Let $S(A_U)$ be the subset of points of A_U which are not smooth points of $\dim r$ for this space.

$S(A_U)$ is a C-analytic set in U of $\dim < r$ (Prop. 3 and 4). The semi-analytic subset of U given by $S_U = S(A_U) \cap \Sigma_U$ has empty interior in Σ_U (since otherwise $\dim S(A_U)$ would be r). If $a \in A_U$ recall that we have set $\hat{I}_a$ = ideal of $\hat{I}(A_U)$ in a.

Now we are ready to take the main step towards the proof of the density theorem: let Σ_r be a strongly-analytic submanifold of $\mathbb{R}^n$.

PROPOSITION 5. *a)* $\overline{\tau(\Sigma_r)}$ *is an r-dimensional semi-analytic subset of* $\mathbb{R}^n \times G^{n,r}$;

b) if Σ_s is a strongly-analytic submanifold of $\mathbb{R}^n$ *of* $\dim s < r$, $\Sigma_s < \Sigma_r$, *then $\tau(\Sigma_r, \Sigma_s)$ is a semi-analytic subset of* $\dim < r$ *in* $\mathbb{R}^n \times G^{n,r}$.

PROOF. Take a point p in the closure of Σ_r, and U, A_U, S_U as before.

$\Sigma_U - S_U$ is semi-analytic in U; that is,

$$(1) \qquad \Sigma_U - S_U = \bigcup_{i=1}^{h} \bigcap_{j=1}^{l} [x \in U : \varphi_{ij}(x) = 0] \cap [x \in U : \psi_{ij}(x) > 0]$$

with φ_{ij}, ψ_{ij} analytic in U.

All the points of $\Sigma_U - S_U$ are smooth for $(A_U, \mathcal{A}_{A_U}/\hat{I}(A_U))$: in these points $\hat{I}(A_U)$ is the sheaf of germs of analytic functions vanishing on Σ_U.

Now, take a set of generators: $\{f\} = \{f_1, \dots, f_s\}$ for $\hat{I}_p$. By coherence, we may take an open nbd. $V \subset U$ of p in which sections $(f_1, \dots, f_s)$ associated to the germs $\{f\}$ still generate the ideals of the sheaf $\hat{I}(A_U)$ over $\Sigma_V = \Sigma_U \cap V$.

If $q \in V$, given $\lambda = (\lambda_1, \dots, \lambda_{n-r})$, $1 \leqslant \lambda_1 < \dots < \lambda_{n-r} \leqslant s$ and $\nu = (\nu_1, \dots, \nu_{n-r})$, $1 \leqslant \nu_1 < \dots < \nu_{n-r} \leqslant n$, let $D_{\lambda\nu}(q)$ be the determinant of the matrix

$$\left. \frac{\partial(f_{\lambda_1}, \dots, f_{\lambda_{n-r}})}{\partial(x_{\nu_1}, \dots, x_{\nu_{n-r}})} \right|_q .$$

When $q \in \Sigma_V - S_V = V \cap (\Sigma_U - S_U)$, some $D_{\lambda\nu}(q)$ must be $\neq 0$ (q is smooth). Take $\mu = (\mu_1, \dots, \mu_{n-r+1})$, $1 \leqslant \mu_1 < \dots < \mu_{n-r+1} \leqslant n$.

With $\mu^{(i)}$ we shall label the $(n-r)$-vector that is obtained omitting the component μ_i in μ. Now, $\forall \mu$ and $\forall \lambda$ let's define the vector

$$\boldsymbol{v}_{\lambda\mu}(q) = (v^1_{\lambda\mu}, \dots, v^n_{\lambda\mu})$$

$$v^j_{\lambda\mu} = \begin{cases} \text{if} \quad j \in \mu, \quad j = \mu_i, \quad v^j_{\lambda\mu} = (-1)^{i-1} D_{\lambda\mu^{(i)}}(q) \\ \text{if} \quad j \notin \mu, \quad v^j_{\lambda\mu} = 0 . \end{cases}$$

