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On the Envelope of Regularity for Solutions of Homogeneous Systems of Linear Partial Differential Operators.

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Introduction.

Given a topological space X with a countable topology and denoting by $C(X)$ the space of complex valued continuous functions on X, it is known that a closed subset F of X is compact if and only if

$$\sup_F |f| < +\infty$$

for every function f in $C(X)$ (this criterion is essentially due to Weierstrass). In general, given a part $\mathcal{S}$ of $C(X)$, the set $C(X, \mathcal{S}) = \{F | F$ is a closed subset of X and $\sup_F |f| < +\infty, \forall f \in \mathcal{S}\}$ is larger than the set of all compact subsets of X. The elementary theory of convexity is interested in the question of giving criteria for $C(X, \mathcal{S})$ to consist of the compact subsets of X only.

A first interesting instance of this question is the following: X is an open set in $\mathbb{C}^n$ and $\mathcal{S}$ is the space $\mathcal{H}(X)$ of holomorphic functions on X. Then the necessary and sufficient condition for $C(X, \mathcal{H}(X))$ to consist of all compact sets, is that X is an open set of holomorphy. This can be viewed as the content of the classical theorem of H. Cartan and P. Thullen (cf. [5]).

Here we are interested in the following general situation: X is an open subset of the numerical space $\mathbb{R}^n$ and $\mathcal{S}$ the set of solutions u of an elliptic system

$$Au = 0 \quad \text{on } X,$$

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where $A: \mathcal{E}^q(X) \to \mathcal{E}^p(X)$ is a $p \times q$ matrix of linear partial differential operators on X (by $\mathcal{E}$ we denote the space of complex valued, C^∞ functions).

In § 1 we develop the theory of the envelope of regularity and give the extension of the theorem of Cartan and Thullen to the case of operators whose coefficients are either constant or real analytic.

In § 2 we undertake the study of domains of regularity for overdetermined systems, with constant coefficients, in one unknown function (the case $q = 1$). The simple cases of differential ideals of dimension zero and of reduced homogeneous differential ideals of dimension one are given directly to cover the theory till the vanishing theorems for cohomology (with values in the sheaf of germs of solutions).

In § 3 we treat general systems in one unknown function.

In the last paragraph the properties of the logarithmic distance from the boundary of a domain of regularity are established until an analog of Levi's convexity condition is found.

This convexity condition of Levi type will be studied in a subsequent paper, where we will deal with the finiteness of the cohomology groups related to such systems of differential equations.

Some of the results of this paper have been treated in two seminars of the R.C.P. 25 at Strasbourg (1).

§ 1. - Elementary convexity theory.

1. – Preliminaries.

(*a*) Let us consider a $p \times q$ matrix $A_0(x, D)$ of differential operators with C^∞ coefficients on $\mathbb{R}^n$ (2).

If $\mathcal{E}$ denotes the sheaf of germs of (complex valued) C^∞ functions on $\mathbb{R}^n$, A_0 defines a linear map:

$$\mathcal{E}^q \xrightarrow{A_0} \mathcal{E}^p .$$

By $\mathcal{O}_{A_0}$ we denote the sheaf of germs of C^∞ solutions of the homogeneous

(1) A. Andreotti - M. Nacinovich, *Théorie élémentaire de la convexité*, R.C.P. 25, November 25, 1976.

A. Andreotti - M. Nacinovich, *Domaines de regularité pour les opérateurs elliptiques à coefficients constants*, R.C.P. 25, May 26, 1977.

(2) By $x = (x_1 \dots x_n)$ we denote cartesian coordinates in $\mathbb{R}^n$ and by $D = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ the symbols of differentiation.

equation $A_0 u = 0$ so that we have an exact sequence of sheaves:

$$0 \rightarrow \mathcal{O}_{A_0} \rightarrow \mathcal{E}^q \xrightarrow{A_0} \mathcal{E}^p .$$

A basis for open sets in the topology of $\mathcal{O}_{A_0}$ is given by the sets:

$$W(\Omega, u) = \{u_y (= \text{germ of } u \text{ at } y),\ \forall y \in \Omega\}$$

with Ω any open subset of $\mathbb{R}^n$ and $u \in \mathcal{E}^q(\Omega)$ a solution of $A_0(x, D)u = 0$ on Ω.

With this topology the natural map

$$\mathcal{O}_{A_0} \xrightarrow{\pi} \mathbb{R}^n$$

is a local homeomorphism. However, the topology of $\mathcal{O}_{A_0}$ may fail to be a Hausdorff topology.

(*b*) If $\mathcal{S}$ is a subsheaf of the sheaf $\mathcal{E}$, at any point x_0 of $\mathbb{R}^n$ we can associate to a germ σ in $\mathcal{S}_{x_0}$ the Taylor series

$$\mathcal{T}_{x_0}(\sigma) = \sum \frac{D^\alpha \sigma(x_0)}{\alpha!} (x - x_0)^\alpha$$

as an element of the ring Φ_{x_0} of formal power series centered at x_0. Thus we obtain a linear map:

$$\mathcal{T}_{x_0} : \mathcal{S}_{x_0} \rightarrow \Phi_{x_0} .$$

We say that the sheaf $\mathcal{S}$ has *the property* (A) *of Aronszajn* if for every x_0 in $\mathbb{R}^n$ the map $\mathcal{T}_{x_0}$ is injective.

In particular, this is the case if $\mathcal{S}$ is a subsheaf of the sheaf $\mathcal{A}$ of (complex valued) real analytic functions on $\mathbb{R}^n$.

