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An Approximate Layering Method for Multi-Dimensional Nonlinear Parabolic Systems of a Certain Type (*).

AVRON DOUGLIS (**)

Abstract. – *A new method is described of solving initial-value problems for PDE systems of the form*

$$(E) \qquad \partial u_i/\partial t + \sum_{j=1}^{d} (\partial/\partial x_j) f_{ij}(x, t, u) + g_i(x, t, u) = \mu_i \Delta u_i\,, \qquad i = 1, \ldots, n\,,$$

in d dimensions, where $d \geq 1$, $\mu_i > 0$. *The method is an outgrowth of ideas previously put forth by N. N. Kuznetsov* [Math. Zametki, **2** (1967), pp. 401-410] *and the author* [Ann. Inst. Fourier Grenoble, **22** (1972), pp. 141-227] *in connection with scalar first-order PDE's. Time t starting from* 0 *is divided into short intervals* $Z_m = \{(m-1)h \leq t \leq mh\}$, $m = 1, 2, \ldots$. *Bounded, measurable values of* $u = (u_1, \ldots, u_n)$ *are supposed to be prescribed at time* $t = 0$, *and the first step is to smooth them. Then in* Z_1 *an approximate solution* $v^1(x, t) = v^1 = (v_1^1, \ldots, v_n^1)$ *of the* first order *system* $(E)_0$ *to which* (E) *reduces when the* μ_i *are replaced by zero is obtained such that at time* $t = 0$, v^1 *coincides with the smoothed initial data. Once* $v^{m-1}(x, t)$ *has been constructed in the time-interval* Z_{m-1}, *its terminal values* $v^{m-1}(x, (m-1)h)$ *are smoothed to do duty as initial data for an approximate solution of* $(E)_0$, $v^m(x, t)$, *in the next following time-interval* Z_m. *In this way, a « layered » function* $v^{(h)}(x, t) = v^m(x, t)$ *for* $(m-1)h \leq t < mh$, $m = 1, 2, \ldots$, *is built up. If the smoothing at each step is carried out appropriately, its effects accumulate in such a way that, for small h,* $u^{(h)}(x, t)$ *will approximate a solution of* (E) *at least for a certain finite interval of time. The derivatives of* $u^{(h)}$ *with respect to x and their difference-quotients with respect to t will be well behaved if the* f_{ij} *and* g_i *are sufficiently smooth.*

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1. - Introduction.

Summary.

In this paper, a new approach is presented to the construction of solutions of multi-dimensional parabolic systems of the form

$$(E)\qquad F_i[u] = \frac{\partial u_i}{\partial t} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\big(f_{ij}(x, t, u)\big) + g_i(x, t, u) = \mu_i \Delta u_i\,, \qquad i = 1, \dots, n,$$

where $\mu_i > 0$, $x = (x_1, \dots, x_d)$, $u = u(x, t) = \big(u_1(x, t), \dots, u_n(x, t)\big)$, and Δ denotes the d-dimensional Laplacian. Systems of equations of this kind arise in diverse contexts, for instance in theories of chemical reactions, thermal diffusion, population growth and diffusion, predator-prey interactions, and, with some (not all) μ_i equal to zero, of enzyme-morphogen interaction, and of nerve excitation.

The present study is restricted to initial-value problems, in which solutions $u(x, t)$ of (E) are demanded in time-space zones $Z^T = \{(x, t)\colon 0 \leqslant t \leqslant T,\ x \in R^d\}$ of appropriate duration T under initial conditions of the form

$$(IC)\qquad u(x, 0) = u^0(x) \quad \text{for } x \in R^d\,.$$

(Real cartesian space of d dimensions is denoted by R^d with $d \geqslant 1$.) Subsequent papers will show how the methods used in these initial-value problems can be adapted to boundary problems for (E) and can be extended to other types of systems of equations, in general leading to convergent calculational schemes. A treatment of the Navier-Stokes equations from this point of view is being worked out at present jointly with E. Fabes.

We assume $f_{ij}(x, t, u)$ and $g_i(x, t, u)$ to be sufficiently smooth and $u^0(x)$ to be bounded and measurable on R^d. To construct functions that will approximate a solution of an initial-value problem (E), (IC), our procedure, in outline, is as follows:

(1) Select a suitable averaging operator S_ε ($\varepsilon > 0$) acting on functions $v(x)$ that are bounded and continuous on R^d. For instance, S_ε might be repeated arithmetical averaging, i.e., $S_\varepsilon = A_\varepsilon{}^b$, where b is a positive integer, and, for each $x \in R^d$, $A_\varepsilon v(x)$ is the arithmetical average of the values of v on a d-dimensional cube of center x and edge length 2ε.

(2) Divide the half-space $R_+^{d+1} = \{(x, t): t \geq 0, x \in R^d\}$ into thin layers

$$Z_m = \{(x, t): (m-1)h \leq t \leq mh, x \in R^d\}, \qquad m = 1, 2, \dots,$$

of « duration », or « thickness », h, $h > 0$.

(3) Approximate the initial function $u^0(x)$ by a smooth function $u^*(x)$ depending on h, approaching $u^0(x)$ in the sense of L^1_{loc} as $h \to 0$, and in absolute value having the same bound as its limit. Then, in particular, $|u^*(x)| \leq M_0$ for $x \in R$, where $M_0 = \sup_{x \in R} |u^0(x)|$. Choose arbitrarily a number M such that $M \geq M_0 + 1$.

(4) For each $i = 1, \dots, n$ determine an averaging parameter ε_i by a condition of the form $\varepsilon_i^2 = \lambda_i h$, in which λ_i is a certain constant proportional to μ_i. In the first layer Z_1, find a (vector) function $u^1(x, t) = \big(u_1^1(x, t), \dots, u_n^1(x, t)\big)$ that, on the lower face of the layer, complies with the initial conditions

$$u_i^1(x, 0) = S_{\varepsilon_i} u_i^*(x), \qquad i = 1, \dots, n,$$

and, in the interior of the layer, satisfies approximately the « layer equations »

$$(E)_0 \qquad\qquad F_i[u] = 0, \qquad i = 1, \dots, n.$$

Then in Z_m, for each $m = 2, 3, \dots$ in turn, find a function $u^m(x, t) = \big(u_1^m(x, t), \dots, u_n^m(x, t)\big)$ that again is an approximate solution of $(E)_0$ and that satisfies at the bottom of the layer the initial conditions

$$u_i^m\big(x, (m-1)h\big) = S_{\varepsilon_i} u_i^m\big(x, (m-1)h\big), \qquad i = 1, \dots, n.$$

Let m_0 be the largest integer, finite or infinite, for which

$$|u^m(x, t)| \leq M \quad \text{for } (x, t) \in Z_m, \qquad m = 1, 2, \dots, m_0,$$

and define in the zone $Z^{m h_0}$ the « layered solution », or « approximate layered solution »,

$$u^{(h)}(x, t) = u^m(x, t) \quad \text{for } (m-1)h \leq t < mh,\ x \in R^d,\ m = 1, 2, \dots, m_0.$$

Provided that, in each Z_m, u^m satisfies the layer equation closely enough for the purpose, $u^{(h)}$ will be found for small h to approximate a solution of (E), (IC).

In more detail, our main results are as follows.

(A) A common zone Z^T ($T > 0$) exists in which, for sufficiently small h, all $u^{(h)}$ are defined and, in absolute value, are $\leqslant M$ (Section 5). For certain special types of systems (E), values of M corresponding to any choice of $T' > 0$ can be determined such that $|u^{(h)}(x, t)| \leqslant M$ on $Z^{T'}$. (An instance is given in Section 5A.)

(B) The derivatives with respect to x of $u^{(h)}(x, t)$ are subject to certain estimates in Z^T provided that $t > 0$ and that h is sufficiently small. The derivatives of k-th order, namely, in absolute value are $\leqslant \sigma_k(M, T)t^{-k/2}$ with constant $\sigma_k(M, T)$ independent of x, t, h (Section 6). Notwithstanding that $u^{(h)}$ is discontinuous across the interfaces between consecutive layers, analogous estimates also hold for first and higher difference-quotients with respect to t of approximate layered solutions and of their derivatives with respect to x (Section 7). Thus, the artificial discontinuities created in the layering method are less disordering than might be feared.

(C) Approximate layered solutions converge to a certain limit $u(x, t)$ as layer thickness h approaches zero. The convergence is uniform in any subset of Z^T having positive distance from the initial plane (Section 8).

(D) The limit $u(x, t)$ is a solution of equations (E) and also satisfies the initial conditions (IC) in a generalized sense (Section 3).

Our presentation departs from the natural order of ideas in two respects. The central and motivating fact that the limit of layered solutions must be a solution of (E) is the first thing proved. The discussion of averaging, being outside the main flow of ideas in the paper, is relegated to an appendix at the end.

Remarks and short illustrations.

(1) *Layering for the heat equation.* The cumulative effects of the repeated averagings performed in a layering process are a manifestation of the central limit theorem. Only a special form of this law is involved, that which pertains to repeated application of a type of one-dimensional averaging operator S_ε. For an operator of that type, which is to act, say, on functions $v(x)$ that are bounded and continuous on the real line R, $S_\varepsilon v(x)$ is an integral expression,

$$S_\varepsilon v(x) = \int k_\varepsilon(\xi - x)\, v(\xi)\, d\xi ,$$

in which $k_\varepsilon(\xi) = \varepsilon^{-1}k(\xi/\varepsilon)$, where $k(\xi)$ is a function on R having the following properties: $k(\xi) \geqslant 0$, $\int k(\xi)\,d\xi = 1$, $\int \xi k(\xi)\,d\xi = 0$, $m_2 = \int \xi^2 k(\xi)\,d\xi < \infty$. (The interval of integration in all cases is R.) Thus $S_\varepsilon v = k_\varepsilon * v$, the star signifying convolution, and, therefore,

$$S_\varepsilon^j v(x) = \int k_\varepsilon^{(j)}(\xi - x)\,v(\xi)\,d\xi \quad \text{for } j = 2, 3, \dots,$$

where $k_\varepsilon^{(j)} = k_\varepsilon * k_\varepsilon * \dots * k_\varepsilon$, k_ε occurring in the convolution product j times. The central limit theorem says that if $\varepsilon \to 0$ and $j \to \infty$ in such a way that $m_2 j \varepsilon^2$ has a finite limit σ^2, then

$$S_\varepsilon^j v(x) \to \int (2\pi)^{-\frac{1}{2}} \sigma^{-1} \exp\left[-(x - \xi)^2/2\sigma^2\right] v(\xi)\,d\xi .$$

The mechanism through which this law acts in a layering process is most visible in connection with the heat equation

$$u_t = \mu u_{xx} \quad (\mu > 0),$$

for convenience taken here in one dimension. A solution is desired in R_+^2 satisfying the initial condition

$$u(x, 0) = u^0(x) \quad \text{for } x \in R$$

with bounded, measurable $u^0(x)$. Select at pleasure a layer height $h > 0$, and determine the averaging parameter ε through the condition $\varepsilon^2 = \lambda h$, where $\lambda = 2\mu/m_2$. In this case, the layer equation is $u_t = 0$ and consequently

$$u^m(x, t) = u^m\big(x, (m-1)h\big) = S_\varepsilon u^{m-1}\big(x, (m-1)h\big) \quad \text{in } Z_m,\ m = 0, 1, \dots,$$

under the convention that $u^0(x, 0) = u^0(x)$ (here we can dispense with u^*). These relations imply that

$$u^m(x, t) = S_\varepsilon^m u^0(x) \quad \text{for} \quad (m-1)h \leqslant t < mh, \quad x \in R,$$

and thus that

$$u^{(h)}(x, t) = S_\varepsilon^m u^0(x) \quad \text{for} \quad (m-1)h \leqslant t < mh, \quad x \in R.$$

Now fixing x, τ, with $\tau > 0$, we can easily find the limit of $u^{(h)}(x, \tau)$ as $h \to 0$. For each value of h, let N denote the integer for which $(N-1)h \leqslant \tau < Nh$.

Then

$$u^{(h)}(x, \tau) = u^N(x, \tau) = S_\varepsilon^N u^0(x)\,,$$

while

$$m_2 N\varepsilon^2 = m_2\lambda N h \to m_2\lambda\tau = 2\mu\tau \quad \text{as } h \to 0\,.$$

From the central limit theorem in the form described, we thus have

$$\lim_{h\to 0} u^{(h)}(x, \tau) = \int (4\pi\mu\tau)^{-\frac{1}{2}} \exp\{-(\xi - x)^2/4\mu\tau\}\, u^0(\xi)\, d\xi\,,$$

the standard representation of the solution of the problem stated.

In an entirely different connection, a somewhat similar construction for the heat equation and, more generally, equations of the form $\partial u/\partial t = (-1)^{k+1}(\partial^{2k}u/\partial x^{2k})$, $k = 1, 2, \dots$, was given in 1953 by I. J. Schoenberg [23, pp. 203-204].

(2) *Layering for scalar conservation laws.* The estimates in this paper all derive from the central limit theorem, and such estimates explode as $\mu \to 0$. In the scalar case ($n = 1$), estimates of an entirely different kind can be made of layered solutions $u^{(h)} \equiv u^{(h)}_{(\mu)}$ of (E). These estimates are uniform with respect to μ as well as h, are passed on to $u_{(\mu)} = \lim_{h\to 0} u^{(h)}_{(\mu)}$, and assure that a null sequence $\{\mu_k\}$ will exist such that $u_{(\mu_k)}$ has a limit u as $k \to \infty$. The final limit u is a weak solution of the limit equation $(E)_0$. Estimates of this second kind are obtainable, for instance, with respect to the variation of layered solutions. Exactly the same means may be used to derive and to apply them as were given in [13] and [4] in connection with scalar equations of first order.

History.

Existence theorems applicable to parabolic systems of type (E) in initial-value or boundary-value problems are already available by various means. M. I. Visik [27] uses something like Galerkin's ideas. Ladyzhenskaya, Solonnikov, and Ural′ceva [15, Theorem 7.1, p. 596] prove a priori parabolic estimates and appeal to the Leray-Schauder fixed-point theorem. S. D. Eidel'man [7], first treating linear systems by means of a fundamental solution and its potentials, handles quasilinear systems by iteration. W. von Wahl [28, 29] makes use of elliptic estimates and semi-groups. (We do not mention the many papers devoted to the scalar case or abstract treatments of evolution equations not specialized to parabolic partial differential equations.) The results arrived at in these several ways are quite general in some respects, but require that the initial data be smooth in one degree or another, while boundedness and measurability suffice in the present treatment.

The subject of greatest interest in this paper is believed, however, to be its method. The idea of using a layering process to construct solutions of parabolic systems goes back to 1972 and was a rather natural extension of layering methods for first-order scalar conservation laws, and for equations of Hamilton-Jacobi type, previously developed by N. N. Kuznetsov [13, 14] and the author [4, 5]. Ensuing work on parabolic layering schemes was restricted for a time to one-dimensional systems (E) with hyperbolic layer equations $(E)_0$ (see [6]), but these limitations were overcome in 1976 by using approximate instead of exact solutions of $(E)_0$.

Layering procedures have an obvious relation to the so-called method of fractional steps (see, in particular, A. Pazy [21, Cor. 5.5, p. 96], J. E. Marsden [19], and Chorin, Hughes, MacCracken, and Marsden [2]). A computational method of fractional steps for the Navier-Stokes equations was put forth by A. Chorin [1] in 1973. In that scheme, Euler's equations and the heat equation are solved numerically in alternating short intervals of time. Marsden [20] justified Chorin's procedure in principle by proving convergence for a parallel construction in which Euler's equations and the heat equation are solved exactly in their respective layers; the question is further discussed in [2]. Perhaps approximate layering methods will provide an entirely different approach to theory and calculation in this problem.

2. - Main notational conventions.

Real d-dimensional Euclidean space is denoted by R^d, a point of R^d, for instance, by $x = (x_1, \dots, x_d)$. Such a point as well as its coordinates are referred to as spatial, a $(d+1)$-st coordinate t as temporal. All points of R^{d+1} considered here are confined to the half-time-space

$$Z_+ = \{(x, t): x \in R^d,\ t \geqslant 0\}$$

or to « zones » or « slabs » in time-space such as

$$Z^T = \{(x, t): x \in R^d,\ 0 \leqslant t \leqslant T\}.$$

All the closed « zones », « slabs », or « layers » in the discussions to follow will be of the type

$$Z_{t_1,t_2} = \{(x, t): x \in R^d,\ t_1 \leqslant t \leqslant t_2\},$$

these including, in particular, the consecutive layers

$$Z_m = Z_{(m-1)h,mh} = \{(x, t): x \in R^d,\ (m-1)h \leqslant t \leqslant mh\}, \qquad m = 1, 2, \dots,$$

for any $h > 0$. The half-open counterparts

$$Z'_m = \{(x, t): x \in R^d,\ (m-1)h < t < mh\}$$

of this sequence of layers also will be considered.

To a bounded, measurable function $v(x)$ on the domain R^d is attached the norm

$$\|v\| = \operatorname*{ess\,sup}_{x \in R^d} |v(x)|, \tag{1}$$

to a bounded, measurable function $v(x, t)$ on a slab Z such as Z^T or Z_{t_1,t_2}, the norms

$$\|v(\cdot, t)\| = \operatorname*{ess\,sup}_{x \in R^d} |v(x, t)|,$$

$$\|v\|_Z = \sup_{(x,t) \in Z} |v(x, t)|;$$

« ess sup » stands for « essential supremum ».

For $M > 0$, $T > 0$, $0 \leqslant t' < t''$, let

$$Z(t', t''; M) = \{(x, t, v): x \in R^d,\ t' \leqslant t \leqslant t'',\ v = (v_1, \dots, v_n),\ |v| \leqslant M\},$$

where $|v| = \max_i |v_i|$, and let $Z(T; M) = Z(0, T; M)$. For a bounded, measurable function $H(x, t, v)$ on $Z(t', t''; M)$, we define

$$\|H(\cdot, t, \cdot)\| = \operatorname*{ess\,sup}_{\substack{x \in R^d \\ |v| \leqslant M}} |H(x, t, v)|,$$

$$\|H\|_{Z(t',t'';M)} = \operatorname*{ess\,sup}_{(x,t,v) \in Z(t',t'';M)} |H(x, t, v)|.$$

The « length » of a vector $V = (V_1, \dots, V_s)$, say of s components, in general is measured by

$$|V| = \max_{i=1}^{s} |V_i|,$$

and the norm of a vector function $V(x) = \big(V_1(x), \dots, V_s(x)\big)$ with bounded, measurable components $V_i(x)$, $i = 1, \dots, s$, is defined by

$$\|V\| = \max_i \|V_i\|.$$

But if V is a function of x, t, the norm considered will relate to a particular time t or to a particular layer Z_{t_1,t_2} and if a function of x, t, v to a particular time or to a zone $Z(t', t''; M)$, as in the scalar case.

The d-dimensional vectors $f_i = (f_{i1}, \dots, f_{id})$ and the array $f = (f_1, \dots, f_n)$ are partial exceptions to the previous rule, for we stipulate

$$|f_i| = \sum_{j=1}^{d} |f_{ij}| \quad \text{and} \quad |f| = \max_i |f_i|$$

with similar understandings for their norms, which pertain again to time t or to some zone $Z(t', t''; M)$.

Partial differentiation with respect to x_i will be indicated by the symbol ∂_i or by means of the subscript x_i and partial differentiation with respect to t by ∂_t or the subscript t. The symbol ∂_x will refer generically to any particular ∂_i, ∂_x^k to any particular partial differentiation of k-th order with respect to $x_1, \dots, x_d$, $\nabla_x^k v$ denoting the array of all partial derivatives of v of k-th order with respect to $x_1, \dots, x_d$. For an n-dimensional vector function $v(x) = \big(v_1(x), \dots, v_n(x)\big)$, v_x will denote the set of derivatives v_{i,x_j}, $i = 1, \dots, n$, $j = 1, \dots, d$, and we write

$$|v_x| = \max_{i,j} |v_{i,x_j}|, \qquad \|v_x\| = \sup_{x \in R^d} |v_x|.$$

Similarly for a vector function of x, t given on some layer Z_{t_1,t_2}.

The vectors $f_i = (f_{i1}, \dots, f_{id})$ and the array $f = (f_1, \dots, f_n)$ again are partial exceptions to the general rules, for we define (using the summation convention)

$$f_{i,x} = f_{ij,x_j} \quad \text{and} \quad f_x = (f_{1\,x}, \dots, f_{n,x})$$

with

$$|f_x| = \max_i |f_{i,x}|.$$

For the array of $n^2 d$ quantities f_{ij,u_k}, we set

$$|f_u| = \max_i \sum_{j,k} |f_{ij,u_k}|.$$

As to $g = (g_1, \dots, g_n)$, we use analogously

$$|g| = \max_i |g_i|, \qquad |g_x| = \max_{i,k} |g_{i,x_k}|, \qquad |g_u| = \max_i \sum_k |g_{i,u_k}|.$$

In all cases, similar conventions are made as to norms on appropriate zones $Z(t', t''; M)$.

In general, for a function $H(\boldsymbol{q})$ of an s-dimensional vector $\boldsymbol{q} = (q_1, \ldots, q_s)$, $H_{\boldsymbol{q}}$ will denote the array of partial derivatives $\{H_{q_1}, \ldots, H_{q_s}\}$. Accordingly, for any s-dimensional vector $\boldsymbol{r} = (r_1, \ldots, r_s)$, by $H_{\boldsymbol{q}}\boldsymbol{r}$ will be meant the scalar product $H_{q_i} r_i$ (summation on i) with similar notation when H is a vector. Consistently with this and previous conventions, by $f_v v_x$ is meant the vector $(f_{ij,v_k} v_{k,x_j})_{i=1,\ldots,n}$ (summation on j and k). Extending this usage, we also shall abbreviate by av_x a system of expressions of the form $\{a_{ijk} v_{j,x_k}\}_{i=1,\ldots,n}$, by bv_t a system of the form $\{b_{i,v_j} v_{j,t}\}_{i=1,\ldots,n}$, and so forth.

If, in (scalar or vector) functions depending upon x, t, v, the argument v is replaced by a (vector) function $w(x, t)$ to produce compounds such as

$$g^*(x, t) = g\big(x, t, w(x, t)\big)\,, \qquad f_i^*(x, t) = f_i\big(x, t, w(x, t)\big)\,,$$
$$f^*(x, t) = f\big(x, t, w(x, t)\big)\,,$$

derivatives of the compound functions are written in such notation as the following:

$$(g)_x = g_x^* \;= (g_{x_k} + g_{u_l} w_{l,x_k})_{k=1,\ldots,d}\,,$$
$$(f_i)_x = f_{i,x}^* = f_{ij,x_j} + f_{ij,u_l} w_{l,x_j}\,,$$
$$(f)_x = \big((f_1)_x, \ldots, (f_n)_x\big)\,,$$

summation again being performed on repeated indices.

In this notation, equation $(E)_0$ is abbreviated as

$$u_t + (f)_x + g = 0\,.$$

3. - Approximate layered solutions and a characterization of their limit as a solution of the full parabolic system.

In this section, we describe the kind of approximate layered solution $u^{(h)}(x, t)$ to be employed in this work and verify for this kind that, if $\lim\limits_{h\to 0} u^{(h)}$ exists, then the limit satisfies the full parabolic system (E), as desired. We begin with some remarks about the two processes that are alternated in a layering procedure: (a) smoothing, and (b) solving the layer equations exactly or approximately.

(a) Smoothing in our layering procedure will be performed by means of averaging operators K_ε transforming any function $v(x)$ bounded and measurable on R^d into

$$\mathrm{K}_\varepsilon v(x) = \int \mathrm{k}(\xi)\, v(x + \varepsilon\xi)\, d\xi = \int \varepsilon^{-d} \mathrm{k}\big((y - x)/\varepsilon\big)\, v(y)\, dy \,.$$

(When integrating over R^d, a single integration sign is used, and the limits of integration are not indicated explicitly. Similarly when integrating over R.) It is assumed that

$$\mathrm{k}(\xi) = k(\xi_1) \dots k(\xi_d) \,,$$

where $k(s)$ is a one-dimensional averaging kernel such that $k(s) \geqslant 0$, $\int k(s)\, ds = 1$, $k(s) = k(-s)$. This one-dimensional kernel additionally is required to be sectionally continuous and sectionally of class C^∞, and also to have certain other properties, all of which pertain, in particular, to the kernel

$$\begin{aligned} k(s) &= \tfrac{1}{2} \quad \text{for } |s| < 1 \\ &= 0 \quad \text{for } |s| > 1 \end{aligned}$$

of arithmetical averaging. Iterated kernels of arithmetical averaging—i.e., the kernels that belong to an arithmetical average of an arithmetical average, and so forth—as well as a Gaussian kernel

$$g(s) = g_b(s) = (2\pi)^{-\frac{1}{2}} b^{-1} \exp\left[- s^2/2b^2\right]$$

also are acceptable for use as $k(s)$.

With reference to a given averaging operator K_ε, we shall refer to ε loosely as the « averaging distance ». When layer height $h > 0$ has been fixed, an individual averaging distance

$$\varepsilon_i = (\lambda_i h)^{\frac{1}{2}}$$

is associated with each μ_i, $i = 1, \dots, n$, where λ_i is a constant depending both on μ_i and on the type of averaging. A vector $\omega(x) = \big(\omega_1(x), \dots, \omega_n(x)\big)$ will be smoothed by applying the operator $\mathrm{S}_i = \mathrm{K}_{\varepsilon_i}$ to its i-th component, the smoothed vector being denoted by $\mathrm{S}\omega(x) = \big(\mathrm{S}_1 \omega_1(x), \dots, \mathrm{S}_n \omega_n(x)\big)$.

(***b***) Exact solutions of the layer equations are obtainable, and can be used in layering, only in special cases, in particular, (i) in the scalar case ($n = 1$), (ii) in case the layer equations are hyperbolic, and $d = 1$, and (iii) in the case in which the layer equations are ordinary differential equations, i.e., $f = 0$. Approximate solutions of the layer equations are more easily and more generally available than exact solutions and by various means, one of which is discussed in Section 4. There, under certain requirements as to S, and with f and g, and their partial derivatives of various orders, assumed bounded in a region $Z(\tau, \tau + \tau'; M)$ $(0 \leqslant \tau < \tau + \tau',\ M > 0)$, an approximate solution $v(x, t)$ of the layer equations is constructed in a sufficiently narrow layer $Z_{\tau,\tau+h}$, $0 < h < \tau'$, to the following specifications. Given a function $\omega(x)$ of class C^{j_0}, $j_0 \geqslant 1$, such that

$$\|\omega\| < M\,, \quad \|\partial_x^j \omega\| \leqslant c_j h^{-j/2}\,, \quad j = 1, \ldots, j_0\,,$$

with certain constants c_j supposed to be sufficiently large, it is required first that

$$v(x, \tau) = \mathrm{S}\omega(x)\,.$$

Secondly, $v(x, t)$ is to be of class C^{j_0} and to be subject to the bounds

$$\|v\|_{Z_{\tau,\tau+h}} \leqslant M\,, \quad \|\partial_x^j v\|_{Z_{\tau,\tau+h}} \leqslant c_j h^{-j/2}\,, \quad j = 1, \ldots, j_0\,.$$

Thirdly, again for sufficiently small h, $v(x, t)$ must satisfy the layer equations approximately in the sense that

$$\|\partial_x^j F[v]\|_{Z_{\tau,\tau+h}} \leqslant \varrho_j h^{(r-j-1)/2}\,, \quad j = 0, 1, \ldots, j_0 - 1\,,$$

where $F[v] = v_t + \big(f(x, t, v)\big)_x$, and where the ϱ_j are constants possibly depending upon M, and r is an integer $\geqslant 2$. Finally the derivatives of v and of $F[v]$ are required to satisfy certain further conditions, which are stated in inequalities (4.28), (4.29), (4.29)* in the conclusion to Theorem 4.3. By means of the construction of Section 4, a positive quantity h^* depending only on M and $M - \|\omega\|$ is produced such that all the foregoing demands (with $j_0 \geqslant 2r$) are met for $0 < h \leqslant h^*$. Without commitment to this particular construction, these properties always will be assumed for the approximate solutions of the layer equations considered in this paper.

In general, the given data $u^0(x)$ have to be replaced by an approximation $u^*(x)$ of class C^{j_0} for which, with sufficiently large constants C_j,

$$\|u^*\| \leqslant M_0\,, \quad \|\partial_x^j u^*\| \leqslant C_j h^{-j/2} \quad \text{for } j = 1, \ldots, j_0\,, \tag{1a}$$

$$\lim_{h \to 0} u^*(x) = u^0(x) \qquad \text{for almost all } x \text{ in } R^d. \tag{1b}$$

(The replacement is not necessary in the case, for instance, in which S is

Gaussian.) To obtain u^*, we apply a Gaussian operator to u^0 or, in case S is arithmetical, apply the iterated averaging S^{j_0} to u_0.

Let M be any constant $\geqslant M_0 + 1$. The first step in layering, after the previous adjustment of the initial data, is to find in Z_1 an exact or approximate solution $u^1(x, t)$ of the layer equations $(E)_0$ satisfying the initial conditions $u^1(x, 0) = Su^*(x)$. (We could just as well define $u^1(x, 0) = u^*(x)$.) The second step is to find in Z_2 an exact or approximate solution $u^2(x, t)$ of equations $(E)_0$ such that $u^2(x, h) = Su^1(x, h)$. Continuing from layer to layer in this way, let m_0 be the largest index for which a chain exists of exact or approximate solutions $u^1(x, t), \dots, u^{m_0}(x, t)$ of equations $(E)_0$ on $Z_1, \dots, Z_{m_0}$, respectively, such that $u^m(x, t)$ is of class C^{j_0} in Z_m and that

$$u^m(x, (m-1)h) = Su^{m-1}(x, (m-1)h)$$

and

$$\|u^m\|_{Z_m} \leqslant M$$

for $m = 1, \dots, m_0$, where $u^0(x, 0) = u^*(x)$. Previous remarks imply that $m_0 \geqslant 1$ if h is sufficiently small; possibly $m_0 = \infty$.

