ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 6, nº 3 (1979), p. 427-445

<http://www.numdam.org/item?id=ASNSP_1979_4_6_3_427_0>

A Uniqueness Theorem for Nonstationary Navier-Stokes Flow Past an Obstacle.

JOHN G. HEYWOOD (*)

1. – Introduction.

In this paper we consider the initial boundary value problem for the Navier-Stokes equations in the case of a three-dimensional exterior domain. Using results given in [11] and [17] [1] concerning the completions of certain classes of solenoidal functions, we show that the difference w of two solutions u and v must satisfy the identity

$$(1) \qquad \frac{1}{2}\frac{d}{dt}\|\nabla w\|^2 + \nu\|P\Delta w\|^2 = \int_\Omega (v\cdot\nabla v - u\cdot\nabla u)\cdot P\Delta w\,dx\,,$$

provided ∇u, ∇u_t, Δu and similar derivatives of v are square-summable over a space-time cylinder. Then, estimating the right-hand side of (1) and integrating with respect to t, we prove the continuous dependence of solutions on their initial values in the Dirichlet norm. Finally, estimates of the same type used to prove this result are shown to provide the basis for an existence theorem. In equation (1) above, P is a projection of $L^2(\Omega)$ onto a subspace of solenoidal functions; our notation is fully explained in section 2.

The results of this paper are obtained without hypotheses on the pressure functions corresponding to u and v, and without assuming that u and v differ from the prescribed velocity at infinity, or from each other, by only square-summable amounts. Thus, in contrast to the uniqueness theorem given in [11, p. 89], which is based on an energy identity, the continuous

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(1) I wish to thank Dr. Chun-Ming Ma for supplying the proof of Lemma 2; this important lemma is presented in his paper [17].

Pervenuto alla Redazione il 9 Maggio 1978.

dependence and uniqueness theorem given here applies to solutions which may possess non-finite wake energies. Although the present theorem requires more regularity of solutions than the theorem in [11], the regularity required is no more than what is known for solutions obtained by the method of Kiselev and Ladyzhenskaya [15, p. 167], or by the method given in the concluding section of this paper. Related uniqueness theorems have been given by Graffi [7], Cannon and Knightly [2], and Rionero and Galdi [20]. These theorems, in contrast to ours, require assumptions concerning the pressure as well as the velocity, but do not require the velocity to tend to a limit at infinity or to satisfy any global integrability conditions. Another uniqueness theorem which should be mentioned here is one given for the Cauchy problem by Cannon and Knightly [3] and Knightly [13]; this theorem is particularly remarkable for being based on an integral representation which yields asymptotic estimates for the solution.

In the final section of this paper we give a method of proving existence which, like the uniqueness theorem, is based on an estimate of the Dirichlet integral rather than the energy integral. In this respect, it is related to existence theorems for bounded domains of Prodi [19] and Shinbrot and Kaniel [21], where such estimates were first used. Even though our estimates for an exterior domain give only local existence (i.e., the solution is shown to exist for only a finite period of time), we have chosen to give them in the specific context of the starting problem (which concerns accelerating a body from rest to a constant terminal velocity). We think these estimates may be of use, eventually, in proving that a solution of the starting problem exists globally and converges to steady state. In any case, giving our estimates in this context will serve to illuminate some of the problem's inherent difficulties. It should be mentioned that a potential theoretic study of the starting problem has been recently initiated by Knightly [14].

Throughout this paper, Ω represents a spatial region filled with fluid and is taken to be an open connected set of R^3, with a bounded (possibly empty) complement Ω^c and a twice continuously differentiable (possibly empty) boundary $\partial\Omega$. Spatial points are denoted by $x = (x_1, x_2, x_3)$ and the time variable by t. Space-time cylinders $\Omega \times (\varepsilon, T)$, with $0 \leqslant \varepsilon < T$, will be denoted by $Q_{\varepsilon,T}$ or simply by Q_T if $\varepsilon = 0$. The initial boundary value problem to be considered is that of finding an R^3-valued vector field $u(x, t)$, which is defined in the closure of a given space-time cylinder Q_T and satisfies the following: there exists a scalar function $p(x, t)$ such that

$$(2a) \qquad u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + b_{\infty t} + f \quad \text{in } Q_T;$$

$$(2b) \qquad \nabla \cdot u = 0 \quad \text{in } Q_T;$$

(2c) $\quad u(x, 0) = a(x) \quad$ for $x \in \Omega$;

(2d) $\quad u(x, t) = b(x, t) \quad$ for $(x, t) \in \partial\Omega \times (0, T)$;

(2e) $\quad u(x, t) \to b_\infty(t) \quad$ as $|x| \to \infty$, for $t \in (0, T)$.

This formulation of the initial boundary value problem is sufficiently general to include the case of flow around a body which undergoes translational acceleration because a time dependent velocity at infinity is allowed. The time derivative of the velocity at infinity appears in equation (2a) as a fictitious force; instead it could have been absorbed into the pressure term. By taking $\Omega = R^3$ and dropping condition (2d), the Cauchy problem is included as a special case of problem (2).

Our results concern what we call *class H_0 solutions* of problem (2), in keeping with the terminology of our papers on the Stokes equations [9, 10]. A class H_0 solution is a vector field $u \in L^1_{\text{loc}}(Q_T)$ which, with its distributional derivatives, satisfies the following conditions:

(3a) ∇u and Δu are square-summable over Q_T, and ∇u_t is square-summable over $Q_{\varepsilon,T}$ for every positive $\varepsilon < T$;

(3b) u satisfies (2b), and a scalar function $p(x, t)$ exists which together with u satisfies (2a);

(3c) $\|\nabla u(t) - \nabla a\|_{L^2(\Omega)} \to 0$ as $t \to 0^+$;

(3d) (2d) is satisfied by the trace of u on $\partial\Omega \times (0, T)$, and (2e) is satisfied in the sense that $\int_0^T dt \int_\Omega |u(x, t) - b_\infty(t)|^2 \cdot |x|^{-2} dx < \infty$.

Regarding (3d), we remark that every function $u \in L^1_{\text{loc}}(Q_T)$, with derivatives ∇u square-summable over Q_T, possesses a limit $u_\infty(t)$ at infinity in the sense that $\int_0^T dt \int_\Omega |u(x, t) - u_\infty(t)|^2 \cdot |x|^{-2} dx < \infty$; see Lemma 3. The limit $u_\infty(t)$ is unique up to a set of t-measure zero because $\int_\Omega |x|^{-2} dx = \infty$.