One verifies [13] that the finite set of real analytic vector fields defined in all V that are obtained in this way, are such that $\forall \lambda, \mu$, $\boldsymbol{v}_{\lambda\mu}(q)$ lies on the tangent plane to the analytic manifold $\Sigma_V - S_V$ in q. Furthermore, the set of all $\boldsymbol{v}_{\lambda\mu}(q)$ span the plane (so we see that $\tau(\Sigma_V - S_V)$ is an analytic submanifold of dim r in $\mathbb{R}^n \times G^{n,r}$).

Now, let $\{\alpha\} = \{\alpha^\sigma, \ \sigma = (\sigma_1, \dots, \sigma_r), \ 1 \leqslant \sigma_1 < \dots < \sigma_r \leqslant n\}$ be the homogeneous coordinates of an r-plane T (consider the usual embedding of $G^{n,r}$ as an analytic submanifold of a real projective space).

A vector $\boldsymbol{w}$ lies on the plane T if and only if the exterior product

$$\boldsymbol{w} \vee T = 0, \text{ that is, } \forall \gamma = (\gamma_1, \dots, \gamma_{r+1}), \ 1 \leqslant \gamma_1 < \dots < \gamma_{r+1} \leqslant n$$

$$(\boldsymbol{w} \vee T)^\gamma = \sum_{k=1}^{r+1} (-1)^{k-1} w^{\gamma_k} \cdot \alpha^{\gamma^{(k)}} = 0 .$$

In this way we see that a set of homogeneous coordinates $\{\alpha\}$ shall belong to a plane T on which $\boldsymbol{v}_{\lambda\mu}(q)$ lies if and only if

$$(2) \qquad \boldsymbol{v}_{\lambda\mu}(q) \vee \{\alpha\} = 0 \qquad \text{that is,} \qquad \forall\gamma,\ \sum_{k=1}^{\mu+1}(-1)^{k-1} v_{\lambda\mu}^{\gamma_k}(q)\cdot\alpha^{\gamma^{(k)}} = 0$$

which means that we have $\binom{n}{r+1}$ equations in $\{\alpha\}$ for each $\boldsymbol{v}_{\lambda\mu}(q)$.

As a consequence of this, if $(x, \alpha) \in V \times G^{n,r}$, from (1) and (2) we have

$$\tau(\Sigma_V - S_V) = \bigcup_{i=1}^{h}\bigcap_{j=1}^{l}[(x, \alpha)\colon \psi_{ij}(x) > 0] \cap [(x, \alpha)\colon \varphi_{ij}(x) = 0] \cap \\ \cap [(x, \alpha)\colon \boldsymbol{v}_{\lambda\mu}(x) \vee \{\alpha\} = 0, \forall\lambda, \mu]$$

and this expression shows that $\tau(\Sigma_V - S_V)$ is a semi-analytic subset in $V \times G^{n,r}$ of $\dim r$.

The closure of this set in $V \times G^{n,r}$ is semi-analytic of $\dim r$ ([8], Prop. 1, p. 76) and coincides with the closure of $\tau(\Sigma_V)$ in $V \times G^{n,r}$. The proof of $a)$ is complete: a subset E of a manifold M is semi-analytic if and only if for each point y of M one finds a nbd. W of y in M for which $W \cap E$ is semi-analytic in W.

To prove $b)$, it suffices to take the points x which belong to the border of Σ_r: we need to verify that in a nbd. of x the limit planes of Σ_r induce a semianalytic set of $\dim < r$ in $\mathbb{R}^n \times G^{n,r}$. We take $V \subset U$, with $x \in V$, as before. By intersecting the closure of $\tau(\Sigma_V - S_V)$ in $V \times G^{n,r}$ with $(\Sigma_s \cap V) \times G^{n,r}$ we get a semi-analytic subset in $V \times G^{n,r}$ of $\dim < r$ ([8] Prop. 5, p. 82). This subset is $\tau(\Sigma_r, \Sigma_s) \cap V \times G^{n,r}$. □

3. – Density theorem.