Another remarkable case in which (A) holds is when $\mathcal{S}$ is the sheaf of germs of solutions of a second order partial differential equation:

$$\sum a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = 0$$

with a_{ij}, b_i and c of class C^∞ and for any x_0 in $\mathbb{R}^n$ and any non zero real vector ξ

$$\operatorname{Re} \sum a_{ij}(x_0) \xi_i \xi_j > 0 .$$

This is the content of a theorem of Aronszajn [3].

PROPOSITION 1. *A subsheaf $\mathcal{S}$ of $\mathcal{E}$ with property (A) has a Hausdorff topology.*

PROOF. We denote by $\pi\colon \mathcal{S} \to \mathbb{R}^n$ the natural projection. We have to prove that any two distinct points α, β in $\mathcal{S}$ have disjoint neighborhoods in $\mathcal{S}$. If $\pi(\alpha) \neq \pi(\beta)$ this is straightforward. So, assume that $\pi(\alpha) = \pi(\beta) = x_0$. In a sufficiently small neighborhood ω of x_0 we can find two sections s, σ in $\Gamma(\omega, \mathcal{S})$ with:

$$s_{x_0} = \alpha \quad \text{and} \quad \sigma_{x_0} = \beta .$$

We consider a fundamental sequence $\{\omega_\nu\}$ of neighborhoods of x_0 in ω, and the neighborhoods of α, β given by $W(\omega_\nu, s|_{\omega_\nu})$ and $W(\omega_\nu, \sigma|_{\omega_\nu})$ respectively. We claim that, for some ν_0,

$$W(\omega_{\nu_0}, s|_{\omega_{\nu_0}}) \cap W(\omega_{\nu_0}, \sigma|_{\omega_{\nu_0}}) = \emptyset .$$

Otherwise we will find a sequence $\{y_\nu\}$ converging to x_0 in ω such that

$$s_{y_\nu} = \sigma_{y_\nu} \quad \text{for every } \nu .$$

But then we have, for any multiindex μ,

$$(D^\mu s)(y_\nu) = (D^\mu \sigma)(y_\nu)$$

and thus, passing to the limit,

$$(D^\mu s)(x_0) = (D^\mu \sigma)(x_0) .$$

By property (A) this implies that $\alpha = s_{x_0} = \sigma_{x_0} = \beta$, which contradicts our assumption. The proof is complete.

We say that the operator $A_0(x, D)$ is *elliptic* if, for every open set Ω in $\mathbb{R}^n$, any distribution solution u of

$$A_0(x, D)\, u = 0 \quad \text{on } \Omega$$

is real analytic.

A *theorem of Petrowski* [9] states that $A_0(x, D)$ is elliptic (in this sense) if:

(i) $A_0(x, D)$ has real analytic coefficients;

(ii) $p \geqslant q$ and, for each minor determinant $M_j(x, \xi)$ $\left(1 \leqslant j \leqslant \binom{p}{q}\right)$ of order q of the matrix $A_0(x, \xi) = \big(a_{ij}(x, \xi)\big)$ we can find an integer $m_j \geqslant$ degree in ξ of $M_j(x, \xi)$ such that for every x in $\mathbb{R}^n$ the system

$$\text{part of degree } m_j \text{ of } M_j(x, \xi) = 0 \qquad 1 \leqslant j \leqslant \binom{p}{q}$$

has no real solution $\xi \neq 0$.

Every elliptic operator with constant coefficients is elliptic in the sense of Petrowski.

But there are elliptic operators with real analytic coefficients which are not of Petrowski type [3].

From Proposition 1 we deduce the following

COROLLARY. *For an elliptic operator $A_0(x, D)$, the sheaf $\mathcal{O}_{A_0}$ has a Hausdorff topology.*

2. – Riemann domains.

(a) Let Y be a differentiable connected manifold (for instance, $Y = \mathbb{R}^n$); by a *Riemann domain over* Y we mean the set of the following data:

(α) a connected topological manifold X;

(β) a continuous map $\omega: X \to Y$ which is a local homeomorphism.

Then X has necessarily the same dimension of Y. Moreover on X there is a unique differentiable structure in which ω becomes a local diffeomorphism.

We will assume that Y has a countable topology [4].

THEOREM 1 (*of Poincaré-Volterra*). *Any Riemann domain $X \xrightarrow{\omega} Y$ over Y has a countable topology.*

PROOF. We assume first that Y is an open subset of $\mathbb{R}^n$. Then $\Omega = \omega(X)$ is open and connected and we can as well assume that $Y = \Omega$. Let I be the set of rational points in Ω and set $A = \omega^{-1}(I)$. For every $\alpha \in X$ define the positive number $\varepsilon(\alpha)$ as follows: $\varepsilon(\alpha) = \sup \{r > 0 |$ the ball with center $\omega(\alpha)$ and radius r is contained in Ω and can be isomorphically lifted via ω to a « ball » in X with center $\alpha\}$.

(3) For a positive integer k, the operator

$$A_0\left(x, y; \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x} + ix^{2k}\frac{\partial}{\partial y}$$

in $\mathbb{R}^2$ is of this sort.

(4) We mean by this that there exists on Y a countable system of open sets such that any open set of Y is a union of open sets of that system.

Because ω is a local homeomorphism, $\varepsilon(\alpha) > 0$, for every α in X, and $\varepsilon(\alpha)$ is also a continuous function of α on X. Let $B(\alpha, \varrho)$ denote the open « ball » of X with center α and radius ϱ for $0 < \varrho < \varepsilon(\alpha)$.