The u^m, $m = 1, \dots, m_0$, are parts, which we assemble into a whole, the « layered solution », or « approximate layered solution », $u^{(h)}$ defined as

$$u^{(h)} = u^m \quad \text{on } Z'_m,\ m = 1, \dots, m_0 .$$

By definition of m_0,

$$\|u^{(h)}\|_{Z^{m_0 h}} \leqslant M . \tag{2}$$

From the properties attributed to approximate solutions of the layer equations and thus, in particular, to the u^m, $u^{(h)}$ will be of class C^{j_0} in each Z'_m, $m = 1, \dots, m_0$, and, if h is sufficiently small, its derivatives will be subject to the bounds

$$\|\partial_x^j u^{(h)}\|_{Z^{m_0 h}} \leqslant c_j h^{-j/2}, \quad j = 1, \dots, j_0 . \tag{2$'$}$$

The conditions

$$u^{(h)}(x, (m-1)h) = u^{(h)}(x, (m-1)h + 0) = Su^{(h)}(x, (m-1)h - 0) \tag{3}$$

will hold upon the layer interfaces. Again for sufficiently small h, in each Z'_m, $u^{(h)}$ will satisfy the layer equations approximately in the sense that

$$u_t^{(h)} + (f(x, t, u^{(h)}))_x + g(x, t, u^{(h)}) = \mathfrak{h}^{(h)}(x, t) , \tag{4}$$

where

(5) $$\|\partial_x^j \mathfrak{h}^{(h)}\|_{Z^{m_0 h}} \leqslant \varrho_j h^{(r-j-1)/2}, \qquad j = 0, 1, \dots, j_0 - 1\,;$$

as before, r is an integer $\geqslant 2$. Certain further conditions result from applying inequalities (4.28), (4.29), (4.29)* to the u^m, $m = 1, \dots, m_0$, individually. These conditions, some of which will be used later to estimate the quantities

$$p_j^{(h)}(t) = t^{j/2}\|\partial_x^j u^{(h)}(\cdot, t)\|\,, \qquad j = 1, \dots, j_0 - 1\,,$$

are that, for $0 < t \leqslant m_0 h$,

(2)* $$p_{j+1}^{(h)}(t) \leqslant X_j h^{-\frac{1}{4}} t^{\frac{1}{4}}\,, \qquad j = 0, 1, \dots, j_0 - 1\,,$$

(5)* $$t^{j/2}\|\partial_x^j \mathfrak{h}^{(h)}(\cdot, t)\| \leqslant Y_j h^{(r-1)/2}\,, \qquad j = 1, \dots, j_0 - 1\,,$$

and, if $j_0 \geqslant r + 1$,

(5)** $$t^{i/2}\|\partial_x^i \mathfrak{h}^{(h)}(\cdot, t)\| \leqslant Y_i' t^{-\frac{1}{2}} h^{r/2}\,, \qquad i = 0, 1, \dots, j_0 - r - 1\,,$$

where X_j and Y_j depend polynomially upon $p_1^{(h)}(t), \dots, p_j^{(h)}(t)$, and $h^{1/2}$; Y_i' depends polynomially upon $p_1^{(h)}(t), \dots, p_{i+r+1}^{(h)}(t)$; and X_j, Y_j, Y_i' also depend upon M, $m_0 h$.

Now that we have described layering, we can explain why it should be expected to be of use in parabolic problems. Postponing all intermediate considerations, for this purpose we make, with reference to some zone Z^{T_0}, $T_0 > 0$, the following hypotheses:

(A) For all sufficiently small, positive h, exact or approximate layered solutions $u^{(h)}(x, t)$ exist on Z^{T_0} and satisfy an inequality of the form $\|u^{(h)}\|_{Z^{T_0}} \leqslant M$, where M is a constant independent of h. Moreover, $\|f\|_{Z(T_0,M)}$, $\|f_x\|_{Z(T_0,M)}$, $\|f_u\|_{Z(T_0,M)}$, $\|g\|_{Z(T_0,M)}$ are finite.

(B) In the layers Z_m' contained in Z^{T_0}, the $u^{(h)}$ are of class C^2 and satisfy equations of the form

$$u_t^{(h)} + \big(f(x, t, u^{(h)})\big)_x + g(x, t, u^{(h)}) = \mathfrak{h}^{(h)}(x, t)\,,$$

where $\lim_{h\to 0} \mathfrak{h}^{(h)} = 0$ uniformly on any compact subregion of Z_{δ,T_0} with any positive $\delta < T_0$.

(C) A constant $M_1(T_0)$ exists such that, for all sufficiently small, positive h, $\|u_x^{(h)}(\cdot, t)\| \leqslant M_1(T_0) t^{-\frac{1}{2}}$ if $0 < t \leqslant T_0$.

(D) As $h \to 0$, the $u^{(h)}(x, t)$ approach a limit $u(x, t)$ for $t > 0$, uniformly on any compact subregion of Z_{δ, T_0}. The limit $u(x, t)$ is continuous in (x, t), of class C^1 with respect to t, and of class C^2 with respect to x, for $0 < t \leqslant T_0$.

Of the various requirements later to be laid upon the averaging kernel $\varepsilon^{-1}k(x/\varepsilon)$, we here demand that the first four absolute moments

$$m_k = \int k(\xi)|\xi|^k d\xi\,, \qquad k = 1, 2, 3, 4\,,$$

be finite.

Our first contention is:

THEOREM 1. *Under assumptions* (A) *to* (D), *and with an averaging operator as described, the limit* $u(x, t)$ *is a solution of the parabolic system of equations* (E).

Later (Theorem 2) we shall also verify that $u(x, t)$ satisfies the initial condition (IC) at almost all points of the initial plane $t = 0$.

The main task in proving Theorem 1 is to show that $u(x, t)$ is a « weak solution » of equations (E) in the following sense.

DEFINITION. A bounded, measurable function $u(x, t)$ is a *weak solution* of (E) if the conditions

$$(6)\quad -\iint_{Z^{T_0}} \{u_i \varphi_t + f_i(x, t, u)\varphi_x - \varphi g_i\}\, dx\, dt = \mu_i \iint_{Z^{T_0}} u_i \Delta\varphi\, dx\, dt \qquad i = 1, \dots, n\,,$$

are satisfied with any (scalar) « test function » $\varphi(x, t)$ of class C^4 and vanishing outside a cylinder

$$C(X_0;\, \delta,\, T_0) = \{(x, t) \in Z_{\delta, T_0} \colon |x| \leqslant X_0\}$$

of finite radius X_0 and base-elevation δ $(0 < \delta < T_0)$.

By $f_i\varphi_x$ is meant the scalar product $f_{ij}\varphi_{x_j}$, summation over j from 1 to d being understood. The vector $(f_i\varphi_x)_{i=1,\dots,n}$ will be denoted by $f\varphi_x$.

PROOF OF THEOREM 1. If $u(x, t)$ is a weak solution of (E), i.e., satisfies (6), and if $u(x, t)$ also satisfies Assumption (D), then it is easily established using integration by parts that $u(x, t)$ also satisfies (E), as claimed. To prove $u(x, t)$ to be a weak solution, we start by multiplying both sides of equation (5) by a test function $\varphi(x, t)$, integrating over Z_m, and then integrating by parts. Let u^m, $m = 1, \dots, m_0$, again denote the « parts » of $u^{(h)}$, and for convenience assume $(m_0 - 1)h < T_0 \leqslant m_0 h$. We obtain

$$\int \varphi(x, mh)\, u^m(x, mh)\, dx - \int \varphi\big(x, (m-1)h\big)\, u^m\big(x, (m-1)h\big)\, dx$$
$$- \iint_{Z_m} \{u^m \varphi_t + f(x, t, u^m)\varphi_x - \varphi g(x, t, u^m)\}\, dx\, dt$$
$$= \iint_{Z_m} \varphi \mathfrak{h}^m\, dx\, dt \qquad \text{for } m = 1, \dots, m_0 .$$

Summing for $m = 1, \dots, m_0$ gives, since φ is supported in $C(X_0; \delta, T_0)$,

$$(7) \qquad - \iint_{C(X_0;\delta,T_0)} \{u^{(h)}\varphi_t + f(x, t, u^{(h)})\varphi_x - g(x, t, u^{(h)})\varphi\}\, dx\, dt = T^{(h)} + \mathfrak{H}^{(h)} ,$$

where

$$T^{(h)} = \sum_1^{m_0} \int \varphi(x, mh)\{u^{m+1}(x, mh) - u^m(x, mh)\}\, dx ,$$

and

$$\mathfrak{H}^{(h)} = \iint_{C(X_0;\delta,T_0)} \varphi \mathfrak{h}^{(h)}\, dx\, dt .$$

In view of Assumption (B),

$$(8) \qquad \lim_{h \to 0} \mathfrak{h}^{(h)} = 0 \qquad \text{uniformly on } C(X_0; \delta, T_0)$$

and, as we intend to prove,

$$(9) \qquad \lim_{h \to 0} T_i^{(h)} = (1/2)\, m_2 \lambda_i \iint_{C(X_0;\delta,T_0)} u_i \Delta\varphi\, dx\, dt ,$$

$T_i^{(h)}$ being the i-th component of $T^{(h)}$, $i = 1, \dots, n$. If (9) is granted, then in view of (8), letting $h \to 0$ in (7) gives (6), but with $m_2\ \lambda_i/2$ in place of μ_i. Consequently, $u(x, t)$ is a weak solution of (E) if λ_i is determined by the condition $m_2 \lambda_i/2 = \mu_i$.

Contention (9) is implied by the following lemma.

LEMMA 1. *For $\varepsilon > 0$, let K_ε denote an averaging operator as previously specified. If $v(x)$ is a bounded, measurable function on R^d and $\varphi(x)$ a function of class C^4 in R^d with compact support, then*

$$(10) \qquad \left| \int \varphi(x)\{\mathrm{K}_\varepsilon v(x) - v(x)\}\, dx - (m_2/2)\varepsilon^2 \int v \Delta\varphi\, dx \right| \leqslant (d^4/24)\, m_4 |\varphi^{(4)}|_L \|v\| \varepsilon^4 ,$$

where

$$|\varphi^{(4)}|_L = \int \max_{i,j,k,l} |\varphi_{x_i x_j x_k x_l}| dx .$$

That this lemma implies (9) is seen as follows. In view of (10), $T_i^{(h)}$ differs from

$$(m_2/2)\,\varepsilon_i^2 \sum_1^{m_0} \int u_i^{(h)}(x, mh)\, \Delta\varphi(x, mh)\, dx$$

by a quantity that approaches 0 with h. The last expression differs by a similarly vanishing quantity from

$$(m_2/2)(\varepsilon_i^2/h) \int_0^{T_0} \int u_i^{(h)}(x, t)\, \Delta\varphi(x, t)\, dx\, dt ,$$

since, by Assumptions (A) and (D), a constant M_1^* exists for which $\|u_i^{(h)}\|_{Z'_m} \leq M_1^*$ for $\delta \leq mh \leq T_0$. Hence, $T_i^{(h)}$ and $(m_2/2)\,\lambda_i \int_0^{T_0} \int u_i^{(h)} \Delta\varphi\, dx\, dt$ differ by a quantity that vanishes with h, while the second of these two converges as $h \to 0$ towards the second member of (9), in view of (A) and (D). Thus, condition (9) follows from the indicated lemma, as asserted.

PROOF OF LEMMA 1. Accompanying any vector $\xi = (\xi_1, \ldots, \xi_d)$ in R^d are the « re-directed » vectors

$$\xi(\beta) = (\beta_1 \xi_1, \ldots, \beta_d \xi_d) ,$$

where $\beta_j = +1$ or -1 for $j = 1, \ldots, d$, and $\beta = (\beta_1, \ldots, \beta_d)$. Since $\mathrm{k}(\xi)$ is even in each coordinate of ξ,

$$\mathrm{k}(\xi) = \mathrm{k}(\xi(\beta)) .$$

By reflection of the appropriate coordinate axes, we thus have

$$\int \mathrm{k}(\xi)\, v(x + \varepsilon\xi)\, d\xi = \int \mathrm{k}(\xi)\, v(x + \varepsilon\xi(\beta))\, d\xi$$

for each of the 2^d « re-directing » vectors β. Summing over all the re-directing vectors and dividing by their number gives

$$\int \mathrm{k}(\xi)\, v(x + \varepsilon\xi)\, d\xi = 2^{-d} \sum_\beta \int \mathrm{k}(\xi)\, v(x + \varepsilon\xi(\beta))\, d\xi .$$

Hence,

$$\int \varphi(x)\Big(\int \mathrm{k}(\xi)v(x+\varepsilon\xi)\,d\xi\Big)dx = 2^{-d}\sum_{\beta}\int\varphi(x)\Big(\int \mathrm{k}(\xi)v\big(x+\varepsilon\xi(\beta)\big)\,d\xi\Big)dx \tag{11}$$

$$= 2^{-d}\sum_{\beta}\int \mathrm{k}(\xi)\Big(\int\varphi(x)v\big(x+\varepsilon\xi(\beta)\big)\,dx\Big)d\xi$$

$$= 2^{-d}\sum_{\beta}\int \mathrm{k}(\xi)\Big(\int\varphi\big(x-\varepsilon\xi(\beta)\big(v(x)\,dx\Big)d\xi$$

$$= \int v(x)\Big(\int \mathrm{k}(\xi)\,2^{-d}\sum_{\beta}\varphi\big(x-\varepsilon\xi(\beta)\big)\,d\xi\Big)dx\,.$$

For fixed β, by Taylor's expansion,

$$\begin{aligned}\varphi\big(x-\varepsilon\xi(\beta)\big) = \varphi(x) &- \varepsilon\sum_{i}\beta_i\xi_i\varphi_{x_i}(x)\\ &+ (\varepsilon^2/2)\sum_{i,j}\beta_i\beta_j\xi_i\xi_j\varphi_{x_ix_j}(x)\\ &- (\varepsilon^3/6)\sum_{i,j,k}\beta_i\beta_j\beta_k\xi_i\xi_j\xi_k\varphi_{x_ix_jx_k}(x)\\ &+ (\varepsilon^4/24)\sum_{i,j,k,l}\beta_i\beta_j\beta_k\beta_l\xi_i\xi_j\xi_k\xi_l\varphi_{ijkl}(x,\xi)\,,\end{aligned} \tag{12}$$

where the indices i, j, k, l are summed from 1 to d, and

$$\varphi_{ijkl}(x,\xi) = 4\int_0^1 \varphi_{x_ix_jx_kx_l}\big(x-\varepsilon(1-t)\xi(\beta)\big)t^3\,dt\,.$$

We need to sum both sides of (12) over all re-directing vectors β. For each set of values of $d-1$ components of β, however, the remaining component can take one of just two values $+1$ and -1. Hence, summing linear or cubic expressions in the β_i, as occur in the terms in (12) of orders ε or ε^3, produces 0 out of total cancellation. Summing the terms of order ε^2 gives

$$(\varepsilon^2/2)\sum_{\beta}\sum_{i,j}\beta_i\beta_j\xi_i\xi_j\varphi_{x_ix_j}(x) = (\varepsilon^2/2)\sum_{i,j}\Big\{\sum_{\beta}\beta_i\beta_j\Big\}\xi_i\xi_j\varphi_{x_ix_j}(x)\,. \tag{13}$$

If $i\neq j$, then $\sum_{\beta}\beta_i\beta_j = 0$ for the same reasons as before. To argue this in greater detail, consider $\sum_{\beta}\beta_1\beta_2$ as representative of such sums. The set of all re-directing vectors $\beta=(\beta_1,\dots,\beta_d)$ can be grouped into pairs, the members of which are identical except in their first components. The two terms in the sum $\sum_{\beta}\beta_1\beta_2$ corresponding to the two members of such a pair cancel: hence, the sum is 0, as asserted.

For $i = j$, the inner summation in (13) is

$$\sum_\beta \beta_i^2 = \sum_\beta 1 = 2^d,$$

there being 2^d terms in the summation. Therefore, the summation (13) reduces to

$$(\varepsilon^2/2)2^d \sum_i \xi_i^2 \varphi_{x_i x_i}(x)$$

and the result of summing (12) over all β to

$$2^{-d} \sum_\beta \varphi(x - \varepsilon\xi(\beta)) = \varphi(x) + (\varepsilon^2/2) \sum_i \xi_i^2 \varphi_{x_i x_i}(x) + \varepsilon^4 r(x, \xi),$$

where

$$|r(x, \xi)| \leqslant (1/24)|\varphi^{(4)}| \sum_{i,j,k,l} |\xi_i||\xi_j||\xi_k||\xi_l| = (1/24)|\varphi^{(4)}| \Big(\sum_i |\xi_i|\Big)^4$$
$$= (1/24)|\varphi^{(4)}| \sum_{i_1+\ldots+i_d=4} \frac{4!}{i_1!\ldots i_d!} |\xi_1|^{i_1} \ldots |\xi_d|^{i_d}$$

with $|\varphi^{(4)}| = |\varphi^{(4)}(x, \xi)| = \max_{p,q,r,s} |\varphi_{pqrs}(x, \xi)|$. Substituting this in (11) shows that

$$\int \varphi(x)\Big(\int \mathrm{k}(\xi) v(x + \varepsilon\xi)\, d\xi\Big) dx = \int v(x)\varphi(x)\, dx + (\varepsilon^2 m_2/2)\int v(x)\Delta\varphi(x)\, dx + \varepsilon^4 R,$$

since $\int \mathrm{k}(\xi)|\xi_i|^k d\xi = m_k$ for all i and k. For the same reason, the coefficient of ε^4, namely

$$R = \int\int v(x)\,\mathrm{k}(\xi)\, r(x, \xi)\, dx\, d\xi,$$

is, in absolute value,

$$\leqslant (1/24)|\varphi^{(4)}|_L \sum_{i_1+\ldots+i_d=4} \frac{4!}{i_1!\ldots i_d!} \int \mathrm{k}(\xi)|\xi_1|^{i_1} \ldots |\xi_d|^{i_d} d\xi .$$

The integral in the last expression is equal to $m_{i_1} \ldots m_{i_d}$ and thus to one of the monomials

$$m_1^4, \quad m_1^2 m_2, \quad m_1 m_3, \quad m_2^2, \quad m_4 .$$

From the fact that none of the quantities m_1, $m_2^{\frac{1}{2}}$, $m_3^{\frac{1}{3}}$ exceeds $m_4^{\frac{1}{4}}$, it follows that each of the five monomials is $\leqslant m_4$. Therefore,

$$|R| \leqslant (1/24)|\varphi^{(4)}|_L m_4 \sum_{i_1+\dots+i_d=4} \frac{4!}{i_1!\dots i_d!} = (1/24)|\varphi^{(4)}|_L m_4 d^4$$

and the lemma is proved.

To justify the initial conditions (IC), we add to the previous hypotheses concerning K_ε the supposition that, for any $v(x)$ integrable on R^d,

$$\|\mathrm{K}_\varepsilon^j G(\delta)v\| \leqslant \mathrm{const}\, \sigma^{*-d}\int |v(x)|dx, \tag{14}$$

where $\sigma^{*2} = j\varepsilon^2 + \delta^2$, and $\mathrm{G}(\delta)v(x) = \int \mathrm{g}_\delta(y-x)v(y)dy$. In the case in which K_ε is Gaussian or arithmetical averaging, such an inequality follows from the remark inserted after Theorem 2 in Section 9. (The remark pertains to $d=1$, but is easily extended to any d.)

THEOREM 2. *Under supposition* (14) *in addition to the hypotheses of Theorem* 1, *we have*

$$\lim_{t\to 0} u(x,t) = u^0(x) \qquad \textit{for almost all } x \textit{ in } R^d. \tag{15}$$

PROOF. Note first that, if $j = [t/h]$, i.e., $jh \leqslant t < (j+1)h$, then

$$\lim_{\substack{h\to 0\\ t\to 0}} \mathrm{S}^j u^* = u^0 \qquad \text{almost everywhere in } R^d. \tag{16}$$

Since $u^*(x) = \mathrm{G}(\delta)u^0(x) = \int \mathrm{g}_\delta(y-x)u^0(y)dy$, in justifying (16) componentwise, it suffices to show that if $\varepsilon = \varepsilon(h)$ and $\delta = \delta(h)$ approach 0 with h, then for any bounded, measurable function $v(x)$ on R^d,

$$\lim_{\substack{h\to 0\\ t\to 0}} \mathrm{K}_\varepsilon^j \mathrm{G}(\delta)v = v \tag{17}$$

at the Lebesgue points of v.

Let $\mathrm{k}^*(\xi) = k^*(\xi_1)\dots k^*(\xi_d)$ denote the kernel of the averaging operator $\mathrm{K}^* = \mathrm{K}_\varepsilon^j \mathrm{G}(\delta)$: $\mathrm{K}^*v(x) = \int \mathrm{k}^*(y-x)v(y)dy$. Then for any $a > 0$

$$\begin{aligned}\mathrm{K}^*v(x) - v(x) &= \int \mathrm{k}^*(y-x)[v(y)-v(x)]dy \\ &= \int_{|y-x|<a\sigma^*} + I_2 = I_1 + I_2,\end{aligned}$$

I_2 being an integral over the complement of the (d-dimensional) interval $|y-x|<a\sigma^*$, which is a union of semi-infinite intervals, on each of which $|y_i - x_i| > a\sigma^*$ for at least one index j.

Let $\eta > 0$. Chebychev's inequality and the boundedness of v show that, if a is sufficiently large, then $|I_2| < \eta/2$. Using inequality (14) in I_1 gives

$$|I_1| \leqslant \text{const}\, \sigma^{*-d} \int\limits_{|y-x|<a\sigma^*} |v(y)-v(x)|\,dy\,,$$

so that $|I_1| < \eta/2$ for sufficiently small σ^* if x is a Lebesgue point of v. In this way, (16) is established.

Since

$$|u(x,t)-u^0(x)| \leqslant |u(x,t)-u^{(h)}(x,t)| + |u^{(h)}(x,t) - S^j u^*(x)| + |S^j u^*(x) - u_0(x)|\,,$$

and in view of (16) and assumption (D), to prove (15) it suffices to justify the inequality

$$\|u^{(h)}(\cdot, t) - S^j u^*\| < M' t^{\frac{1}{2}} \tag{18}$$

with suitable constant M' depending on T_0. To do so, we consider the equations

$$u_t^m + \big(f(x,t,u^m)\big)_x + g(x,t,u^m) = \mathfrak{h}^m(x,t) \quad \text{in } Z_m \tag{19$_m$}$$

satisfied by the «parts» $u^m(x,t)$, $m = 1, \ldots, m_0$, of any layered solution $u^{(h)}(x,t)$. Taking a particular time τ for which $0 < \tau < T_0$ and $j = [\tau/h] > 1$, we apply S^{j-m} to both sides of $(19)_m$ for $1 \leqslant m \leqslant j$ and then integrate with respect to t from $(m-1)h$ to mh. Since $u^m(x,(m-1)h) = Su^{m-1}(x,(m-1)h)$, we obtain

$$\begin{aligned} S^{j-m}u^m(x,mh) - S^{j-m+1}u^{m-1}\big(x,(m-1)h\big) \\ = -\int\limits_{(m-1)h}^{mh} S^{j-m}\big[\big(f(x,t,u^m)\big)_x + g(x,t,u^m)\big]\,dt \\ + \int\limits_{(m-1)h}^{mh} S^{j-m}\mathfrak{h}^m(x,t)\,dt\,, \qquad m = 1, \ldots, j-1\,. \end{aligned}$$

Besides these, we have similarly

$$\begin{aligned} u^j(x,\tau) - Su^{j-1}\big(x,(j-1)h\big) \\ = -\int\limits_{(j-1)h}^{\tau} \big[\big(f(x,t,u^j)\big)_x + g(x,t,u^j)\big]\,dt + \int\limits_{(j-1)h}^{\tau} \mathfrak{h}^j(x,t)\,dt\,, \end{aligned}$$

and adding all j relations, we obtain

$$u^j(x,\tau)-\mathrm{S}^j u^*(x)=-\sum_{m=1}^{j-1}\int_{(m-1)h}^{mh}\mathrm{S}^{j-m}\big[\big(f(x,t,u^m)\big)_x+g(x,t,u^m)\big]\,dt$$
$$-\int_{(j-1)h}^{\tau}\big[\big(f(x,t,u^j)\big)_x+g(x,t,u^j)\big]\,dt$$
$$+\sum_{m=1}^{j-1}\int_{(m-1)h}^{mh}\mathrm{S}^{j-m}\mathfrak{h}^m(x,t)\,dt+\int_{(j-1)h}^{\tau}\mathfrak{h}^j(x,t)\,dt\,.$$

This implies the needed inequality (18), in view of assumptions (A), (B), (C) and Theorem 2 is thus proved.

4. - A type of approximate solution in thin layers of first-order partial differential equations.

In Section 3, we listed the properties needed in this paper of approximate solutions of the layer equations

$$F[u]\equiv u_t+\big(f(x,t,u)\big)_x+g(x,t,u)=0\,.$$

The approximate solutions of interest exist in a layer $Z_{\tau,\tau+h}$ and satisfy initial conditions of the form

$$v(x,\tau)=w(x)\,, \tag{1}$$

where the initial data are averages,

$$w(x)=\mathrm{S}\omega(x)\,, \tag{2}$$

with averaging distances ε_i proportional to $h^{\frac{1}{2}}$. Under appropriate hypotheses, we shall here show that approximate solutions as desired can be obtained in the form

$$v(x,t)=w(x)+\sum_{j=1}^{r}b_j(x)(t-\tau)^j, \tag{3}$$

in which $b_1(x),\dots,b_r(x)$ are determined by the requirement that, for $\tau>0$,

$$F[v]=O\big((t-\tau)^r\big)\quad\text{in } Z_{\tau,\tau+h}\text{ as } t-\tau\to 0\ (^1)\,. \tag{4}$$

(1) There are surely many ways of obtaining approximate solutions. We mention in particular a new method of F. Treves [24] as possibly useful in the present context.

A new stipulation concerning S is made, that

$$\|\partial_x^i S\omega\| \leqslant \alpha_i \|\omega\| h^{-i/2}, \quad i = 1, \ldots, r+1,$$

with certain constants $\alpha_1, \ldots, \alpha_{r+1}$. This condition is satisfied, in particular, by arithmetical averaging repeated $r+1$ times.

Rather strong differentiability assumptions are called for in the present method. With M a constant for which $\|\omega\| < M$, we shall assume that, in $Z(\tau, \tau+h; M)$,

$$a(x, t, u) = f_u(x, t, u) \quad \text{and} \quad c(x, t, u) = f_x(x, t, u) + g(x, t, u)$$

have bounded partial derivatives of orders $\leqslant r + j^0$, where $j^0 \geqslant r$. We shall also assume $\omega(x)$ to be of class C^{j^0} and the conditions

$$\|\partial_x^j \omega\| \leqslant N\beta_j h^{-j/2}, \quad j = 1, \ldots, j^0,$$

to hold, where $N \geqslant M$, and the β_j are certain positive constants depending on $\alpha_1, \ldots, \alpha_r$. Defining $w(x)$ by (2), and using condition (4) to determine the coefficients in (3), we shall find concerning the resulting function $v(x, t)$ that, if h is sufficiently small, then $\|v\|_{Z_{\tau,\tau+h}} \leqslant M$,

$$\|\partial_x^j F[v]\|_{Z_{\tau,\tau+h}} \leqslant \varrho_j h^{(r-j-1)/2}, \quad j = 0, 1, \ldots, j^0 - 1,$$

and

$$\|\partial_x^j v\|_{Z_{\tau,\tau+h}} \leqslant N\beta_j h^{-j/2}, \quad j = 1, \ldots, j^0,$$

the constants ϱ_j depending on bounds in $Z(\tau, \tau+h; M)$ for the partial derivatives of a and c of orders $\leqslant j + r$.

These results will be consequences of Theorems 1 and 2 to follow. Further properties, which are of importance in estimating the derivatives $\partial_x^k v$, are given in Theorem 3.

Theorem 1 is not concerned directly with ω or with S and requires only the assumptions that

$$\|w\| = a_0 < M \tag{5}$$

and

$$\|\partial_x^k w\| \leqslant a_k h^{-k/2} \quad \text{for } k = 1, \ldots, j^0 + r, \tag{6}$$

where $a_1, \ldots, a_{j_0+r}$ are constants.

THEOREM 1. *Let $0 \leqslant \tau < \tau + h$. Under assumptions* (5), (6), *the coefficients $b_1(x), \ldots, b_r(x)$ determined by condition* (4) *satisfy inequalities of the form*

$$\|\partial_x^j b_k\| \leqslant \varrho_{jk} h^{-(j+k)/2} \tag{7}$$

for $j = 0, 1, \ldots, j^0$, $k = 1, \ldots, r$. Thus, for $0 \leqslant t - \tau \leqslant h$,

$$\|\partial_x^j v(\cdot, t)\| \leqslant \left\{a_j + \sum_{k=1}^{r} \varrho_{jk} h^{k/2}\right\} h^{-j/2} . \tag{7$'$}$$

If $h \leqslant \frac{1}{4}$ and $a_0 + 2\varrho^* h^{\frac{1}{2}} \leqslant M$, where $\varrho^* = \max_{k=1,\ldots,r} \varrho_{0k}$, then $\|v\|_{Z_{\tau,\tau+h}} \leqslant M$. If in addition, $h^{\frac{1}{2}} \leqslant 1/4\varrho^*$, then

$$\|\partial_x^j F[v]\|_{Z_{\tau,\tau+h}} \leqslant \varrho_j h^{(r-j-1)/2} , \quad j = 0, 1, \ldots, j^0 - 1 . \tag{8}$$

The ϱ_{jk} depend just on bounds in $Z(\tau, \tau + h; M)$ for the partial derivatives of $a(x, t, u)$ and $c(x, t, u)$ of orders up to $r + j - 1$, the ϱ_j on bounds for these derivatives of orders up to $r + j$. The constants $a_1, \ldots, a_{r+j}$ also are involved.

PROOF. If $v(x, t)$ is given by (3) with any coefficients $b_j(x)$, Taylor's expansion of $F[v]$ will produce an expression of the form

$$F[v] = \sum_{k=0}^{r-1} B_k(x, t, b_0, b_1, \ldots, b_r)(t - \tau)^k + F_r[v] , \tag{9}$$

in which $b_0 = w$. Then the b_j, $j = 1, \ldots, r$, are determined by the conditions

$$B_0 = 0 , \quad B_1 = 0, \ldots, B_{r-1} = 0 . \tag{10}$$

Although this procedure is well known, we sketch it to call attention to what is essential in proving contentions (7) and (8). To this end, we introduce a concept of « weight » as follows. The weight $j + k$ is attached to $\partial_x^k b_j$, $j = 0, 1, \ldots, r$, $k = 0, 1, \ldots$. The weight of a product of weighted quantities is to be the sum of the weights of the quantities, the weight of a sum of weighted quantities the greatest of the weights of the summands. The weight 0 is attributed to any constant and also to $a(\cdot, \cdot, \cdot)$ and $c(\cdot, \cdot, \cdot)$, and to their first and higher partial derivatives, evaluated at $(x, t, w(x))$ or $(x, t, v(x, t))$. (At the same time, $v(x, t)$ might be endowed with the weight 0, for instance by giving to $t - \tau$ a suitable negative weight such as -1.) It follows, in particular, that if p is a polynomial of weight j in the derivatives

$\partial_x^k w$ (recall $w = b_0$), the coefficients in p being quantities of weight 0, then under assumptions (6) $\|p\| < \text{const}\, h^{-j/2}$.