It may be noticed that without assumptions about the prescribed initial and boundary values, conditions (3) do not imply the initial condition (2c). Condition (2c) is assured, in the sense that $\|u(t) - a\|_{L^6(\Omega)} \to 0$ as $t \to 0^+$, if one assumes either that $\lim_{t\to 0} \int_{\partial\Omega} |b(x, t) - a(x)|^2 ds = 0$, or that $\lim_{t\to 0} b_\infty(t) = a_\infty$, where a_∞ satisfies $\int_\Omega |a(x) - a_\infty|^2 \cdot |x|^{-2} dx < \infty$. Neither of these assumptions is needed for the following continuous dependence and uniqueness theorem, which is our main result. However, for this theorem we assume that $\int_0^T |b_\infty|^2 dt < \infty$ and $\int_0^T \|b\|^2_{W_2^{3/2}(\partial\Omega)} dt < \infty$.

THEOREM. *Let u and v be two class H_0 solutions of problem* (2). *Suppose the prescribed boundary values and forces b, b_∞, f are the same for both u and v, and that the initial values u_0 and v_0, which are permitted to be different, satisfy*

$$\|\nabla u_0 - \nabla v_0\|^2 \leqslant (4T)^{-\frac{1}{4}} \exp[-\mu - H(T)], \tag{4}$$

where μ is a constant depending only on ν and Ω, and where

$$H(t) = \mu\int_0^t \{\|P\Delta u\|^2 + \|\nabla u\|^2 + \|b - b_\infty\|^2_{W_2^{3/2}(\partial\Omega)} + |b_\infty|^2 + 1\} d\tau .$$

Then, for all $t \in [0, T]$,

$$\|\nabla u(t) - \nabla v(t)\|^2 \leqslant \|\nabla u_0 - \nabla v_0\|^2 \exp\left[\frac{\mu t}{T} + H(t)\right]. \tag{5}$$

The proof of this theorem is given in section 3, following some preliminary lemmas in section 2. The related existence theory is presented in section 4.

2. – Preliminaries.

The notation of this paper is similar to that of our papers [9, 10]. Function spaces consisting of R^3-valued functions are denoted with bold-faced letters. In particular, the spaces $\boldsymbol{L}^p(G)$ and $L^p(G)$ consist of functions, vector and scalar-valued respectively, which are p-th–power summable over a given region G of space or space-time. The spaces $\boldsymbol{L}^p_{\text{loc}}(G)$ and $L^p_{\text{loc}}(G)$ consist of functions locally p-th–power summable over G. $\boldsymbol{C}_0^\infty(G)$ is the space of smooth vector-valued functions with compact support in G. If X is a Banach space, $L^2(t_1, t_2; X)$ is the space of measurable X-valued functions $\varphi(t)$, which are defined on the interval (t_1, t_2) and satisfy $\int_{t_1}^{t_2} \|\varphi\|_X^2 dt < \infty$. In keeping with the usual notation of vector analysis, the i-th components of $u\cdot\nabla v$, Δu and ∇p are $\sum_{j=1}^3 u_j\, \partial v_i/\partial x_j$, $\sum_{j=1}^3 \partial^2 u_i/\partial x_j^2$ and $\partial p/\partial x_i$ respectively; also $u\cdot v = \sum_{i=1}^3 u_i v_i$, $\nabla u : \nabla v = \sum_{i,j=1}^3 \partial u_i/\partial x_j \cdot \partial v_i/\partial x_j$ and $\nabla\cdot u = \sum_{i=1}^3 \partial u_i/\partial x_i$.

Some frequently needed norms and seminorms are, for $p \geqslant 1$ and $G \subset R^3$:

$$\|u\|_{p,G} = \left(\sum_{i=1}^{3} \int_G |u_i|^p \, dx\right)^{1/p}$$

$$\|\nabla u\|_{p,G} = \left(\sum_{i,j=1}^{3} \int_G \left|\frac{\partial u_i}{\partial x_j}\right|^p dx\right)^{1/p}$$

$$\|D^2 u\|_{p,G} = \left(\sum_{i,j,k=1}^{3} \int_G \left|\frac{\partial^2 u_i}{\partial x_j \, \partial x_k}\right|^p dx\right)^{1/p}.$$

The subscripts to these norms and seminorms will be suppressed when they take the values $p = 2$ or $G = \Omega$; thus $\|\cdot\| = \|\cdot\|_{2,\Omega}$, $\|\cdot\|_p = \|\cdot\|_{p,\Omega}$, and $\|\cdot\|_G = \|\cdot\|_{2,G}$. The spaces $W_2^1(G)$ and $W_2^2(G)$ consist of R^3-valued functions with finite norms

$$\|u\|_{W_2^1(G)} = \left(\|u\|_G^2 + \|\nabla u\|_G^2\right)^{\frac{1}{2}}$$

and

$$\|u\|_{W_2^2(G)} = \left(\|u\|_G^2 + \|\nabla u\|_G^2 + \|D^2 u\|_G^2\right)^{\frac{1}{2}},$$

respectively. If ∂G is k-times continuously differentiable, functions $u \in W_2^k(G)$ possess boundary values (or « traces ») in $W_2^{k-\frac{1}{2}}(\partial G)$. Denoting these boundary values also by u, the inequality

$$\|u\|_{W_2^{k-\frac{1}{2}}(\partial G)} \leqslant C_G \|u\|_{W_2^k(G)} \tag{6}$$

holds, with a constant C_G depending only on G; conversely, assuming a little more regularity of ∂G, it is known that every function in $W_2^{k-\frac{1}{2}}(\partial G)$ can be extended to a function in $W_2^k(G)$ satisfying the reverse of inequality (6), with a different constant. These results as well as definitions of $W_2^{k-\frac{1}{2}}(\partial G)$ and its associated norm can be found in [18, pp. 81-104].

We denote some spaces of solenoidal functions as follows:

$$\begin{aligned} \boldsymbol{D}(\Omega) &= \{\varphi : \varphi \in \boldsymbol{C}_0^\infty(\Omega) \text{ and } \nabla \cdot \varphi = 0\}, \\ \boldsymbol{J}(\Omega) &= \text{Completion of } \boldsymbol{D}(\Omega) \text{ in norm } \|\varphi\|, \\ \boldsymbol{J}_0(\Omega) &= \text{Completion of } \boldsymbol{D}(\Omega) \text{ in norm } \|\nabla\varphi\|. \end{aligned}$$

The projection P mentioned in section 1 is the orthogonal projection of $\boldsymbol{L}^2(\Omega)$ onto its subspace $\boldsymbol{J}(\Omega)$. The following lemma was proved in [11], where it was used to prove a uniqueness theorem for the steady Stokes equations.

The assumption that Ω contains a complete neighborhood of infinity is used in its proof; as was pointed out in [11], the result is not true for some types of domains.