A *stratification* $\Sigma = \{\Sigma_i^j\}$ of a set $X \subset \mathbb{R}^n$ is a partition of X into a locally finite family of differentiable submanifolds of $\mathbb{R}^n$, the *strata* $\Sigma_i^j (i = \dim \Sigma_i^j)$.

Moreover, it is required that the frontier condition holds. We shall say that X is a *stratified space* if to the set $X \subset \mathbb{R}^n$ is associated a well determined stratification Σ. Sometimes, for clearness, we shall write $\{X, \Sigma\}$ instead of X.

A *Morse function* over a stratified set X (that is: $\{X, \Sigma\}$) is a C^k function $f\colon X \to \mathbb{R}$, $2 \leqslant k \leqslant \infty$, such that:

1) $f_{|\Sigma_i^j}$ has no degenerate critical points, $\forall i, j$ with $i > 0$;

2) if $\Sigma_s < \Sigma_r$ (we shall drop the upper index j from now on), $\forall (p, H) \in \tau(\Sigma_r, \Sigma_s)$ the linear mapping $(Df)_p: \mathbb{R}^n \to \mathbb{R}$ does not vanish on H.

These conditions imply that the set of 0-strata of X and of critical points of f is discrete. As we have already pointed out, the concept of Morse function is strictly related to the particular stratification that has been fixed for X.

THEOREM 1. *Let X be a stratified space in $\mathbb{R}^n$ with strongly-analytic strata. Then, Morse functions over this stratification are dense in $C^k(X, \mathbb{R})$, $2 \leqslant k \leqslant \infty$.*

EXAMPLE. The conclusion of the theorem is false if we drop the hypothesis that the strata are semi-analytic. Take the space $X \subset \mathbb{R}^2$, made of two strata: $\Sigma_0 = \{0\}$ and $\Sigma_1 = \{$the spiral defined in polar coordinates by the equation $r = e^{-\theta^2}\}$. One verifies the following facts [5]:

A) the stratification $\Sigma = \{\Sigma_0, \Sigma_1\}$ of X satisfies Whitney's conditions *a* and *b*;

B) Σ_1 is not a semi-analytic subset of $\mathbb{R}^2$.

One sees that there are no Morse functions on this stratification: the limit planes of Σ_1 in $\{0\}$ represent all directions in $\mathbb{R}^2$.

PROOF OF THEOREM 1:

I) We shall first prove an assertion of existence of Morse functions over X.

Let f be a C^k function over X $(2 \leqslant k \leqslant \infty)$.

We embed X in $\mathbb{R}^{n+1}$ with the mapping $h: \mathbb{R}^n \to \mathbb{R}^{n+1}$, $h(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n))$. $h(X)$ is a stratified space, whose strata are C^k-diffeomorphic to the analytic strata of X.

PROPOSITION 6. *The set of points p in $\mathbb{R}^{n+1}$ for which the function $L_p: \mathbb{R}^n \to \mathbb{R}$, $L_p(x) = \|p - h(x)\|^2$ (for the euclidean norm), is a Morse function over X, is dense and residual in $\mathbb{R}^{n+1}$.*

Proposition 6 is established by means of the following lemmas:

LEMMA 5. *Let $C_1 \subset \mathbb{R}^{n+1}$ be the set of points p which are focal points of some stratum of $h(X)$. Then C_1 has Lebesgue measure 0 in $\mathbb{R}^{n+1}$.*

PROOF. See [11]. □

LEMMA 6. *Let $C_2 \subset \mathbb{R}^{n+1}$ be the set of points p for which L_p has differential $(DL_p)_q$ vanishing over some limit plane of some stratum Σ_r of X, in some point $q \in \partial\Sigma_r$. Then C_2 has Lebesgue measure zero in $\mathbb{R}^{n+1}$.*

PROOF. Take any two strata Σ_r and Σ_s with $\Sigma_s \subset \partial\Sigma_r$.