As a preliminary to making the Taylor expansion of $F[v] = v_t + a(x, t, v)v_x + c(x, t, v)$, we first verify that

$$\text{(11)} \qquad \partial_t^i c(x, t, v(x, t)) = \sum_{j=i}^{L_i} p_{ij}(x, t, v(x, t); b_1, \dots, b_r)(t-\tau)^{j-i} \qquad \text{for } i = 0, 1, 2, \dots, j^0,$$

where p_{ij} is a polynomial—more exactly, p_{ij} is a vector whose components are polynomials—in $b_1, \dots, b_r$ of weight at most j with coefficients that are bounded functions of x, t, $v(x, t)$, and where $L_i < \infty$. Analogously,

$$\text{(12)} \qquad \partial_t^i[a(x, t, v(x, t))v_x(x, t)] = \sum_{m=i}^{M_i} q_{im}(t-\tau)^{m-i} \qquad \text{for } i = 0, 1, \dots,$$

where $M_i < \infty$, and q_{im} is a polynomial in $b_1, \dots, b_r, b_{0,x}, b_{1,x}, \dots, b_{r,x}$ of weight at most $m + 1$ and with coefficients that are bounded functions of x, t, $v(x, t)$. Each term in the polynomial q_{im} contains at least one x-derivative of some b_j, i.e., at least one component of some gradient vector $b_{j,x}$.

Identity (11) is proved by mathematical induction. In the case $i = 1$, we have

$$\begin{aligned} \partial_t c &= c_t + c_u \dot{v} \\ &= c_t + c_u \sum_{j=1}^{r} b_j j(t-\tau)^{j-1}, \end{aligned}$$

so that $p_{11} = c_t + c_u b_1$, and $p_{1j} = jc_u b_j$ for $j > 1$. The weight of p_{1j} is j, and thus (11) is verified in the case $i = 1$. If (11) holds for an integer $i \geqslant 1$, then

$$\begin{aligned} \partial_t^{i+1} c(x, t, v) &= \partial_t\left\{\sum_{j=i}^{L_i} p_{ij}(t-\tau)^{j-i}\right\} \\ &= \sum_{j=i}^{L_i}\left\{p_{ij,t} + p_{ij,v}\sum_{k=1}^{r} k b_k (t-\tau)^{k-1}\right\}(t-\tau)^{j-i} + \sum_{j=i+1}^{L_i} p_{ij}(j-i)(t-\tau)^{j-i-1}. \end{aligned}$$

It is not necessary to collect the coefficients of the various powers of $t-\tau$ to verify the contention as to their weights. For instance, in the double summation, the coefficient of $(t-\tau)^{j+k-i-1}$ is $kp_{ij,v}b_k$, and its weight, which is equal to the weight of $p_{ij,v}$ plus the weight of b_k, is at most $j + k$. This accords with the rule stated in connection with (11), and the other coefficients also obey this rule, as is seen in a similar way. Thus (11) is proved.

To justify (12), Leibnitz's rule is applied to express $\partial_t^i(av_x)$ as a sum of the terms

$$(13)\qquad \binom{i}{k}(\partial_t^{i-k}a)(\partial_t^k v_x)\,, \qquad k=0,\ldots,i\,.$$

By (11),

$$\partial_t^{i-k}a=\sum_{s=i-k}^{m_{i-k}} Q_{i-k,s}(t-\tau)^{s-i+k}\,,$$

Q_{js} being a polynomial in $b_1,\ldots,b_r$ of weight at most s, and m_{i-k} being finite. Since $v_x=\sum_0^r b_{j,x}(x)(t-\tau)^j$, the product (13) becomes

$$\binom{i}{k}\sum_{s=i-k}^{m_{i-k}}\sum_{j=k}^{r} j(j-1)\ldots(j-k+1)\,b_{j,x}Q_{i-k,s}(t-\tau)^{s+j-i}\,.$$

The weight of $b_{j,x}Q_{i-k,s}$ being $\leqslant j+1+s$, it is clear that if the double summation is rearranged to take the form of the right member of (12), then the weight of q_{im}, the coefficient of $(t-\tau)^{m-i}$, must be $\leqslant m+1$, as asserted. Contentions (11) and (12) thus are both proved.

By (11),

$$\partial_t^i c(x,t,v(x,t))\big|_{t=\tau}=p_{ii}(x,\tau,w(x);b_1,\ldots,b_r)\ ,$$

and Taylor's expansion of $c(x,t,v(x,t))$ in powers of $t-\tau$ takes the form

$$c(x,t,v(x,t))=c(x,\tau,w(x))+\sum_1^{r-1}P_i(t-\tau)^i+c_r\,,$$

where $P_i=p_{ii}(x,\tau,w(x);b_1,\ldots,b_r)/i!$ is a polynomial in $b_1,\ldots,b_r$ of weight at most i with coefficients that are functions of x, τ, $w(x)$. The remainder c_r is expressible as

$$(14)\quad c_r=[(r-1)!]^{-1}\int_\tau^t (t-s)^{r-1}\,\partial_s^r c(x,s,v(x,s))\,ds=\int_\tau^t (t-s)^{r-1}\sum_{j=r}^{L_r}P_{rj}(s-\tau)^{j-r}ds$$

by (11), where $P_{rj}=p_{rj}/(r-1)!$ is a polynomial of weight at most j in the quantities b_0, $b_1,\ldots,b_r$ and their derivatives with coefficients that are functions of x, s, $v(x,s)$.

Similarly, by use of (12), we have

$$a(x,t,v)\,v_x(x,t)=a(x,\tau,w)\,w_x+\sum_1^{r-1}Q_i(t-\tau)^i+A_r\,,$$

where $Q_i = q_{ii}/i!$ is a polynomial in $b_1, \ldots, b_r, b_{0,x}, \ldots, b_{r,x}$ of weight at most $i + 1$ with coefficients that are functions of x, τ, w, and where

(15) $$A_r = [(r-1)!]^{-1}\int_\tau^t (t-s)^{r-1} \sum_{m=r}^{M_r} q_{rm}(s-\tau)^{m-r} ds .$$

It is important to note that each term of Q_i contains at least one derivative $b_{i,x}$.

The expansions of c and av_x, and the equality $v_t = \sum_{j=0}^{r-1} (j+1) b_{j+1}(t-\tau)^j$ give Taylor's expansion of $F[v]$:

$$F[v] = v_t + a(x, t, v) v_x + c(x, t, v)$$
$$= \sum_0^{r-1} B_i(t-\tau)^i + F_r[v] ,$$

where

$$B_0 = b_1 + a(x, \tau, w) w_x + c(x, \tau, w)$$
$$B_i = (i+1) b_{i+1} + Q_i + P_i , \quad i = 1, \ldots, r-1 ,$$

and

$$F_r = A_r + c_r .$$

We now verify that conditions (10) can be used to determine $b_1, \ldots, b_r$ recursively and that b_j so determined is a polynomial of weight j in the derivatives of w with coefficients that are bounded functions of x, τ, w. This work out for b_1 immediately. In the case $i > 0$, the condition $B_i = 0$ is equivalent to

(16) $$(i+1) b_{i+1} = - Q_i - P_i .$$

Since P_i is of weight $\leqslant i$, P_i contains no b_j with $j > i$. Since Q_i is of weight $\leqslant i + 1$, Q_i contains no b_j with $j > i + 1$ and also contains no derivative of $b_{i+1}, \ldots, b_r$. Furthermore, each term of Q_i contains some derivative $b_{j,x}$ and, being of weight $\leqslant i + 1$, cannot also contain b_{i+1}. Consequently, Q_i is free of $b_{i+1}, \ldots, b_r$ and of their derivatives, so that (16) is a recursion stating that

$$b_1 = S_1(b_{0,x}) ,$$
$$b_{i+1} = S_{i+1}(b_1, \ldots, b_i, b_{0,x}, b_{1,x}, \ldots, b_{i,x}) , \quad i = 1, \ldots, r-1 ,$$

where each S_j is a polynomial of weight $\leqslant j$ in the indicated arguments, its coefficients being functions of x, τ, w. From this recursion, an argument of mathematical induction, which we omit, shows that for $i = 1, \ldots, r$,

$$b_i = T_i(w_x, w_{xx}, \ldots, \partial_x^i w) , \tag{17}$$

where T_i is a polynomial of weight i with coefficients that are functions of x, τ, w. From (17), $\partial_x^j b_i$ is obtained as a polynomial in $w_x, w_{xx}, \ldots, \partial_x^{j+i} w$ of weight $i + j$ and with coefficients that again are functions of x, τ, w. Under assumptions (6), conditions (7), and therefore (7)′, follow from this.

Conditions (7)′ for $j = 0$ imply that, for $0 \leqslant t - \tau \leqslant h$,

$$|v(x, t)| \leqslant |w(x)| + \varrho^* \sum_1^r h^{k/2}$$

$$\leqslant a_0 + 2\varrho^* h^{1/2}$$

since we are assuming $\sqrt{h} \leqslant \frac{1}{2}$. (Recall $a_0 = \|w\|$, and $\varrho^* = \max_{k=1,\ldots,r} \varrho_{0k}$.) Hence,

$$|v(x, t)| \leqslant M \qquad \text{on } Z_{\tau,\tau+h} ,$$

and $a(x, t, u)$, $c(x, t, u)$, and the partial derivatives of these functions of the orders entering into our process, are bounded accordingly on $Z_{\tau,\tau+h}$.

Before proceeding to (8), we must consider the effect of differentiating with respect to x an expression of the form

$$P \equiv \sum_0^J p_j (t - \tau)^j , \tag{18}$$

in which p_j is a polynomial of weight at most $j + \zeta$ with coefficients that are of weight 0 and are (smooth) functions of x, t, $v(x, t)$. We shall conclude that a derivative of this expression of first order with respect to x is a sum of the same form in which the coefficient of $(t - \tau)^j$ is a polynomial of weight at most $j + \zeta + 1$. To this end consider a typical term of p_j, say

$$A(x, t, v)\, G_j(b) ,$$

in which $G_j(b)$ denotes a product of b_j's and their derivatives of weight at most $j + \zeta$. Differentiating this term with respect to x gives a sum

$$A(x, t, v)\, G_{j+1}(b) + A_x(x, t, v)\, G_j(b) + A_v(x, t, v) \sum_{k=0}^r b_{k,x}(t - \tau)^k G_j(b) ,$$

in which G_{j+1} is a polynomial of weight at most $j + \zeta + 1$. This result is of the form $\sum_{k=0}^{r} P_{j,k}(t-\tau)^k$ with polynomials $P_{j,k}$ of weight at most $j + k + \zeta + 1$. Hence, $\partial_x p_j$ also is of this form, so that, in form,

$$P_x = \sum_{k=0}^{J} \sum_{j=0}^{r} P_{j,k}(t-\tau)^{j+k},$$

with $P_{j,k}$ again denoting a polynomial of weight at most $j + k + \zeta + 1$ in the b_l and their derivatives. Arranging the right-hand side in powers of $t-\tau$ gives a representation of P_x similar to (18), except that the coefficient of $(t-\tau)^l$ will be a polynomial of weight $l + \zeta + 1$, $l = 0, \ldots, J$, as contended.

The previous remark is applied to the Taylor remainders c_r and A_r given in (14) and (15). With respect to the first of these, we find

$$\partial_x^i c_r = \int_\tau^t (t-s)^{r-1} \sum_{j=r}^{L'_r} P_{irj}(s-\tau)^{j-r}\, ds,$$

where $L'_r < \infty$, and P_{irj} is a polynomial of weight at most $i + j$ in the quantities $b_0, \ldots, b_r$ and their derivatives with coefficients that again are (smooth) functions of x, s, $v(x, s)$. In view of (7), $|P_{irj}| \leqslant \text{const}\; h^{-(i+j)/2}$, so that

$$\begin{aligned} |\partial_x^i c_r| &\leqslant \text{const} \sum_{j=r}^{L_r} h^{-(i+j)/2}(t-\tau)^j \\ &\leqslant \text{const} \sum_{j=r}^{L'_r} h^{(j-i)/2} \\ &\leqslant \text{const}\; h^{(r-i)/2} \end{aligned}$$

for $h \leqslant \frac{1}{4}$, $0 \leqslant t-\tau \leqslant h$. In an exactly similar way,

$$|\partial_x^i A_r| \leqslant \text{const}\; h^{(r-i-1)/2}.$$

Since $F[v] = F_r[v] = A_r + c_r$, by (9) and (10), the foregoing estimates imply that

$$|\partial_x^i F[v]| \leqslant \text{const}\; h^{(r-1-i)/2},$$

justifying (8). Theorem 1 is thus proved, except for a count of the derivatives of $a(x, t, u)$ and $c(x, t, u)$ involved. Scrutinizing the previous arguments shows that derivatives of orders up to $i-1$ enter into b_i, derivatives of orders up to $r-1$ into $v(x,t)$, and derivatives of orders up to r into $F_r[v]$. This information underlies the statements made in the theorem about differentiability.

To use Theorem 1 in a layering process, it is of course necessary that inequalities of type (6) be fulfilled in every layer. This will be the case trivially for averaging kernels $k(x)$ of sufficient differentiability, but less stringent conditions will do as well. In fact, concerning S, it suffices for this purpose that constants $\alpha_1, \dots, \alpha_r$ exist such that

$$|\partial_x^i \mathrm{S}\omega| \leqslant \alpha_i \|\omega\| h^{-i/2}, \quad i=1,\dots,r, \tag{19}$$

for any bounded, measurable $\omega(x)$.

The proposition to follow is stated in terms of the numbers

$$A = \max(\alpha_1, \dots, \alpha_r), \quad A_0 = A+1, \quad \text{and} \quad A_m = AA_{m-1}+1, \quad m=1,2,\dots,$$

recursively defined.

THEOREM 2. *Suppose $a(x,t,u)$ and $c(x,t,u)$ to have bounded partial derivatives in $Z(\tau, \tau+h; M)$ of orders up to $(q+2)r$, where q is an integer $\geqslant 0$. In the construction described in Theorem* 1, *let $w(x)$ be given by* (2) *with* S *subject to* (19) *and thus to*:

$$\|\partial_x^i \mathrm{S}\omega\| \leqslant A \|\omega\| h^{-i/2}, \quad i=1,\dots,r. \tag{20}$$

If $\|\omega\| \leqslant N$, $\omega(x) \in C^{(q+1)r}$, and

$$\|\partial_x^{mr+i}\omega\| \leqslant NA_m h^{-(mr+i)/2}, \quad i=1,\dots,r, \quad m=0,1,\dots,q, \tag{21}$$

then $v(x,t) \in C^{(q+1)r}$, and for sufficiently small h,

$$\|\partial_x^{mr+i} v\|_{Z_{\tau,\tau+h}} \leqslant NA_m h^{-(mr+i)/2} \tag{22}$$

for the same indices, $i=1,\dots,r$, $m=0,1,\dots,q$.

REMARK. Inequality (22) justifies the assertions at the beginning of this section concerning $\partial_x^j v$ for $j=1,\dots,j^0$ in the case in which $j^0=(q+1)r$. To obtain the other cases, in which $qr \leqslant j^0 < (q+1)r$ for some positive integer q, only trivial changes need to be made in the following proof.

PROOF. Apply ∂_x^{mr+i} to (3) for $i=1,\dots,r$, $m=0,1,\dots,q$. In view of (17) and the remarks subsequent to (18), the result is of the form

$$\partial_x^{mr+i} v = \partial_x^{mr+i} w + \sum_{j-1}^{r} T_{j,mr+i}(\partial_x w, \dots, \partial_x^{mr+i+j} w)(t-\tau)^j, \tag{23}$$

where $T_{j,k}$ denotes a polynomial in $\partial_x w, \ldots, \partial_x^{j+k} w$ of weight $j+k$ with coefficients that are bounded functions of x, τ, w. The coefficients can be estimated in absolute value by quantities that depend on N, T only, it being understood that $0 \leqslant \tau < \tau + h \leqslant T$. For the derivatives of w of orders up to $(q+1)r$, we have from (21) the estimates

$$(24) \quad \|\partial_x^{mr+i} w\| \leqslant \|\partial_x^{mr+i} \omega\| \leqslant N A_m h^{-(mr+i)/2}, \quad i = 1, \ldots, r, \quad m = 0, 1, \ldots, q.$$

For the derivatives of orders $(q+1)r+1$ to $(q+2)r$, assumptions (20) (with $i = r$) and (21) give

$$(25) \quad \begin{aligned} \|\partial_x^{(m+1)r+i} w\| &= \|\partial_x^r S \partial_x^{mr+i} \omega\| \\ &\leqslant N A A_m h^{-((m+1)r+i)/2}, \qquad i = 1, \ldots, r, \; m = 0, 1, \ldots, q. \end{aligned}$$

By use of (24) and (25) we have estimates of the form

$$(26) \quad \|T_{j,mr+i}(\partial_x w, \ldots, \partial_x^{mr+i+j} w)\| \leqslant A_{j,m}(A_0, \ldots, A_m; N, T) h^{-(mr+i+j)/2},$$

where $A_{j,m}$ is a polynomial in $A_0, \ldots, A_m$ with coefficients that depend upon N, T. It follows from (26) that, if we require h to be such that

$$\sum_{j=1}^{r} A_{j,m} h^{j/2} < N,$$

then

$$\|\partial_x^{mr+i} v\|_{Z_{\tau,\tau+h}} \leqslant \|\partial_x^{mr+i} w\| + N h^{-(mr+i)/2}.$$

In the case $m = 0$, we use the estimate

$$\|\partial_x^i w\| \leqslant A N h^{-i/2},$$

arising from (20), to obtain

$$\|\partial_x^i v\|_{Z_{\tau,\tau+h}} \leqslant N(A+1) h^{-i/2} = N A_0 h^{-i/2}, \quad i = 1, \ldots, r.$$

In the case $m > 0$, applying (25) with m replacing $m+1$ gives

$$\|\partial_x^{mr+i} w\| \leqslant N A A_{m-1} h^{-(mr+i)/2},$$

and thus

$$\|\partial_x^{mr+i} v\|_{Z_{\tau,\tau+h}} \leqslant N(A A_{m-1} + 1) h^{-(mr+i)/2},$$

these results verifying (22) completely. Theorem 2 is proved.

The following results will be used in obtaining bounds for the derivatives of layered solutions, which ultimately will be of the form $\|\partial_x^k u^{(h)}(\cdot, t)\| \leqslant t^{-k/2} P_k$ in each zone in which $u^{(h)}$ is known to exist and be bounded.

THEOREM 3. *Suppose* (19) *to hold for* $i = 1, \dots, r+1$. *Suppose also that for some* $k \geqslant 1$,

$$\|\partial_x^l \omega\| \leqslant W_l \tau^{-l/2} \quad \text{for } l = 1, 2, \dots, k. \tag{27}$$

Then for $h \leqslant \tau$,

$$\|\partial_x^{k+1} v\|_{Z_{\tau,\tau+h}} \leqslant X_k \tau^{-k/2} h^{-1/2} \tag{28}$$

and

$$\|\partial_x^k F[v]\|_{Z_{\tau,\tau+h}} \leqslant Y_k \tau^{-k/2} h^{(r-1)/2}, \tag{29}$$

where X_k *and* Y_k *depend polynomially on* $W_1, \dots, W_k$, $\tau^{1/2}$, $h^{1/2}$, *and also depend on bounds in* $Z(\tau, \tau+h; M)$ *for the derivatives of* $a(x, t, u)$ *and* $c(x, t, u)$ *of orders up to* $r+k$. *If* $0 \leqslant j \leqslant k-r-1$, *then*

$$\|\partial_x^j F[v]\|_{Z_{\tau,\tau+h}} \leqslant Y_j' \tau^{-(j+1)/2} h^{r/2}, \tag{29*}$$

with Y_j' *depending polynomially on* $W_1, \dots, W_{j+r+1}$, $\tau^{1/2}$, *and also depending on bounds in* $Z(\tau, \tau+h; M)$ *for the derivatives of* a *and* c *of orders up to* $j+r$.

PROOF. The derivatives of $w = S\omega$ of orders $\leqslant k$ can be estimated by means of $W_1\tau^{-1/2}, \dots, W_k\tau^{-k/2}$, the derivatives of w of orders $k+1, \dots, k+r+1$ by means of $\alpha_l W_k \tau^{-k/2} h^{-l/2}$ for $l = 1, \dots, r+1$, respectively. These estimates will imply bounds for $\partial_x^{k+1} b_j$, which, being of weight $j+k+1$, is a sum of terms consisting of products of the type

$$(\partial_x w)^{p_1} (\partial_x^2 w)^{p_2} \dots (\partial_x^{j+k+1} w)^{p_{j+k+1}} \tag{30}$$

multiplied by bounded coefficients, where

$$p_1 + 2p_2 + \dots + (j+k+1) p_{j+k+1} \leqslant j+k+1. \tag{31}$$

In (30), $(\partial_x^l w)^{p_l}$ is to be interpreted as a product of p_l derivatives of w of l-th order, the p_l derivatives not necessarily being alike. Each such derivative, in absolute value, is $\leqslant C\varphi_l(h)\psi_l(\tau)$, where

$$\begin{aligned} \varphi_l(h) &= 1 && \text{if } l \leqslant k, \\ &= h^{-(l-k)/2} && \text{if } k < l \leqslant r+k+1, \end{aligned}$$

$$\begin{aligned} \psi_l(\tau) &= \tau^{-l/2} \qquad \text{if } l \leqslant k, \\ &= \tau^{-k/2} \qquad \text{if } k < l \leqslant r + k + 1, \end{aligned}$$

and where C is a generic constant to signify possibly different values in different usages. Thus, in absolute value, the product (30) is

$$\begin{aligned} &\leqslant C \prod_{l=1}^{j+k+1} \varphi_l(h)^{p_l} \psi_l(\tau)^{p_l} \\ &= C\tau^{-\frac{1}{2}[p_1+2p_2+\ldots+kp_k+k(p_{k+1}+\ldots+p_{k+j+1})]} h^{-\frac{1}{2}(p_{k+1}+2p_{k+2}+\ldots+(j+1)p_{k+j+1})}. \end{aligned}$$

This is of the form $Ch^{-X/2}\tau^{-Y/2}$, where

$$X + Y = p_1 + 2p_2 + \ldots + (k + j + 1)p_{k+j+1}\,.$$

By (31), $X + Y \leqslant j + k + 1$, and our previous estimate of (30) is seen to be

$$\begin{aligned} &\leqslant Ch^{-X/2}\tau^{-\frac{1}{2}(j+k+1-X)} \\ &= C\tau^{-(j+k+1)/2}(\tau/h)^{X/2}\,. \end{aligned} \tag{32}$$

Again by (31),

$$\begin{aligned} X &= p_{k+1} + 2p_{k+2} + \ldots + (j+1)p_{k+j+1} \\ &\leqslant \left(\frac{j+1}{k+j+1}\right)[(k+1)p_{k+1} + \ldots + (k+j+1)p_{k+j+1}] \leqslant j+1\,. \end{aligned}$$

Hence $(\tau/h)^{X/2} \leqslant (\tau/h)^{(j+1)/2}$ for $h \leqslant \tau$, and substitution in (32) shows any quantity (30) to be, in absolute value, $\leqslant C\tau^{-k/2}h^{-(j+1)/2}$. Consequently,

$$\|\partial_x^{k+1} b_j\| \leqslant C\tau^{-k/2}h^{-(j+1)/2}\,, \qquad j = 1, \ldots, r\,, \tag{33}$$

for $h \leqslant \tau$, and

$$\begin{aligned} \|\partial_x^{k+1} v\|_{Z_{\tau,\tau+h}} &\leqslant \|\partial_x^{k+1} w\| + \sum_{j=1}^{r} \|\partial_x^{k+1} b_j\| h^j \\ &\leqslant \alpha_1 W_k \tau^{-k/2} h^{-\frac{1}{2}} + C\tau^{-k/2}(1 + h^{\frac{1}{2}} + h + \ldots) \\ &\leqslant C\tau^{-k/2} h^{-\frac{1}{2}}\,, \end{aligned}$$

this justifying (28).

To prove (29), we must estimate $\partial_x^k c_r$ and $\partial_x^k A_r$, but only the latter is considered in detail. From (15) and the remark made in connection with (18),

$$\partial_x^k A_r = [(r-1)!]^{-1} \int_\tau^t (t-s)^{r-1} \sum_{m=r}^{M_r} q_{krm}(s-\tau)^{m-r}\, ds\,. \tag{34}$$

Here q_{krm} is a polynomial of weight at most $k+m+1$ in the coefficients $b_0, b_1, \ldots, b_r$ and their derivatives of orders $\leqslant k+1$. Since b_j for $j=1, \ldots, r$ is a polynomial in the derivatives of w of orders $\leqslant j$, it follows that q_{krm} is a polynomial in the derivatives of w of orders $\leqslant r+k+1$. The previous methods show from this that

$$h^m \|q_{krm}\| \leqslant C\tau^{-(k+m+1)/2}(\tau/h)^{\frac{1}{2}((r+1)/(r+k+1))(m+k+1)} h^m$$
$$= C\tau^{-k/2}(h/\tau)^{(k(m-r))/(2(r+k+1))} h^{(r-1)/2+(m-r)/2};$$

to arrive at the final equality requires some elementary calculations. It follows from this and (34) that

$$\|\partial_x^k A_r\| \leqslant C\tau^{-k/2} h^{(r-1)/2}(1 + h^{\frac{1}{2}} + h + \ldots)$$
$$\leqslant C\tau^{-k/2} h^{(r-1)/2} \quad \text{for } h \leqslant \tau .$$

Since $\partial_x^k c_r$ is subject to a milder estimate, inequality (29) follows. Contention (29)* is established by similar means, this completing the proof of Theorem 3.

5. - A common domain independent of layer height, and a common bound, for approximate layered solutions.

It is essential that, for sufficiently small h, layered solutions $u^{(h)}$ exist in a common zone. As will be seen in this section, this will be so if the S_i are Gaussian, or arithmetical, or result from one or more repetitions of arithmetical averaging. More generally, the smoothing operators in this discussion are required to commute with differentiation and to satisfy certain inequalities. Again, as in Section 3, let $S = (S_1, \ldots, S_n)$ act on vector functions $v(x) = (v_1(x), \ldots, v_n(x))$ to produce $Sv(x) = (S_1 v_1(x), \ldots, S_n v_n(x))$. Also, let $S^j v(x) = (S_1^j v_1(x), \ldots, S_n^j v_n(x))$. Recall that $S_i = K_{\varepsilon_i}$, and let

$$\varepsilon_* = \min_i \varepsilon_i, \qquad \varepsilon^* = \max_i \varepsilon_i$$
$$\lambda_* = \varepsilon_*^2/h, \qquad \lambda^* = \varepsilon^{*2}/h, \qquad \mu_* = \min_i \mu_i,$$
$$\Lambda = \varepsilon^*/\varepsilon_* = \sqrt{\lambda^*/\lambda_*}\ .$$

The requirement of commuting with differentiation is that for all $v(x) \in C^1$,

$$\partial_x(Sv) = S(\partial_x v), \qquad \partial_t(Sv) = S(\partial_t v)\ . \tag{1}$$

The inequalities referred to state that for all bounded, continuous n-dimensional vectors $v(x)$ on R^d,

$$\|\partial_x S^j v\| \leqslant s_1 (j\varepsilon_*^2)^{-\frac{1}{2}} \|v\| \quad \text{for } j = 1, 2, \ldots, \tag{2}$$

and

$$\|\partial_x^2 S^j v\| \leqslant s_2 (j\varepsilon_*^2)^{-1} \|v\| \quad \text{for } j = 2, 3, \ldots, \tag{2$'$}$$

where s_1 and s_2 are absolute constants. These follow (in some cases with different constants) from the one-dimensional inequalities (9.1), (9.2).