Let

$$\mathcal{B} = \{B(\alpha, \varrho) | \alpha \in A,\ 0 < \varrho < \varepsilon(\alpha),\ \varrho \in \mathbb{Q}\} .$$

Since $\varepsilon(\alpha)$ is a continuous function one verifies first that every open set $\Sigma \subset X$ is the union of all $B(\alpha, \varrho)$ contained in it:

$$\Sigma = \bigcup_{\substack{B(\alpha,\varrho)\in\mathcal{B} \\ B(\alpha,\varrho)\subset\Sigma}} B(\alpha, \varrho) .$$

Therefore $\mathcal{B}$ is a basis for open sets. It will be enough to show that $\mathcal{B}$ is a countable set and, for that, that A is countable. Fix an element $\alpha_0 \in A$. For any $\alpha \in A$ we can find a continuous arc

$$\gamma\colon [0, 1] \to X \qquad \text{with } \gamma(0) = \alpha_0,\ \gamma(1) = \alpha$$

because X is connected and thus arcwise connected.

If is not restrictive to assume that $\sigma = \omega \circ \gamma\colon [0, 1] \to \Omega$ has the following two properties:

i) σ is a broken line joining $\omega(\alpha_0)$ to $\omega(\alpha)$.

ii) the edges of this broken line are all in I.

Now, given an arc

$$\sigma\colon [0, 1] \to \Omega$$

with $\sigma(0) = \omega(\alpha_0)$, there exists at most a unique lifting $\gamma\colon [0, 1] \to X$ with $\gamma(0) = \alpha_0$. This because ω is a local homeomorphism.

Therefore A is in one to one correspondence with a subset of the set Λ of all broken lines with starting point in $\omega(\alpha_0)$ and edges in I. Any one of these broken lines is determined by the sequence of its edges. Thus

$$\operatorname{card}(\Lambda) = \operatorname{card} \bigcup_{m=1}^{\infty} (\mathbb{Q}^n)^m .$$

Hence Λ is countable and thus A is also countable.

We now drop the assumption that Y is an open subset of $\mathbb{R}^n$. As Y has a countable topology we can find a proper imbedding

$$J\colon Y \to \mathbb{R}^N$$

for some sufficiently large N.

We can also extend this imbedding to a diffeomorphism

$$\lambda\colon Y \times D^{N-n} \to \Omega \subset \mathbb{R}^N$$

where n denotes the dimension of Y, $D^{N-n} = \{t \in \mathbb{R}^{N-n} | \Sigma t_i^2 < 1\}$, and Ω is a connected open tubular neighborhood of $J(Y)$ in $\mathbb{R}^N$. The natural map

$$\omega \times id\colon X \times D^{N-n} \to Y \times D^{N-n} \simeq \Omega$$

is a local homeomorphism. By the previous argument $X \times D^{N-n}$ has a countable topology. Therefore the closed subset $X \times \{0\}$ of $X \times D^{N-n}$ has also a countable topology (5).

Given two Riemann domains over the same manifold Y:

$$X \xrightarrow{\omega} Y \qquad Z \xrightarrow{\pi} Y,$$

a *morphism* of the first into the second is a differentiable map $f\colon X \to Z$ which is a local homeomorphism and makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ & {\scriptstyle\omega}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & Y & \end{array}$$

commutative.

(*b*) Given a vector space V over $\mathbb{C}$ and a differentiable manifold X we can consider the space $J^k(X, V)$ of k-jets of X into V as a fiber space over X, via the source map $\alpha\colon J^k(X, V) \to X$. The fibers are the typical spaces $F^k(n, p)$ of k-jets at 0 of $\mathbb{R}^n$ into $V = \mathbb{C}^p$. This is a vector bundle, as $F^k(n, p)$ inherits from the vector space structure of $V = \mathbb{C}^p$ a vector space structure compatible with the source map $\alpha\colon J^k(X, V) \to X$. The dual bundle,

$$T^k(X, V) = \big(J^k(X, V)\big)^*$$

(called the k-th tangent bundle of X relative to V) is the bundle whose sections are the differential operators of order k over X on functions with values in V.

(5) A slight modification of the first part of the argument would give a direct proof of the following general statement: *let X, Y be arcwise connected and locally simply connected topological spaces. Let $X \to Y$ be a local homeomorphism. If Y has a countable topology then X has also a countable topology.*

The first part of the argument given above is sufficient for the applications we have in mind.

Now, if $\lambda: X \to Y$ is a local diffeomorphism, it establishes a natural map

$$\lambda_*: J^k(X, V) \to J^k(Y, V)$$

by transplanting every k-jet α over $x \in X$ to a k-jet β over $\lambda(x) \in Y$ in such a way that

$$\alpha = \beta \circ \lambda$$

(as this equation has a unique solution).

Dually we obtain a natural lifting

$$\lambda^*: T^k(Y, V) \to T^k(X, V)$$

of k-th order differential operators over Y to k-th order differential operators over X.