The main fact we use is Prop. 5; $\tau(\Sigma_r, \Sigma_s)$ is a semi-analytic set of dim $< r$ and so it can be expressed as $\bigcup_i S_i$ with S_i analytic submanifolds of $\mathbb{R}^n \times G^{n,r}$ of $\dim l_i < r$. Then we may proceed as in the proof of th. 3 in [1]. □

If $p \notin C_1 \cup C_2$, L_p is a Morse function over X and so Prop. 6 is proved.

II) Now we shall prove density. Let K be a compact subset of $\mathbb{R}^n$, and $A(K)$ the set of functions φ in $C^k(X, \mathbb{R})$ for which one can find an open set $U \supset X \cap K$, such that φ is a Morse function over $X \cap U$.

PROPOSITION 7. *$A(K)$ is open and dense in $C^k(X, \mathbb{R})$, $k \geqslant 2$.*

PROOF. Openness results from the considerations of part 4 of this article. Let's see density.

Take any $f \in C^k(X, \mathbb{R})$, and define h, C_1, C_2 as before. Let χ be a C^∞ function with compact support, $\chi \colon \mathbb{R}^n \to [0, 1]$, $\chi = 1$ over a nbd. of K. Let $C \in \mathbb{R}$; take $p = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1} - C)$, $p \notin C_1 \cup C_2$. $g = (L_p - C^2)/2C$ is a Morse function over X, and by suitably choosing the ε_i, C, one has that the derivatives of $\psi = \chi \cdot (g - f)$ may be taken as small as needed up to any finite order s [11].

Now set $\varphi = f + \psi$; φ is a Morse function over an open nbd. of $X \cap K$; and is as close as needed to f. □

By taking a locally finite compact covering $\{K_i\}$ of X and observing that $C^k(X, \mathbb{R})$ is a Baire space, one achieves the proof of the theorem.

Using openness of Morse functions (see part 4) it's easy to show, with the aid of small perturbations near the critical points, the following

COROLLARY. *Morse functions with distinct critical values (i.e., $f(p_1) \neq f(p_2)$ if $p_1 \neq p_2$ are critical points or 0-strata) are dense in $C^k(X, \mathbb{R})$ and in $C_0^k(X, \mathbb{R})$, $2 \leqslant k \leqslant \infty$.*

4. – Stability of Morse functions.

Let $X \subset \mathbb{R}^n$ be any stratified space. Morse functions are an open set in $C^k(X, \mathbb{R})$, $2 \leqslant k \leqslant \infty$.

In fact we can show more than that: let f be a Morse function on X. Give a family $\{U_i\}_{i \in I}$ of nbd.s of the critical points p_i of f in the strata of X, with $\overline{U}_i \cap \overline{U}_j = \emptyset$ if $i \neq j$, each U_i being contained in some chart of the stratum. It is possible to find a convex nbd. N of f in $C^k(X, \mathbb{R})$ such that every $g \in N$ has one nondegenerate critical point q_i in each U_i and no other critical points, and g is a Morse function for X.

The proof of this involves nothing more than local finiteness of the strata and the usual methods for showing openness of sets of functions (see [6] Chap. 2, par. 1).

If f is a *proper* Morse function with *distinct critical values*, the nbd. N may be chosen in such a way that it is made of proper Morse functions with distinct critical values. Since N is convex, if $g \in N$, it is possible to fix $\varepsilon > 0$ so that the function $F_t(x) = tg + (1-t)f$, $t \in I = (-\varepsilon, 1+\varepsilon)$ belongs to N and so is a Morse function over X.