LEMMA 1. *In order for a function* $u \in \boldsymbol{L}^1_{\mathrm{loc}}(\Omega)$ *to belong to* $\boldsymbol{J}_0(\Omega)$, *it is necessary and sufficient that its first derivatives be square-summable over* Ω *and that* $\nabla\cdot u = 0$ *in* Ω, $u = 0$ *on* $\partial\Omega$, *and* $\int_\Omega |u(x)|^2\cdot|x|^{-2}\,dx < \infty$.

We need two further spaces of solenoidal functions which were introduced in [9]. Their elements may be regarded as certain solutions of the inhomogeneous Stokes equations.

DEFINITION. $\boldsymbol{K}_0(\Omega)$ *is the set of all* $u \in \boldsymbol{J}_0(\Omega)$ *such that* $\int_\Omega \nabla u : \nabla\varphi\,dx = -\int_\Omega f\cdot\varphi\,dx$ *holds for some* $f \in \boldsymbol{D}(\Omega)$ *and all* $\varphi \in \boldsymbol{J}_0(\Omega)$.

Clearly, at most one f corresponds to each $u \in \boldsymbol{K}_0(\Omega)$, and therefore a map $\tilde{\Delta}: \boldsymbol{K}_0(\Omega) \to \boldsymbol{J}(\Omega)$ is well defined by setting $\tilde{\Delta}u = f$. Adopting this notation, we have

$$\int_\Omega \nabla u : \nabla\varphi\,dx = -\int \tilde{\Delta}u\cdot\varphi\,dx \tag{7}$$

for all $u \in \boldsymbol{K}_0(\Omega)$ and $\varphi \in \boldsymbol{J}_0(\Omega)$. It can be easily shown that the map $\tilde{\Delta}: \boldsymbol{K}_0(\Omega) \to \boldsymbol{J}(\Omega)$ is closable.

DEFINITION. $\boldsymbol{H}_0(\Omega)$ *is the domain of the closure of the map* $\tilde{\Delta}: \boldsymbol{K}_0(\Omega) \to \boldsymbol{J}(\Omega)$ *or, equivalently, the completion of* $\boldsymbol{K}_0(\Omega)$ *in the norm* $\|u\|_{\boldsymbol{H}_0(\Omega)} = \left(\|\nabla u\|^2 + \|\tilde{\Delta}u\|^2\right)^{\frac{1}{2}}$. *The extension of* $\tilde{\Delta}$ *to* $\boldsymbol{H}_0(\Omega)$ *is denoted again by* $\tilde{\Delta}$.

The proof of identity (1) hinges largely on the following result due to Ma [17]; Ma's proof utilizes, again, the assumption that Ω is an exterior domain.

LEMMA 2. *In order for a function* $u \in \boldsymbol{J}_0(\Omega)$ *to belong to* $\boldsymbol{H}_0(\Omega)$, *it is necessary and sufficient that* $\Delta u \in \boldsymbol{L}^2(\Omega)$. *In this case* $\tilde{\Delta}u = P\Delta u$.

In the next lemma, due to Finn [5, p. 368], we let $E_r = \{x: |x| > r\}$, where $r \geqslant 0$ is arbitrary.

LEMMA 3. *If* $u \in \boldsymbol{L}^1_{\mathrm{loc}}(E_r)$ *and if* ∇u *is square-summable over* E_r, *then there exists a constant vector* u_∞ *such that*

$$\int_{E_r} |u(x) - u_\infty|^2\cdot|x|^{-2}\,dx \leqslant 6\,\|\nabla u\|^2_{E_r}\,. \tag{8}$$

As u_∞ is uniquely determined, it will be regarded as the generalized limit of the function u at infinity. Below, a number of different constants will all be denoted by C, usually with subscripts to indicate what they depend upon.

We let $\Omega_r = \{x \in \Omega\colon |x| < r\}$, where $r > r^* \equiv \max_{x\in\partial\Omega} |x|$. Setting $A_r = \{x\colon r^* < |x| < r\}$, (8) implies $\|u - u_\infty\|^2_{A_r} \leqslant 6r^2\|\nabla u\|^2$. Using a type of Poincaré inequality [18, p. 113], namely $\|\varphi\|^2_{\Omega_r} \leqslant C_{\Omega,r}(\|\nabla\varphi\|^2_{\Omega r} + \|\varphi\|^2_{A_r})$, one thus obtains

$$\|u - u_\infty\|_{\Omega_r} \leqslant C_{\Omega,r}\|\nabla u\| . \tag{9}$$

This leads to the following:

LEMMA 4. *If $u \in L^1_{\text{loc}}(\Omega)$ and if ∇u is square-summable over Ω, then*

$$\|u - u_\infty\|_6 \leqslant C_\Omega\|\nabla u\| . \tag{10}$$

Because of (9), $u - u_\infty$ can be extended to a function v which is defined in all R^3 and satisfies $\|\nabla v\|_{R^3} \leqslant C_\Omega\|\nabla u\|$; see [18, p. 75]. Further, v can be approximated in norm $\|\nabla\cdot\|_{R^3}$ by smooth functions with compact supports, see [5, p. 368], and therefore satisfies the Sobolev inequality

$$\|v\|_{6,R^3} \leqslant C\|\nabla v\|_{R^3} . \tag{11}$$

Clearly, (10) follows from (11).

LEMMA 5. *If $u \in L^1_{\text{loc}}(\Omega)$, if both ∇u and Δu are square-summable over Ω, and if $\nabla\cdot u = 0$ in Ω, then*

$$\|D^2 u\| \leqslant C_\Omega\big(\|P\Delta u\| + \|\nabla u\| + \|u - u_\infty\|_{W_2^{3/2}(\partial\Omega)}\big) . \tag{12}$$

This lemma will be proved in several steps. First, we can show

$$\|D^2 u\|_{E_r} \leqslant C_{\Omega,r}\big(\|P\Delta u\| + \|\nabla u\|\big) , \tag{13}$$

by referring to Lemmas 3 and 7 of our paper [11]. Setting $v = u - u_\infty$, one finds as a consequence of inequalities (26), (32) and (33) in [11] that

$$\|D^2 v\|_{E_r} \leqslant C_{\Omega,r}\big(\|P\Delta v\| + \|\nabla v\| + \|(\Delta\zeta)v\|_{\Omega_r}\big) ,$$

where ζ is a scalar cut-off function which equals one for $|x| \geqslant r$ and vanishes for $|x| \leqslant (r + r^*)/2$. The inequality (9) for v implies $\|(\Delta\zeta)v\|_{\Omega_r} \leqslant C_{\Omega,r}\|\nabla v\|$.

Since the derivatives of u and v are the same, we obtain (13).