That the $u^{(h)}$ exist in a common zone will follow from limitations upon their growth implied by certain integral relations we now derive. Holding h fixed at first, let $u^m(x, t)$, $m = 1, \ldots, m_0$, denote the parts of the approximate layered solution $u^{(h)}(x, t)$, as in Section 3. For each m, $u^m(x, t)$ is, in particular, a C^1-solution in Z_m of an equation of the form

$$u_t^m + \left(f(x, t, u^m)\right)_x + g(x, t, u^m) = \mathfrak{h}^m, \tag{3}$$

where

$$\|\mathfrak{h}^m\|_{Z_m} \leqslant \varrho_0 h^{\frac{1}{2}}. \tag{4}$$

For $0 < \tau \leqslant m_0 h$, define

$$m_1 = [\tau/h] = \text{largest integer } m \text{ such that } mh \leqslant \tau\,;$$

thus $u^{(h)} = u^{m_1+1}$ in $Z_{(m_1 h, \tau)}$. Now apply S^{m_1+1-m} to the two members of (3) to obtain

$$S^{m_1+1-m} u_t^m + S^{m_1+1-m}\Big\{\Big(f(x, t, u^m(x, t))\Big)_x + g(x, t, u^m(x, t))\Big\} = S^{m_1+1-m}\mathfrak{h}^m \quad \text{for } m = 1, \ldots, m_1. \tag{5$_m$}$$

In view of (1),

$$\int_{(m-1)h}^{mh} S^j u_t^m(x, t)\, dt = S^j u^m(x, mh) - S^j u^m(x, (m-1)h)$$

for any positive integer j, while by construction

$$u^m(x, (m-1)h) = S u^{m-1}(x, (m-1)h).$$

Hence, by integrating both sides of $(5)_m$ with respect to t from $(m-1)h$

to mh we have

$$\text{(6)}\quad \mathrm{S}^{m_1+1-m}u^m(x, mh) = \mathrm{S}^{m_1+2-m}u^{m-1}\big(x, (m-1)h\big)$$
$$-\int_{(m-1)h}^{mh}\Big\{\mathrm{S}^{m_1+1-m}\Big(f\big(x, t, u^m(x, t)\big)\Big)_x + \mathrm{S}^{m_1+1-m}g\big(x, t, u^m(x, t)\big)\Big\}dt + \mathfrak{H}^m$$
$$\text{for } m = 1, \ldots, m_1,$$

where by (4)

$$\|\mathfrak{H}^m\| = \Big\|\int_{(m-1)h}^{mh}\mathrm{S}^{m_1+1-m}\mathfrak{h}^m\,dt\Big\| \leqslant \varrho_0 h^{\frac{3}{2}}.$$

In the equation for $m = 1$, we have used the convention that $u^0(x, 0) = u^*(x)$. Similarly to (6), we also have

$$\text{(6)}'\quad u^{m_1+1}(x, \tau) = \mathrm{S}u^{m_1}(x, m_1h)$$
$$-\int_{m_1h}^{\tau}\big\{f\big(x, t, u^{m_1+1}(x, t)\big)_x + g\big(x, t, u^{m_1+1}(x, t)\big)\big\}\,dt + \mathfrak{H}^{m_1+1}(x, \tau)$$

with $\|\mathfrak{H}^{m_1+1}(\cdot, \tau)\| \leqslant \varrho_0 h^{\frac{1}{2}}(\tau - m_1h)$. Adding the m_1+1 relations (6) and (6)′ and making appropriate cancellations gives

$$\text{(7)}\quad u^{m_1+1}(x, \tau) = \mathrm{S}^{m_1+1}u^*(x)$$
$$-\sum_{m=1}^{m_1}\int_{(m-1)h}^{mh}\mathrm{S}^{m_1+1-m}\Big\{\Big(f\big(x, t, u^m(x, t)\big)\Big)_x + g\big(x, t, u^m(x, t)\big)\Big\}dt$$
$$-\int_{m_1h}^{\tau}\Big\{\Big(f\big(x, t, u^{m_1+1}(x, t)\big)\Big)_x + g\big(x, t, u^{m_1+1}(x, t)\big)\Big\}dt + \mathfrak{J}^{(h)}(x, \tau),$$

where $\|\mathfrak{J}^{(h)}(\cdot, \tau)\| \leqslant \varrho_0 \tau h^{\frac{1}{2}}$. In view of the definition of $u^{(h)}$, this implies the estimate

$$\text{(8)}\quad \|u^{(h)}(\cdot, \tau)\| \leqslant \|\mathrm{S}^{m_1+1}u^*\| + \int_{m_1h}^{\tau}\|f_x^{(h)}(\cdot, t) + g^{(h)}(\cdot, t)\|\,dt$$
$$+\sum_{m=1}^{m_1}\int_{(m-1)h}^{mh}\|\mathrm{S}^{m_1+1-m}\big(f_x^{(h)}(\cdot, t) + g^{(h)}(\cdot, t)\big)\|\,dt + \varrho_0\tau h^{\frac{1}{2}},$$

where $f^{(h)}(x, t) = f\big(x, t, u^{(h)}(x, t)\big)$, $g^{(h)}(x, t) = g\big(x, t, u^{(h)}(x, t)\big)$.

We interrupt our present argument with the remark, which will be of use in Section 6, that a similar procedure to the foregoing provides estimates of derivatives of $u^{(h)}$ with respect to x. It is necessary to assume $u^{(h)}$ to be of class C^{j_0}, $j_0 \geq 2$, in each Z'_m and to satisfy conditions (3.2) to (3.5). The functions $f(x, t, u)$ and $g(x, t, u)$ also must be sufficiently smooth. (If $u^{(h)}$ is constructed by the method of Section 4, it is simplest to require that $j_0 = (q+1)r$, q being any nonnegative integer, and that $f(x, t, u)$ and $g(x, t, u)$, in $Z(0, (m_0+1)h; M)$, have continuous, bounded partial derivatives of orders up to $(q+2)r+1$ and $(q+2)r$, respectively.) Under these assumptions, ∂_x^k can be applied to the two members of equation (3) with $k \leq j_0 - 1$. After that, S^{m_1+1-m} is applied and then exactly the same steps followed as previously had led to (8). In view of (3.4) and (3.5)*, the outcome here is the inequality

$$(8)_k \qquad \|\partial_x^k u^{(h)}(\cdot, \tau)\| \leq \|S^{m_1+1}\partial_x^k u^*\| + \int_{m_1 h}^{\tau} \|\partial_x^k (f_x^{(h)} + g^{(h)})\| dt$$

$$+ \sum_{m=1}^{m_1} \int_{(m-1)h}^{mh} \|S^{m_1+1-m}\partial_x^k (f_x^{(h)} + g^{(h)})\| dt + \sum_{m=1}^{m_1} \mathfrak{H}_k^m + \mathfrak{H}_k(\tau) \qquad \text{for } k = 1, \dots, j_0 - 1$$

with

$$\mathfrak{H}_k^m = \int_{(m-1)h}^{mh} \|\partial_x^k S^{m_1+1-m}\mathfrak{h}^{(h)}(\cdot, t)\| dt,$$

$$\mathfrak{H}_k(\tau) = \int_{m_1 h}^{\tau} \|\partial_x^k \mathfrak{h}^{(h)}(\cdot, t)\| dt.$$

Our present aim is to use (8) to obtain an estimate of

$$U(t) = \sup_{0 \leq s \leq t} \|u^{(h)}(\cdot, s)\|$$

that is independent of h.

Let $F(t, v)$, $F^*(t, v)$, $F_1(t, v)$, $G(t, v)$, $G_1(t, v)$ be positive continuous functions for $t \geq 0$, $v \geq 0$, nondecreasing as t increases or v increases, such that for $x \in R$, $0 \leq t \leq T$, $|u| \leq v$,

$$|f(x, t, u)| \leq F(t, v), \qquad |f_u(x, t, u)| \leq F^*(t, v), \qquad |f_x(x, t, u)| \leq F_1(t, v),$$

$$|g(x, t, u)| \leq G(t, v), \qquad |g_x(x, t, u)|, \qquad |g_u(x, t, u)| \leq G_1(t, v).$$

In case f has bounded, continuous derivatives with respect to x, u of order $k \geq 2$, also let $F_k(t, v)$ be a positive continuous function, nondecreasing in both arguments, that is an upper bound for the absolute values of the derivatives of f with respect to x, u of orders up to k. Let $G_k(t, v)$ play a similar role with respect to the derivatives of orders not greater than k of $g(x, t, u)$.

In addition, let $U_0(\tau) = \|S^{m_1+1}u^*\|$ for $m_1 h \leqslant \tau < (m_1+1)h$.

We shall use these quantities to estimate the terms on the right side of (8). First, with reference in (8) to the integral from $m_1 h$ to τ, we have

$$\|g^{(h)}(\cdot, t)\| \leqslant \|g^{(h)}(\cdot, t)\| \leqslant G(t, U(t)) \leqslant G(\tau, U(\tau)) ,$$

and by (3.2)′

$$\|\partial_x f^{(h)}(\cdot, t)\| \leqslant F_1(\tau, U(\tau)) + F^*(\tau, U(\tau))\, C_1 h^{-\frac{1}{2}} .$$

Concerning the summands in the summation occurring in (8), we have obviously

$$\|S^{m_1+1-m} g^{(h)}(\cdot, t)\| \leqslant G(t, U(t)) .$$

Furthermore, by applying properties (1) and (2),

$$\begin{aligned}\|S^{m_1+1-m}\partial_x f^{(h)}(\cdot, t)\| &= \|\partial_x S^{m_1+1-m} f^{(h)}(\cdot, t)\| \\ &\leqslant s_1(m_1+1-m)^{-\frac{1}{2}}\varepsilon_*^{-1}\|f^{(h)}(\cdot, t)\| \\ &\leqslant s_1(m_1+1-m)^{-\frac{1}{2}}\varepsilon_*^{-1}F(t, U(t)) .\end{aligned}$$

But for $m \leqslant m_1$ and $(m-1)h \leqslant t < \tau < (m_1+1)h$, we have (crudely)

$$(9) \qquad (m_1+1-m)h \geqslant \tfrac{1}{2}\{(m_1+1)h - (m-1)h\} > \tfrac{1}{2}(\tau - t) .$$

Hence, for $(m-1)h \leqslant t < \tau$,

$$\|S^{m_1+1-m}\partial_x f^{(h)}(\cdot, t)\| \leqslant \sqrt{2}\, s_1 \lambda_*^{-\frac{1}{2}}(\tau - t)^{-\frac{1}{2}} F(t, U(t)) .$$

Substituting from these inequalities into (8) shows that for $0 \leqslant \tau < < (m_0+1)h$, $\|u^{(h)}(\cdot, \tau)\|$ is not greater than the right hand member of the relation

$$(10) \qquad \begin{aligned} U(\tau) \leqslant U_0(\tau) &+ h\Big\{F_1(\tau, U(\tau)) + F^*(\tau, U(\tau))\, C_1 h^{-\frac{1}{2}} + G(\tau, U(\tau))\Big\} \\ &+ \int_0^\tau \{\sqrt{2}\, s_1 \lambda_*^{-\frac{1}{2}}(\tau - t)^{-\frac{1}{2}} F(t, U(t)) + G(t, U(t))\}\, dt + \varrho_0 \tau h^{\frac{1}{2}} .\end{aligned}$$

For that reason, this relation (10) holds for all $h > 0$ and all τ for which $0 \leqslant \tau \leqslant T$ and $U(\tau) \leqslant M$.

From relation (10), it is easy to show that all layered solutions $u^{(h)}(x, t)$ exist in a common zone if h is sufficiently small. Assumptions appropriate to the construction of Section 4 are made as to the smoothness of $f(x, t, u)$ and $g(x, t, u)$.

THEOREM 1. *Given* $\|u^0\| \leqslant M_0 < \infty$, *let* $M \geqslant M_0 + 1$ *and* $T > 0$. *In* $Z(0, T; M)$, *suppose* $f(x, t, u)$ *and* $g(x, t, u)$ *to have bounded partial derivatives of respective orders up to* 3 *and* 2. *Let* $u^*(x)$ *be an approximation to* $u^0(x)$ *of class* C^2 *satisfying conditions* (3.1*a*,*b*) *with* $j_0 = 2$, *and in carrying out a layering procedure, use averaging operators that satisfy conditions* (1), (2). (*Gaussian averaging, arithmetical averaging, and repeated arithmetical averaging are among the admitted operators.*) *Under these hypotheses, positive constants* T_0, h_0 *exist such that all layered solutions* $u^{(h)}(x, t)$ *for which* $0 < h \leqslant h_0$ *exist on* Z^{T_0} *and obey the restrictions* $\|u^{(h)}\|_{Z^{T_0}} \leqslant M$, $\|u_x^{(h)}\|_{Z^{T_0}} \leqslant C_1 h^{-\frac{1}{2}}$, *where* C_1 *is a constant depending on* M, T.

PROOF. Recall that $U_0(\tau) \leqslant \|u^0\| \leqslant M_0$ and $h\varepsilon_*^{-1} = \lambda_*^{-1}\varepsilon_*$. For this reason, positive h_0, T_0 exist such that when, in the right member of (10), (1) h is replaced by h_0, (2) the functions of $(\tau, U(\tau))$ are replaced by the corresponding functions of (T, M), (3) the functions of $(t, U(t))$ occurring in the integrand are replaced by the corresponding functions of (T, M), and (4) the upper limit of integration is changed from τ to T_0, the new expression obtained is $\leqslant M$. Since the replacements made can only increase the value of the right hand side of (10), it follows that $U(\tau) \leqslant M$, and thus that $\|u^{(h)}(\cdot, \tau)\| \leqslant M$, for $0 \leqslant \tau \leqslant T_0$, $0 < h \leqslant h_0$, as demanded. The estimate of $u_x^{(h)}$ follows immediately from the construction.

Layered solutions $u^{(h)}(x, t)$ with arbitrarily small h cannot in general be expected to exist outside limited belts Z^T. This is because the solutions of parabolic initial value problems under hypotheses (*a*) to (*d*) in general have restricted domains of existence. (See A. Friedman [8], H. Fujita [9], R. T. Glassey [10], H. A. Levine [16, 17], Levine and Payne [18], M. Tsutsumi [25, 26].) Under special circumstances, restrictions on the domains of existence, however, will disappear. In fact, a family of layered solutions $u^{(h)}(x, t)$ will exist on a band of given height T if positive numbers M_T, h_T can be found such that, first, f and g have bounded derivatives of the requisite orders in $Z(T; M_T)$, and, secondly, it is known by some means that $U(t) \leqslant M_T$ for $0 \leqslant t \leqslant T$, $0 < h \leqslant h_T$. One condition under which a priori bounds M_T exist for layered solutions is discussed in the appendix to this section. Other conditions will appear in papers to follow.

We remark that a priori bounds have been given for actual solutions of certain quasi-linear parabolic systems (see T. D. Wentzell [31], A. Jeffrey [11], Ladyzhenskaya, Solonnikov, Ural'ceva [15, Chapter 7], and W. von Wahl [30]). It seems likely that the layered solutions too of such systems, if of form (E), will be bounded a priori.

5A. - Appendix to Section 5 - The existence of layered solutions over long periods of time. The sublinear case.

Here we consider the situation in which (in the notation of Section 5) $F(t, v)$ and $G(t, v)$ are linear functions of v. The result we prove is the following.

THEOREM 1. *Suppose the hypotheses of Theorem* 4.1 *to hold for all* $T > 0$, $M > 0$, *and again require the averaging operators used in the layering procedure to satisfy conditions* (5.1), (5.2). *In addition, suppose that for* $x \in R$, $t \geqslant 0$, *and* $|u| \leqslant v$,

$$|f(x, t, u)| \leqslant \alpha + \beta v\,, \qquad |g(x, t, u)| \leqslant \alpha + \beta v\,,$$

with positive constants α, β. *Then for any* $T > 0$, *a number* h_T *exists such that, if* $0 < h \leqslant h_T$, *all layered solutions* $u^{(h)}$ *exist on* Z^T. *If* $\theta > 0$, *it is possible to determine* h_T *so that, if* $0 < h \leqslant h_T$,

$$\|u^{(h)}(\cdot, t)\| \leqslant (1 + \theta)\, M_0 p(2\pi s_1^2 \beta^2 \lambda_*^{-1} t)\,, \tag{1}$$

where the function $p(t)$ *is representable by a certain series*

$$p(t) = 1 + \sum_{k=1}^{\infty} c_k t^{k/2}\,,$$

converging for $t \geqslant 0$. *The series is described in the lemma of this section.*

PROOF. Let $M = M_T$ denote the number represented by the right side of (1) when t is replaced by T. In the light of the new hypotheses, inequality (5.10) can be rewritten as

$$U(\tau) \leqslant M_0 + C(T, M) h^{\frac{1}{2}} + 2\sqrt{2}\, s_1 \lambda_*^{-\frac{1}{2}} \alpha \tau^{\frac{1}{2}} + \alpha\tau + 2 s_1 \lambda_*^{-\frac{1}{2}} \beta \int_0^\tau (\tau - t)^{-\frac{1}{2}}\, U(t)\, dt + \beta \int_0^\tau U(t)\, dt\,, \tag{2}$$

the quantity $C(T, M)$ being $\geqslant 1$ and independent of h, τ. With any $h > 0$, this relation holds for all τ for which $0 \leqslant \tau < T$ and $U(t) \leqslant M$.

Taking $\theta > 0$, require h_T to be such that

$$h_T \leqslant \theta^2 \; C(T, M)^{-2} \; M_0^2 \, .$$

Then for $0 < h \leqslant h_T$, we have $C(T, M)h^{\frac{1}{2}} \leqslant \theta M_0$, and by referring to (2) and using standard reasoning, we have $U(\tau) \leqslant P(\tau)$, where $P(\tau)$ is a solution of the equality

$$P(\tau) = (1+\theta)M_0 + 2\sqrt{2}\, s_1 \lambda_*^{-\frac{1}{2}} \alpha \tau^{\frac{1}{2}} + \alpha\tau + \sqrt{2}\, s_1 \lambda_*^{-\frac{1}{2}} \beta \int_0^\tau (\tau - t)^{-\frac{1}{2}} P(t)\, dt + \beta \int_0^\tau P(t)\, dt \, .$$

Under the substitution $P(\tau) = (1+\theta) M_0 p(2\pi s_1^2 \beta^2 \lambda_*^{-1} \tau)$, this equality takes the equivalent form

$$p(\sigma) = 1 + a_1 \sigma^{\frac{1}{2}} + a_2 \sigma + \pi^{-\frac{1}{2}} \int_0^\sigma (\sigma - s)^{-\frac{1}{2}} p(s)\, ds + b \int_0^\sigma p(s)\, ds \, , \tag{3}$$

where

$$a_1 = 2\alpha / \{(1+\theta) M_0 \beta \sqrt{\pi}\}\, , \qquad a_2 = \alpha\lambda_* / \{2\pi(1+\theta) M_0 s_1^2 \beta^2\}\, , \qquad b = \lambda_* / (2\pi s_1^2 \beta)\, .$$

It will be proved soon in a lemma that $p(\sigma) < \infty$ for $\sigma > 0$, and it will follow that if $0 < h \leqslant h_T$, then $M = P(T)$ is an upper bound for $U(\tau)$, and hence for $|u^{(h)}(x, t)|$, in Z^T. Hence, in particular, $u^{(h)}(x, t)$ exists in Z^T and also satisfies (1). Thus the theorem is proved.

As A. Pazy has pointed out, it is not necessary to use $p(\sigma)$ in order to derive an a priori bound for $U(\tau)$ on an arbitrary zone. Instead, first obtain from Theorem 5.1 positive constants δ, M' such that $U(\tau) \leqslant M'$ for $0 \leqslant \tau \leqslant \delta$. For $\delta < \tau \leqslant T$, make the substitution

$$\int_0^\tau (\tau - t)^{-\frac{1}{2}}\, U(t)\, dt = \int_0^{\tau-\delta} + \int_{\tau-\delta}^{\tau} \leqslant \delta^{-\frac{1}{2}} \int_0^\tau U(t)\, dt + 2\delta^{\frac{1}{2}}\, U(\tau)\, .$$

As a result, (2) is transformed into a more usual type of inequality from which an a priori bound follows by standard means.

We have to characterize $p(\sigma)$, however, for later applications as well as the previous use.

LEMMA 1. *An integral equation of the form*

$$p(\sigma) = 1 + \sum_{q=1}^{Q} a_q \sigma^{q/2} + \pi^{-\frac{1}{2}} \int_0^\sigma (\sigma - s)^{-\frac{1}{2}} p(s)\, ds + b \int_0^\sigma p(s)\, ds \tag{4}$$

with $b > -\frac{1}{4}$ has a unique solution on the interval $\sigma \geqslant 0$. The solution can be represented as a series

$$p(\sigma) = 1 + \sum_{j=1}^{\infty} c_j \sigma^{j/2} \tag{5}$$

converging absolutely for $\sigma \geqslant 0$.

PROOF. To solve (4), it suffices to determine a series of the form (5) that satisfies (4) formally and that converges absolutely for $\sigma \geqslant 0$. Formally substitute series (5) into (4) and use the rule

$$\begin{aligned} \pi^{-\frac{1}{2}} \int_0^{\sigma} (\sigma - s)^{-\frac{1}{2}} s^{\alpha} ds &= \pi^{-\frac{1}{2}} \sigma^{\alpha + \frac{1}{2}} \int_0^1 s^{\alpha} (1 - s)^{-\frac{1}{2}} ds \\ &= \pi^{-\frac{1}{2}} \sigma^{\alpha+\frac{1}{2}} B(\alpha + 1, \tfrac{1}{2}) = \sigma^{\alpha+\frac{1}{2}} \Gamma(\alpha + 1)/\Gamma(\alpha + \tfrac{3}{2}) \qquad (\alpha > -1), \end{aligned} \tag{6}$$

$B(r, s)$ and $\Gamma(r)$ denoting Euler's beta function and gamma function, respectively. Under the conventions $c_0 = 1$, $c_{-1} = 0$, we arrive at the recursions

$$c_q = a_q + \Gamma((q + 1)/2)\{\Gamma((q + 2)/2)\}^{-1} c_{q-1} + (2b/q) c_{q-2}, \qquad q = 1, \dots, Q,$$

and

$$c_j = \Gamma((j + 1)/2)\{\Gamma(j + 2)/2)\}^{-1} c_{j-1} + (2b/j) c_{j-2}, \qquad j \geqslant Q + 1.$$

These recursions are simplified by writing them in terms of the new variables

$$d_j = \Gamma((j + 2)/2) c_j, \qquad j = 1, 2, \dots, \tag{7}$$

the second set, in particular, becoming

$$d_j = d_{j-1} + b d_{j-2}, \qquad j \geqslant Q + 1, \tag{8}$$

since $\Gamma((j + 2)/2) = (j/2)\Gamma(j/2)$. The solutions of the system (8) are linear combinations of the two solutions $d_k = r^k$, where r is determined by the condition $r^k = r^{k-1} + b r^{k-2}$. This condition is equivalent to the quadratic equation $r^2 - r - b = 0$, the two roots of which are

$$r_1 = \{1 - (1 + 4b)^{\frac{1}{2}}\}/2, \qquad r_2 = \{1 + (1 + 4b)^{\frac{1}{2}}\}/2.$$

Thus, the solutions of (8) are quantities of the form

$$d_j = \alpha_1 r_1^j + \alpha_2 r_2^j, \qquad j \geqslant Q + 1.$$

The constants α_1 and α_2 appropriate to our recursions must satisfy the conditions

$$d_{Q+1} = \alpha_1 r_1^{Q+1} + \alpha_2 r_2^{Q+1} = d_Q + b d_{Q-1},$$
$$d_{Q+2} = \alpha_1 r_1^{Q+2} + \alpha_2 r_2^{Q+2} = d_{Q+1} + b d_Q = (1+b) d_Q + b d^{Q-1},$$

with values of d_Q and d_{Q-1} obtained from the first Q recursions, which pertain to $d_1, \ldots, d_Q$. With these constants, we then have

$$c_j = \{\Gamma((j+2)/2)\}^{-1}(\alpha_1 r_1^j + \alpha_2 r_2^j), \qquad j \geqslant Q+1,$$

and it follows from Stirling's formula that the series (5) converges absolutely for $\sigma \geqslant 0$, as contended.

It is still to be shown that the integral equation (4) has no other solution than the function $p(\sigma)$ just constructed. Therefore, consider any solution $y(s)$ of the homogeneous integral equation corresponding to (4). If $|y(s)| \leqslant y_0$ in the interval $[0, \delta]$, it is elementary to verify that, if δ is sufficiently small, then in fact $y = 0$. If $[0, s_0]$ is the largest interval on which $y(s) = 0$, similar reasoning shows that s_0 is not finite. Since this means that $y(s)$ is identically zero, the solution of (4) is unique, as asserted.

We conclude with a lemma very like the previous that will be applied in section 8.

LEMMA 2. *An integral equation of the form*

$$P(\tau) = A + B\tau^{\frac{1}{2}}\int_0^\tau (\tau - t)^{-\frac{1}{2}} t^{-\frac{1}{2}} P(t)\, dt + C\tau^{\frac{1}{2}}\int_0^\tau t^{-\frac{1}{2}} P(t)\, dt, \tag{9}$$

in which A, B, C are nonnegative constants, and $B > 0$, has a unique solution for $\tau \geqslant 0$. The solution is representable as

$$P(\tau) = A p(\pi B^2 \tau), \tag{10}$$

where $p(\sigma)$ satisfies the equation

$$p(\sigma) = 1 + (\sigma/\pi)^{\frac{1}{2}}\int_0^\sigma (\sigma - s)^{-\frac{1}{2}} s^{-\frac{1}{2}} p(s)\, ds + c\sigma^{\frac{1}{2}}\int_0^\sigma s^{-\frac{1}{2}} p(s)\, ds \tag{11}$$

with $c = C/\pi B^2$ and is given by a series of the form

$$p(\sigma) = 1 + \sum_{k=1}^{\infty} c_k \sigma^{k/2} \tag{12}$$

converging absolutely for $\sigma \geqslant 0$.

PROOF. Under the substitution (10), it is immediately seen that (9) and (11) are equivalent. Uniqueness follows in the same way as in the previous lemma. To justify the series expansion for $p(\sigma)$, again it suffices to determine a series of the form (12) that satisfies (11) formally and that converges absolutely for $\sigma \geqslant 0$. In a formal substitution of (12) into (11), rule (6) shows that $c_1 = \sqrt{\pi}$ and that

$$c_k = c_{k-1}\Gamma(k/2)/\Gamma\big((k+1)/2\big) + \big(2c/(k-1)\big)\, c_{k-2}$$

for $k \geqslant 2$, where $c_0 = 1$. Since the quantities

$$d_k = c_k \Gamma\big((k+1)/2\big)$$

satisfy relations

$$d_k = d_{k-1} + c d_{k-2} \qquad \text{for } k \geqslant 2\,,$$

which are identical in form to (8), arguments given in the proof of Lemma 1 will show here that the series (12) does converge absolutely for $\sigma \geqslant 0$, as asserted.

6. - Estimates of spatial derivatives of approximate layered solutions.

Consider an approximate layered solution $u^{(h)}(x, t)$, as described in Section 3, that is of class C^{j_0}, $j_0 \geqslant 2$, in each half-open layer Z'_m, $m = 1, \ldots, m_0$, and satisfies conditions (3.2) to (3.5) as well as (3.2)′, (3.2)*, (3.5)*. Assume $f(x, t, u)$ to be of class C^{j_0} in $Z(0, T; M)$, where $T = m_0 h$, and assume $g(x, t, u)$ to be of class $C^{j_0} - 1$. Assume also that constants F^*, F_k, G_l exist such that, on $Z(0, T; M)$,

(a) $\|f_u\| \leqslant F^*$,

(b) the partial derivatives of $f(x, t, u)$ of orders 0 through k are, in absolute value, $\leqslant F_k$, $k = 0, \ldots, j_0$,

(c) the partial derivatives of $g(x, t, u)$ of orders 0 through l are, in absolute value, $\leqslant G_l$, $l = 0, \ldots, j_0 - 1$.

In this section, we shall establish the following consequences.

THEOREM 1. *Under the foregoing hypotheses, positive quantities* $h_k = h_k(M, T)$ *and* $\sigma_k = \sigma_k(M, T)$, $k = 1, \ldots, j_0 - 1$, *exist such that, if* $r \geqslant k$ *and*

$0 < h \leqslant h_k$, *then*

$$\|\partial_x^k u^{(h)}(\cdot, t)\| \leqslant \sigma_k t^{-k/2} \qquad \text{for } 0 < t \leqslant T .$$

PROOF. It suffices to find bounds on $(0, T]$ for

$$P_k(t) = \sup_{0 \leqslant s \leqslant t} \left\{ s^{k/2} \|\partial_x^k u^{(h)}(\cdot, s)\| \right\} .$$

The desired bounds will result from inequalities $(5.8)_k$ after estimations are made of the individual terms of the right-hand members.

At first it is convenient to consider only those τ for which

$$m_1 \geqslant 4k + 4 . \tag{1}$$

In view of (1), and because

$$\tau \leqslant (m_1 + 1) h \leqslant (m_1 + 1) \lambda_*^{-1} \varepsilon^{*2} ,$$

inequality (9.3) implies

$$\begin{aligned} \|\mathrm{S}^{m_1+1} \partial_x^k u^*\| &= \|\partial_x^k \mathrm{S}^{m_1+1} u^*\| \\ &\leqslant s_k \varepsilon_*^{-k} (m_1 + 1)^{-k/2} \|u^*\| \\ &\leqslant s_k \Lambda^k \lambda_*^{-k/2} \tau^{-k/2} \|u^0\| . \end{aligned} \tag{2}$$

By (3.5) in the case $j = 0$, by (9.3), and by (1) and (3.5)*,

$$\begin{aligned} \mathfrak{H}_k^m &\leqslant s_k \varrho_0 \varepsilon_*^{-k} (m_1 + 1 - m)^{-k/2} h^{(r+1)/2} \qquad \text{for } m \leqslant m_1 + 1 - 2k , \\ &\leqslant Y_k [(m-1) h]^{-k/2} h^{(r+1)/2} \leqslant 2^{k/2} Y_k \tau^{-k/2} h^{(r+1)/2} \\ &\qquad\qquad \text{for } m_1 + 1 - 2k < m \leqslant m_1 . \end{aligned} \tag{3}$$

Similarly,

$$\mathfrak{H}_k(\tau) \leqslant Y_k (m_1 h)^{-k/2} h^{(r+1)/2} \leqslant (8/7)^{k/2} Y_k \tau^{-k/2} h^{(r+1)/2} . \tag{4}$$

By (3.2)*,

$$\begin{aligned} &\int_{m_1 h}^{\tau} \|\partial_x^k (f_x^{(h)} + g^{(h)})\| \, dt \\ &\qquad \leqslant (8/7)^{k/2} \{ \tau^{-(k+1)/2} (F_{k+1} + G_k) V_k h + F_1 W_k h^{\frac{1}{2}} \tau^{-k/2} \} , \end{aligned} \tag{5}$$

where V_k and W_k depend polynomially upon $\tau^{\frac{1}{2}}$, $P_1(\tau), \ldots, P_k(\tau)$, and also depend upon M. For $m = 1, \ldots, m_1$, from (9.1) we have by use of (5.9)

$$(6) \quad \int_{(m-1)h}^{mh} \|S^{m_1+1-m} \partial_x^k (f_x^{(h)}(\cdot, t) + g^{(h)}(\cdot, t)\| \, dt$$
$$\leqslant s_1 \varepsilon_*^{-1} (m + 1 - m_1)^{-\frac{1}{2}} \int_{(m-1)h}^{mh} \|\partial_x^{k-1} (f_x^{(h)}(\cdot, t) + g^{(h)}(\cdot, t)\| \, dt$$
$$\leqslant \sqrt{2}\, s_1 \lambda_*^{-\frac{1}{2}} \int_{(m-1)h}^{mh} \{\|\partial_x^{k-1} f_x^{(h)}(\cdot, t)\| + \|\partial_x^{k-1} g^{(h)}(\cdot, t)\|\} (\tau - t)^{-\frac{1}{2}} \, dt .$$

For values of m for which

$$(1)_m \qquad m \leqslant m_1 + 1 - 2k ,$$

with the help of (9.3), we also obtain the alternative estimates

$$(7) \quad \int_{(m-1)h}^{mh} \|S^{m_1+1-m} \partial_x^k f_x^{(h)}(\cdot, t)\| \, dt \leqslant s_{k+1} \varepsilon_*^{-(k+1)} (m_1 + 1 - m)^{-(k+1)/2} F_0 h$$

and

$$(8) \quad \int_{(m-1)h}^{mh} \|S^{m_1+1-m} \partial_x^k g^{(h)}(\cdot, t)\| \, dt \leqslant s_k \varepsilon_*^{-k} (m_1 + 1 - m)^{-k/2} G_0 h .$$

Let $m_2 = [m_1/2]$. In view of (1), $m_2 + 1 < m_1 - 2k + 1$, and m satisfies $(1)_m$, in particular, if $m \leqslant m_2 + 1$. We shall use (6) in $(5.8)_k$ for $m = m_2 + 2, \ldots, m_1$, and we shall use (7) and (8) for $m = 1, \ldots, m_2 + 1$. We also substitute from (2) to (5). Since

$$\sum_{m=m_2+2}^{m_1} \int_{(m-1)h}^{mh} = \int_{(m_2+1)h}^{m_1 h} \leqslant \int_{\tau/2}^{\tau} ,$$

and since, if $\psi \geqslant 0$, then

$$\tau^{k/2} \int_{\tau/2}^{\tau} \psi(t) \, dt \leqslant 2^{k/2} \int_{\tau/2}^{\tau} t^{k/2} \psi(t) \, dt \leqslant 2^{k/2} \int_0^{\tau} t^{k/2} \psi(t) \, dt ,$$

the result may be written

$$(9)\qquad \tau^{k/2}\|\partial_x^k u^{(h)}(\cdot,\tau)\| \leqslant s_k \Lambda^k \lambda_*^{-k/2}\|u^0\|$$

$$+ 2^{(k+1)/2} s_1 \lambda_*^{-\frac{1}{2}}\int_0^\tau \{\|\partial_x^{k-1} f_x^{(h)}(\cdot,t)\| + \|\partial_x^{k-1} g^{(h)}(\cdot,t)\|\} t^{k/2}(\tau-t)^{-\frac{1}{2}}\,dt$$

$$+ s_{k+1}\tau^{k/2}\varepsilon_*^{-(k+1)} F_0 h \sum_{m=1}^{m_2+1} (m_1+1-m)^{-(k+1)/2}$$

$$+ s_k \tau^{k/2}\varepsilon_*^{-k} G_0 h \sum_{m=1}^{m_2+1}(m_1+1-m)^{-k/2}$$

$$+ s_k \varrho_0 \tau^{k/2}\varepsilon_*^{-k} h^{(r+1)/2}\sum_{m=1}^{m_1+1-2k}(m_1+1-m)^{-k/2} + R_k\,,$$

where

$$(10)\qquad R_k = R_k(P_1(\tau),\ldots,P_k(\tau),\tau,h)$$

$$= (2k)2^{k/2}\{Y_k h^{(r+1)/2} + (F_{k+1}+G_k)V_k h\tau^{-\frac{1}{2}} + F_1 W_k h^{\frac{1}{2}}\}\,,$$

(9) holding for $4(k+1)h \leqslant \tau \leqslant T$, $k \geqslant 1$. (For such τ, $h\tau^{-\frac{1}{2}} \leqslant h^{\frac{1}{2}}/[4(k+1)]$ in (10).) The remainder R_k depends polynomially upon its indicated arguments.