As t varies, the critical points of $F_t\colon X \to \mathbb{R}$ lie on connected differentiable curves $x_i(t)$ in $U_i \times I \subset X \times I$. Consider the mapping $\Phi\colon X\times I \to \mathbb{R}\times I$ defined by $\Phi(x,t) = (tg(x) + (1-t)f(x), t) = (F_t(x), t)$. The images $C_i(t) = \Phi(x_i(t))$ of the curves of critical points are disjoint differentiable curves in $\mathbb{R}\times I$.

From now on suppose the stratification of X satisfies Whitney's conditions a and b. In this case we shall show that f and g have the same topological type. In the next pages we will verify that the sequence

$$X\times I \xrightarrow{\Phi} \mathbb{R}\times I \xrightarrow{\pi} I$$

where π is the projection on I, is « *locally trivial* » [10]. This fact, along with the connectedness of I, shall give us an equivalence between f and g. We shall establish in this way the following

THEOREM 2. *Morse functions with distinct critical values are stable in* $C_0^k(X, \mathbb{R})$, $3 \leqslant k \leqslant \infty$, *whenever* X *is a stratified space for which hold Whitney's conditions* a *and* b.

EXAMPLE. If the stratification does not satisfy Whitney's conditions, stability of Morse functions may not occur. Take the « Cayley umbrella », $X = \{x^2 = zy^2 \subset C^3\}$.

It is a real analytic subset in $\mathbb{R}^6$. Stratify it with strata $\Sigma_2 = \{$the points of X which lie on the complex z-axis$\}$, $\Sigma_4 = X - \{$the complex z-axis$\}$.

This is a stratification with strongly analytic strata, since the points of Σ_4 are all the regular points of X. It is not a Whitney stratification though: condition a is not verified at $\{0\}$. For any sequence of points of Σ_4 contained in the complex y-axis the limit planes are normal to Σ_2 in $\{0\}$.

Let p be a point $\neq \{0\}$ belonging to the complex y-axis. It is readily checked that the function $l_p(x) = \|p - x\|^2$ is a Morse function over $\{X, \Sigma\}$; $l_{p|\Sigma_2}$ has a critical point (a minimum) at $\{0\}$.

Fix a nbd. N of $l_p(x)$ in $C_0^k(X, \mathbb{R})$. Take a function $g \in N$, and take it equal to $l_q(x)$ over the ball $B = \{\|x\| \leqslant 2\|p-q\|\}$, for some q close to p but not lying on the complex y-axis. By the preceding considerations, if N is narrow g is a Morse function with distinct critical values. $g_{|\Sigma_2}$ has a minimum near $\{0\}$; moreover, for $g_{|\Sigma_2}$,$\{0\}$ is a regular point. g and f are not topologically equivalent: it is not possible to find homeomorphisms $h_1: X \to X$ and $h_0: \mathbb{R} \to \mathbb{R}$ that give us the equivalence.

In fact, one should have $h_1(\Sigma_2) = \Sigma_2$ and $h_1(0) = 0$ (see Remark 1). And so, if h_1 and h_0 existed, $\{0\}$ should be a critical point for both functions ([11], th. 3.2 pag. 14).

REMARK 1. The number $\#$ of irreducible components of a germ (X, x_0) of a complex analytic set X at x_0 is a topological invariant.

$\#$ is equal to the number of connected components of the set of regular points of X: $\operatorname{Reg} X = \{X - \operatorname{Sing} X\}$, in a small nbd. U of x_0.

If $\operatorname{Reg}_T X$ is the set of points at which X is a topological manifold (and this set may not coincide with $\operatorname{Reg} X$, see [2]), we want to see that $\#$ is equal to the number $*$ of the connected components of $\operatorname{Reg}_T X$.

It could only happen that $\# > *$. We shall show that this is impossible by verifying that X is necessarily irreducible at any point $x \in \operatorname{Sing} X \cap \cap \operatorname{Reg}_T X$.