In particular, if $Y = \mathbb{R}^n$ and $x_1, \ldots, x_n$ are cartesian coordinates on Y, for any Riemann domain $\omega: X \to R^n$ we can consider the lifting

$$\omega^*\left(\frac{\partial}{\partial x_i}\right) \qquad 1 \leqslant i \leqslant n$$

of the n vector fields $\partial/\partial x_i$ over $\mathbb{R}^n$. These give to X a parallelizable structure and for any differential operator

$$E = \sum_{|\alpha| \leqslant k} c_\alpha(x) D^\alpha \quad \text{on } \mathbb{R}^n \qquad \left(D = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)\right)$$

we can consider its lifting $\omega^* E$ to X given by

$$\omega^* E = \sum_{|\alpha| \leqslant k} c_\alpha(\omega(y)) \mathfrak{D}^\alpha, \quad \text{where} \quad \mathfrak{D} = \left(\omega^*\left(\frac{\partial}{\partial x_1}\right), \ldots, \omega^*\left(\frac{\partial}{\partial x_n}\right)\right).$$

As $y_i = \omega^*(x_i)$, $1 \leqslant i \leqslant n$, are local coordinates everywhere on X the operators $\omega^*(\partial/\partial x_i)$ $(1 \leqslant i \leqslant n)$ are the partial derivations with respect to the set of these local coordinates.

3. – Partial completion (elliptic operators).

(a) By a domain in $\mathbb{R}^n$ we mean an open and connected set. Let Ω be a domain in $\mathbb{R}^n$ and let $A_0(x, D)$ be an elliptic operator on $\mathbb{R}^n$; we set

$$\mathcal{H}(\Omega) = \Gamma(\Omega, \mathcal{O}_{A_0});$$

this is the space of solutions on Ω of the homogeneous equation

$$A_0 u = 0 \, .$$

A function $u \in \mathcal{H}(\Omega)$ defines a section

$$F_u \colon \Omega \to \mathcal{O}_{A_0}$$

by associating to each point $y \in \Omega$ the germ u_y of u at y.

Let $\tilde{\Omega}_u$ be the connected component of $F_u(\Omega)$ in $\mathcal{O}_{A_0}$ and denote by $\omega \colon \tilde{\Omega}_u \to \mathbf{R}^n$ the natural projection induced on $\tilde{\Omega}_u$ by the natural projection $\pi \colon \mathcal{O}_{A_0} \to \mathbf{R}^n$.

Since $\mathcal{O}_{A_0}$ has a Hausdorff topology, $\tilde{\Omega}_u$ is a connected Hausdorff topological space and ω is a local homeomorphism. Therefore $\tilde{\Omega}_u$ acquires a differentiable structure in which ω becomes a local diffeomorphism.

Moreover for every point $\alpha \in \tilde{\Omega}_u$ we can define

$$U(\alpha) = \alpha\big(\omega(\alpha)\big) \in \mathbf{C}^p \, .$$

We obtain in this way a function $U \colon \tilde{\Omega}_u \to \mathbf{C}^p$ which extends the function $\omega^* u$ on $F_u(\Omega)$ and which satisfies the equation $(\omega^* A_0)\, U = 0$.

Moreover let $\hat{\Omega} \xrightarrow{\hat{\omega}} \mathbf{R}^n$ be a Riemann domain over $\mathbf{R}^n$ provided with a section

$$\hat{F}_u \colon \Omega \to \hat{\Omega}$$

with the property that on $\hat{\Omega}$ there exists a function $\hat{U} \colon \hat{\Omega} \to \mathbf{C}^p$ such that

(α) $(\omega^* A_0)\, \hat{U} = 0$;

(β) $\hat{U} | \hat{F}_u(\Omega) = \hat{\omega}^* u$.

Then a unique natural map $\lambda \colon \hat{\Omega} \to \tilde{\Omega}_u$ is defined (by associating to each point β of $\hat{\Omega}$ the germ of $\hat{U}$ at β in $\mathcal{O}_{A_0}$ and thus a point of $\tilde{\Omega}_u$) which makes the diagramm

$$\begin{array}{ccc} \hat{\Omega} & \xrightarrow{\lambda} & \tilde{\Omega} \\ & {\scriptstyle\hat{\omega}} \searrow \quad \swarrow {\scriptstyle\omega} & \\ & \mathbf{R}^n & \end{array}$$

commutative and induces an isomorphism from $\hat{F}_u(\Omega)$ onto $F_u(\Omega)$ for which $\hat{U} = \lambda^* U$.

We say then that $\hat{\Omega} \xrightarrow{\hat{\omega}} \mathbf{R}^n$ is a *u-completion* of Ω. We summarize these remarks with the

PROPOSITION 2. *Every* $u \in \mathcal{H}(\Omega)$ *defines a section* $F_u\colon \Omega \to \mathcal{O}_{A_0}$. *The connected component* $\tilde{\Omega}_u$ *of* $F_u(\Omega)$ *in* $\mathcal{O}_{A_0}$ *with its natural projection* $\omega\colon \tilde{\Omega} \to \mathbb{R}^n$ *has the following properties*:

i) $\tilde{\Omega}_u \xrightarrow{\omega} \mathbb{R}^n$ *is a Riemann domain over* $\mathbb{R}^n$. *In particular from Poincaré-Volterra's theorem it follows that all fibers of* ω *are at most countable.*

ii) *There is an analytic function* $U\colon \tilde{\Omega}_u \to \mathbb{C}^p$ *such that*

(α) $(\omega^* A_0)\, U = 0$;

(β) $U|F_u(\Omega) = \omega^* u$.

iii) *For any u-completion* $\{\hat{\Omega} \xrightarrow{\hat{\omega}} \mathbb{R}^n, \hat{F}_u, \hat{U}\}$ *of* Ω *we have a uniquely defined commutative diagramm*

$$\begin{array}{ccc} \hat{\Omega} & \xrightarrow{\lambda} & \tilde{\Omega} \\ & {}_{\hat{\omega}}\searrow \quad \swarrow_{\omega} & \\ & \mathbb{R}^n & \end{array}$$

in which $\lambda^* U = \hat{U}$, λ *being an isomorphism from* $\hat{F}_u(\Omega)$ *onto* $F_u(\Omega)$. *Then* $\{\tilde{\Omega} \xrightarrow{\omega} \mathbb{R}^n, F_u, U\}$ *is the maximal u-completion of* Ω, *in the sense that every u-completion factors uniquely through it.*

(*b*) What has been said for a single function $u \in \mathcal{H}(\Omega)$ can be repeated with only slight changes if we replace the element u by a part $\mathcal{S} \subset \mathcal{H}(\Omega)$.