In the bounded region Ω_r, again letting $v = u - u_\infty$, we use the estimate

$$\|v\|_{W_2^2(\Omega_r)} \leqslant C_{\Omega,r}\big(\|P\Delta v\|_{\Omega_r} + \|v\|_{W_2^{3/2}(\partial\Omega_r)}\big) \tag{14}$$

of Solonikov; see [15, p. 78]. It can be shown that the fractional derivative norm over $\partial\Omega_r$ satisfies

$$\|\cdot\|_{W_2^{3/2}(\partial\Omega_r)} \leqslant C_{\Omega,r}\big(\|\cdot\|_{W_2^{3/2}(\partial\Omega)} + \|\cdot\|_{W_2^{3/2}(\partial E_r)}\big),$$

and thus (14) implies

$$\|D^2 u\|_{\Omega_r} \leqslant C_{\Omega,r}\big(\|P\Delta u\| + \|u - u_\infty\|_{W_2^{3/2}(\partial\Omega)} + \|u - u_\infty\|_{W_2^{3/2}(\partial E_r)}\big).$$

This together with (13) implies (12), if we observe that

$$\|u - u_\infty\|_{W_2^{3/2}(\partial E_r)} \leqslant C_r\|u - u_\infty\|_{W_2^2(A)} \leqslant C_{\Omega,r}\big(\|P\Delta u\| + \|\nabla u\|\big).$$

Here $A = \{x : r < |x| < r + 1\}$; in the first step we have used the inequality (6), and in the second step the inequalities (13) and (9), the latter with r replaced by $r + 1$.

LEMMA 6. *Under the hypotheses of Lemma* 5, *there holds*

$$\sup_{x\in\Omega} |u(x) - u_\infty| \leqslant C_\Omega\big(\|P\Delta u\| + \|\nabla u\| + \|u - u_\infty\|_{W_2^{3/2}(\partial\Omega)}\big). \tag{15}$$

Since $\partial\Omega$ is of class C^2, there is a family F of geometrically similar cones, each contained in Ω, such that each point of Ω is the vertex of one of them. The inequalities (10) and (12) imply

$$\sup_{K\in F} \|u - u_\infty\|_{W_2^2(K)} \leqslant C_\Omega\big(\|P\Delta u\| + \|\nabla u\| + \|u - u_\infty\|_{W_2^{3/2}(\partial\Omega)}\big),$$

from which (15) follows by a well-known Sobolev inequality.

LEMMA 7. *Under the hypotheses of Lemma* 5, *there holds*

$$\|\nabla u\|_3 \leqslant C_\Omega\big(\|P\Delta u\|^{\frac12} \cdot \|\nabla u\|^{\frac12} + \|\nabla u\| + \|u - u_\infty\|_{W_2^{3/2}(\partial\Omega)}\big). \tag{16}$$

To prove (16) we only need to know that functions $\varphi \in W_2^1(\Omega)$ satisfy

$$\|\varphi\|_3 \leqslant C_\Omega\big(\|\nabla\varphi\|^{\frac12} \cdot \|\varphi\|^{\frac12} + \|\varphi\|\big), \tag{17}$$

because one can then obtain (16) by substituting ∇u for φ and using (12). For $\varphi \in C_0^\infty(R^3)$, and by a density argument for $\varphi \in W_2^1(R^3)$, the inequality

$$\|\varphi\|_{3,R^3} \leqslant C\|\nabla\varphi\|_{R^3}^{\frac{1}{2}} \cdot \|\varphi\|_{R^3}^{\frac{1}{2}}$$

is well known [6, p. 24]. This implies (17) because it is possible to extend functions φ defined in Ω to functions $E\varphi$ defined in R^3, in such a way that $\|E\varphi\|_{R^3} \leqslant C_\Omega\|\varphi\|$ and $\|\nabla(E\varphi)\|_{R^3} \leqslant C_\Omega(\|\nabla\varphi\| + \|\varphi\|)$; see [18, p. 76].

LEMMA 8. *Suppose u is a class H_0 solution of the initial boundary value problem* (2), *and that the prescribed boundary values and limit at infinity satisfy $\int_0^T \|b\|^2_{W_2^{3/2}(\partial\Omega)} dt < \infty$ and $\int_0^T |b_\infty|^2 dt < \infty$. Then, for every positive $\varepsilon < T$, $\int_{Q_{\varepsilon,T}} |u \cdot \nabla u|^2 dx\, dt < \infty$.*

The conditions (3*a*) imply $\sup_{\varepsilon<t<T} \|\nabla u(t)\| < \infty$. Therefore, in view of conditions (3*a*) and (3*d*), the inequality (15) implies

$$\int_{Q_{\varepsilon,T}} |u \cdot \nabla u|^2 dx\, dt \leqslant \int_\varepsilon^T \sup_{x\in\Omega} |u(x,t)|^2 \int_\Omega |\nabla u|^2 dx\, dt$$
$$\leqslant C_\Omega \sup_{\varepsilon<t<T} \|\nabla u\|^2 \int_\varepsilon^T \left(\|P\Delta u\|^2 + \|\nabla u\|^2 + \|b - b_\infty\|^2_{W_2^{3/2}(\partial\Omega)} + |b_\infty|^2\right) dt < \infty .$$

LEMMA 9. *Let w be the difference of two class H_0 solutions of problem* (2), *both of which satisfy conditions* (3*d*) *with the same values on $\partial\Omega$ and at infinity. Then $w \in L^2(0, T; \boldsymbol{H}_0(\Omega))$ and, for every positive $\varepsilon < T$, $w_t \in L^2(\varepsilon, T; \boldsymbol{J}_0(\Omega))$. In particular, $w_t \in \boldsymbol{L}^2_{\mathrm{loc}}(Q_T)$.*

Lemmas 1 and 2 immediately imply $w \in L^2(0, T; \boldsymbol{H}_0(\Omega))$, in view of conditions (3*a*) and (3*d*). Similarly, Lemma 1 will imply $w_t \in L^2(\varepsilon, T; \boldsymbol{J}_0(\Omega))$, once w_t is shown to belong to $\boldsymbol{L}^1_{\mathrm{loc}}(Q_T)$ and to tend to zero on $\partial\Omega$ and at infinity. To prove the distributional derivative w_t is a locally summable function, one must use, in addition to (3*a*), either the boundary condition or the condition at infinity. We will consider first the case $\Omega = R^3$, for which there is no boundary condition to work with. Clearly, w_t belongs to $\boldsymbol{L}^2_{\mathrm{loc}}(Q_T)$ and tends to zero at infinity, if, for every positive $\varepsilon < T/2$,

$$\int_\varepsilon^{T-\varepsilon} \int_{R^3} |w_{\varrho t}|^2 \cdot |x|^{-2} dx\, dt < C_\varepsilon \tag{18}$$

holds uniformly as $\varrho \to 0$, where w_ϱ represents a mollification of w with respect to the time variable (or the space and time variables) of radius ϱ.