There will be a nbd. V of x in X, $V \subset \operatorname{Reg}_T X$. Suppose one could find two irriducible components of X at x, say X_1 and X_2, so that $M = \{X_1 - \operatorname{Sing} X\} \cup V \cup \{X_2 - \operatorname{Sing} X\}$ is a connected manifold of dim $2n$.

$V \cap \operatorname{Sing} X$ has topological dim $\leqslant 2n-2$ and it cannot disconnect M.

PROOF OF THEOREM 2. The main technical fact that is needed is the following result, known as Thom's second «Isotopy Lemma» [9], [10]:

LEMMA 7. *Let* $X_i \xrightarrow{f_i} \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{\pi} Y$ *be a sequence of* C^2 *manifolds and mappings; if* $0 \leqslant j \leqslant i$ *let* $A_j \subset X_j$ *be a closed (or locally closed) subset of* X_j *with a* C^2 *Whitney stratification* $\Sigma^j = \{\Sigma_s^k\}^j$, $0 \leqslant s \leqslant \dim \Sigma^j$.

Suppose $f_j(A_j) \subset A_{j-1}$, *and that*

1) *every stratum* Σ_s^k *of* A_j *is mapped submersively by* f_j *to a stratum of* A_{j-1};

2) *Thom's condition* a_{f_j} [10] *is satisfied for pairs of strata in* A_j, $j \geqslant 1$;

3) *every stratum of* A_0 *is mapped submersively by* π *to* Y;

4) $f_j: A_j \to A_{j-1}$ *is proper, and so is* $\pi: A_0 \to Y$. *Whenever all these conditions are fulfilled the sequence*

$$A_i \xrightarrow{f_i} \dots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

is «locally trivial» over Y *with respect to* π.

We are going to give a stratification of $X \times I = A_1 \subset X_1 = \mathbb{R}^n \times I$ and one of $\mathbb{R} \times I = A_0 = X_0$ in such a way that the resulting sequence of spaces and mappings satisfies the conditions of Lemma 7. First of all, we remark that in our case the condition that π be proper can be dropped. The conclusion of Lemma 7 will still be true, since the only purpose of this condition is to make sure that, when one constructs a suitable lifting (by π) over A_0 for the unit vector fields $\partial_1 \dots \partial_m$ ($m = \dim Y$) in a chart of a nbd. U of a point of Y, this lifting results globally integrable in $\pi^{-1}(U)$ [9], [10]. In the present case π is the projection from $\mathbb{R} \times I$ to I: it is not proper, but a globally integrable stratified lifting for the unit vector field over I is readily obtainable, as will be evident from the stratification we shall fix on $\mathbb{R} \times I$.

If the stratification of X is Whitney regular, then so is the product stratification $X \times I$. Let's form new strata in $X \times I$, by taking the curves $x_i(t)$ of the critical points of the functions $F_t(x)$ as t varies in I.

Analogously, we introduce new strata in $\mathbb{R} \times I$ by taking the segments $\Phi(\Sigma_0, t)$ (where Σ_0 is a 0-stratum of X), and the curves of critical values $C_i(t)$.

Since $x_i(t)$ and $C_i(t)$ are differentiable submanifolds of the strata of $X \times I$ and $\mathbb{R} \times I$, the new stratifications (say: $\{X \times I\}'$ and $\{\mathbb{R} \times I\}'$) shall be Whitney regular.

The new sequence we get, $\{X \times I\}' \xrightarrow{\Phi} \{\mathbb{R} \times I\}' \xrightarrow{\pi} I$, still does not verify the 1) of Lemma 7. So we must introduce again new strata by taking the intersections with $\{X \times I\}'$ of the inverse images $\Phi^{-1}(C_i(t))$ for each disjoint curve $C_i(t)$. Let $\{X \times I\}''$ be the stratification we obtain.