In this last instance the sheaf $\mathcal{O} = \mathcal{O}_{A_0}$ must be replaced by the sheaf $\mathbf{O}_{\mathcal{S}}$ of germs of maps of $\mathbb{R}^n$ into $(\mathbb{C}^p)^{\mathcal{S}}$, having all their components in $\mathcal{O}_{A_0}$.

An element $f_{x_0} \in \mathbf{O}_{\mathcal{S},x_0}$ is therefore a collection $\{f_\sigma\}_{\sigma\in\mathcal{S}}$ of germs $f_\sigma \in \mathcal{O}_{A_0,x_0}$, *all defined in a sufficiently small but common neighborhood of* x_0.

Let us consider $\forall \sigma \in \mathcal{S}$ a copy $\mathcal{O}_{A_0}(\sigma)$ of $\mathcal{O}_{A_0}$ and consider the fiber product over X of all these copies $\prod_{\sigma\in\mathcal{S}} \mathcal{O}_{A_0}(\sigma)$.

We have a natural inclusion

$$\mathbf{O}_{\mathcal{S}} \hookrightarrow \prod_{\sigma\in\mathcal{S}} \mathcal{O}_{A_0}(\sigma)\,.$$

The topology of $\mathbf{O}_{\mathcal{S}}$ is defined in the usual way as the topology of the sheaf of germs of maps of $\mathbb{R}^n$ into $(\mathbb{C}^p)^{\mathcal{S}}$ with components in $\mathcal{O}_{A_0}$. Therefore the topology of $\mathbf{O}_{\mathcal{S}}$ is a Hausdorff topology. A natural section

$$F_{\mathcal{S}}\colon \Omega \to \mathbf{O}_{\mathcal{S}}$$

is then defined by

$$F_{\mathcal{S}}(y) = \prod_{\sigma\in\mathcal{S}} \sigma_y \quad \text{for } y \in \Omega\,.$$

If $\tilde{\Omega}_{\mathrm{S}}$ denotes the connected component of $F_{\mathrm{S}}(\Omega)$ in $\mathbf{O}_{\mathrm{S}}$ we obtain, setting $\omega = \pi|\tilde{\Omega}_{\mathrm{S}}$ ($\pi\colon \mathbf{O}_{\mathrm{S}} \to \mathbb{R}^n$ being the natural projection), *a Riemann domain*

$$\tilde{\Omega}_{\mathrm{S}} \xrightarrow{\omega} \mathbb{R}^n$$

having the following property (*Proposition* 3):

there exists a section $F\colon \Omega \to \tilde{\Omega}_{\mathrm{S}}$ *such that* $\forall g \in \mathrm{S}$ *there exists an analytic function* $G = G_g$ *on* $\tilde{\Omega}_{\mathrm{S}}$ (*with values in* $\mathbb{C}^p$) *such that*

(α) $(\omega^* A_0) G = 0$;

(β) $G|F_{\mathrm{S}}(\Omega) = \omega^* g$.

A Riemann domain $\hat{\Omega}_{\mathrm{S}} \xrightarrow{\hat{\omega}} \mathbb{R}^n$ endowed with a section $\hat{F}_{\mathrm{S}}\colon \Omega \to \hat{\Omega}$ such that $\forall g \in \mathrm{S}$ properties (α) and (β) above specified are satisfied by $\{\hat{\Omega}_{\mathrm{S}} \xrightarrow{\hat{\omega}} \mathbb{R}^n, \hat{F}_{\mathrm{S}}, \{\hat{G}_g\}_{g\in\mathrm{S}}\}$ will be called an *S-completion of Ω*.

The Riemann domain $\{\tilde{\Omega}_{\mathrm{S}} \xrightarrow{\omega} \mathbb{R}^n, F_{\mathrm{S}}, \{G_g\}_{g\in\mathrm{S}}\}$ satisfies the following universal property:

for every S-completion $\{\hat{\Omega} \xrightarrow{\hat{\omega}} \mathbb{R}^n, \hat{F}_{\mathrm{S}}, \{\hat{G}_g\}_{g\in\mathrm{S}}\}$ *of Ω there is a uniquely defined map* $\lambda\colon \hat{\Omega} \to \tilde{\Omega}_{\mathrm{S}}$ *such that we have a commutative diagramm*

$$\begin{array}{ccc} \hat{\Omega} & \xrightarrow{\lambda} & \tilde{\Omega} \\ & {\scriptstyle\hat{\omega}}\searrow \quad \swarrow{\scriptstyle\omega} & \\ & \mathbb{R}^n & \end{array}$$

moreover λ is an isomorphism of $\hat{F}_{\mathrm{S}}(\Omega)$ *onto* $F_{\mathrm{S}}(\Omega)$, *and* $\forall g \in \mathrm{S}$, $\hat{G}_g = \lambda^* G_g$.