Now, from condition (3a), one obtains $\int_\varepsilon^{T-\varepsilon}\int_{R^3} |\nabla w_{\varrho t}|^2\,dx\,dt < C_\varepsilon$ for small values of ϱ. Thus, by Lemma 3, there is a function $\mu(t;\varrho)$ such that $\int_\varepsilon^{T-\varepsilon}\int_{R^3} |w_{\varrho t}(x,t) - \mu(t;\varrho)|^2\cdot|x|^{-2}\,dx\,dt < C_\varepsilon$. To get (18), we will show $\mu(t;\varrho)$ vanishes almost everywhere as a function of t, for each fixed ϱ. For any values of t_1 and t_2 satisfying $\varepsilon < t_1 < t_2 < T-\varepsilon$, we obtain through the Schwarz inequality:

$$\begin{aligned} C_\varepsilon &> \int_{R^3}|x|^{-2}\int_{t_1}^{t_2}|w_{\varrho t} - \mu|^2\,dt\,dx \\ &\geqslant \int_{R^3}|x|^{-2}(t_2-t_1)^{-1}\Big\{\int_{t_1}^{t_2}\big(w_{\varrho t}(x,t) - \mu(t;\varrho)\big)\,dt\Big\}^2 dx \\ &= (t_2-t_1)^{-1}\int_{R^3}|x|^{-2}\Big\{w_\varrho(x,t_2) - w_\varrho(x,t_1) - \int_{t_1}^{t_2}\mu(t,\varrho)\,dt\Big\}^2 dx\,. \end{aligned}$$

Using the condition at infinity (3d), one obtains $\int_{R^3}|w_\varrho(x,t)|^2\cdot|x|^{-2}\,dx < \infty$ for every t. So, by the triangular inequality,

$$\int_{R^3}|x|^{-2}\Big\{\int_{t_1}^{t_2}\mu(t;\varrho)\,dt\Big\}^2 dx < \infty\,.$$

This implies $\mu = 0$ almost everywhere because t_1 and t_2 are arbitrary and $\int_{R^3}|x|^{-2}\,dx = \infty$.

If $\partial\Omega$ is not empty, we set $w \equiv 0$ in Ω^c, and the arguments above remain clearly valid; in particular, w_t and ∇w_t are seen to be locally square summable over $R^3\times(0,T)$. Since $w_t \equiv 0$ in Ω^c, the trace of w_t on $\partial\Omega$ vanishes for almost every t.

3. – Proof of the continuous dependence and uniqueness theorem.

Let u and v be two class H_0 solutions of problem (2), let $w = v - u$ be their difference, and let q be the difference of their associated pressures. Then equation (2a) implies

$$w_t + (v\cdot\nabla v - u\cdot\nabla u) = -\nabla q + \nu\Delta w\,.$$

Each term in this equation is locally integrable over $\Omega\times(0, T]$, as a result of Lemmas 8 and 9. Thus, for every solenoidal vector field $f\in \boldsymbol{C}_0^\infty(\Omega\times[0, T])$, and for every positive $\varepsilon < T$, there holds

$$\int\limits_{Q_{\varepsilon,T}}[w_t + (v\cdot\nabla v - u\cdot\nabla u) - \nu\Delta w]\cdot f\,dx\,dt = 0\,. \tag{19}$$

The pressure term is absent from (19) because it vanishes through an integration by parts.

It was proven in Lemma 9 that $w\in L^2(0, T; \boldsymbol{H}_0(\Omega))$. Because $\boldsymbol{K}_0(\Omega)$ is dense in $\boldsymbol{H}_0(\Omega)$, this implies the existence of a sequence of solenoidal functions $f_n\in \boldsymbol{C}_0^\infty(\Omega\times[0, T])$ such that

$$\int\limits_{Q_T}(f_n - P\Delta w)^2\,dx\,dt \to 0\,, \qquad \int\limits_{Q_T}(\nabla\varphi_n - \nabla w)^2\,dx\,dt \to 0$$

as $n\to\infty$, where φ_n is the unique element of $L^2(0, T; \boldsymbol{J}_0(\Omega))$ which satisfies

$$\int\limits_{Q_T}\Delta\psi\colon \nabla\varphi_n\,dx\,dt = -\int\limits_{Q_T}\psi\cdot f_n\,dx\,dt\,,$$

for every $\psi\in L^2(0, T; \boldsymbol{J}_0(\Omega))$. One may insert these functions f_n into equation (19) and let $n\to\infty$. Since $w_t\in L^2(\varepsilon, T; \boldsymbol{J}_0(\Omega))$, as proven in Lemma 9, one has

$$\int\limits_{Q_{\varepsilon,T}}w_t\cdot f_n\,dx\,dt = -\int\limits_{Q_{\varepsilon,T}}\nabla w_t\colon \nabla\varphi_n\,dx\,dt \to -\int\limits_{Q_{\varepsilon,T}}\nabla w_t\colon \nabla w\,dx\,dt\,.$$

Remembering also the result of Lemma 8, that $u\cdot\nabla u$, $v\cdot\nabla v\in \boldsymbol{L}^2(Q_{\varepsilon,T})$, one can pass to the limit in (19), obtaining

$$\int\limits_{Q_{\varepsilon,T}}[\nabla w_t\colon \nabla w - (v\cdot\nabla v - u\cdot\nabla u)\cdot P\Delta w + \nu(P\Delta w)^2]\,dx\,dt = 0\,.$$

Equation (1) follows now, because ε is arbitrary.

REMARK. The uniqueness theorems based on energy estimates given in [11] and [8] depend upon inserting into equation (19) a sequence of solenoidal functions $f_n\in \boldsymbol{C}_0^\infty(\Omega\times[0, T])$ such that

$$\int\limits_{Q_T}(\nabla f_n - \nabla w)^2\,dx\,dt \to 0\,, \qquad \int\limits_{Q_T}(f_n - w)^2\,dx\,dt \to 0$$

as $n \to \infty$. For several types of domains, including that of a three-dimensional exterior domain, and for w, $\nabla w \in \boldsymbol{L}^2(Q_T)$, the existence of such a sequence of functions was proven in [11]. The resulting uniqueness theorem is valid for all solutions u of problem (2), such that $u - b_\infty$, $u_t - b_{\infty t}$, and ∇u are square-summable over Q_T and $u - b_\infty \in L^\infty(0, T; \boldsymbol{L}^4(\Omega))$.