To estimate $P_1(t)$, $0 < t \leqslant T$, take $k=1$ in (9). Respecting the first summation on the right-hand side, we have

$$\varepsilon_*^{-2} h \sum_{m=1}^{m_2+1}(m_1+1-m)^{-1} < \varepsilon_*^{-2} h \log[m_1/(m_1-m_2-1)] < \lambda_*^{-1}\log 4\,,$$

since $m_1 - m_2 - 1 \geqslant m_1/2 - 2 > m_1/4$. Similarly,

$$\varepsilon_*^{-1} h \sum_{m=1}^{m_2+1}(m_1+1-m)^{-\frac{1}{2}} < 3\lambda_*^{-\frac{1}{2}}\tau^{\frac{1}{2}}\,,$$

$$\varepsilon_*^{-1} h^{(r+1)/2}\sum_{m=1}^{m_1}(m_1+1-m)^{-\frac{1}{2}} < 3\lambda_*^{-\frac{1}{2}}\tau^{\frac{1}{2}} h^{(r-1)/2}\,.$$

Respecting the integral in the second member of (9), the integrand in the case $k=1$ can be estimated by $\{(F_1+G_0)t^{\frac{1}{2}} + F^* P_1(t)\}(\tau-t)^{-\frac{1}{2}}$ and the integral itself thus by

$$F^*\int_0^\tau P_1(t)(\tau-t)^{-\frac{1}{2}}\,dt + (\pi/2)(F_1+G_0)\tau\,.$$

Using these results, we deduce from (9) that

$$P_1(\tau) \leqslant \alpha_1 \|u^0\| + \alpha_{11}\tau^{\frac{1}{2}} + \alpha_{12}\tau + \gamma_1 \int_0^\tau P_1(t)(\tau - t)^{-\frac{1}{2}} dt + R_1 \tag{11}$$

for $8h \leqslant \tau \leqslant T$, where

$$\begin{aligned}
\alpha_1 &= s_1 \lambda_*^{-\frac{1}{2}}, \\
\alpha_{11} &= s_2 F_0 \lambda_*^{-1} \log 4, \\
\alpha_{12} &= s_1 \lambda_*^{-\frac{1}{2}} \{\pi F_1 + (\pi + 3) G_0 + 3\varrho_0 h^{(r-1)/2}\}, \\
\gamma_1 &= 2 s_1 \lambda_*^{-\frac{1}{2}} F^*.
\end{aligned}$$

For $0 < \tau \leqslant 8h$, (3.2)′ implies that $P_1(\tau) \leqslant 8^{\frac{1}{2}} C_1$. Hence, if the first term in the second member of (11) is replaced by $\alpha^* = \max\left(\alpha_1 \|u^0\|, 8^{\frac{1}{2}} C_1\right)$, the resulting integral inequality will hold for $0 < \tau \leqslant T$.

Consider the function $P_1^*(t)$ satisfying the relation

$$P_1^*(\tau) = \alpha^* + 1 + \alpha_{11}\tau^{\frac{1}{2}} + \alpha_{12}\tau + \gamma_1 \int_0^\tau P_1^*(t)(\tau - t)^{-\frac{1}{2}} dt. \tag{12}$$

A substitution of the form $P_1^*(t) = (\alpha^* + 1) p(\alpha t)$ with suitable constant α changes (12) into an inequality of the form

$$p(\sigma) \leqslant 1 + \beta_{11}\sigma^{\frac{1}{2}} + \beta_{12}\sigma + \pi^{-\frac{1}{2}} \int_0^\sigma p(s)(\sigma - s)^{-\frac{1}{2}} ds,$$

where $\sigma = \alpha\tau$, $s = \alpha t$. As was shown in Section 5*A*, such an inequality implies an a priori bound for $p(\sigma)$ on any finite interval. Consequently, $P_1^*(\tau)$ is bounded on any finite interval. Let P_1^{**} be an upper bound for $P_1^*(\tau)$ for $0 < \tau \leqslant T$, and determine $h_1 = h_1(M, T)$ by the conditions that $h_1 \leqslant h_2$ (described in Theorem 5.1) and

$$R_1(P_1^{**}, T, h_1) < 1.$$

Since $P_1(0) \leqslant P_1^*(0)$, since $P_1^*(t)$ satisfies (12), and since $P_1(t)$ satisfies an inequality like (11), but with α^* in place of α_1, it follows that, if $0 < h \leqslant h_1$, then $P_1(\tau) < P_1^*(\tau)$ for $0 < \tau \leqslant T$. Thus, Theorem 1 is proved in the case $k = 1$.

In a mathematical induction, now suppose that, for some integer $k \geqslant 2$, constants $\sigma_1, \ldots, \sigma_{k-1}$ have been established such that

$$P_l(t) \leqslant \sigma_l \tag{13}$$

for $l = 1, \ldots, k-1$. The case $k = 2$ has just been treated.

Conditions (13) imply of course that

$$\|\partial_x^l u^{(h)}(\cdot, t)\| \leqslant \sigma_l t^{-l/2}, \quad l = 1, \ldots, k-1 . \tag{13$'$}$$

We shall incorporate assumptions (13) into (9) and then use arguments like the preceding to arrive at a bound for $P_k(t)$, completing the induction. The three terms in the second member of (9) containing summations are, in toto, $\leqslant \sum_{l=1}^{k} \alpha_{kl} \tau^{l/2}$ with suitable constants α_{kl}; the condition $r \geqslant k$ is used in connection with the third summation. Estimating the integral in (9) is based on the fact that, for any sufficiently differentiable function $f(x, v)$ and any vector function $v(x) = \big(v_1(x), \ldots, v_n(x)\big)$, the k-th derivative $\partial_x^k f\big(x, v(x)\big)$ is a sum of terms of the form

$$a\big(x, v(x)\big)\left(\prod_{i=1}^{l_1} (\partial_x^{k_{1i}} v_1)^{j_{1i}}\right) \cdots \left(\prod_{i=1}^{l_n} (\partial_x^{k_{ni}} v_n)^{j_{ni}}\right),$$

where $\sum_{m=1}^{n} \sum_{i=1}^{l_m} k_{mi} j_{mi} \leqslant k$, and where $(\partial_x^l v_m)^j$ symbolizes the product of j derivatives of v_m of l-th order. The k-th derivative $\partial_x^k v_m$ occurs only in the term $f_{v_m}\big(x, v(x)\big)\, \partial_x^k v_m$. These remarks and (13)$'$ give us, in particular,

$$t^{k/2} \|\partial_x^k f^{(h)}(\cdot, t)\| \leqslant F^* P_k(t) + \sum_{l=0}^{k} m_{kl} t^{l/2}$$

and

$$t^{k/2} \|\partial_x^{k-1} g^{(h)}(\cdot, t)\| \leqslant \sum_{l=1}^{k} n_{kl} t^{l/2},$$

where the m_{kl} are functions of $F_k, \sigma_1, \ldots, \sigma_{k-1}$, and the n_{kl} functions of $G_{k-1}, \sigma_1, \ldots, \sigma_{k-1}$. The integral in (9) is estimated by means of these inequalities and the fact that

$$\begin{aligned}\int_0^\tau t^\alpha (\tau - t)^{-\frac12} dt &= \tau^{\alpha + \frac12} \int_0^1 s^\alpha (1 - s)^{-\frac12} ds \\ &= B(\alpha + 1, \tfrac12)\, \tau^{\alpha + \frac12} \\ &= \pi^{\frac12} \{\Gamma(\alpha + 1)/\Gamma(\alpha + \tfrac32)\}\, \tau^{\alpha + \frac12}\end{aligned}$$

for $\alpha > -1$, where B and Γ are Euler's beta and gamma functions, respectively.

Now we replace the second member of (9) by the estimate arising from the foregoing considerations, and we replace the first member by $P_k(\tau)$. The result is a relation of the form

$$(14) \qquad P_k(\tau) \leqslant \alpha_k \|u^0\| + \sum_{l=0}^{k+1} \alpha_{kl} \tau^{l/2} + \gamma_k \int_0^\tau P_k(t)(\tau - t)^{-\frac{1}{2}} dt + R_k ,$$

in which $\alpha_k = s_k \lambda_*^{-k/2}$, $\gamma_k = 2^{(k+1)/2} s_1 \lambda_*^{-\frac{1}{2}} F^*$, and α_{kl} depend upon k, λ_*, F_k, G_{k-1}, $\sigma_1, \ldots, \sigma_{k-1}$. Inequality (14) is good only for $\tau \geqslant 4(k+1)h$.

The rest of the argument to establish that $P_k(t)$ has a bound for $0 < t \leqslant T$ is parallel to the reasoning that followed the derivation of (11) and need not be given. Thus, in effect, the proof of Theorem 1 is complete.

7. - Difference quotients with respect to t.

We now consider difference-quotients of various orders for $u^{(h)}(x, t)$ with respect to t. These « time-difference-quotients » will be seen to be equal approximately to certain functions of the spatial derivatives $\partial_x^j u^{(h)}$. The functions are the same as those which relate time-derivatives to spatial derivatives of solutions of the limit equation,

$$(1) \qquad v_t + \big(f(x, t, v)\big)_x + g(x, t, v) = \boldsymbol{\mu}\{\partial_1^2 v + \ldots + \partial_d^2 v\} ,$$

in which $\partial_\alpha = \partial / \partial x_\alpha$, $\alpha = 1, \ldots, d$. Respecting the bold-face symbol $\boldsymbol{\mu}$ the convention is adopted that for any n-dimensional vector $w = (w_1, \ldots, w_n)$,

$$\boldsymbol{\mu} w = (\mu_1 w_1, \ldots, \mu_n w_n) .$$

(The μ_i are the coefficients associated with the equation for u_i in the system (E).) Bold-face symbols $\boldsymbol{\lambda}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^2$, and so forth, will be used in what follows analogously to $\boldsymbol{\mu}$.

In the formal process of constructing the functions referred to, we shall attribute sufficient differentiability to $v(x, t)$, a presumed solution of (1), as well as to $f(x, t, u)$, $g(x, t, u)$.

It will be convenient in this section to denote by $\nabla_x^k v$ the array consisting of all spatial derivatives $\partial_x^k v_i$ of k-th order with respect to $x_1, \ldots, x_d$

of all the components $v_1, \ldots, v_n$ of v. (Thus, ∇_x^2 here does not represent the Laplacian.) Let $\boldsymbol{q}_k$ be a vector having the same number of components as $\nabla_x^k v$, $k = 0, 1, \ldots$. The functions of $\nabla_x^k v$ desired are n dimensional vector functions

$$H_k(x, t, \boldsymbol{q}_0, \ldots, \boldsymbol{q}_{2k}) = (H_{k1}(x, t, \boldsymbol{q}_0, \ldots, \boldsymbol{q}_{2k}), \ldots, H_{kn}(x, t, \boldsymbol{q}_0, \ldots, \boldsymbol{q}_{2k}))$$

such that a smooth solution of (1) will satisfy

$$\partial_t^k v = H_k(x, t, v, \nabla_x v, \ldots, \nabla_x^{2k} v)\,. \tag{2}$$

The first of these functions comes from equation (1), according to which

$$\begin{aligned} v_t &= - f_x(x, t, v) - f_v(x, t, v)\nabla_x v - g(x, t, v) + \mu(\partial_1^2 v + \ldots + \partial_d^2 v) \\ &\equiv H_1(x, t, v, \nabla_x v, \nabla_x^2 v)\,; \end{aligned}$$

$H_1(x, t, \boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{q}_2)$ is the function obtained from the second member of this equation when v, $\nabla_x v$, $\nabla_x^2 v$ are replaced by $\boldsymbol{q}_0$, $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, respectively.

Modifying previous notation, let $(l) = (l_1, \ldots, l_d)$ signify a multi-index of « order » $l = l_1 + \ldots + l_d$ with $\partial_x^{(l)} = \partial_1^{l_1} \ldots \partial_d^{l_d}$. Along with $H_1(x, t, \boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{q}_2)$ we must also consider the functions $H_{1.(l)}(x, t, \boldsymbol{q}_0, \ldots, \boldsymbol{q}_{l+2})$ for which, by successive differentiations with respect to the x_α of the equation $v_t = H_1(x, t, v, \nabla_x v, \nabla_x^2 v)$ and application of the chain rule we have

$$\partial_x^{(l)} v_t = H_{1.(l)}(x, t, v, \nabla_x v, \ldots, \nabla_x^{l+2} v) \tag{3}$$

with $H_{1.(0)} = H_1$. By $H_{1.l}$ we shall mean the aggregate of all $H_{1.(l)}$ for multi-indices of order l.

For $k > 1$, each function $H_k(x, t, v, \nabla_x v, \ldots, \nabla_x^{2k} v)$ results from formally differentiating $H_{k-1}(x, t, v, \nabla_x v, \ldots, \nabla_x^{2k-2} v)$ with respect to t, using the chain rule, and substituting $H_{1.(l)}(x, t, v, \nabla_x v, \ldots, \nabla_x^{l+2} v)$ for $\partial_x^{(l)} v_t$. In the notation described in Section 2,

$$\begin{aligned} H_k(x, t, v, \nabla_x v, \ldots, \nabla_x^{2k} v) &= H_{k-1.t}(x, t, v, \nabla_x v, \ldots, \nabla_x^{2k-2} v) \\ &+ \sum_{j=0}^{2k-2} H_{k-1.\boldsymbol{q}_j}(x, t, v, \nabla_x v, \ldots, \nabla_x^{2k-2} v) H_{1.j}(x, t, v, \nabla_x v, \ldots, \nabla_x^{j+2} v)\,, \quad k = 2, 3, \ldots \end{aligned} \tag{4}$$

The difference-quotients we wish to consider are of the type

$$\Delta \psi(x, t) = \frac{\psi(x, t + ph) - \psi(x, t)}{ph}$$

with $p = \pm 1, \pm 2, \ldots$, ψ being an arbitrary function of x, t. For $t > 0$ and any positive integer N, let Δ^N denote an operator of the form $\prod_{i=1}^{N} \Delta_{p_i}$, where $p_1, \ldots, p_N$ are nonzero integers such that

$$(5) \qquad p_i h,\ (p_i + p_j)h,\ (p_i + p_j + p_k)h, \ldots, (p_1 + p_2 + \ldots + p_N)h \geqslant -t/2$$

for all distinct indices $i, j, k, \ldots = 1, 2, \ldots, N$. Our main aim is to estimate $\Delta^N \partial_x^l u^{(h)}$ in a band Z^T in which

$$(6) \qquad \|\nabla_x^j u^{(h)}(\cdot, t)\| \leqslant \sigma_j t^{-j/2} \quad \text{for } j = 0, 1, 2, \ldots, r,\ r \geqslant 2N + l\,,$$

the first member in (6) signifying the maximum of $\sup_{x \in R} |\partial_x^{(j)} u^{(h)}(x, t)|$ for all multi-indices (j) having order j. For $u^{(h)}$ constructed by the method of Section 4 and satisfying the condition $\|u^{(h)}\|_{Z^T} \leqslant \sigma_0$, bounds of type (6) with $j \geqslant 1$ require f_u and $f_x + g$ to have bounded, continuous derivatives in $Z(0, T; \sigma_0)$ of orders $\leqslant 2r$, where r is the degree of the polynomial introduced in (4.3), and where $2N + l \leqslant r$ (see Section 6). In the forthcoming discussion, if the method of Section 4 is again used, these differentiability conditions will again suffice with respect to any $\Delta^k \partial_x^l u^{(h)}$ for which $2k + l \leqslant r$. Whatever the construction of $u^{(h)}$, we shall always assume with reference to a fixed, given integer $r \geqslant 2$, that f and g are differentiable enough to justify considering $\partial_x^l \Delta^k u^{(h)}$ for $2k + l \leqslant r$.

The theorems to follow are formulated with respect to approximate layered solutions $u^{(h)}(x, t)$ as described in Section 3. Such functions satisfy within each layer approximate layer equations of the form

$$(7) \qquad u_t^{(h)}(x, t) = G^{(h)}(x, t) + \mathfrak{h}^{(h)}(x, t)$$

where, according to (3.4), (3.5),

$$G^{(h)}(x, t) = G\big(x, t, u^{(h)}(x, t), \nabla_x u^{(h)}(x, t)\big)\,,$$

$$G(x, t, u, \nabla_x u) = -f_x(x, t, u) - f_u(x, t, u) \nabla_x u - g(x, t, u)\,,$$

and

$$(8) \qquad \|\partial_x^j \mathfrak{h}^{(h)}\|_{Z^T} \leqslant \varrho_j h^{(r-j-1)/2}, \qquad 0 \leqslant j \leqslant r - 1\,.$$

THEOREM 1. *Let $u^{(h)}(x, t)$ be an approximate layered solution of equations* (1) *in a zone Z^T, satisfying conditions* (6), (7), (8) *in each layer. For $0 < t \leqslant T$, let $p_1, \ldots, p_N$ be nonzero integers satisfying condition* (5) *as well as inequalities of the form*

$$\eta_* \leqslant |p_i h| \leqslant \eta^* \quad \text{for } i = 1, \ldots, N\,,$$

where $0 < \eta_* \leqslant \eta^* < 1$. *If* $\eta_* \geqslant h$, *then a quantity* $\sigma_{N,l}$ *independent of* x, t, h *exists such that*

$$\|\Delta^N \partial_x^{(l)} u^{(h)}(\cdot, t)\| \leqslant \sigma_{N,l} t^{-N-l/2} \tag{9}$$

for multi-indices (l) *of order* l *for which* $2N + l \leqslant r$.

THEOREM 2. *Under the same hypotheses as in Theorem* 1,

$$\Delta^N u^{(h)} = H_N^{(h)} + r_N^{(h)}\,, \tag{10}$$

where $H_N^{(h)}(x, t)$ *is the function that results from replacing* $\boldsymbol{q}_j$ *by* $\nabla_x^j u^{(h)}(x, t)$, $j = 0, \ldots, 2k$, *in* $H_N(x, t, \boldsymbol{q}_0, \ldots, \boldsymbol{q}_{2N})$. *If* $\eta_* \geqslant Eh^{\frac{1}{4}}$, *where* E *is a positive constant, then a quantity* $\sigma'_{N,l}$ *independent of* x, t, h *exists such that*

$$\|\nabla_x^l r_N^{(h)}(\cdot, t)\| \leqslant \sigma'_{N,l} \eta^* t^{-N-1-l/2}\,. \tag{11}$$

Several preliminaries precede the proofs of these theorems. In the first place, it will be helpful to express symbolically a relation such as (9) or (11) of the form $|w(x, t)| \leqslant C\Phi(t, h)$, C being a constant independent of x, h, t, as

$$w(x, t) \sim \Phi(t, h)\,.$$

Secondly, as in Section 4, a concept of « weight » again is convenient. We attribute the weight 0 to any bounded function of x, t, $u^{(h)}(x, t)$ and, for $t > 0$, $l \geqslant 0$, $N \geqslant 0$, the weight $l/2 + N$ to any member of the aggregate $\Delta^N \nabla_x^l u^{(h)}(x, t)$. The weight of a product of weighted quantities is to be the sum of the weights of the quantities, the weight of a sum of weighted quantities the greatest of the weights of the summands. These conventions are to apply even to expressions in which the functions $u^{(h)}(x, t)$ and their spatial derivatives and time-difference-quotients appear in different places with different (positive) values of the « time-argument » t. The conventions also are to apply to integrals of such expressions with respect to the time-argument on intervals $[t', t'']$ for which $0 < t' < t''$. Under these conventions, if, for instance, $p^{(h)}(x, t + s\theta)$ is a polynomial of weight j in quantities that belong to the aggregates $\Delta^N \nabla_x^l u^{(h)}(x, t + s\theta)$ for $l, N \geqslant 0$, $t > 0$, $t + s > 0$, and $0 \leqslant \theta \leqslant 1$, the coefficients in the polynomial having weight 0, then $\int_0^1 p^{(h)}(x, t + s\theta)\, d\theta$ also has weight j. If $t + s \geqslant t/2$ and inequalities (5) hold, then the polynomial described and its integral are both $\sim t^{-j}$.

The members of the aggregate $H_{1,j}^{(h)}(x, t)$ have weight $1 + j/2$, the members of $\nabla_x^j G^{(h)}$ weight $(j + 1)/2$. By mathematical induction from (4), $H_k^{(h)}(x, t)$ has weight k.

The last preliminary is a formula for the deviation of a function from its average of a kind given in [4]. As in Section 3, but now dropping ε as an index, let K be an averaging operator, defined say for functions $v(x)$ bounded and continuous on R, by

$$\mathrm{K}v(x) = \int \mathrm{k}(\xi)\, v(x + \varepsilon\xi)\, d\xi$$

with

$$\mathrm{k}(\xi) = k(\xi_1) \dots k(\xi_d)\,.$$

The one-dimensional kernel $k(s)$ is required to be sectionally continuous and to satisfy the conditions $k(s) \geqslant 0$, $\int k(s)\,ds = 1$, $k(s) = k(-s)$. The desired formula for $\mathrm{K}u - u$ is a consequence of the one-dimensional result

$$\int k(s)\, v(r + \varepsilon s)\, ds - v(r) = \varepsilon^2 (d^2/dr^2) \int \left\{ \int_{|s|}^{\infty} ds' \int_{s'}^{\infty} k(s'')\, ds'' \right\} v(r + \varepsilon s)\, ds\,,$$

holding for bounded, continuous functions $v(s)$ on R, and easily verifiable by carrying out the indicated differentiations (see [4, p. 177]). For any bounded, continuous function $v(x) = v(x_1, \dots, x_d)$ on R^d, and for each index $\alpha = 1, \dots, d$, define

$$K_\alpha v(x) = \int k(s)\, v(x_1, \dots, x_{\alpha-1}, x_\alpha + \varepsilon s, x_{\alpha+1}, \dots, x_d)\, ds\,,$$

$$M_\alpha v(x) = \int \left\{ \int_{|s|}^{\infty} ds' \int_{s'}^{\infty} k(s'')\, ds'' \right\} v(x_1, \dots, x_{\alpha-1}, x_\alpha + \varepsilon s, x_{\alpha+1}, \dots, x_d)\, ds\,.$$

For these one-dimensional transformations, $\mathrm{K} = K_1 \dots K_d$, and, in view of the previous identity,

$$K_\alpha v - v = \varepsilon^2 \partial_\alpha^2 M_\alpha v \quad \text{(no summation)}\,. \tag{12}$$

Note that, if $v \in C^2$, then $\partial_\alpha^2 M_\alpha v = M_\alpha \partial_\alpha^2 v$.

PROPOSITION 1. *Define*

$$J_\beta = M_\beta K_{\beta+1} \dots K_d \quad \text{for } \beta = 1, \dots, d-1\,,$$

$$J_d = M_d\,,$$

the domain of each J_α being the space of functions $v(x)$ that are bounded and continuous on R^d. Then

$$\mathrm{K} - \mathrm{I} = \varepsilon^2 \sum_{\alpha=1}^{d} \partial_\alpha^2 J_\alpha\,, \tag{13}$$

where I *is the identity operator on the same space of functions* $v(x)$. *For* $v(x)$ *of class* C^2,

$$(13)' \qquad (\mathrm{K}-\mathrm{I})\, v = \varepsilon^2 \sum_{\alpha=1}^{d} J_\alpha \, \partial_\alpha^2 v \, .$$

REMARK. J_α is a $(d-\alpha+1)$-dimensional integral operator:

$$J_\alpha v(x) = \int \dots \int j_\alpha(s_\alpha, \dots, s_d) v(x_1, \dots, x_{\alpha-1}, x_\alpha + \varepsilon s_\alpha, \dots, x_d + \varepsilon s_d) \, ds_\alpha \dots ds_d \, ,$$

where

$$j_\alpha(s_\alpha, \dots, s_d) = \left\{ \int_{|s_\cdot|}^{\infty} ds' \int_{s'}^{\infty} k(s'') \, ds'' \right\} k(s_{\alpha+1}) \dots k(s_d) \, .$$

(It is understood that for $\alpha = 1$, v has arguments $x_1 + \varepsilon s_1, \dots, x_d + \varepsilon s_d$, and that for $\alpha = d$, $k(s_{\alpha+1})$ is to be replaced by 1.) Note that

$$\int j_\alpha(s_\alpha, \dots, s_d) \, ds_\alpha \dots ds_d = (\tfrac{1}{2}) \int s^2 k(s) \, ds \equiv m_2/2 \, .$$

PROOF OF PROPOSITION 1. It follows from (12) that

$$\varepsilon^2 \partial_\beta^2 J_\beta = K_\beta K_{\beta+1} \dots K_d - K_{\beta+1} \dots K_d \qquad \text{for } \beta = 1, \dots, d-1 \, ,$$

and

$$\varepsilon^2 \partial_d^2 J_d = K_d - \mathrm{I} \, .$$

When the second members of these equalities are substituted for the summands in (13), the resulting sum telescopes to $K_1 \dots K_d - \mathrm{I}$. Thus, (13) is verified; it results also as a special case of formulas (1, 5) in [4], pp. 176-179.

Proof of Theorem 1.

With fixed $t > 0$, let $m_1 = [t/h]$. To justify (9) first in the case $N = 1$, consider

$$\Delta_{p_1} u^{(h)}(x, t) = (p_1 h)^{-1} \{ u^{(h)}(x, t + p_1 h) - u^{(h)}(x, t) \} \, .$$

Since $u^{(h)}(x, t)$ varies smoothly as t increases from $(m-1)h$ to $mh - 0$, but is discontinuous from $mh - 0$ to $mh + 0$, m being any positive integer $< m_0$, we have from (7)

$$(14)\qquad p_1 h \Delta_{p_1} u^{(h)}(x,t) = p_1 h \int_0^1 G^{(h)}(x, t+\theta_1 p_1 h)\, d\theta_1 + p_1 h \int_0^1 \mathfrak{h}^{(h)}(x, t+\theta_1 p_1 h)\, d\theta_1 + \sum_{m=1}^{p_1} \{u^{(h)}(x, m_1+m)h) - u^{(h)}(x, (m_1+m)h - 0\} .$$

By (3.3), the summation in (14) is equal to $\sum_{m=1}^{p_1} \{\mathrm{S}u^{(h)}(x, (m_1+m)h - 0) - u^{(h)}(m_1+m)h - 0\}$ and thus, by Proposition 1 just proved, to $\boldsymbol{\varepsilon}^2 \sum_{m=1}^{p_1} \sum_{\alpha=1}^{d} J_\alpha \cdot \partial_\alpha^2 u^{(h)}(x, (m_1+m)h - 0)$. This and the previous calculations give the result

$$(15)\qquad \Delta_{p_1} u^{(h)}(x,t) = \int_0^1 G^{(h)}(x, t+\theta_1 p_1 h)\, d\theta_1 + \int_0^1 \mathfrak{h}^{h}(x, t+\theta_1 p_1 h)\, d\theta_1 + (\boldsymbol{\lambda}/p_1) \sum_{m=1}^{p_1} \sum_{\alpha=1}^{d} J_\alpha \partial_\alpha^2 u^{(h)}(x, (m_1+m)h - 0) .$$

To the extent that $u^{(h)}$ is differentiable, any spatial differentiation ∂_x will commute both with J_α and with the other integral operators in (15). Hence, in view of (6) and (8), applying ∂_x^l to (15) shows that

$$(16)\qquad \partial_x^l \Delta_{p_1} u^{(h)}(x,t) \sim t^{-1-l/2} + h^{(r-1-l)/2} \leqslant t^{-1-l/2}\{1 + T^{1+l/2}\}$$

for $l + 2 \leqslant r$, this implying the case $N = 1$ of inequality (9).