In other words, $\tilde{\Omega}_{\mathrm{S}} \xrightarrow{\omega} \mathbb{R}^n$ is the maximal S-completion (or the S-envelope) of Ω. In particular if $\mathrm{S} = \mathcal{K}(\Omega)$ we speak of the *envelope of regularity* of Ω with respect to $A_0(x, D)$.

(*c*) If $\mathrm{S} \subset \mathrm{S}' \subset \mathcal{K}(\Omega)$ we have a natural map

$$\mathbf{O}_{\mathrm{S}'} \to \mathbf{O}_{\mathrm{S}}$$

and thus a natural morphism of domination between the S and S′ envelopes of Ω

$$\begin{array}{ccc} \tilde{\Omega}_{\mathrm{S}'} & \longrightarrow & \tilde{\Omega}_{\mathrm{S}} \\ & \searrow \quad \swarrow & \\ & \mathbb{R}^n & \end{array}$$

($\tilde{\Omega}_{S'}$ « dominates » $\tilde{\Omega}_S$). Thus *the envelope of regularity dominates every* S*-envelope of* Ω*, and every* $\mathcal{H}(\Omega)$*-completion of* Ω *factors through it.*

4. – Domains of regularity in $\mathbb{R}^n$.

Any domain Ω of $\mathbb{R}^n$ which coincides with its regularity envelope will be called a *domain of regularity.* By this we mean that the section

$$F_{\mathcal{H}(\Omega)}: \Omega \to \tilde{\Omega}_{\mathcal{H}(\Omega)}$$

establishes an isomorphism of Ω *onto* its envelope of regularity $\tilde{\Omega}_{\mathcal{H}(\Omega)}$.

Let $\Delta \subset \tilde{\Delta}$ be two domains in $\mathbb{R}^n$ and suppose that $\Delta \subset \Omega$. Then $\tilde{\Delta}$ is called a $\mathcal{H}(\Omega)|\Delta$*-completion* of Δ if

$$\operatorname{Im}\{\mathcal{H}(\tilde{\Delta}) \xrightarrow{r^{\tilde{\Delta}}_{\Delta}} \mathcal{H}(\Delta)\} \supset \mathcal{H}(\Omega)|\Delta .$$

PROPOSITION 4. *A domain* $\Omega \subset \mathbb{R}^n$ *is a domain of regularity if and only if, for every choice of a domain* $\Delta \subset \Omega$ *and of an* $\mathcal{H}(\Omega)|\Delta$*-completion* $\tilde{\Delta}$ *of* Δ*, we have necessarily that* $\tilde{\Delta} \subset \Omega$.

PROOF. Assume that $\Omega \xrightarrow{\sim} \tilde{\Omega}$. The natural map $\lambda: \tilde{\Delta} \to \tilde{\Omega}$ is injective. Thus, as $\Omega \simeq \tilde{\Omega}$, we must have $\tilde{\Delta} \subset \Omega$.

Conversely if $\Omega \subsetneq \tilde{\Omega}$ we can construct $\Delta \subset \tilde{\Delta}$ satisfying the condition that $\tilde{\Delta}$ is a $\mathcal{H}(\Omega)|\Delta$-completion of Δ and $\tilde{\Delta} \not\subset \Omega$.

For this choose $x_0 \in \Omega$, $x_1 \in \tilde{\Omega} - \Omega$ and a continuous path

$$\gamma: [0, 1] \to \tilde{\Omega}$$

with $\gamma(0) = x_0$, $\gamma(1) = x_1$. Let $t_0 \in [0, 1]$ be the first point in $0 \leqslant t \leqslant 1$ with $\gamma(t_0) \notin \Omega$. Take an ε-neighborhood L_ε of the closed set $L = \gamma([0, t_0])$. Then $L_\varepsilon \not\subset \Omega$ for $\varepsilon > 0$ and, if ε is small, L_ε is a $\mathcal{H}(\Omega)$-completion of the ε-ball $B(x_0, \varepsilon)$ around x_0:

$$L_\varepsilon = \mathcal{H}(\Omega)|B(x_0, \varepsilon)\text{-completion of } B(x_0, \varepsilon) .$$

This proves our assertion, as L_ε is one-sheeted if ε is small.

5. – The $\bar{\partial}$-suspension of an operator.

(*a*) Given a $p \times q$ matrix $A_0(x, D)$ of linear partial differential operators with real analytic (complex valued) coefficients on $\mathbb{R}^n$, we can find an open neighborhood U of $\mathbb{R}^n$ in $\mathbb{C}^n$ to which it extends with holomorphic coefficients. If $x = (x_1, \ldots, x_n)$ are cartesian coordinates in $\mathbb{R}^n$, we denote by $z_j = x_j + iy_j$ holomorphic coordinates in $\mathbb{C}^n$, identifying $\mathbb{R}^n$ to the real subspace $\{z \in \mathbb{C}^n | y = 0\}$. If the operator A_0 is defined by

$$A_0 = \sum c_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

then it is extended by

$$\tilde{A}_0 = \sum c_\alpha(z) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

where $c_\alpha(z)$ are holomorphic (matrix valued) functions on U which reduce to $c_\alpha(x)$ if restricted to $\mathbb{R}^n$.