Since $v\cdot\nabla v - u\cdot\nabla u = w\cdot\nabla w + w\cdot\nabla u + u\cdot\nabla w$, equation (1) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2 + \nu\|P\Delta w\|^2 = \tag{20}$$
$$= \int_\Omega w\cdot\nabla w\cdot P\Delta w\,dx + \int_\Omega w\cdot\nabla u\cdot P\Delta w\,dx + \int_\Omega u\cdot\nabla w\cdot P\Delta w\,dx\,.$$

The first term on the right side of (20) can be estimated using Hölder's inequality and Lemmas 4 and 7. For any $\alpha > 0$, one obtains

$$\begin{aligned}\Big|\int_\Omega w\cdot\nabla w\cdot P\Delta w\,dx\Big| &\leqslant \|w\|_6\cdot\|\nabla w\|_3\cdot\|P\Delta w\| \leqslant \\ &\leqslant C_\Omega\|\nabla w\|\big(\|P\Delta w\|^{\frac12}\|\nabla w\|^{\frac12} + \|\nabla w\|\big)\|P\Delta w\| \\ &\leqslant C_\alpha\|\nabla w\|^4 + C_\alpha\|\nabla w\|^6 + \alpha\|P\Delta w\|^2 \leqslant C_\alpha\|\nabla w\|^2 + C_\alpha\|\nabla w\|^6 + \alpha\|P\Delta w\|^2\,.\end{aligned}$$

The constants C_α, here and below, depend on Ω as well as α. In estimating the term $\|\nabla w\|^{\frac32}\|P\Delta w\|^{\frac32}$, we used Young's inequality $ab \leqslant a^p/p + b^q/q$ with $p = 4$ and $q = \frac43$. The second term on the right side of (20) can also be estimated using Hölder's inequality and Lemmas 4 and 7. For any $\alpha > 0$, there holds

$$\begin{aligned}\Big|\int_\Omega w\cdot\nabla u\cdot P\Delta w\,dx\Big| &\leqslant \|w\|_6\cdot\|\nabla u\|_3\cdot\|P\Delta w\| \\ &\leqslant C_\alpha\|w\|_6^2\|\nabla u\|_3^2 + \alpha\|P\Delta w\|^2 \\ &\leqslant C_\alpha\|\nabla w\|^2\big(\|P\Delta u\|^2 + \|\nabla u\|^2 + \|b - b_\infty\|^2_{W_2^{3/2}(\partial\Omega)}\big) + \alpha\|P\Delta w\|^2\,.\end{aligned}$$

The third term on the right side of (20) can be estimated using Lemma 6. For any $\alpha > 0$, one obtains

$$\begin{aligned}\Big|\int_\Omega u\cdot\nabla w\cdot P\Delta w\,dx\Big| &\leqslant \sup_\Omega|u|\cdot\|\nabla w\|\cdot\|P\Delta w\| \\ &\leqslant C_\alpha\big(\|P\Delta u\|^2 + \|\nabla u\|^2 + \|b - b_\infty\|^2_{W_2^{3/2}(\partial\Omega)} + |b_\infty|^2\big)\|\nabla w\|^2 + \alpha\|P\Delta w\|^2\,.\end{aligned}$$

Using these estimates for terms on the right side of (20), and setting $\alpha = \nu/3$, we obtain

$$\frac{d}{dt}\|\nabla w\|^2 \leqslant 4\mu\|\nabla w\|^6 + h(t)\|\nabla w\|^2 , \tag{21}$$

where

$$h(t) = \mu\{\|P\Delta u\|^2 + \|\nabla u\|^2 + \|b - b_\infty\|^2_{W_2^{3/2}(\partial\Omega)} + |b_\infty|^2 + 1\},$$

and μ is a constant depending only on ν and Ω.

Conditions ($3a$) and ($3c$) imply that $\|\nabla w(t)\|^2$ is continuous on $[0, T]$ and absolutely continuous on $(0, T]$, as a function of t. Over any subinterval I of $(0, T]$ on which $\|\nabla w(t)\|$ is everywhere positive, (21) implies

$$\frac{d}{dt}\log\|\nabla w\|^2 \leqslant 4\mu\|\nabla w\|^4 + h(t) .$$

If $t_1 < t_2$ are numbers lying in such an interval I, one clearly has

$$\|\nabla w(t_2)\|^2 \leqslant \|\nabla w(t_1)\|^2 \exp\int_{t_1}^{t_2}[4\mu\|\nabla w\|^4 + h(t)]\,dt .$$

From this it can be seen that

$$\|\nabla w(t)\|^2 \leqslant \|\nabla w(0)\|^2 \exp\int_0^t[4\mu\|\nabla w\|^4 + h(\tau)]\,d\tau \tag{22}$$

holds for all $t \in [0, T]$, regardless of whether $\|\nabla w(t)\|$ is everywhere positive or not. To verify this, let t_2 represent an arbitrary point of $(0, T]$ with $\|\nabla w(t_2)\| > 0$, if there are any such points, and then pass to the limit as t_1 tends to the left boundary of the largest subinterval I of $(0, T]$, which contains t_2 and on which $\|\nabla w\|$ is positive.

Now suppose $\|\nabla w(0)\| < A$, where A is any given positive number, and let $[0, T_A]$ be the largest subinterval of $[0, T]$ on which $\|\nabla w(t)\| \leqslant A$. Then using (22) and setting

$$H(t) = \int_0^t h(\tau)\,d\tau ,$$

one obtains

$$\|\nabla w(t)\|^2 \leqslant \|\nabla w(0)\|^2 \exp[4\mu A^4 t + H(t)] \tag{23}$$

for $t \in [0, T_A]$. Clearly $T_A = T$ if

$$\|\nabla w(0)\|^2 \exp[4\mu A^4 T + H(T)] \leqslant A . \tag{24}$$

Choosing $A = (4T)^{-\frac{1}{4}}$, our theorem follows at once.

4. – Existence theory. The starting problem.

The estimates used in the last section to prove uniqueness can be used in some situations to prove the local existence of a class H_0 solution, even when the external forces are not square-summable. This observation may prove useful in dealing with the « starting problem », which is to demonstrate that a steady wake develops at $t \to \infty$, if a body initially at rest and surrounded by a fluid at rest is smoothly accelerated to some constant (slow) velocity and then held at that velocity indefinitely. Methods based on a similar observation have proved useful in treating the starting problem for the Stokes equations [9, 17]. To date, the starting problem for the Navier-Stokes equations has been solved only in special cases where the prescribed boundary values and forces are such that the limiting steady wake possesses finite energy [8]. Here, we will examine the starting problem for a body with rigid boundaries to which fluid adheres; this is a case for which the expected steady wake possesses infinite energy [4].