In preparation for the general case of inequality (9), we show now that if $\Delta_{p_2}, \Delta_{p_3}, \ldots, \Delta_{p_N}$ be applied serially to $G^{(h)}(x, t+\theta_1 p_1 h)$, i.e., Δ_{p_2} to $G^{(h)}(x, t+\theta_1 p_1 h)$, Δ_{p_3} to $\Delta_{p_2} G^{(h)}(x, t+\theta_1 p_1 h)$, and so forth, then at each step the weight of the expression is increased by 1 and, furthermore, the time-arguments of the expression at each step are $\geqslant t/2$. As already noted, $G^{(h)}(x, t+\theta_1 p_1 h)$ has weight $\frac{1}{2}$, and $t + \theta_1 p_1 h \geqslant t/2$ because $p_1 h \geqslant -t/2$ by (5). From the integral form of the mean value theorem, we have

$$(17)\qquad \Delta_{p_2} G^{(h)}(x, t+\theta_1 p_1 h) = \int_0^1 \{G_t^* + G_{q_0}^* \Delta_{p_2} u^{(h)}(x, t+\theta_1 p_1 h) + G_{q_1}^* \nabla_x \Delta_{p_2} u^{(h)}(x, t+\theta_1 p_1 h)\}\, d\theta_2 ,$$

the starred derivatives of $G(x, t, u)$ having the arguments

$$x, \quad t + \theta_1 p_1 h + \theta_2 p_2 h ,$$
$$\theta_2 u^{(h)}(x, t+\theta_1 p_1 h + p_2 h) + (1-\theta_2) u^{(h)}(x, t+\theta_1 p_1 h) ,$$
$$\theta_2 \nabla_x u^{(h)}(x, t+\theta_1 p_1 h + p_2 h) + (1-\theta_2) \nabla_x u^{(h)}(x, t+\theta_1 p_1 h)$$

in place of x, t, q_0, q_1, respectively. In view of assumption (5), the time-arguments in (17) are all $\geqslant t/2$. It is also clear from (17) that $\Delta_{p_2} G^{(h)}(x, t + \theta_1 p_1 h)$ has weight $\frac{3}{2}$. Next, consider $\Delta_{p_3}\Delta_{p_2} G^{(h)}(x, t + \theta_1 p_1 h)$. To calculate this, we apply Δ_{p_3} termwise to the second member of (17), using the identity $\Delta_p\big(v(\sigma)w(\sigma)\big) = v(\sigma + ph)\Delta_p w(\sigma) + w(\sigma)\Delta_p v(\sigma)$, and representing $\Delta_{p_3} G_t^*$, $\Delta_{p_3} G_{q_0}^*$, $\Delta_{p_3} G_{q_1}^*$ by integral formulas analogous to (17). The result is an expression in which a polynomial of weight $\frac{5}{2}$ in

$$\Delta_{p_2} u^{(h)}, \quad \Delta_{p_3} u^{(h)}, \quad \Delta_{p_3}\Delta_{p_2} u^{(h)}, \quad \nabla_x \Delta_{p_2} u^{(h)}, \quad \nabla_x \Delta_{p_3} u, \quad \nabla_x \Delta_{p_2}\Delta_{p_3} u^{(h)},$$

evaluated at various time-arguments involving θ_1, the coefficients in the polynomial having time-arguments that depend upon three parameters $\theta_1, \theta_2, \theta_3$, is integrated with respect to θ_2, θ_3. Thus, $\Delta_{p_3}\Delta_{p_2} G^{(h)}(x, t + \theta_1 p_1 h)$ has weight $\frac{5}{2}$, and, because of (5), the time-arguments involved in it are all $\geqslant t/2$. Similar reasoning in a mathematical induction will show that for each $\nu = 1, \ldots, N$, the $(\nu - 1)$-st order time-difference-quotient $\Delta^{\nu-1} G^{(h)} \cdot$ $\cdot (x, t + \theta_1 p_1 h) = \Delta_{p_\nu}\Delta_{p_{\nu-1}} \cdots \Delta_{p_2} G^{(h)}(x, t + \theta_1 p_1 h)$ has weight $\nu - \frac{1}{2}$ and, in fact, results from $\nu - 1$ integrations with respect to $\nu - 1$ parameters $\theta_2, \ldots, \theta_\nu$ of a polynomial of weight $\nu - \frac{1}{2}$ in time-difference-quotients of orders up to $\nu - 1$ of $u^{(h)}$ and $\nabla_x u^{(h)}$. The $\nu - 1$ parameters enter into the coefficients of the polynomial, and all time-arguments are $\geqslant t/2$. Since ∂_x commutes with the integrations performed upon the polynomials referred to, each application of ∂_x to $\Delta^{\nu-1} G^{(h)}(x, t + \theta_1 p_1 h)$ increases the weight of the expression by $\frac{1}{2}$. As a result,

$$\partial_x^l \Delta_{p_\nu}\Delta_{p_{\nu-1}} \cdots \Delta_{p_2} G^{(h)}(x, t + \theta_1 p_1 h)$$

is a quantity of weight $(l - 1)/2 + \nu$; the time-arguments that occur in this expression are all $\geqslant t/2$.

Having this last result, we can now finish the proof of (9) by mathematical induction. The case $N = 1$ having been established in (16), let (9) be granted for $N = \nu - 1$, $\nu \geqslant 2$. To justify (9) in the case $N = \nu$, we apply $\partial_x^l \Delta_{p_\nu} \cdots \Delta_{p_2}$ to both sides of (15). The first term on the right becomes

$$\int_0^1 \partial_x^l \Delta_{p_\nu} \cdots \Delta_{p_2} G^{(h)}(x, t + \theta_1 p_1 h)\, d\theta_1 \tag{18}$$

with the integrand an expression of the sort just described of weight $(l - 1)/2 + \nu$. The time-difference-quotients of $u^{(h)}$ that enter this expression are of orders up to $\nu - 1$, and their time-arguments are $\geqslant t/2$. Since (9) is assumed in the case $N = \nu - 1$, we can conclude that (18) is $\sim t^{-\nu-(l-1)/2}$. The second term on the right-hand side of (15) becomes in absolute value, when the indicated operator is applied,

$$\left| \int_0^1 \Delta_{p_\nu} \cdots \Delta_{p_2} \partial_x^l \mathfrak{h}^{(h)}(x, t + \theta_1 p_1 h)\, d\theta_1 \right| \\ \leqslant 2^{\nu-1} \eta_*^{1-\nu} \max_{s \geqslant t/2} \| \partial_x^l \mathfrak{h}^{(h)}(\cdot, s) \| \leqslant 2^{\nu-1} \varrho_l \eta_*^{1-\nu} h^{(r-l-1)/2} .$$

Concerning the third term in the right-hand member of (15), the operator applied commutes with J_α to give

$$(\boldsymbol{\lambda}/p_1) \sum_{m=1}^{p_1} \sum_{\alpha=1}^{d} J_\alpha \partial_x^{l+2} \Delta_{p_\nu} \ldots \Delta_{p_2} u^{(h)}(x, (m_1 + m)h - 0) ,$$

an expression of weight $l/2 + \nu$ again containing time-difference-quotients of $u^{(h)}$ of orders at most $\nu - 1$ and with time-arguments $\geqslant t/2$. Therefore, this expression $\sim t^{-\nu-l/2}$, and in view of the previous estimates pertaining to the first two terms in the second member of (15), we find that

$$\partial_x^l \Delta^2 u^{(h)}(x, t) \sim t^{-\nu-(l-1)/2} + \eta_*^{1-\nu} h^{(r-l-1)/2} + t^{-\nu-l/2} \\ \leqslant t^{-\nu-l/2} \{1 + T^{\frac{1}{2}} + T^{\nu+l/2} \eta_*^{1-\nu} h^{(r-l-1)/2}\} .$$

Since $2\nu + l \leqslant r$, $h \leqslant \eta_* < 1$, and $2 \leqslant \nu \leqslant N$, we have

$$\eta_*^{1-\nu} h^{(r-l-1)/2} \leqslant h^{\frac{1}{2}} < 1 ,$$

(9) following for $N = \nu$. The mathematical induction thus is complete and inequality (9) completely justified.

Proof of Theorem 2.

The first and main step in this proof is to show that

$$\Delta_{p_1} u^{(h)}(x, t) = H_1^{(h)}(x, t) + r_1^{(h)}(x, t) , \tag{19}$$

where

$$\partial_x^l \Delta^k r_1^{(h)}(x, t) \sim p_1 h t^{-2-k-l/2} \tag{20}$$

for k, $l = 0, 1, \ldots, 2k + l \leqslant r$. Since

$$G^{(h)}(x, t) = H_1^{(h)}(x, t) - \boldsymbol{\mu} \sum_{\alpha=1}^{d} \partial_\alpha^2 u^{(h)}(x, t) ,$$

it is clear that (15) implies a relation of the form (19) with

$$r_1^{(h)} = E_1 + E_2 + E_3\,, \tag{21}$$

where

$$E_1 = \int_0^1 \{G^{(h)}(x, t + \theta_1 p_1 h) - G^{(h)}(x, t)\}\, d\theta_1\,,$$

$$E_2 = (1/p_1) \sum_{m=1}^{p_1} \sum_{\alpha=1}^{d} \{\boldsymbol{\lambda} J_\alpha \partial_\alpha^2 u^{(h)}(x, (m_1 + m)h - 0) - \boldsymbol{\mu} \partial_\alpha^2 u^{(h)}(x, t)\}\,,$$

$$E_3 = \int_0^1 \mathfrak{h}^{(h)}(x, t + \theta_1 p_1 h)\, d\theta_1\,.$$

With respect to E_1, we have

$$\begin{aligned} G^{(h)}(x, t + \theta_1 p_1 h) - G^{(h)}(x, t) &= \int_0^1 \{\theta_1 p_1 h G_t^* + G_{q_0}^*[u^{(h)}(x, t + \theta_1 p_1 h) - u^{(h)}(x, t)] \\ &\quad + G_{q_1}^*[\nabla_x u^{(h)}(x, t + \theta_1 p_1 h) - \nabla_x u^{(h)}(x, t)]\}\, d\theta\,, \end{aligned} \tag{22}$$

the starred derivatives of G here having the arguments

$$x\,, \quad t + \theta\theta_1 p_1 h\,, \quad \theta u^{(h)}(x, t + \theta_1 p_1 h) + (1 - \theta) u^{(h)}(x, t)\,,$$
$$\theta \nabla_x u^{(h)}(x, t + \theta_1 p_1 h) + (1 - \theta) \nabla_x u^{(h)}(x, t)\,.$$

Let $m = [\theta_1 p_1]$. In the case, for instance, in which both $t + \theta_1 p_1 h$ and $t + mh \in [(k-1)h, kh]$ for some integer k, we can use (7) in the layer $[t + mh, t + \theta_1 p_1 h]$ to obtain

$$\begin{aligned} u^{(h)}(x, t + \theta_1 p_1 h) - u^{(h)}(x, t) &= [u^{(h)}(x, t + mh) - u^{(h)}(x, t)] + [u^{(h)}(x, t + \theta_1 p_1 h) - u^{(h)}(x, t + mh)] \\ &= mh\Delta_m u^{(h)}(x, t) + \int_{mh}^{\theta_1 p_1 h} G^{(h)}(x, t + \sigma)\, d\sigma + \int_{mh}^{\theta_1 p_1 h} \mathfrak{h}^{(h)}(x, t + \sigma)\, d\sigma\,, \end{aligned}$$

as well as the accompanying relation obtained by applying ∂_x to the first and third members of this one. The complementary case to the one just considered leads to entirely similar results. Hence, rewriting (22) according to these results and substituting the outcome in the integral for E_1 produces a form to which the operator $\partial_x^l \Delta^k$ can be applied in the same way as in the discussion of (15). The effect in the present instance is that

$$\partial_x^l \Delta^k E_1 \sim p_1 h t^{-k-l/2-\frac{3}{2}} + \eta_*^{-k} h^{(r-l+1)/2} . \tag{23}$$

With reference to E_2, for each $\alpha = 2, \dots, d$, let us set $x' = x'_\alpha = (x_1, \dots, x_{\alpha-1})$, $x'' = x''_\alpha = (x_\alpha, \dots, x_d)$, and write $x = (x', x'')$ on the understanding that x' will be empty in the case $\alpha = 1$. At the same time, let $s'' = (s_\alpha, \dots, s_d)$ and write the formula in the remark after Proposition 1 as

$$J_\alpha v(x) = \int j_\alpha(s'')\, v(x', x'' + \varepsilon s'')\, ds'' .$$

The kernel j_α is even, i.e., $j_\alpha(s'') = j_\alpha(-s'')$, and for this reason

$$\begin{aligned} J_\alpha v(x) &= \int j_\alpha(s'')(\tfrac{1}{2})\{v(x', x'' + \varepsilon s'') + v(x', x'' - \varepsilon s'')\}\, ds'' \\ &= (m_2/2) v(x) + \int j_\alpha(s'')\{(\tfrac{1}{2})[v(x', x'' + \varepsilon s'') + v(x', x'' - \varepsilon s'')] - v(x)\}\, ds'' \\ &= (m_2/2) v(x) + (\varepsilon^2/2) \int j_\alpha(s'') \sum_{\beta,\gamma=\alpha}^{d} s_\beta s_\gamma \cdot \\ &\quad \cdot \left(\int_0^1 \{\partial_\beta \partial_\gamma v(x', x'' + \varepsilon \varrho s'') + \partial_\beta \partial_\gamma v(x', x'' - \varepsilon \varrho s'')\}(1-\varrho)\, d\varrho \right) ds'' , \end{aligned}$$

the last equality resulting from a second-order Taylor expansion with integral remainder. Using this in E_2 gives

$$E_2 = \boldsymbol{\mu} E'_2 + \boldsymbol{\lambda} E''_2 ,$$

where

$$E'_2 = (1/p_1) \sum_{m=1}^{p_1} \sum_{\alpha=1}^{d} \{\partial_\alpha^2 u^{(h)}(x, (m_1 + m) h - 0) - \partial_\alpha^2 u^{(h)}(x, t)\} ,$$

$$\begin{aligned} E''_2 = (\boldsymbol{\varepsilon}^2/2p_1) \sum_{m=1}^{p_1} \sum_{\alpha=1}^{d} \int j_\alpha(s'') \sum_{\beta,\gamma=\alpha}^{d} \Bigg(\int_0^1 \{ &\partial_\alpha^2 \partial_\beta \partial_\gamma u^{(h)}(x', x'' + \boldsymbol{\varepsilon} \varrho s'', (m_1 + m) h - 0) + \\ &+ \partial_\alpha^2 \partial_\beta \partial_\gamma u^{(h)}(x', x'' - \boldsymbol{\varepsilon} \varrho s'', (m_1 + m) h - 0)\}(1-\varrho)\, d\varrho \Bigg) s_\beta s_\gamma\, ds'', \end{aligned}$$

since $\boldsymbol{\mu} = \boldsymbol{\lambda} m_2/2$. (The bold-face symbol $\boldsymbol{\varepsilon}$ is employed in arguments of $u^{(h)}$

to signify that ε_i will occur in $u_i^{(h)}$, $i = 1, \dots, n$.) We have

$$\begin{aligned} E_2' = (1/p_1) \sum_{m=1}^{p_1} \sum_{\alpha=1}^{d} \{\partial_\alpha^2 u^{(h)}(x, (m_1 + m)h - 0) - \partial_\alpha^2 u^{(h)}(x, m_1 h - 0)\} \\ + \sum_{\alpha=1}^{d} [\partial_\alpha^2 u^{(h)}(x, m_1 h - 0) - \partial_\alpha^2 u^{(h)}(x, m_1 h)] \\ + \sum_{\alpha=1}^{d} [\partial_\alpha^2 u^{(h)}(x, m_1 h) - \partial_\alpha^2 u^{(h)}(x, t)] \\ = e_1 + e_2 + e_3 , \end{aligned}$$

and

$$e_1 = (1/p_1) \sum_{m=1}^{p_1} \sum_{\alpha=1}^{d} mh\Delta_m \partial_\alpha^2 u^{(h)}(x, m_1 h - 0) ,$$

$$e_2 = -\boldsymbol{\varepsilon}^2 \sum_{\alpha,\beta=1}^{d} J_\beta \partial_\beta^2 \partial_\alpha^2 u^{(h)}(x, m_1 h - 0) ,$$

$$e_3 = -\sum_{\alpha=1}^{d} \int_{m_1 h}^{t} \{\partial_\alpha^2 G^{(h)}(x, s) + \partial_\alpha^2 \mathfrak{h}^{(h)}(x, s)\} \, ds .$$

The last formulas show that

$$\begin{aligned} \partial_l^x \Delta^k e_1 &\sim p_1 h t^{-k-2-l/2} , \\ \partial_x^l \Delta^k e_2 &\sim h t^{-k-2-l/2} , \\ \partial_x^l \Delta^k e_3 &\sim h t^{-k-\frac{3}{2}-l/2} + \eta_*^{-k} h^{(r-l-1)/2} , \end{aligned}$$

the result for e_3 depending on the fact that for any $\psi(s)$ and any permissible integer q

$$\Delta_q \int_{m_1 h}^{t} \psi(s) \, ds = \int_{m_1 h}^{t} \Delta_q \psi(s) \, ds .$$

Therefore, $\partial_x^l \Delta^k E_2' \sim p_1 h t^{-k-2-l/2} + \eta_*^{-k} h^{(r-l-1)/2}$. Since $\partial_x^l \Delta^k E_2'' \sim h t^{-k-2-l/2}$, as follows from the previous expression for E_2'', we conclude that

$$\partial_x^l \Delta^k E_2 \sim p_1 h t^{-k-2-l/2} + \eta_*^{-k} h^{(r-l-1)/2} . \tag{24}$$

Again by familiar arguments,

$$\partial_x^l \Delta^k E_3 \sim \eta_*^{-k} h^{(r-l-1)/2} ,$$

this, (23), (24), and (21) showing that for $k \geqslant 1$,

$$\partial_x^l \Delta^k r_1^{(h)}(x, t) \sim p_1 h t^{-2-k-l/2} + \eta_*^{-k} h^{(r-l-1)/2} \leqslant \eta^* t^{-2-k-l/2}\{1 + T^{2+k+l/2} E^{-k}\},$$

since $2k + l \leqslant r$, $\eta_* \geqslant E h^{\frac{1}{2}}$. This verifies (20), the first step in a mathematical induction.

As the second step, assume that for some integer $N \geqslant 2$,

$$\Delta^{N-1} u^{(h)} = H_{N-1}^{(h)} + r_{N-1}^{(h)}, \tag{25}$$

where

$$\partial_x^l \Delta^k r_{N-1}^{(h)}(x, t) \sim \eta^* t^{-N-k-l/2} \quad \text{for } k, l \geqslant 0,\ 2k + l + 2N \leqslant r. \tag{26}$$

With $p = p_N$ apply $\Delta = \Delta_p$ to both sides of (25) to obtain

$$\Delta^N u^{(h)} = \Delta H_{N-1}^{(h)} + \Delta r_{N-1}^{(h)}. \tag{27}$$

By means of a second-order Taylor expansion with integral remainder, we have

$$\begin{aligned} \Delta H_{N-1}^{(h)}(x, t) &= (1/ph)\{H_{N-1}^{(h)}(x, t + ph) - H_{N-1}^{(h)}(x, t)\} \\ &= H_{N-1,t} + \sum_{j=0}^{2N-2} H_{N-1,q_j} \nabla_x^j \Delta u^{(h)}(x, t) + phI_N, \end{aligned} \tag{28}$$

where

$$I_N = \int_0^1 \Big\{ H_{N-1,tt}^* + 2 \sum_{i=0}^{2N-2} [\nabla_x^i \Delta u^{(h)}(x, t)] H_{N-1,tq_i}^* + \sum_{i,j=0}^{2N-2} [\nabla_x^i \Delta u^{(h)}(x, t)][\nabla_x^j \Delta u^{(h)}(x, t)] H_{N-1,q_i q_j}^* \Big\} (1-\theta)\, d\theta,$$

the unstarred derivatives of H_{N-1} having the arguments

$$x, \quad t, \quad \nabla_x^i u^{(h)}(x, t), \quad i = 0, 1, \ldots, 2N-2,$$

and the starred derivatives the arguments

$$t + \theta ph, \quad \theta \nabla_x^i u^{(h)}(x, t + ph) + (1-\theta) \nabla_x^i u^{(h)}(x, t), \quad i = 0, 1, \ldots, 2N-2.$$

In view of (4), (28) can be written as

$$\Delta H_{N-1}^{(h)} = H_N^{(h)} + phI_N + J_N, \tag{29}$$

where

$$J_N = \sum_{j=0}^{2N-2} H_{N-1.q_j}(x, t, u^{(h)}(x, t), \ldots, \nabla_x^{2N-2} u^{(h)}(x, t)) \{\nabla_x^j \Delta u^{(h)}(x, t) - H_{1.j}^{(h)}(x, t)\} .$$

We now estimate $\partial_x^l \Delta^k J_N$. The definition of $H_{1.(l)}$ implies that $\partial_x^{(l)} H_1^{(h)}(x, t) = H_{1.(l)}^{(h)}(x, t)$, the last expression denoting the function of x, t obtained from $H_{1.(l)}(x, t, q_0, \ldots, q_{l+2})$ by replacing q_j by $\nabla_x^j u^{(h)}(x, t)$, $j = 0, 1, \ldots, l + 2$. Hence equation (10) in the case $N = 1$ (or equation (19)) implies that

$$\partial_x^{(j)} \Delta u^{(h)} = H_{1.(j)}^{(h)} + \partial_x^{(j)} r_1^{(h)},$$

which implies that

$$J_N = \sum_{j=0}^{2N-2} H_{N-1.q_j}(x, t, \ldots, \nabla_x^{2N-2} u^{(h)}(x, t)) \nabla_x^j r_1^{(h)}(x, t) . \tag{30}$$

Therefore, $\partial_x^l \Delta^k J_N$ is a sum of terms of the form

$$(\partial_x^{l'} \Delta^{k'} H_{N-1.q_j})(\partial_x^{l-l'} \Delta^{k-k'} \partial_x^j r_1^{(h)}) \tag{31}$$

with the previously indicated arguments and with $0 \leqslant l' \leqslant l$, $0 \leqslant k' \leqslant k$. Since $H_{N-1}^{(h)}$ has weight $N-1$, the weight of $H_{N-1.q_j}$ thus being $\leqslant N - 1 - j/2$, and since applying ∂_x adds $\frac{1}{2}$ to the weight of an expression, and applying Δ adds 1, we find that $\partial_x^{l'} \Delta^{k'} H_{N-1.q_j}$ has weight $\leqslant N - 1 - j/2 + l'/2 + k'$. Therefore,

$$\partial_x^{l'} \Delta^{k'} H_{N-1.q_j} \sim t^{-N+1-k'+j/2-l'/2},$$

so that, in view of (20), the term (31) $\sim \eta^* t^{-N-1-k-l/2}$. Hence

$$\partial_x^l \Delta^k J_N \sim \eta^* t^{-N-1-k-l/2} . \tag{32}$$

Similar calculations show that

$$\partial_x^l \Delta^k I_N \sim t^{-N-1-k-l/2} .$$

Thus, from (29),

$$\Delta H_{N-1}^{(h)} = H_N^{(h)} + r_N'^{(h)},$$

where

$$\partial_x^l \Delta^k r_N'^{(h)} \sim \eta^* t^{-N-1-k-l/2} . \tag{33}$$

Then by (25)

$$\Delta^N u^{(h)} = \Delta H_{N-1}^{(h)} + \Delta r_{N-1}^{(h)} = H_N^{(h)} + r_N^{(h)},$$

where $r_N^{(h)} = r_N'^{(h)} + \Delta r_{N-1}^{(h)}$, (33) and (26) showing that

$$\partial_x^l \Delta^k r_N^{(h)} \sim \eta^* t^{-N-1-k-l/2} .$$

This completes the mathematical induction and thus justifies Theorem 2.

8. - The convergence of layered solutions.

If the initial data are bounded, it will be seen in this section that layered solutions converge uniformly as $h \to 0$ in any layer $Z_{\delta,T}$, $0 < \delta < T$. The rate of convergence will be estimated. Our main result is the following:

THEOREM 1. *Let Z^T be a slab on which approximate layered solutions $u^{(h)}$ produced by Gaussian or arithmetical averaging have uniformly bounded norms $\|u^{(h)}\|_{Z^T}$, say for $0 < h \leqslant h_0$ $(h_0 > 0)$. Suppose that $f(x, t, v)$, $g(x, t, v)$ satisfy hypotheses (a) to (c) of Section 6 with $j_0 = j^0 + r$, $j^0 \geqslant r$, the integer r $(r \geqslant 2)$ indicating the degree of approximation of the solution, as in Theorem 4.1. Under these conditions, a constant $C(T)$ exists such that, if $0 < h' < h \leqslant \min(h_0, 8^{-3})$, then*

$$\|u^{(h)}(\cdot, t) - u^{(h')}(\cdot, t)\|$$
$$\leqslant C(T)[t - h^{\frac{2}{3}} \log(T/h)]^{-\frac{1}{2}} \{h^{\frac{2}{3}} \log(T/h) + h^{(r-1)/2}\} \quad \textit{for } h^{\frac{2}{3}} \log(T/h) < t \leqslant T .$$

This theorem of course implies that a function $u(x, t)$ exists such that

$$\|u^{(h)}(\cdot, t) - u(\cdot, t)\|$$
$$\leqslant C(T)[t - h^{\frac{2}{3}} \log(T/h)]^{-\frac{1}{2}} \{h^{\frac{2}{3}} \log(T/h) + h^{(r-1)/2}\} \quad \text{for } h^{\frac{2}{3}} \log(T/h) < t \leqslant T ,$$

in proof of our contention.

REMARK. By using Petrov's more refined estimate referred to in connection with Theorem 9.3, the quantity $\log(T/h)$ occurring in the previous inequalities can be replaced by $[\log(T/h)]^{\frac{2}{3}}$.

To justify the theorem stated, we begin with equation (5.7), giving the integral relation

$$(1) \quad u_i^{(h)}(x, \tau) = \mathrm{S}_i^{(m_1+1)} u_i^*(x) - \sum_{m=1}^{m_1} \int_{(m-1)h}^{mh} \{\mathrm{S}_i^{m_1+1-m} f_{i,x}^{(h)}(x, t) + \mathrm{S}_i^{m_1+1-m} g_i^{(h)}(x, t)\} \, dt$$
$$- \int_{m_1 h}^{\tau} \{f_{i,x}^{(h)}(x, t) + g_i^{(h)}(x, t)\} \, dt + \mathfrak{J}_i^{(h)}(x, \tau) , \qquad i = 1, \dots, n ,$$

where $f^{(h)}(x, t) = f(x, t, u^{(h)}(x, t))$, and similarly for $g^{(h)}(x, t)$; recall that from Theorem 4.1 $\|\mathfrak{J}^{(h)}(\cdot, \tau)\| \leqslant \varrho_0 \tau h^{(r-1)/2}$, where r is an integer $\geqslant 2$ specifying the exactness with which the layer equations are solved. The required estimation is obtained by comparing this and the analogous relation pertaining to $u^{(h')}(x, \tau)$.

To carry out this comparison effectively, it will be helpful to approximate $S_i^{m_1+1-m}$ by the Gaussian operator $G(\tau - t; \mu_i)$ where

$$G(s; \mu)\ v(x, t) \equiv [G(s; \mu)\ v](x, t) \equiv \int g(x - \xi, s; \mu) v(\xi, t) d\xi$$

with

$$g(x, s; \mu) = (2\pi)^{-d/2}(2\mu s)^{-d/2} \exp\{-x^2/4\mu s\}, \qquad x^2 = \sum_{i=1}^{d} x_i^2 .$$

Let τ_* and η be numbers such that

$$(2a) \qquad 8h \leqslant \tau_* < 1 ,$$

$$(2b) \qquad \tau_* < T ,$$

and

$$(2c) \qquad 2h \leqslant \eta \leqslant \tau_*/2 .$$

The main step in making the comparison required is to reduce (1) for values of τ in the interval

$$(2d) \qquad \tau_* \leqslant \tau < T$$

to a more convenient form as follows.

THEOREM 2. *Let*

$$(2e) \qquad \begin{aligned} \tau^* &= \tau_* && \text{if } \tau - \eta \leqslant \tau_* \\ &= \tau - \eta && \text{if } \tau - \eta > \tau_* . \end{aligned}$$

Then

$$u_i^{(h)}(x, \tau) = G(\tau; \mu_i) u_i^*(x) - \int_{\tau_*}^{\tau^*} G(\tau - t; \mu_i)\{f_{i.x}^{(h)}(x, t) + g_i^{(h)}(x, t)\}\, dt + E_i^{(h)}(x, \tau) + \mathfrak{J}_i^{(h)}(x, \tau) ,$$

where

$$\|E_i^{(h)}(\cdot, \tau)\| \leqslant C(T)\tau^{-\frac{1}{2}}\{\eta + \tau_* + [\log(T/h)]^{\frac{1}{2}} h \tau_*^{-\frac{1}{2}}\} \qquad \textit{for } \tau_* \leqslant \tau \leqslant T ,$$

$C(T)$ representing a constant depending on T. If $8h^{\frac{1}{2}} \leqslant \log(T/h)$, then deter-

mining η and τ_* so that

$$2\eta = \tau_* \geqslant h^{\frac{2}{3}} \log (T/h)$$

assures conditions (2a, c) and gives us the estimate

$$\|E_i^{(h)}(\cdot, \tau)\| \leqslant C(T)\tau^{-\frac{1}{2}}\tau_* \quad \text{for } \tau_* \leqslant \tau \leqslant T ,$$

$C(T)$ here representing a different constant from before. Values of h less than 8^{-3} will do.

(If Petrov's estimate referred to is used, then $\log (T/h)$ occurs in place of $[\log (T/h)]^{\frac{3}{2}}$ in the first estimate of $\|E_i^{(h)}(\cdot, \tau)\|$. The form of the second estimate is not changed, but in it we take $2\eta = \tau_* \geqslant [h \log (T/h)]^{\frac{2}{3}}$.)