For every open subset $\tilde{\Omega}$ of U we can consider the operator

$$A = \tilde{A}_0 \oplus \bar{\partial} \colon \mathcal{E}^q(\tilde{\Omega}) \to \mathcal{E}^p(\tilde{\Omega}) \oplus \left(\mathcal{E}^{0,1}(\tilde{\Omega})\right)^q$$

where $\mathcal{E}^{0,1}(\tilde{\Omega})$ denotes the space of C^∞ forms of type $(0, 1)$ on $\tilde{\Omega}$ and $\bar{\partial}$ is the exterior differentiation with respect to antiholomorphic coordinates.

The operator A so defined on U is elliptic (even if A_0 is not so) and is called *the $\bar{\partial}$-suspension* of A_0 (cf. [2]).

For every open subset $\tilde{\Omega}$ of U, we define the space

$$\mathcal{K}_A(\tilde{\Omega}) = \{u \in \mathcal{E}^q(\tilde{\Omega}) | \bar{\partial} u = 0 \text{ and } \tilde{A}_0 u = 0\}\,.$$

This is the Kernel of A over $\tilde{\Omega}$, i.e. the space of q-tuples u of holomorphic functions on $\tilde{\Omega}$ which satisfy the holomorphic differential equation

$$A_0\left(z, \frac{\partial}{\partial z}\right) u = \sum c_\alpha(z) \frac{\partial^{|\alpha|} u}{\partial z^\alpha} = 0 \qquad \text{on } \tilde{\Omega}\,.$$

(*b*) An operator $A_0(x, D) \colon \mathcal{E}^q(\mathbb{R}^n) \to \mathcal{E}^p(\mathbb{R}^n)$ with C^∞ coefficients on $\mathbb{R}^n$ is called hypoelliptic if, for any open subset Ω of $\mathbb{R}^n$ and any solution $u \in \mathcal{D}'(\Omega)^q$ of the equation

$$A_0(x, D)\, u = 0 \qquad \text{on } \Omega$$

we have necessarily $u \in \mathcal{E}^q(\Omega)$. Note that every elliptic operator is also hypoelliptic.

Let $\{K_i\}_{i \in N}$ be an exaustive sequence of compacts in Ω, i.e. we assume that:

i) $K_i \subset \mathring{K}_{i+1}$ $i = 0, 1, 2, \ldots$

ii) $\cup K_i = \Omega$.

For $u \in \mathcal{E}^q(\Omega)$, we define the seminorms

$$p_i(u) = \sup_{K_i} |u(x)| \qquad i = 0, 1, 2, \ldots$$

(where $|\cdot|$ is a norm in $\mathbb{C}^q$).

LEMMA. *If A_0 is hypoelliptic, then the Schwartz topology on $\mathcal{H}(\Omega)$ is also defined by the seminorms p_i. The space $\mathcal{H}(\Omega)$ is complete.*

PROOF. The space $\mathcal{H}(\Omega)$ with the Schwartz topology of uniform convergence of functions together with all derivatives on compact sets is a complete (Fréchet) space. It is also complete under the topology defined by the seminorms p_i, because $A_0(x, D)$ is hypoelliptic by assumption. Denote by $\mathcal{H}_{\{p_i\}}(\Omega)$ the space $\mathcal{H}(\Omega)$ endowed with this topology; by a theorem of Banach the identity map

$$\mathcal{H}(\Omega) \text{ with the Schwartz topology} \xrightarrow{\text{id}} \mathcal{H}_{\{p_i\}}(\Omega)\,,$$

which is obviously continuous, is a topological isomorphism.

(*c*) We can now prove the following

THEOREM 2. *Let $A_0(x, D)$ be an elliptic operator with C^∞ coefficients on an open set $\Omega \subset \mathbb{R}^n$. There exists a lower semicontinuous positive function*

$$\varrho_\Omega \colon \Omega \to \mathbb{R}^+ \qquad \left(\mathbb{R}^+ = \{t \in \mathbb{R} | t > 0\}\right)$$

with the following properties

i) $\varrho_\Omega(x) \leqslant n^{-\frac{1}{2}} \operatorname{dist}(x, \partial\Omega)\ \ \forall x \in \Omega$;

ii) $\forall x_0 \in \Omega$ *and* $\forall u \in \mathcal{H}(\Omega)$, *the Taylor series $\mathcal{T}_{x_0} u$ of u centered at x_0 is convergent in the polycylinder of $\mathbb{C}^n$*

$$|z_i - x_{0_i}| < \varrho_\Omega(x_0) \qquad 1 \leqslant i \leqslant n\,.$$

PROOF. Set $\|x\| = \sup_{1 \leqslant i \leqslant n} |x_i|$ and let δ be the distance based on that norm (instead of the euclidean one). Thus $\forall x_0 \in \Omega$ we set

$$\delta(x_0, \partial\Omega) = \inf_{w \in \partial\Omega} \|w - x_0\|\,.$$

For every positive integer m such that $1/m < \delta(x_0, \partial\Omega)$ we consider the space

$$A(x_0, m) = \left\{u \in \mathcal{H}(\Omega) | \mathcal{T}_{x_0} u \text{ is convergent in } \|z - x_0\| < \frac{1}{m}\right\}.$$

This is a $\mathbb{C}$-vector subspace of $\mathcal{H}(\Omega)$; every element $u \in A(x_0, m)$ admits a well defined analytic extension $\tilde{u}$ to $\Omega \cup \{z \in \mathbb{C}^n | \|z - x_0\| < 1/m\}$ because we have taken the precaution of requiring that $1/m < \delta(x_0, \partial\Omega)$.