WHITNEY REGULARITY OF $\{X \times I\}''$: We distinguish two types of strata:

type A): in this case Σ is a stratum of $\{X \times I\}' - \Phi^{-1}\Big(\bigcup_i C_i(t)\Big)$;

type B): Σ is a stratum of $\{X \times I\}' \cap \Big\{\Phi^{-1}\Big(\bigcup_i C_i(t)\Big)\Big\}$.

I) Whitney's conditions hold for all couples of strata of type B) which are not curves $x_i(t)$, because they are obtained by transversal intersection of two Whitney stratifications:

$$\Big\{X \times I - \bigcup_i x_i(t)\Big\} \quad \text{and} \quad \Big\{\Phi^{-1}\Big(\bigcup_i C_i(t)\Big) - \bigcup_i x_i(t)\Big\},$$

in a nbd. W of $X \times I - \bigcup_i x_i(t)$ in $\mathbb{R}^n \times I - \bigcup_i x_i(t)$.

(Transversality results since $D\Phi$ has rank 2 over $\Big\{X \times I - \bigcup_i x_i(t)\Big\}$ and 1 on the tangent planes to the hypersurfaces $\Big\{\Phi^{-1}\Big(\bigcup_i C_i(t)\Big)\Big\} \cap W$).

When a stratum Σ_r of type B lies in a stratum Σ_{r+1} of $X \times I$, Whitney regularity of the couple $\Big\{\Sigma_r, \Sigma_{r+1} - \Phi^{-1}\Big(\bigcup_i C_i(t)\Big)\Big\}$ is obvious. The only thing that is left to verify is the incidence between a stratum of type B) and a curve $x_i(t)$ of critical points.

II) If a curve $x_i(t)$ is contained in a stratum $M\times I$ of $X\times I$, consider a stratum

$$\Sigma \in \{M\times I \cap \Phi^{-1}(C_i(t))\} - \{x_i(t)\}\,.$$

One sees that the couple $\{x_i(t), \Sigma\}$ verifies Whitney's conditions by means of the following Lemma, which shows that $\{M\times I\}\cap\{\Phi^{-1}(C_i(t))\}$ is locally diffeomorphic, in the points of $x_i(t)$, to the product of a cone with I, hence is Whitney regular.

LEMMA 8. *Let* $\dim M = r$, *with coordinates* $(x_1, \ldots, x_r, t)$ *defined in* $U_i\times I$. *Suppose* $\Phi\in C^k(X\times I, \mathbb{R}\times I)$, $3\leqslant k\leqslant\infty$. *Then there is a change of coordinates in* $\mathbb{R}\times I$, *and* $\forall t_0$ *one may find a change of coordinates* $(x_1, \ldots, x_r, t)\to$ $\to(u_1, \ldots, u_r, t)$ *in a nbd. of* $x_i\,(t_0)$ *in* $U_i\times I\subset M\times I$, *that express* $\Phi_{|M\times I}$ *in this nbd. in the form* $\Phi(u_1, \ldots, u_r, t) = (\pm\, u_1^2 \pm \ldots \pm\, u_r^2, t)$.

PROOF. One gives charts in $U_i\times I$ and $\mathbb{R}\times I$ that take $x_i(t)\to(0\in\mathbb{R}^r)\times I$ and $C_i(t)\to(0\in\mathbb{R})\times I$.

The rest of the proof is patterned on the usual proof of Morse's Lemma, adjusted to the case of the function $F\colon M\times I\to\mathbb{R}$, $F(x, t) = F_t(x)$, for which we now have $F(0\times I) = 0$. □

REMARK 2. Lemma 8 shows that if $3\leqslant k\leqslant\infty$ the critical points of F_t along one connected curve $x_i(t)$ have all the same index. Thus, $\forall g\in N$ the index of q_i coincides with that of the corresponding critical point p_i of f. Of course this does not depend on Whitney's conditions.