To prove this proposition, we shall in effect make a succession of changes in the second member of (1), each change producing an error to be incorporated into $E_i^{(h)}$. In stating the relevant estimates, we shall use C as a generic absolute constant—or constant depending on $\lambda_1, \ldots, \lambda_n$ (which are constants)—representing individual constants that may be different in different places. We shall use $C(T)$ generically to indicate quantities, such as bounds in Z^T for functions of $x, t, u^{(h)}(x, t)$, that depend upon T.

We also shall drop the subscript i attached to u, f, g, ε, μ, λ, S.

The changes to be made are of five kinds. Let

$$m_* = [\tau_*/h] + 1 , \qquad m^* = [\tau^*/h] ,$$

where $[x]$ denotes the greatest integer $\leqslant x$; then

$$\tau^* - h < m^* h \leqslant \tau^* , \qquad \tau_* < m_* h \leqslant \tau_* + h .$$

The first change is to delete from the summation in the right-hand side of (1) all integrals corresponding to indices m such that $m \leqslant m_*$. The error resulting from this change, in absolute value, is

$$E_I \leqslant \sum_{m=1}^{m_*} \int_{(m-1)h}^{mh} \left\{\|\nabla_x \mathrm{S}^{m_1+1-m} f^{(h)}(\cdot, t)\| + \|g^{(h)}(\cdot, t)\|\right\} dt ,$$

since $\partial_x \mathrm{S}^j = \mathrm{S}^j \partial_x$, and $\|\mathrm{S}^j v\| \leqslant \|v\|$. The part of E_{I} pertaining to $\|g^{(h)}(\cdot, t)\|$ can be estimated by $C(T) m_* h$. To handle the other part of E_{I}, it is useful to distinguish two cases: (i) $\tau_* \leqslant \tau \leqslant 2\tau_*$, and (ii) $2\tau_* < \tau \leqslant T$. In the first case, inequality (6.1) implying that

$$\|\nabla_x \mathrm{S}^{m_1+1-m} f^{(h)}(\cdot, t)\| \leqslant C(T) t^{-\frac{1}{2}} ,$$

we have

$$\sum_{m=1}^{m_*}\int_{(m-1)h}^{mh}\|\nabla_x S^{m_1+1-m} f^{(h)}(\cdot, t)\|dt \leqslant C(T)(m_* h)^{\frac12} \leqslant C(T)\tau_*^{\frac12} \leqslant C(T)\,\tau_*\,\tau^{-\frac12}.$$

In the second case, by (9.1) we have

$$\sum_{m=1}^{m_*}\int_{(m-1)h}^{mh}\|\nabla_x S^{m_1+1-m} f^{(h)}(\cdot, t)\|dt \leqslant C(T)\,h\varepsilon^{-1}\sum_{m=1}^{m_*}(m_1+1-m)^{-\frac12}$$
$$< C(T)\,h^{\frac12}\{m_1^{\frac12}-(m_1-m_*)^{\frac12}\}\,,$$

while

$$m_1^{\frac12}-(m_1-m_*)^{\frac12} = m_*/[m_1^{\frac12}+(m_1-m_*)^{\frac12}] < m_*/\{2(m_1-m_*)^{\frac12}\}$$
$$< h^{-\frac12}(\tau_*+h)/\{2(\tau-\tau_*-2h)^{\frac12}\} < \frac{3\sqrt{3}}{4}\,h^{-\frac12}\,\tau_*\,\tau^{-\frac12},$$

in view of (2a). Thus again

$$\sum_{m=1}^{m_*}\int_{(m-1)h}^{mh}\|\nabla_x S^{m_1+1-m} f^{(h)}(\cdot, t)\|dt \leqslant C(T)\,\tau^{-\frac12}\,\tau_*\,.$$

Since $m_* h \leqslant \tau_* + h \leqslant (9/8)\,\tau_* \leqslant (9/8)\,T^{\frac12}\tau^{-\frac12}\tau_*$, we conclude finally that

$$E_{\mathrm{I}} \leqslant C(T)\,\tau^{-\frac12}\,\tau_*\,. \tag{3}$$

The second change in the right-hand side of (1) is to delete the integral from $m_1 h$ to τ and also to delete from the summation all integrals corresponding to indices m for which $m > m^*$; in consequence of the first two changes, all integrals thus are deleted if $\tau \leqslant \tau_* + \eta$.

The « error » produced by the second change, i.e., the difference between the modified and the original quantities, is, in absolute value,

$$E_{\mathrm{II}} \leqslant \int_{\tau^*-h}^{\tau}\{\|\nabla_x f^{(h)}(\cdot, t)\| + \|g^{(h)}(\cdot, t)\|\}\,dt\,.$$

By Theorem 6.1, and since $\tau-\eta-h \leqslant \tau^*-h$,

$$E_{\mathrm{II}} \leqslant C(T)\int_{\tau-\eta-h}^{\tau}(t^{-\frac12}+1)\,dt \leqslant C(T)\{\tau^{\frac12}-(\tau-\eta-h)^{\frac12}+\eta+h\}\,, \tag{4$'$}$$

from which it will be seen that

$$E_{\mathrm{II}} \leqslant C(T)\tau^{-\frac{1}{2}}\eta\,. \tag{4}$$

In fact,

$$\tau^{\frac{1}{2}} - (\tau - \eta - h)^{\frac{1}{2}} = (\eta + h)/[\tau^{\frac{1}{2}} + (\tau - \eta - h)^{\frac{1}{2}}] < (\eta + h)/\{2(\tau - \eta - h)^{\frac{1}{2}}\},$$

while $\eta + h \leqslant 3\eta/2$ by (2c), and $\eta + h \leqslant 5\tau_*/8 \leqslant 5\tau/8$ by (2a, c, d). Therefore,

$$\tau^{\frac{1}{2}} - (\tau - \eta - h)^{\frac{1}{2}} < 3^{\frac{1}{2}}\eta\tau^{-\frac{1}{2}}\,.$$

Substituting this into (4)$'$ and also using the inequality $\eta + h \leqslant 3\eta/2 \leqslant \leqslant (\frac{3}{2})T^{\frac{1}{2}}\tau^{-\frac{1}{2}}\eta$, we obtain (4).

The third modification we make in the right-hand side of (1) is to replace S^{m_1+1-m} in the undeleted terms of the summation by the Gaussian operator $\mathrm{G}\big((m_1 + 1 - m)h; \mu\big)$. Noting that $\mathrm{G}(jh; \mu) = \mathrm{G}_j$ (defined as in (9.4)), we have from Theorem 9.3 that

$$\begin{aligned}\|\mathrm{S}^{m_1+1-m}v - \mathrm{G}\big((m_1 + 1 - m)h; \mu\big)v\| &\\ &< 3B\|v\|[\log\{(m_1 + 1 - m)/3\}^{\frac{1}{2}}]^{\frac{1}{2}}(m_1 + 1 - m)^{-1}\\ &= 2^{-\frac{1}{2}}3B\|v\|L'(m_1 + 1 - m)\,,\end{aligned}$$

where $L(y) = (2/3)[\log(y/3)]^{\frac{3}{2}}$ for $y \geqslant 1$. (Since

$$L''(y) = \{1 - 2\log(y/3)\}/\{2y^2[\log(y/3)]^{\frac{1}{2}}\} < 0 \qquad \text{for } y \geqslant 5$$

L is concave for $y \geqslant 5$.) Hence, the total absolute error E_{III} committed in carrying out these replacements is

$$\begin{aligned}&= 0 \qquad \text{if } \tau \leqslant \tau_* + \eta\\ &\leqslant C(T)\sum_{m=m_*}^{m_*} L'(m_1 + 1 - m)\int_{(m-1)h}^{mh}(t^{-\frac{1}{2}} + 1)\,dt \qquad \text{if } \tau > \tau_* + \eta\,.\end{aligned}$$

In the last expression, since $m \geqslant m_* > 8$, and $h < T/m$, the value of the integral corresponding to the index m is

$$\begin{aligned}I_m &< h^{\frac{1}{2}}(m - 1)^{-\frac{1}{2}} + h\\ &< (8/7)^{\frac{1}{2}}h^{\frac{1}{2}}\{m^{-\frac{1}{2}} + h^{\frac{1}{2}}\}\\ &< C(T)h^{\frac{1}{2}}m^{-\frac{1}{2}}\end{aligned}$$

with $C(T) = (8/7)^{\frac{1}{2}}\{1 + T^{\frac{1}{2}}\}$ in this instance. Therefore,

$$
\begin{aligned}
(5) \qquad E_{\mathrm{III}} &= 0 \quad \text{if } \tau \leqslant \tau_* + \eta \\
&< C(T)h^{\frac{1}{2}} \sum_{m=m_*}^{m^*} L'(m_1 + 1 - m)m^{-\frac{1}{2}} \quad \text{if } \tau > \tau_* + \eta .
\end{aligned}
$$

It will follow from this that

$$
(6) \qquad E_{\mathrm{III}} < C(T)[\log(T/3h)]^{\frac{3}{2}} h\tau_*^{-\frac{1}{2}} \quad \text{for } \tau \geqslant \tau_* .
$$

In fact, concavity implying that $L'(N+1) < L(N+1) - L(N)$ for $N \geqslant 5$, we have

$$
\begin{aligned}
\sum_{m_*}^{m^*} L'(m_1 + 1 - m)m^{-\frac{1}{2}} &< m_*^{-\frac{1}{2}} \sum_{m^*}^{m^*} L'(m_1 + 1 - m_1) \\
&< m_*^{-\frac{1}{2}} \sum_{m_*}^{m^*} \{L(m_1 + 1 - m) - L(m_1 - m)\} \\
&< m_*^{-\frac{1}{2}} L(m_1 + 1 - m_*) \\
&< C[\log(T/3h)]^{\frac{3}{2}} h^{\frac{1}{2}} \tau_*^{-\frac{1}{2}} ,
\end{aligned}
$$

since

$$
L(m_1 + 1 - m) = (\tfrac{2}{3})\{\log[(m_1 + 1 - m_*)/3]\}^{\frac{3}{2}} < (\tfrac{2}{3})[\log(T/3h)]^{\frac{3}{2}} ,
$$

and $(m_* h)^{-\frac{1}{2}} < \tau_*^{-\frac{1}{2}}$. Inequality (6) follows.

The next change to be made pertains to the integrals of the form

$$
\int_{(m-1)h}^{mh} \mathrm{G}\big((m_1 + 1 - m)h; \mu\big) v(x, t)\, dt
$$

that have just been introduced, in which $v(x, t)$ stands for $\nabla_x f^{(h)}(M, t)$ or $g^{(h)}(x, t)$. In these integrals, namely, we replace $\mathrm{G}\big((m_1 + 1 - m)h; \mu\big)$ by $\mathrm{G}(\tau - t; \mu)$. Since, as a short calculation shows,

$$
\int |\mathrm{g}_t(x, t; \mu)|\, dx \leqslant d/t ,
$$

we have

$$
\int |\mathrm{g}\big(x, (m_1 + 1 - m)h; \mu\big) - \mathrm{g}(x, \tau - t; \mu)|\, dx \leqslant dh/(\tau - t - h)
$$

$$
\text{for } (m-1)h \leqslant t \leqslant mh, \; m \geqslant 2 .
$$

Hence, the total absolute error owing to the last set of replacements does not exceed

$$E_{\mathrm{IV}} < C(T)h\int_{\tau_*}^{\tau^*}(\tau - t - h)^{-1}[t^{-\frac{1}{2}} + 1]\,dt$$

with $\tau^* = \tau - \eta$ and $\tau > \tau_* + \eta$.

By an elementary integration,

$$\int_{\tau_*}^{\tau^*} t^{-\frac{1}{2}}(\tau - t - h)^{-1}\,dt = (\tau - h)^{-\frac{1}{2}} \log \frac{(\tau - h)^{\frac{1}{2}} + t^{\frac{1}{2}}}{(\tau - h)^{\frac{1}{2}} - t^{\frac{1}{2}}}\Bigg|_{t=\tau_*}^{\tau^*}$$
$$< (\tau - h)^{-\frac{1}{2}} \log\{[(\tau - h)^{\frac{1}{2}} + \tau^{*\frac{1}{2}}]/[(\tau - h)^{\frac{1}{2}} - \tau^{*\frac{1}{2}}]\}\,,$$

while

$$\begin{aligned}(\tau - h)^{\frac{1}{2}} - \tau^{*\frac{1}{2}} &= (\tau - h)^{\frac{1}{2}} - (\tau - \eta)^{\frac{1}{2}} \\ &= (\eta - h)/\{(\tau - h)^{\frac{1}{2}} + (\tau - \eta)^{\frac{1}{2}}\} \\ &> (\eta - h)/2\tau^{\frac{1}{2}} > (\tfrac{1}{4})\eta\tau^{-\frac{1}{2}}\,,\end{aligned}$$

and $(\tau - h)^{\frac{1}{2}} + \tau^{*\frac{1}{2}} < 2\tau^{\frac{1}{2}}$. Thus, the previous integral is estimated by

$$(8/7)^{\frac{1}{2}}\tau^{-\frac{1}{2}} \log(8\tau/\eta) < 2\tau^{-\frac{1}{2}} \log(8T/\eta)\,,$$

and we have

$$(7) \qquad E_{\mathrm{IV}} \leqslant C(T)h\left\{2\tau^{-\frac{1}{2}} \log(8T/\eta) + \log\frac{\tau - h - \tau_*}{\tau - h - \tau^*}\right\}$$
$$< C(T)\,h\tau^{-\frac{1}{2}} \log(8T/\eta)\,.$$

The fifth and final alteration to be made in the right-hand side of (1) is to replace $\mathrm{S}^{m_1+1}u^*(x)$ by $\mathrm{G}(\tau;\mu)u^*(x)$. The resulting error, in absolute value, is

$$\begin{aligned}(8) \qquad E_{\mathrm{V}} &\leqslant \|\mathrm{S}^{m_1+1}u^* - \mathrm{G}((m_1+1)h;\mu)\,u^*\| + \|\mathrm{G}((m_1+1)h;\mu)\,u^* - \mathrm{G}(\tau;\mu)\,u^*\| \\ &\leqslant 3B\|u^*\|[\log(T/3h)^{\frac{1}{2}}]^{\frac{1}{2}}h(\tau - h)^{-1} + d\|u^*\|h/\tau \\ &< C\|u^*\|[\log(T/3h)]^{\frac{1}{2}}h\tau^{-1} \\ &< C\|u^0\|[\log(T/3h)]^{\frac{1}{2}}h\tau^{-\frac{1}{2}}\tau_*^{-\frac{1}{2}}\,.\end{aligned}$$

The individual estimates (3, 4, 6, 7, 8) show that the total error

$E \equiv E_i^{(h)}(x, \tau)$ resulting from all five changes satisfies the inequality

$$|E| \leqslant C(T)\tau^{-\frac{1}{2}}\{\tau_* + \eta + [\log(T/3h)]^{\frac{3}{2}} h\tau_*^{-\frac{1}{2}}\tau^{\frac{1}{2}} + h\log(8T/\eta) + [\log(T/3h)]^{\frac{1}{2}} h\tau_*^{-\frac{1}{2}}\}$$
$$< C(T)\tau^{-\frac{1}{2}}\{\tau_* + \eta + [\log(T/h)]^{\frac{3}{2}} h\tau_*^{-\frac{1}{2}}\} \qquad \text{for } \tau_* \leqslant \tau \leqslant T.$$

This is the first inequality in Theorem 2, and the second follows from it directly.

We are now ready to derive the estimate in Theorem 1 of $\|u^{(h)}(\cdot, t) - u^{(h')}(\cdot, t)\|$. Suppose $0 < h' < h < 8^{-3}$, and set $2\eta = \tau_* = h^{\frac{2}{3}}\log(T/h)$. Theorem 2 shows with respect to both $u^{(h)}$ and $u^{(h')}$ that

$$(9) \quad \|E_i^{(h)}(\cdot, \tau)\| < C(T)\tau^{-\frac{1}{2}}h^{\frac{2}{3}}\log(T/h), \quad \|E_i^{(h')}(\cdot, \tau)\| < C(T)\tau^{-\frac{1}{2}}h^{\frac{2}{3}}\log(T/h),$$

for $h^{\frac{2}{3}}\log(T/h) \leqslant \tau \leqslant T$ (the function $s^{\frac{2}{3}}\log(T/s)$ increases with s for $T/s > e^{\frac{3}{2}}$). From the representations of $u^{(h)}(x, \tau)$ and $u^{(h')}(x, \tau)$ given by Theorem 2, subtraction gives

$$(10) \quad u_i^{(h)}(x, \tau) - u_i^{(h')}(x, \tau) = -\int_{\tau_*}^{\tau^*} \mathrm{G}(\tau - t; \mu_i)[f_{i.x}^{(h)}(x, t) - f_{i.x}^{(h')}(x, t)]\,dt$$
$$-\int_{\tau_*}^{\tau^*} \mathrm{G}(\tau - t; \mu_i)[g_i^{(h)}(x, t) - g_i^{(h')}(x, t)]\,dt$$
$$+ \{E_i^{(h)}(x, \tau) - E_i^{(h')}(x, \tau)\} + \{\mathfrak{J}_i^{(h)}(x, \tau) - \mathfrak{J}_i^{(h')}(x, \tau)\} = I_1 + I_2 + E + \mathfrak{J}.$$

We consider this relation only for $\tau_* \leqslant \tau \leqslant T$, for which values we have by (9),

$$(11) \qquad \|E\| \leqslant \|E_i^{(h)}\| + \|E_i^{(h')}\| \leqslant C(T)\tau^{-\frac{1}{2}}h^{\frac{2}{3}}\log(T/h).$$

With respect to I_1, using the rule that $\mathrm{G}(t; \mu)\,\partial_x v = \partial_x(\mathrm{G}(t; \mu)v)$ gives us

$$|I_1| = \left|\int_{\tau_*}^{\tau^*}\left\{\int \nabla_x \mathrm{g}(x - \xi, \tau - t; \mu_i)[f_i^{(h)}(\xi, t) - f_i^{(h')}(\xi, t)]\,d\xi\right\}dt\right|$$
$$\leqslant \int_{\tau_*}^{\tau}\left\{\int |\nabla_x \mathrm{g}(x - \xi, \tau - t; \mu_i)|F_0|u^{(h)}(\xi, t) - u^{(h')}(\xi, t)|\,d\xi\right\}dt$$
$$\leqslant F_0(\pi\mu_i)^{-\frac{1}{2}}\int_{\tau_*}^{\tau}\|u^{(h)}(\cdot, t) - u^{(h')}(\cdot, t)\|(\tau - t)^{-\frac{1}{2}}\,dt,$$

and, similarly,

$$|I_2| \leqslant G_1\int_{\tau_*}^{\tau}\|u^{(h)}(\cdot, t) - u^{(h')}(\cdot, t)\|\,dt.$$

These estimates and (10) imply that for $\tau_* \leqslant \tau \leqslant T$,

$$\|u^{(h)}(\cdot,\tau)-u^{(h')}(\cdot,\tau)\| \leqslant F_0(\pi\mu_*)^{-\frac{1}{2}}\int_{\tau_*}^{\tau}\|u^{(h)}(\cdot,t)-u^{(h')}(\cdot,t)\|(\tau-t)^{-\frac{1}{2}}dt \\ +G_1\int_{\tau_*}^{\tau}\|u^{(h)}(\cdot,t)-u^{(h')}(\cdot,t)\|\,dt+E(\tau)+\mathfrak{J}(\tau), \tag{12}$$

where $E(\tau)$ is subject to (11), and $\|\mathfrak{J}\| \leqslant 2\varrho_0\tau h^{(r-1)/2}$.

We use (12) to estimate

$$q(t)=(t-\tau_*)^{\frac{1}{2}}\|u^{(h)}(\cdot,t)-u^{(h')}(\cdot,t)\|$$

for $\tau_* < t < T$. Multiplying both sides of (12) by $(\tau-\tau_*)^{\frac{1}{2}}$ and referring to the previous estimates of $\|E\|$ and $\|\mathfrak{J}\|$ leads to the relation

$$q(\tau) \leqslant F_0(\mu_*\pi)^{-\frac{1}{2}}(\tau-\tau_*)^{\frac{1}{2}}\int_{\tau_*}^{\tau}(\tau-t)^{-\frac{1}{2}}(t-\tau_*)^{-\frac{1}{2}}q(t)\,dt \\ +G_1(\tau-\tau_*)^{\frac{1}{2}}\int_{\tau_*}^{\tau}(t-\tau^*)^{-\frac{1}{2}}q(t)\,dt+C(T)h^{\frac{1}{2}}\log(T/h)+C(T)h^{(r-1)/2} \tag{13}$$

for $\tau_* \leqslant \tau \leqslant T$. In terms of new independent variables

$$s=t-\tau_*,\qquad \sigma=\tau-\tau_*,\qquad 0\leqslant s\leqslant\sigma,$$

and a new dependent variable

$$Q(s)=q(s+\tau_*)=q(t),$$

inequality (13) can be stated as follows:

$$Q(\sigma) \leqslant F_0(\mu^*\pi)^{-\frac{1}{2}}\sigma^{\frac{1}{2}}\int_0^{\sigma}(\sigma-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}Q(s)\,ds+G_1\sigma^{\frac{1}{2}}\int_0^{\sigma}s^{-\frac{1}{2}}Q(s)\,ds \\ +C(T)h^{\frac{1}{2}}\log(T/h)+C(T)h^{(r-1)/2} \tag{14}$$

for $0 \leqslant \sigma \leqslant T-\tau_*$.

We shall estimate $Q(\sigma)$ by means of the inequality $Q(\sigma) \leqslant P(\sigma)$, where $P(\sigma)$ is a function that satisfies a relation identical in form to (14), except that equality replaces inequality, and P replaces Q. It follows from Lemma 2, Section 5A that

$$Q(\sigma) \leqslant P(\sigma) \leqslant C(T)\{h^{\frac{2}{3}} \log (T/h) + h^{(r-1)/2}\}$$

and thus that

$$q(\tau) \leqslant C(T)\{h^{\frac{2}{3}} \log (T/h) + h^{(r-1)/2}\} \qquad \text{for } h^{\frac{2}{3}} \log (T/h) < \tau \leqslant T ,$$

as Theorem 1 asserts.

APPENDIX

9. - Estimates for repeated averages.

The smoothing step in the method of layering was carried out by means of averaging operators, powers of which were required to satisfy certain inequalities. The operators of Gaussian and of arithmetical averaging both will be shown in this appendix to be of the requisite kind.

Only the one-dimensional case is treated in detail, since, in multi-dimensional averaging, the kernels are taken simply to be products of one-dimensional kernels. By arithmetical smoothing, or averaging, of a bounded, measurable function $v(x)$ on the real line R, we mean the transformation defined, for any $\varepsilon > 0$, by

$$Av(x) = A_\varepsilon v(x) = (2\varepsilon)^{-1}\int_{-\varepsilon}^{\varepsilon} v(x + \xi)\, d\xi \, ;$$

by Gaussian smoothing, or averaging, the transformation defined by

$$Gv(x) = G_\varepsilon v(x) = (3/2\pi)^{\frac{1}{2}} \varepsilon^{-1}\int v(\xi) \exp [- 3(x - \xi)^2/2\varepsilon^2]\, d\xi \, .$$

(The kernels in these two operators have equal variances, namely $\varepsilon^2/3$.) In integrations over R, the limits of integration are omitted.

As previously, for a bounded, measurable function $v(x)$ on R, let

$$\|v\| = \operatorname*{ess\,sup}_{x \in R} |v(x)| \, .$$

The most vital estimates in the layering procedure are those of Theorem 1.

THEOREM 1. *If S represents either Gaussian or arithmetical averaging, then constants $s_1, s_2, \ldots$ exist such that*

$$\|(d/dx)S^j v\| \leqslant s_1 \varepsilon^{-1} j^{-\frac{1}{2}} \|v\| \quad \text{for } j = 1, 2, \ldots, \tag{1}$$

$$\|(d^2/dx^2)S^j v\| \leqslant s_2 \varepsilon^{-2} j^{-1} \|v\| \quad \text{for } j = 2, 3, \ldots, \tag{2}$$

$$\|(d^k/dx^k)S^j v\| \leqslant s_k \varepsilon^{-k} j^{-k/2} \|v\| \quad \text{for } k \geqslant 2,\ j \geqslant 2k, \tag{3}$$

for all bounded, continuous $v(x)$ on R.

Proofs of (1), (2) for Gaussian averaging.

By the reproducing property of normal distributions, the j-th power G^j of G is equal to the Gaussian operator G_j defined by

$$G_j v(x) = \int g_j(x - \xi) v(\xi)\, d\xi \tag{4}$$

with

$$g_j(x) = (3/2\pi j)^{\frac{1}{2}} \varepsilon^{-1} \exp[-3x^2/2j\varepsilon^2], \quad j = 1, 2, \ldots .$$

(See, for instance, H. Cramer [3], equation (17.3.2), p. 212.) Since the integral in (4) can be differentiated under the sign of integration, inequalities of the form (1), (2) will be apparent when we have proved that

$$\int |g_j'(y)|\, dy = (6/\pi)^{\frac{1}{2}} j^{-\frac{1}{2}} \varepsilon^{-1}, \tag{5}$$

$$\int |g_j''(y)|\, dy = 12(2\pi e)^{-\frac{1}{2}} j^{-1} \varepsilon^{-2}. \tag{6}$$

To do so, using the abbreviation $\sigma = (j\varepsilon^2/3)^{\frac{1}{2}}$, note that

$$g_j'(y) = -\sigma^{-2} y g_j(y), \quad g_j''(y) = \sigma^{-4}(y^2 - \sigma^2) g_j(y).$$

Thus, $\int |g_j'(y)|\, dy = 2\int_{-\infty}^{0} g_j'(y)\, dy = 2g_j(0)$, this verifying (5). Secondly,

$$\int |g_j''(y)|\, dy = \sigma^{-4}\int (y^2 - \sigma^2) g_j(y)\, dy + 2\sigma^{-4}\int_{-\sigma}^{\sigma} (\sigma^2 - y^2) g_j(y)\, dy = I_1 + I_2.$$

We have $I_1 = 0$, because $\int g_j(y)\, dy = 1$, and $\int y^2 g_j(y)\, dy = \sigma^2$. The substitution $y = \sigma z$ gives

$$I_2 = 2(2\pi)^{-\frac{1}{2}}\sigma^{-2}\int_{-1}^{1}(1 - z^2)\exp[-z^2/2]\,dz\,,$$

while by partial integration

$$\int_{-1}^{1} z^2 \exp[-z^2/2]\,dz = -2\exp[-\tfrac{1}{2}] + \int_{-1}^{1}\exp[-z^2/2]\,dz\,.$$

Thus, $I_2 = 4(2\pi e)^{-\frac{1}{2}}\sigma^{-2}$, and (6) follows at once.

Proofs of (1), (2) for arithmetical averaging.

Define

$$\begin{aligned} a_1(x) &= 1/2\varepsilon \qquad \text{for } |x| < \varepsilon\,,\\ &= 0 \qquad\quad\ \ \text{for } |x| > \varepsilon\,, \end{aligned}$$

so that $Av(x) = \int a_1(\xi)v(x+\xi)\,d\xi$. Then define, recursively,

$$a_{j+1}(x) = \int a_1(\xi)a_j(x-\xi)\,d\xi = Aa_j(x) \qquad \text{for } j = 1, 2, \dots;$$

in terms of these kernels,

$$A^j v(x) = \int a_j(\xi)v(x+\xi)\,d\xi = \int a_j(\eta - x)v(\eta)\,d\eta \qquad \text{for } j = 1, 2, \dots\,.$$

Each kernel $a_j(x)$ is of compact support and has sectionally continuous derivatives of order $j-1$. Hence,

$$(7)\qquad |(d/dx)A^j v(x)| = \left|\int a_j'(\eta - x)v(\eta)\,d\eta\right| \leqslant \|v\|\int |a_j'(y)|\,dy \qquad \text{for } j = 2, 3, \dots;$$

similarly,

$$(8)\qquad |(d^2/dx^2)A^j v(x)| \leqslant \|v\|\int |a_j''(y)|\,dy \qquad \text{for } j = 3, 4, \dots\,.$$

Except for the lowest values of j, we shall derive (1) and (2) from these and suitable estimates of the integrals of $|a_j'|$ and $|a_j''|$. (For the exceptional values—$j = 1$ in (1) and $j = 2$ in (2)—direct calculations are easily given to the same effect.) The following result is needed.

LEMMA 1. *The kernels $a_j(x)$ are even, i.e., $a_j(-x) = a_j(x)$. For $j \geqslant 2$,*

$$a_j'(x) \leqslant 0 \quad \text{for } x \geqslant 0 . \tag{9}$$

For $j \geqslant 3$, a positive number α_j exists such that

$$\begin{aligned} a_j''(x) &< 0 \quad \text{for } 0 \leqslant x < \alpha_j , \\ &\geqslant 0 \quad \text{for } \alpha_j \leqslant x . \end{aligned} \tag{10}$$

PROOF. Their definitions imply through mathematical induction that the a_j are even. Therefore, in particular, $a_j'(-x) = -a_j'(x)$, $a_j''(-x) = a_j''(x)$. Inequalities (9) and (10) for $j = 2, 3$ can be verified from the explicit formulas

$$\begin{aligned} a_2(x) &= 0 && \text{for} \quad x \leqslant -2\varepsilon , \\ &= (2\varepsilon)^{-1}(1 + x/2\varepsilon) && \text{for } -2\varepsilon \leqslant x \leqslant 0 , \\ a_3(x) &= 0 && \text{for} \quad x \leqslant -3\varepsilon , \\ &= x^2/16\varepsilon^3 + 3x/8\varepsilon^2 + 9/16\varepsilon && \text{for } -3\varepsilon \leqslant x \leqslant -\varepsilon , \\ &= -x^2/8\varepsilon^3 + 3/8\varepsilon && \text{for } -\varepsilon \leqslant x \leqslant 0 . \end{aligned}$$

These are obtained by elementary calculations. To justify the inequalities for higher values of j, mathematical induction is used based on the following formulas:

$$a_{j+1}'(x) = (d/dx)Aa_j(x) = (2\varepsilon)^{-1}\{a_j(x+\varepsilon) - a_j(x-\varepsilon)\} \quad \text{for } j \geqslant 2 , \tag{11}$$

$$a_{j+1}''(x) = (2\varepsilon)^{-1}\{a_j'(x+\varepsilon) - a_j'(x-\varepsilon)\} \quad \text{for } j \geqslant 3 . \tag{12}$$

If (9) holds for some particular index $j \geqslant 2$, then $a_j(x)$ is a monotonically increasing function from $-\infty$ to 0 and a monotonically decreasing function from 0 to $+\infty$. This and (11) show that $a_{j+1}'(x)$, which is 0 for large negative x, becomes first positive, then negative, and finally 0, as x increases. Since a_{j+1}' is odd, the transition from positive to negative values of $a_{j+1}'(x)$ takes place at $x = 0$. Thus, contention (9) is proved for the index $j+1$, and, by mathematical induction, therefore is valid for all $j \geqslant 2$.