The starting problem can be considered as a special case of the initial boundary value problem (2) by prescribing the data appropriately. In order to find a solution of problem (2), or of the starting problem in particular, the first step is to choose an extension into $\Omega \times [0, T]$ of the boundary values $b(x, t)$ prescribed on $\partial\Omega \times [0, T]$. We will denote the chosen extension, as well as the prescribed boundary values, by $b(x, t)$. We require the extension b to have derivatives ∇b, Δb, ∇b_t square-summable over Q_T, to be solenoidal, to have trace on $\partial\Omega \times [0, T]$ equal to the prescribed boundary values, and to satisfy $\int_\Omega \big(b(x, t) - b_\infty(t)\big)^2 |x|^{-2} dx < \infty$ for all $t \in [0, T]$. Then the solution of problem (2) can be reasonably sought in the form $u = v + b$, with $v \in L^2\big(0, T; \boldsymbol{H}_0(\Omega)\big)$.

It is known from the investigations [16], [5], [1] and [8], that the exterior stationary problem

$$\left\{\begin{array}{ll} w\cdot\nabla w = -\nabla p + \nu \Delta w\,, & \nabla\cdot w = 0 \text{ in } \Omega \\ w = 0 \quad \text{on } \partial\Omega\,, & w(x) \to w_\infty \text{ as } |x| \to \infty \end{array}\right. \tag{25}$$

has a solution $w(x)$ with ∇w square-summable over Ω, and that this solution is unique and stable (as a time dependent motion) provided w_∞ is sufficiently small. Thus, if w_∞ is sufficiently small, $w(x)$ can be expected to arise as the steady limit of a time dependent solution representing flow around a rigid body Ω^c, which is accelerated from rest to the constant velocity $-w_\infty$. Let us suppose the body's velocity is given by $-\zeta(t) w_\infty$ relative to the inertial frame in which the fluid is initially at rest, where $\zeta(t)$

is a smooth function of $t \in [0, \infty)$, which vanishes for t near zero and equals one after the initial period of acceleration. The appropriate data for problem (2) is then $b_\infty(t) \equiv \zeta(t) w_\infty$, $f(x, t) \equiv 0$, $a(x) \equiv 0$, and $b(x, t) = 0$ on $\partial\Omega \times [0, \infty)$. As the extension of the boundary values into $\Omega \times [0, \infty)$ we choose $b(x, t) = \zeta(t) w(x)$; this function meets all the required conditions.

Inserting $u = v + b$ into equation (2a), one obtains

$$v_t + v\cdot\nabla v + b\cdot\nabla v + v\cdot\nabla b = -\nabla p + \nu \Delta v + g\,, \tag{26}$$

where $g = \nu\Delta b - b\cdot\Delta b + b_{\infty t} - b_t$. We seek v by a variant of the Galerkin method. As basis functions, we choose a sequence $\{a^l(x)\}$ which is contained in $\boldsymbol{K}_0(\Omega)$, complete in $\boldsymbol{H}_0(\Omega)$, and orthonormal in $\boldsymbol{J}_0(\Omega)$. As n-th approximate solution we take the solution

$$v^n(x, t) = \sum_{l=1}^{n} c_{ln}(t)\, a^l(x)$$

of the initial value problem: $v^n(x, 0) = 0$ and

$$\begin{aligned}(\nabla v_t^n, \nabla a^l) + \nu(\tilde{\Delta} v^n, \tilde{\Delta} a^l) &= \\ &= (v^n\cdot\nabla v^n, \tilde{\Delta} a^l) + (b\cdot\nabla v^n, \tilde{\Delta} a^l) + (v^n\cdot\nabla b, \tilde{\Delta} a^l) - (g, \tilde{\Delta} a^l)\,,\end{aligned} \tag{27}$$

for $t \geqslant 0$ and $l = 1, 2, \ldots, n$. The brackets here indicate integration over Ω. The equations (27) are obtained, formally, by multiplying equation (26) through by $\tilde{\Delta} a^l$, integrating the result over Ω, and using the identity (7).

REMARK. The differential equations for approximate solutions, used when the existence theory is based on energy estimates, are obtained by multiplying (26) through by a^l rather than by $P\Delta a^l$. For problems in a bounded domain Ω, one can choose eigenfunctions of the operator $P\Delta$ as basis functions and thereby obtain at once estimates both of energy type and of the type given here; see Prodi [19].

Taking linear combinations of the equations (27), and of the t-derivatives of these equations, one obtains the following identities for the Galerkin approximations (the superscript n is suppressed):

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \nu\|\tilde{\Delta} v\|^2 = (v\cdot\nabla v, \tilde{\Delta} v) + (b\cdot\nabla v, \tilde{\Delta} v) + (v\cdot\tilde{\Delta} b, \tilde{\Delta} v) - (g, \tilde{\Delta} v)\,, \tag{28}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla v_t\|^2 + \nu\|\tilde{\Delta} v_t\|^2 &= (v_t\cdot\nabla v, \tilde{\Delta} v_t) + (v\cdot\nabla v_t, \tilde{\Delta} v_t) + \\ &+ (b_t\cdot\nabla v, \tilde{\Delta} v_t) + (b\cdot\nabla v_t, \tilde{\Delta} v_t) + (v_t\cdot\nabla b, \tilde{\Delta} v_t) + (v\cdot\nabla b_t, \tilde{\Delta} v_t) - (g_t, \tilde{\Delta} v_t)\,.\end{aligned} \tag{29}$$

The first three terms on the right side of (28) can be estimated in much the same manner as were the terms on the right side of (20). For any $\alpha > 0$, one obtains

$$|(v\cdot\nabla v,\ \tilde{\Delta} v)| \leqslant C_\alpha \|\nabla v\|^2 + C_\alpha \|\nabla v\|^6 + \alpha\|\tilde{\Delta} v\|^2,$$
$$|(b\cdot\nabla v,\ \tilde{\Delta} v)| \leqslant C_\alpha \sup_\Omega |b|^2 \|\nabla v\|^2 + \alpha\|\tilde{\Delta} v\|^2,$$
$$|(v\cdot\nabla b,\ \tilde{\Delta} v)| \leqslant C_\alpha \|\nabla b\|_3^2 \|\nabla v\|^2 + \alpha\|\tilde{\Delta} v\|^2 .$$

The term $(g, \tilde{\Delta} v)$ may appear to present a difficulty because g is not square-summable. However, the steady solution w can be written as $w = w_1 + w_2$, with w_1 a function that vanishes in a neighborhood of $\partial\Omega$ and equals w_∞ in a neighborhood of infinity, and with $w_2 \in \boldsymbol{J}_0(\Omega)$. Thus one can write $g = g_1 + g_2$, where $g_1 \equiv \nu\zeta\Delta w - \zeta^2 w\cdot\nabla w + \zeta_t(w_\infty - w_1)$ is bounded in $\boldsymbol{L}^2(\Omega)$ as a function of t, and $g_2 \equiv -\zeta_t w_2$ is bounded in $\boldsymbol{J}_0(\Omega)$ as a function of t. Using the identity $(g_2, \tilde{\Delta} v) = -(\nabla g_2, \nabla v)$, one obtains

$$|(g,\ \tilde{\Delta} v)| \leqslant \tfrac{1}{2}\|\nabla g_2\|^2 + C_\alpha\|g_1\|^2 + \tfrac{1}{2}\|\nabla v\|^2 + \alpha\|\tilde{\Delta} v\|^2 .$$