III) Now take two strata Y and Z in the stratification of X: $Y<Z$, $\dim Z = h$. Consider a curve $x_i(t)\subset Y\times I$. $x_i(t)\subset\Phi^{-1}(C_i(t))$, and the last is a n-submanifold in a nbd. of $X\times I$. Set $\Sigma = \Phi^{-1}(C_i(t))\cap Z\times I$.

We must show condition b for the incidence of a stratum of Σ with $x_i(t)$. Let $P_m\in\Sigma$ and $Q_m\in x_i(t)$ be two sequences converging to $Q\in x_i(t)$. Assume that $\lim\limits_{m\to\infty}\widehat{P_mQ_m} = l$ in $\mathbb{R}P^n$, and that the tangent planes T_m to Σ in P_m converge to a limit plane T. Condition b amounts to saying that $T\supset l$.

By passing to a subsequence if necessary, we may suppose the tangent planes τ_m to $Z\times I$ in P_m converge to a limit τ, and the n-dimensional tangent planes σ_m to $\Phi^{-1}(C_i(t))$ in P_m converge to a limit σ.

$\tau\supset l$ for $X\times I$ is a Whitney stratification; $\sigma\supset l$ since $\Phi^{-1}(C_i(t))$ is a regular hypersurface in a nbd. of $X\times I$. Thus $l\subset\tau\cap\sigma$.

What is left to show is that $\sigma\cap\tau$ coincides with $T = \lim T_m$. Let $\boldsymbol{v}(t)$ be the normal vector to the curve $C_i(t)$.

For any $P_m = (x_m, t_m) \in \Sigma$ we define the linear map

$$\lambda_{P_m}: \tau_m \to \mathbb{R}, \qquad \lambda_{P_m}(\boldsymbol{w}) = \langle \boldsymbol{v}(t_m), (D\Phi)(P_m)|_{\tau_m}\boldsymbol{w}\rangle$$

for any vector $\boldsymbol{w}$ in τ_m, where $\langle\, ,\rangle$ denotes the usual scalar product. Then $T_m = \ker \lambda_{P_m}$. But $(D\Phi)(Q)$ over τ has maximal rank, and by the continuity of $D\Phi$, we have that, given

$$\lambda_Q: \tau \to \mathbb{R}, \qquad \lambda_Q(\boldsymbol{w}) = \langle \boldsymbol{v}(t_0), (D\Phi)(Q)|_{\tau}\boldsymbol{w}\rangle,$$

$\ker \lambda_Q$ (that is, by definition, $\sigma \cap \tau$) $= \lim \ker \lambda_{P_m} = \lim T_m = T$.

THOM'S CONDITION a_Φ: Let us take two strata in $X \times I$, of the type $U = Y \times I$ and $V = Z \times I$ with $Y < Z$. We' re given a sequence $P_m = (x_m, t_m) \in V$ converging to $(y_0, t_0) \in U$, such that $\lim \ker (D\Phi|_V)(P_m) = \tau$.

We may suppose that the tangent planes T_m to V in (x_m, t_m) converge to a limit T, and we call T_Z the $\mathbb{R}^n$-component of this limit (that is, if π^* is the tangent mapping of the projection $\pi: \mathbb{R}^n \times I \to \mathbb{R}^n$, $T_Z = \lim \pi^*(T_m)$).

T_Z contains the tangent plane to Y at y_0.

$$\tau = \lim \ker (D\Phi|_V)(P_m) = \lim \ker (DF_{t_m}/Dx)(x_m)\,;$$

τ has codim 1 in T_Z.

Over τ, $(DF_{t_0}/Dx)(y_0)$ is 0 for continuity. Then $\ker (DF_{t_0}/Dx)(y_0) \subset \tau$ since otherwise $(DF_{t_0}/Dx)(y_0)$ would be null over all T_Z, contradicting the hypothesis that all F_t are Morse functions on X.

The proof that condition a_Φ is verified for all other types of strata in $\{X \times I\}''$ follows smoothly from analogous or simpler considerations.

By applying Lemma 8, Theorem 2 is now established.

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