Let us now suppose (10) to hold for a particular index $j \geqslant 3$. This and the oddness show that, as x increases, $a_j'(x)$ increases monotonically from 0 to a positive maximum attained at $x = -\alpha_j$ ($\alpha_j > 0$), decreases from this maximum m_j to the minimum $-m_j$ at $x = \alpha_j$, and from that point increases to the ultimate value 0. The graph of $a_j'(x)$ is illustrated in Figure 1.

Because of the shape of this graph, it follows from (12) that, as x increases, $a''_{j+1}(x)$, again 0 for large negative x, first becomes positive, then negative, next positive, and ultimately 0 again. By symmetry, this function must change its sign at just two points $\pm a_{j+1}$, and contention (10) thus holds for for the index $j+1$. With this statement, proof of the lemma is complete.

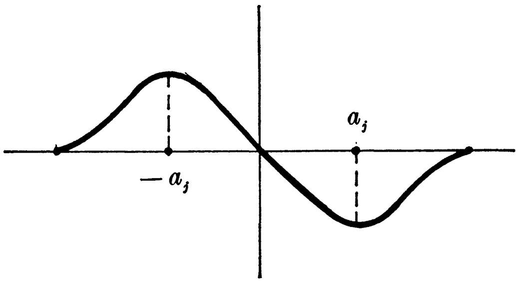

Figure 1

The foregoing lemma implies that

$$\int |a_j'(x)|\,dx = 2a_j(0)\,, \tag{13}$$

$$\int |a_j''(x)|\,dx = 4\|a_j'\|\,. \tag{14}$$

In fact, by (9),

$$\int |a_j'(x)|\,dx = \int_{-\infty}^{0} a_j'(x)\,dx - \int_{0}^{\infty} a_j'(x)\,dx = 2a_j(0)\,,$$

and by (10)

$$\int_0^\infty |a_j''(x)|\,dx = -\int_0^{a_j} a_j''(x)\,dx + \int_{a_j}^{\infty} a_j''(x)\,dx = -\,2a_j'(a_j) = 2\|a_j'\|\,.$$

(The fact that a_j'' is even also is used.)

Inequalities (1), (2) result immediately from (7), (8), (13), (14), and the estimates of $a_j(0) = \|a_j\|$ and of $\|a_j'\|$ in Theorem 2 to be given soon.

Proof of (3).

After (1) has been established, the following arguments produce constants s_k for $k \geqslant 2$ serviceable in (2) and (3). The determinations of the s_k

thus obtained are rather crude, but the reasoning employed is much simpler than in the discussion of s_2 just concluded. First, mathematical induction is used to prove the following.

CLAIM. *If j, l, and r are integers, $r \geqslant 2$, and $j \geqslant lr$, then*

$$\|(d^l/dx^l)S^j v\| \leqslant s_1^l \varepsilon^{-l} r^{-l/2} \|v\| . \tag{15}$$

For $l = 1$, this is true by (1). Then let (15) hold for $j \geqslant lr$, where $l \geqslant 1$. If $j \geqslant (l+1)r$, it follows that

$$\begin{aligned}\|(d^{l+1}/dx^{l+1})S^j v\| &= \|(d/dx)S^r\{(d^l/dx^l)S^{j-r}v\}\| \\ &\leqslant s_1 \varepsilon^{-1} r^{-\frac{1}{2}} \|(d^l/dx^l)S^{j-r}v\| \\ &\leqslant s_1 \varepsilon^{-1} r^{-\frac{1}{2}} s_1^l \varepsilon^{-l} r^{-l/2} \|v\| ,\end{aligned}$$

a result confirming (15) for $l+1$ in place of l and thus completing the induction.

With claim (15) established, let $r = [j/k]$, r thus being an integer such that $r \leqslant j/k < r+1$. Then, in particular, $j \geqslant kr$, and by (15),

$$\|(d^k/dx^k)S^j v\| \leqslant s_1^k \varepsilon^{-k} r^{-k/2} \|v\| . \tag{16}$$

On the other hand, $r > (j-k)/k$, so that

$$r^{-k/2} < k^{k/2}(j-k)^{-k/2} = k^{k/2} j^{-k/2} \{j/(j-k)\}^{k/2} .$$

By assumption, $r \geqslant 2$, and therefore $j/(j-k) < (r+1)k/(r-1)k \leqslant 3$. Hence (16) implies (3) with $s_k = (3k)^{k/2} s_1^k$. Thus Theorem 1 is proved, except for estimates needed of $\|a_j\|$ and $\|a_j'\|$, which will be justified now.

Let $\sigma_j = (j/3)^{\frac{1}{2}} \varepsilon$.

THEOREM 2. *For all integers j, l with $j \geqslant 2$, $0 \leqslant l < j$,*

$$\|\partial_x^l a_j\| \leqslant c_{j,l} \sigma_j^{-l-1} , \tag{17}$$

where the constants $c_{j,l}$ have limits

$$\lim_{j\to\infty} c_{j,0} = (2\pi)^{-\frac{1}{2}}, \qquad \lim_{j\to\infty} c_{j,1} = c_1 = \pi^{-1} ,$$

$$\begin{aligned}\lim_{j\to\infty} c_{j,l} = c_l &= (2\pi)^{-\frac{1}{2}} 1\cdot 3\cdot 5\cdot \ldots (l-1) \qquad &&\textit{for } l = 2, 4, 6, \ldots , \\ &= \pi^{-1}\cdot 2\cdot 4\cdot \ldots (l-1) &&\textit{for } l = 3, 5, 7, \ldots .\end{aligned}$$

This theorem will follow from properties of the Fourier transforms of the kernels in question. The Fourier transform of $a_1(x)$ is

$$\hat{a}_1(s) = \int \exp[ixs] a_1(x)\, dx = (\varepsilon s)^{-1} \sin \varepsilon s ,$$

and since $a_j(x)$ is the convolution of $a_1(x)$ by itself j times, we have for the Fourier transform of $a_j(x)$,

$$\hat{a}_j(s) = (\varepsilon s)^{-j} (\sin \varepsilon s)^j .$$

Furthermore, as is well known,

$$\hat{g}_j(s) = \int \exp[ixs] g_j(x)\, dx = \exp[-j\varepsilon^2 s^2/6] .$$

Let

$$y(s) = y_j(s) = (\varepsilon s)^{-j} (\sin \varepsilon s)^j ,$$

$$z(s) = z_j(s) = \exp[-j\varepsilon^2 s^2/6] .$$

The following result will be very helpful.

LEMMA 2. *For* $0 < \varepsilon s < \pi/2$, *a function* $\alpha(s)$ *exists such that* $0.006\,987 < < \alpha(s) < 0.0969$ *and*

(18) $$y_j(s) = z_j(s) \exp\left(-\alpha(s) \sigma_j^2 \varepsilon^2 s^4\right) .$$

PROOF. By Taylor's expansion of $\sin t$, we have

(19) $$t^{-1} \sin t = 1 - t^2/6 + t^4 A(t) ,$$

where

(20) $$A(t) \equiv t^{-4}(t^{-1} \sin t - 1 + t^2/6)$$

(21) $$= 1/5! - \{(t^2/7! - t^4/9!) + \ldots\} ,$$

the pairs of terms in parentheses in (21) all having positive derivatives if $t < 6$. Thus, if $0 < t < \pi/2$, we have $A(\pi/2) < A(t) < A(0)$, i.e.,

$$a_1 < A(t) < a_2 \quad \text{for } 0 < t < \pi/2 ,$$

where, by (20),

$$a_1 = (2/\pi)^4 (2/\pi - 1 + \pi^2/24) > 0.007\,86 ,$$

$$a_2 = 1/120 < 0.008\,34 .$$

Since, for $|u| < 1$, $\log(1-u) = -u - u^2/2(1-\theta u)^2$ with $0 < \theta < 1$, we have by (19)

$$\log[t^{-1}\sin t] = \log[1 - t^2/6 + t^4 A(t)]$$
$$= -t^2/6 + t^4 A(t) - B[-t^2/6 + At^4]^2,$$

where $B = (\frac{1}{2})[1 - \theta(t^2/6 - At^4)]^{-2}$; evidently, $B > \frac{1}{2}$ and $B < (\frac{1}{2})(1-\pi^2/24)^{-2}$, i.e.,

$$0.500 < B < 1.443 \qquad \text{for } 0 < t < \pi/2 .$$

Thus,

$$\log(t^{-1}\sin t) = -t^2/6 - Ct^4,$$

where $C = -A + B(-1/6 + At^2)^2$. In view of the estimates previously obtained for A and B,

$$0.002\,329 < C < 0.0323 \qquad \text{for } 0 < t < \pi/2 .$$

It follows that for $0 < \varepsilon s < \pi/2$

$$\log y_j(s) = j\{-\varepsilon^2 s^2/6 - C\varepsilon^4 s^4\}$$

and thus that

$$y_j(s) = z_j(s)\exp[-3C\sigma_j^2\varepsilon^2 s^4],$$

which coincides with (18) with $\alpha(s) = 3C$. Thus, the lemma is proved.

Proof of Theorem 2.

First consider the case $l = 0$. The Fourier integral inversion $a_j(x) = (1/2\pi)\int \hat{a}_j(s)\exp[-ixs]\,ds$ implies that

$$\|a_j\| \leqslant (2\pi)^{-1}\int \hat{a}_j(s)\,ds = \pi^{-1}\int_0^\infty |y_j(s)|\,ds$$
$$= \pi^{-1}\Big\{\int_0^{\pi/2\varepsilon} + \int_{\pi/2\varepsilon}^\infty\Big\}.$$

Using the estimate $y_j(s) \leqslant z_j(s)$ in $(0, \pi/2\varepsilon)$, as follows from (18), and the estimate $|y_j(s)| \leqslant (\varepsilon s)^{-j}$ in $(\pi/2\varepsilon, \infty)$ gives

$$\|a_j\| \leqslant \pi^{-1}\int_0^{\pi/2\varepsilon} z_j(s)\,ds + \pi^{-1}\int_{\pi/2\varepsilon}^\infty (\varepsilon s)^{-j}\,ds .$$

The first term on the right-hand side is $< (2\pi)^{-\frac{1}{2}}\sigma_j^{-1}$, and the second term is

$$\begin{aligned} &= (j-1)^{-1}\pi^{-1}\varepsilon^{-1}(2/\pi)^{j-1} \\ &= (1/2\sqrt{3})j^{\frac{1}{2}}(j-1)^{-1}(2/\pi)^j\sigma_j^{-1} \quad \text{for } j\geqslant 2\,. \end{aligned}$$

Combining the estimates for the two terms gives (17) in the case $l=0$.

The norms of derivatives of the a_j are estimated similarly. The Fourier transform of the l-th derivative being given by

$$\mathfrak{F}[a_j^{(l)}](s) = (-\,is)^l\hat{a}_j(s)\,,$$

we have $\|a_j^{(l)}\| \leqslant (2\pi)^{-1}\int |s|^l y_j(s)\,ds$, and in the same manner as before,

$$\begin{aligned} \|a_j^{(l)}\| &\leqslant \pi^{-1}\int_0^{\pi/2\varepsilon} s^l z_j(s)\,ds + \pi^{-1}\int_{\pi/2\varepsilon}^{\infty} s^l(\varepsilon s)^{-j}\,ds \\ &\leqslant (c_l + c_{j,l'})\sigma_j^{-l-1} \qquad \text{for } j\geqslant l+2\,, \end{aligned}$$

where

$$c_{j,l'} = (\tfrac{1}{2})(j/3)^{(l+1)/2}(j-l-1)^{-1}(2/\pi)^{j-l}\,.$$

Thus, Theorem 2 is proved.

In earlier versions of this paper, Theorem 2 in the case $l=0$, but with a larger constant in place of $c_{l,0}$, was proved by use of the inequality

$$t^{-1}\sin t \leqslant 1-(t/\pi)^2 \quad \text{for } |t|\leqslant\pi\,.$$

The inequality follows from Euler's product representation and was suggested for this application by P. Mikulski and R. Syski.

REMARK. Consider the operator $K^* = K_\varepsilon^j K_{\varepsilon'}^{\prime j'}$, where K_ε and $K'_{\varepsilon'}$ are operators of Gaussian or of arithmetical averaging with kernels $k_\varepsilon(\xi)$ and $k'_{\varepsilon'}(\xi)$, respectively. The kernel $k_j(\xi)$ of the operator K_ε^j is of course the convolution of $k_\varepsilon(\xi)$ with itself j times, and similarly for the kernel $k'_{j'}(\xi)$ of $K_{\varepsilon'}^{\prime j'}$. The kernel $k^*(\xi)$ of K^* is the convolution of k_j with $k'_{j'}$. Let σ^2 and σ'^2 denote the respective variances of the distributions with densities $k_\varepsilon(\xi)$, $k'_{\varepsilon'}(\xi)$: for instance, $\sigma^2 = \int k_\varepsilon(\xi)\xi^2 d\xi$. The variance of the distribution with density $k^*(\xi)$ is $\sigma^{*2} = j\sigma^2 + j'\sigma'^2$, and we have

$$\|K^* v\| \leqslant \text{const}\,\sigma^{*-1}\int |v(x)|\,dx$$

for any $v(x)$ integrable on R. This follows from the inequality

$$|K^* v(x)| = \left|\int k^*(y-x)v(y)\,dy\right| \leqslant \|k^*\| \int |v(y)|\,dy$$

and an estimate of $\|k^*\|$ obtained by use of (18) as in the proof of Theorem 2.

The next results are used in showing that layered solutions converge as layer heights approach zero.

THEOREM 3. *Constants A and B exist such that*

$$\|g_j - a_j\| \leqslant A\sigma_j^{-3}\varepsilon^2 \quad \text{for } j \geqslant 1 \tag{22}$$

and that

$$\int |a_j(x) - g_j(x)|\,dx < B[\log(\sigma_j/\varepsilon)]^{\frac{1}{2}}(\varepsilon/\sigma_j)^2\,, \tag{23}$$

where we can take $A = 0.40$, $B = 4.1$.

REMARK. In a far more thorough analysis than that which follows, Petrov [22] gives general asymptotic expansions that pertain to $a_j - g_j$ (Theorem 15, p. 206) and to $\int |a_j - g_j|\,dx$ (Theorem 18, p. 212) for large j. His results imply inequalities (22) and (23), but without the logarithm. In Section 8, in which the inequalities are applied, the improvement from using Petrov's result is, however, rather slight. Hence, and for the sake of completeness, we use Theorem 3 as stated, its proof being comparatively direct.

Proof of Theorem 3.

First we must estimate $\|g_j - a_j\|$. Let σ abbreviate σ_j. Then using the Fourier inversion formula as before, and referring to the previous lemma, we have

$$\begin{aligned}2\pi\|g_j - a_j\| &\leqslant \int |y_j(s) - z_j(s)|\,ds \\ &= 2\int_0^{\pi/2\varepsilon} \exp[-\sigma^2 s^2/2]\{1 - \exp(-\alpha(s)\sigma^2\varepsilon^2 s^4)\}\,ds + 2\int_{\pi/2\varepsilon}^{\infty} |y_j(s) - z_j(s)|\,ds \\ &= 2I_1 + 2I_2\,.\end{aligned}$$

To estimate I_1, we use the inequality $1 - \exp[-x] \leqslant x$, which holds for $x \geqslant 0$, to obtain

$$I_1 < \alpha^* \sigma^2 \varepsilon^2 \int_0^\infty \exp[-\sigma^2 s^2/2] s^4 ds$$
$$= (\tfrac{3}{2})(2\pi)^{\frac{1}{2}} \alpha^* \sigma^{-3} \varepsilon^2 ,$$

where $\alpha^* = \max_{0 \leqslant s \leqslant \pi/2} |\alpha(s)| < 0.097$.

Concerning I_2, we have

$$I_2 < \int_{\pi/2\varepsilon}^\infty y(s)\, ds + \int_{\pi/2\varepsilon}^\infty z(s)\, ds = I_2' + I_2'' .$$

By the well-known rule

$$\int_x^\infty \exp[-\xi^2/2c^2] d\xi < \int_x^\infty \exp[-x\xi/2c^2] d\xi \tag{24}$$
$$= (2c^2/x) \exp[-x^2/2c^2] \quad \text{for } x > 0 ,$$

we have

$$I_2'' < (4/\pi) \sigma^{-2} \varepsilon \exp(-\pi^2 \sigma^2/8\varepsilon^2)$$

and, since

$$\varepsilon^{-\alpha} \exp(-C\varepsilon^{-\beta}) \leqslant (\alpha/Ce\beta)^{\alpha/\beta} \tag{25}$$

for positive α, β, C, ε,

$$I_2'' < 8\pi^{-2} \exp[-\tfrac{1}{2}] \sigma^{-3} \varepsilon^2 .$$

In addition,

$$I_2' < \int_{\pi/2\varepsilon}^\infty (\varepsilon s)^{-j} ds$$
$$= (j-1)^{-1} \varepsilon^{-1} (2/\pi)^{j-1}$$
$$= (\pi/6)[j/(j-1)] \sigma^{-2} \varepsilon (2/\pi)^{3\sigma^2/\varepsilon^2} ,$$

since $j = 3\sigma^2/\varepsilon^2$. In this result replacing $2/\pi$ by $\exp[-\log(\pi/2)]$ and again referring to (25) shows that

$$I_2' < (\pi/6)[6e \log(\pi/2)]^{-\frac{1}{2}} [j/(j-1)] \sigma^{-3} \varepsilon^2 ,$$

and in view of the previous finding for I_2'',

$$I_2 < \{8\pi^{-2} \exp[-\tfrac{1}{2}] + (\pi/3)[6e \log(\pi/2)]^{-\frac{1}{2}}\} \sigma^{-3} \varepsilon^2 \quad \text{for } j \geqslant 2 .$$

In sum, therefore,

$$\|g_j - a_j\| \leqslant \pi^{-1}(I_1 + I_2) < A\sigma^{-3}\varepsilon^2 \qquad \text{for } j \geqslant 2$$

with

$$A = \pi^{-1}\{(\tfrac{3}{2})(2\pi)^{\frac{1}{2}}\alpha^* + 8\pi^{-2}\exp[-\tfrac{1}{2}] + (\pi/3)[6e\log(\pi/2)]^{-\frac{1}{2}}\} < 0.40$$

in verification of (22).

In proving (23), we shall require a serviceable estimate of

$$A_j(x) = \int_{-\infty}^{x} a_j(\xi)\,d\xi$$

for large negative x, an estimate we shall obtain by comparing $A_j(x)$ with

$$G_j(x) = \int_{-\infty}^{x} g_j(\xi)\,d\xi\,.$$

Since $\hat{a}_j(s) \equiv y_j(s)$ is even, the Fourier inversion formula for a_j takes the form

$$a_j(\xi) = \pi^{-1}\int_0^{\infty} y_j(s)\cos s\xi\,ds\,.$$

Therefore, integrating with respect to ξ on an interval $(0, x)$ gives

$$A_j(x) - \tfrac{1}{2} = \pi^{-1}\int_0^{\infty} y_j(s)s^{-1}\sin sx\,ds\,.$$

A similar formula holds for $G_j(x)$, and by subtracting the two formulas we obtain for the quantity

$$R_j \equiv A_j - G_j$$

the expression

$$R_j(x) = \pi^{-1}\int_0^{\infty} r_j(s)s^{-1}\sin sx\,ds\,,$$

in which

$$r_j(s) = y_j(s) - z_j(s)\,.$$

Consequently,

$$|R_j(x)| \leqslant \pi^{-1}\int_0^\infty |r_j(s)|s^{-1}ds = \pi^{-1}\left\{\int_0^{\pi/2\varepsilon} + \int_{\pi/2\varepsilon}^\infty\right\} = \pi^{-1}\{J_1 + J_2\}.$$

To estimate J_1, again by use of (18) and the inequality $1 - e^{-x} \leqslant x$ for $x \geqslant 0$, we obtain

$$J_1 < \alpha^* \sigma^2 \varepsilon^2 \int_0^\infty \exp[-\sigma^2 s^2/2] s^3 ds = 2\alpha^* \sigma^{-2}\varepsilon^2.$$

Concerning J_2, we have

$$J_2 < \int_{\pi/2\varepsilon}^\infty y_j(s)s^{-1}ds + \int_{\pi/2\varepsilon}^\infty z_j(s)s^{-1}ds = J_2' + J_2'',$$

while

$$J_2' < \int_{\pi/2\varepsilon}^\infty (\varepsilon s)^{-j}s^{-1}ds = j^{-1}(2/\pi)^j = (\tfrac{1}{3})\sigma^{-2}\varepsilon^2(2/\pi)^j < (\tfrac{1}{3})\sigma^{-2}\varepsilon^2,$$

and

$$J_2'' = \int_{\pi/2\varepsilon}^\infty \exp[-\sigma^2 s^2/2]s^{-1}ds < (2/\pi)\varepsilon \int_{\pi/2\varepsilon}^\infty \exp[-\pi\sigma^2 s/4\varepsilon]ds$$
$$= (8/\pi^2)\sigma^{-2}\varepsilon^2 \exp(-\pi^2\sigma^2/8\varepsilon^2) < (8/\pi^2)\sigma^{-2}\varepsilon^2.$$

In sum,

$$|R_j(x)| < A_1\varepsilon^2\sigma^{-2}$$

with

$$A_1 = \pi^{-1}\{2\alpha^* + \tfrac{1}{3} + 8/\pi^2\} < 0.426.$$

This result implies, in particular, that for $x > 0$

$$A_j(-x) \leqslant G_j(-x) + A_1\varepsilon^2\sigma^{-2},$$

while, as follows from rule (24),

$$G_j(-x) < 2(2\pi)^{-\frac{1}{2}}(\sigma/x)\exp[-x^2/2\sigma^2] \quad \text{for } x > 0. \tag{26}$$

Thus,

$$A_j(-x) < 2(2\pi)^{-\frac{1}{2}}(\sigma/x)\exp[-x^2/2\sigma^2] + A_1\varepsilon^2\sigma^{-2} \quad \text{for } x > 0. \tag{27}$$

This and (22) permit us to estimate

$$I = \int_0^\infty |a_j - g_j|\,dx = \int_0^X + \int_X^\infty = I_1 + I_2\,,$$

in which we take $X = 2[\log(\sigma/\varepsilon)]^{\frac{1}{2}}\sigma$. By (22),

$$I_1 \leqslant 2A[\log(\sigma/\varepsilon)]^{\frac{1}{2}}(\varepsilon/\sigma)^2\,,$$

and by (26) and (27),

$$\begin{aligned} I_2 &< A_j(-X) + G_j(-X) \\ &< A_1(\varepsilon/\sigma)^2 + 2(2\pi)^{-\frac{1}{2}}[\log(\sigma/\varepsilon)]^{-\frac{1}{2}}(\varepsilon/\sigma)^2 \\ &< A_2(\varepsilon/\sigma)^2 \end{aligned}$$

where $A_2 = A_1 + 2(2\pi)^{-\frac{1}{2}}$ if $j \geqslant 3$. In sum, therefore,

$$\int |a_j - g_j|\,dx = 2I < A_3[\log(\sigma/\varepsilon)]^{\frac{1}{2}}(\varepsilon/\sigma)^2 \quad \text{for } j \geqslant 3 \tag{28}$$

with $A_3 = 2(2A + A_2) < 4.1$, as stated in (23).

The final theorem is concerned with L^1 norms, the estimates in this theorem pertaining to bounded, measurable functions $v(x)$ that either are periodic on R of period $P > 0$ or else are integrable on R. We set

$$\begin{aligned} |v|_L &= \int_0^P |v(x)|\,dx \quad \text{in the periodic case}\,, \\ &= \int |v(x)|\,dx \quad \text{in the integrable case}\,. \end{aligned}$$

THEOREM 4. *Let $S = S_\varepsilon$ denote the operator of Gaussian or arithmetical averaging. If $v(x)$ is integrable on R, then a constant s_0 exists such that*

$$\|S^j v\| \leqslant s_0 \varepsilon^{-1} j^{-\frac{1}{2}} |v|_L \quad \textit{for } j \geqslant 1\,. \tag{29}$$

If $v(x)$ is periodic of period P, an inequality of the form (29) *still holds, but with s_0 replaced by a quantity of the form constant $+\ \varepsilon j^{\frac{1}{2}}$. For $v(x)$ of either type, a constant s_1 exists such that*

$$\|(d/dx) S^j v\| \leqslant s_1' \varepsilon^{-2} j^{-1} |v|_L \quad \textit{for } j \geqslant 3\,; \tag{30}$$

in addition,

$$|(d/dx)\,S^j v|_L \leqslant s_1 \varepsilon^{-1} j^{-\frac{1}{2}} |v|_L \quad \text{for } j \geqslant 1\,. \tag{31}$$

Let $k_j(\xi)$ denote the kernel of S^j:

$$S^j v(x) = \int k_j(\xi - x)\, v(\xi)\, d\xi\,. \tag{32}$$

(For arithmetic averaging, $k_j(\xi) = a_j(\xi)$; for Gaussian averaging, $k_j(\xi) = g_j(\xi)$.)

Proof of (29).

Inequality (29) is immediate in the case of integrable $v(x)$, since

$$\|k_j\| = k_j(0) \leqslant s^* \varepsilon^{-1} j^{-\frac{1}{2}} \tag{33}$$

with a suitable constant s^*, see (17) with $l = 0$.

If $v(x)$ is periodic of period $P > 0$, we have from (32)

$$\begin{aligned} S^j v(x) &= \sum \int_{mP+x}^{(m+1)P+x} k_j(\xi - x)\, v(\xi)\, d\xi = \sum \int_x^{P+x} k_j(\xi + mP - x)\, v(\xi)\, d\xi \\ &= \sum \int_0^P k_j(\xi + mP)\, v(x + \xi)\, d\xi\,, \end{aligned} \tag{34}$$

the summation being over all integers m. For $0 \leqslant \xi \leqslant P$, of course $mP \leqslant \xi + mP \leqslant (m+1)P$, and since $k_j(y)$ is even and decreases as y increases starting from 0,

$$\begin{aligned} k_j(\xi + mP) &\leqslant k_j\big((m+1)P\big) \qquad \text{for } m < 0\,, \\ &\leqslant k_j(mP) \qquad \text{for } m \geqslant 0\,. \end{aligned}$$

Therefore,

$$\sum k_j(\xi + mP) \leqslant \sum_{m=-\infty}^{-1} k_j\big((m+1)P\big) + \sum_{m=0}^{\infty} k_j(mP) = 2 \sum_{m=0}^{\infty} k_j(mP)\,,$$

evenness accounting for the second equality. Because $k_j(\xi)$ decreases for $\xi > 0$,

$$\sum_{m=1}^{\infty} k_j(mP) < \int_0^{\infty} k_j(\xi)\, d\xi = \tfrac{1}{2}\,.$$

Therefore,

$$\sum k_j(\xi + mP) < 2k_j(0) + 1 ,$$

and

$$\|S^j v\| \leqslant \{2k_j(0) + 1\}|v|_L \leqslant \{2s^* \varepsilon^{-1} j^{-\frac{1}{2}} + 1\}|v|_L .$$

This formula justifies (29) in the periodic case.

Proof of (30).

By (32),

$$(d/dx)\, S^j v(x) = -\int k_j'(\xi - x)\, v(\xi)\, d\xi \, . \tag{35}$$

Hence, for integrable $v(x)$,

$$\|(d/dx)\, S^j v\| \leqslant \|k_j'\|\, |v|_L ,$$

inequality (30) following from the expression for k_j' in the case $S = G_\varepsilon$ and from (17) in the case $S = A_\varepsilon$.

For periodic $v(x)$ of period P, we obtain from (35), analogously to (34),

$$(d/dx)\, S^j v(x) = -\sum \int_{mP+x}^{(m+1)P+x} k_j'(\xi - x)\, v(\xi)\, d\xi = -\int_0^P \sum k_j'(\xi + mP)\, v(x + \xi)\, d\xi , \tag{36}$$

the summation again being over all integers m. Since $k_j'(\xi)$ is an odd function,

$$\sum k_j'(\xi + mP) = k_j'(\xi) + \sum_{m=1}^{\infty} \{k_j'(\xi + mP) - k_j'(mP)\} + \sum_{m=-\infty}^{-1} \{k_j'(\xi + mP) - k_j'(mP)\} ,$$

while for $0 \leqslant \xi \leqslant P$

$$|k_j'(\xi + mP) - k_j'(mP)| = \Big| \int_{mP}^{\xi + mP} k_j''(s)\, ds \Big| \leqslant \int_{mP}^{(m+1)P} |k_j''(s)|\, ds \, .$$

Hence,

$$\Big|\sum k_j'(\xi + mP)\Big| < \|k_j'\| + \int_{-\infty}^{\infty} |k_j''(s)|\, ds \leqslant 5\|k_j'\|$$

by (14) in the case $S = A$ and by the analogous equality in the case $S = G$. Using this in (36) proves

$$\|(d/dx)S^j v\| \leqslant 5\|k_j'\|\,|v|_L\,,$$

from which and from (17) pertaining to A and the analogous estimate pertaining to G, (30) follows.

Proof of (31).

Let $j \geqslant 2$. Again starting from (35), we have for integrable $v(x)$,

$$\int |(d/dx)S^j v(x)|\,dx \leqslant \int\int |k_j'(\xi - x)|\,|v(\xi)|\,d\xi\,dx \leqslant |v|_L \int |k_j'(\xi)|\,d\xi \leqslant 2k_j(0)\,|v|_L\,,$$

(31) following from this.

In the periodic case,

$$\int_0^P |(d/dx)S_j v(x)|\,dx \leqslant \int_0^P \int |k_j'(\zeta)|\,|v(x + \zeta)|\,d\zeta\;dx \leqslant |v|_L \int |k_j'(\zeta)|\,d\zeta = 2k_j(0)\,|x|_L\,,$$

(31) again being the consequence.

For $j = 1$, the previous methods again suffice in the case $S = G$, while if $S = A$ the formula $(d/dx)Av(x) = (2\varepsilon)^{-1}[v(x + \varepsilon) - v(x - \varepsilon)]$ is used.

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