By combining these estimates for terms on the right side of (28) and setting $\alpha = \nu/8$, one finds that

$$\frac{d}{dt}\|\nabla v\|^2 + \nu\|\tilde{\Delta} v\|^2 \leqslant C_{\Omega,\nu,w_\infty}\|\nabla v\|^2 + C_{\Omega,\nu}\|\nabla v\|^6 + \|\nabla g_2\|^2 + C_\nu\|g_1\|^2 . \tag{30}$$

The form of this differential inequality implies the existence of an interval $[0, T^*]$ and of a constant C, both independent of the order n of the approximation, such that

$$\|\nabla v(t)\| < C \quad \text{for } t \in [0, T^*], \qquad \text{and} \quad \int_0^{T^*} \|\tilde{\Delta} v\|^2 dt < C . \tag{31}$$

Equation (29) can now be used to estimate $\|\nabla v_t\|$. For terms on the right side of (29), we have

$$|(v_t\cdot\nabla v,\ \tilde{\Delta} v_t)| \leqslant \|v_t\|_6\|\nabla v\|_3\|\tilde{\Delta} v_t\| \leqslant C_\alpha\{\|\nabla v\|^2 + \|\tilde{\Delta} v\|^2\}\|\nabla v_t\|^2 + \alpha\|\tilde{\Delta} v_t\|^2,$$
$$|(v\cdot\nabla v_t,\ \tilde{\Delta} v_t)| \leqslant \sup_\Omega|v|\cdot\|\nabla v_t\|\,\|\tilde{\Delta} v_t\| \leqslant C_\alpha\{\|\nabla v\|^2 + \|\tilde{\Delta} v\|^2\}\|\nabla v_t\|^2 + \alpha\|\tilde{\Delta} v_t\|^2,$$
$$|(b_t\cdot\nabla v,\ \tilde{\Delta} v_t)| \leqslant C_\alpha \sup_\Omega |b_t|^2\cdot\|\nabla v\|^2 + \alpha\|\tilde{\Delta} v_t\|^2,$$
$$|(b\cdot\nabla v_t,\ \tilde{\Delta} v_t) \leqslant C_\alpha \sup_\Omega |b|^2\cdot\|\nabla v_t\|^2 + \alpha\|\tilde{\Delta} v_t\|^2,$$

$$|(v_t\cdot\nabla b,\ \tilde{\Delta} v_t)|\leqslant C_\alpha\|\nabla b\|_3^2\|\nabla v_t\|^2+\alpha\|\tilde{\Delta} v_t\|^2,$$

$$|(v\cdot\nabla b_t,\ \tilde{\Delta} v_t)|\leqslant C_\alpha\|\nabla b_t\|_3^2\|\nabla v\|^2+\alpha\|\tilde{\Delta} v_t\|^2,$$

$$|(g_t,\ \tilde{\Delta} v_t)|\quad\leqslant\tfrac{1}{2}\|\nabla g_{2t}\|^2+C_\alpha\|g_{1t}\|^2+\tfrac{1}{2}\|\nabla v_t\|^2+\alpha\|\tilde{\Delta} v_t\|^2 .$$

Using these estimates for terms in (29) and setting $\alpha=\nu/14$, one obtains

$$\frac{d}{dt}\|\nabla v_t\|^2+\nu\|\tilde{\Delta} v_t\|^2\leqslant C_{\Omega,\nu}\{\|\nabla v\|^2+\|\tilde{\Delta} v\|^2+C_{\Omega,w_\infty}\}\|\nabla v_t\|^2+ \\ +C_{\Omega,\nu,w_\infty,\zeta}\|\nabla v\|^2+\|\nabla g_{2t}\|^2+C_\nu\|g_{1t}\|^2 .$$

This inequality can be integrated over the same interval $[0, T^*]$ on which the estimates (31) are valid, yielding the further estimates:

$$(32)\qquad \|\nabla v_t(t)\|<C\quad\text{ for } t\in[0,T^*]\,,\qquad\text{and}\quad \int_0^{T^*}\|\tilde{\Delta} v_t\|^2dt<C\,.$$

Since $\|\nabla v_t(0)\|=0$ for each Galerkin approximation, the constant C is again independent of n. The estimates (31) and (32) for $\|\nabla v\|$, $\int_0^{T^*}\|\tilde{\Delta} v\|^2dt$, and $\|\nabla v_t\|$ are enough to ensure that a subsequence of the approximations converges to a function $v\in L^2(0,T^*;\boldsymbol{H}_0(\Omega))$ with $v_t\in L^2(0,T^*;\boldsymbol{J}_0(\Omega))$, and that $u=v+b$ is a class H_0 solution of the starting problem, on the interval $0\leqslant t\leqslant T^*$. We remark that by taking still another linear combination of equations (27) one obtains

$$\frac{\nu}{2}\frac{d}{dt}\|\tilde{\Delta} v\|^2+\|\nabla v_t\|^2=(v\cdot\nabla v,\ \tilde{\Delta} v_t)+(b\cdot\nabla v,\ \tilde{\Delta} v_t)+(v\cdot\nabla b,\ \tilde{\Delta} v_t)-(g,\ \tilde{\Delta} v_t)\ ,$$

which leads to the estimate $\|\tilde{\Delta} v(t)\|<C$ on $[0, T^*]$.

A local existence theorem for strong solutions for the starting problem can also be based on energy type estimates, if the extension b is chosen so that g is square-summable. For instance, one can take $b(x,t)=\zeta(t)\,w_1(x)$, where as before $w_1(x)$ is a solenoidal function which vanishes in a neighborhood of $\partial\Omega$ and equals w_∞ in a neighborhood of infinity. Then, setting $v=u-b$, the norm $\|v(t)\|$ will be finite for every t; it can be shown to grow at most linearly by a method of Hopf [12]. However, $\|v(t)\|$ must necessarily grow infinite as $t\to\infty$, and g will not vanish after the initial period of acceleration. The advantage in choosing b as before and of relying primarily on estimates of the Dirichlet integral is that g then vanishes after the initial period of acceleration and also $\|\nabla v(t)\|$, the principal quantity being estimated, is expected to tend to zero as $t \to \infty$. Even if it could be merely shown that $\int_{\tau}^{\tau+1} \|\nabla v(t)\|^2 dt < \delta$ holds for all $\tau > 0$ and for a sufficiently small value of δ, (30) would yield a global existence theorem.

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