Set

$$\tilde{K}_i = K_i \cup \left\{ \|z - x_0\| \leqslant \frac{1}{m} - \frac{1}{i} \right\}$$

and, for every $u \in A(x_0, m)$, set

$$\|u\|_i = \sup_{\tilde{K}_i} |\tilde{u}| .$$

Then the sequence of seminorms $\|u\|_i$ makes the space $A(x_0, m)$ a complete space.

Consider the restriction map

$$r_m \colon A(x_0, m) \to \mathcal{H}(\Omega) .$$

This is continuous and $\mathcal{H}(\Omega) = \cup \operatorname{Im} r_m$. Thus for at least one $m = m(x_0)$ we must have that $\operatorname{Im} r_{m(x_0)}$ is of second category. By Banach open mapping theorem then $r_{m(x_0)}$ must be surjective (and thus a topological isomorphism). We can therefore define

$$\varrho_\Omega(x_0) = \sup \{\varrho > 0 | \varrho < \delta(x_0, \partial\Omega) \text{ and } \forall u \in \mathcal{H}(\Omega) \text{ the Taylor series } \mathcal{T}_{x_0} u \text{ converges in } \|z - x_0\| < \varrho\} .$$

If $\|x - x_0\| < \varrho_\Omega(x_0)$, then

$$\varrho_\Omega(x) \geqslant \varrho_\Omega(x_0) - \|x - x_0\| .$$

This proves that ϱ_Ω is lower semicontinuous. Since $\varrho_\Omega(x_0) \leqslant \delta(x_0, \partial\Omega)$ and we have $\delta(x_0, \partial\Omega) \leqslant n^{-\frac{1}{2}} \operatorname{dist}(x_0, \partial\Omega)$, the theorem is completely proved.

Remark. In the course of the previous proof we have shown that the map $A\big(x_0, (1/\lambda)\varrho_\Omega(x_0)\big) \xrightarrow{r} \mathcal{H}(\Omega)$ is a topological isomorphism, for every $\lambda \geqslant 1$.

In particular *for every $\lambda > 1$ we can find a compact set $B(x_0, \lambda) \subset \Omega$ and a constant $c(x_0, \lambda) > 0$ such that* $\forall u \in \mathcal{H}(\Omega)$

$$\sup_{\|z - x_0\| < (1/\lambda)\varrho_\Omega(x_0)} |\tilde{u}(z)| \leqslant c(x_0, \lambda) \sup_{x \in B(x_0, \lambda)} |u(x)| .$$

We deduce from this remark the following useful

Corollary 1. *$\forall x_0 \in \Omega$ and every $\lambda > 1$ there exist constants $\varepsilon > 0$, $c(x_0, \lambda) > 0$*

and a compact set $B(x_0, \lambda)$ *such that* $\forall u \in \mathcal{H}(\Omega)$ *we have*

$$|D^\alpha u(x)| \leqslant \alpha!\, \lambda^{|\alpha|} c(x_0, \lambda) \frac{\sup_{x \in B(x_0, \lambda)} |u(x)|}{\varrho_\Omega(x_0)^{|\alpha|}}$$

for all x *with* $\|x - x_0\| < \varepsilon$.

COROLLARY 2. *Let* $A_0(x, D)$ *be an elliptic operator with real analytic coefficients defined in an open set* Ω *of* $\mathbb{R}^n$. *Then we can find an* Ω*-connected neighborhood* U *of* Ω *in* $\mathbb{C}^n$ *in which the operator* A_0 *can be* $\bar{\partial}$*-suspended into a holomorphic operator* A *and such that the natural restriction map*

$$r_\Omega^U : \mathcal{H}_A(U) \to \mathcal{H}_{A_0}(\Omega)$$

is an isomorphism $\left(\mathcal{H}_A(U) = \{u \in \Gamma(U, \mathcal{O}^q) | Au = 0\}\right)$.

REMARK. We can always replace the function ϱ_Ω of the theorem by the function $\varrho_\Omega(x) = \sup\{\varrho(x) | \varrho$ verifies conditions i) and ii) of theorem 2$\}$. This is the best one verifying the same conditions i) an ii). It is still lower semicontinuous (as supremum of a class of lower semicontinuous functions). We will call this function ϱ_Ω *pseudodistance from the boundary.*

If U is an Ω-connected neighborhood of Ω in $\mathbb{C}^n$, as in corollary 2, for many practical purposes it will be sufficient to take for $\varrho_\Omega(x)$ the function

$$\varrho_\Omega(x) = \inf_{w \in \partial U} \|w - x\| \quad \text{where} \quad \|z\| = \sup_{1 \leqslant i \leqslant n} |z_i| .$$

In this way $\varrho_\Omega(x)$ is also continuous.

6. – Convexity theory. Operators with constant coefficients.

(*a*) We will assume that $A_0(D)$ is a matrix-operator with constant coefficients.

If K is a compact subset of an open set Ω we set for $u \in \mathcal{E}^q(\Omega)$,

$$\|u\|_K = \sup_{x \in K} |u(x)|$$

where $|\cdot|$ denotes a norm in the space $\mathbb{C}^q$.

For every $c \geqslant 1$ *and every compact set* $K \subset \Omega$ *we set*

$$\hat{K}_\Omega(c) = \{x \in \Omega \,\big|\, |u(x)| \leqslant c \|u\|_K \ \forall u \in \mathcal{H}(\Omega)\} \ (^6) .$$

(6) Sometimes we will write $\hat{K}(c)$ instead of $\hat{K}_\Omega(c)$.

REMARK. – *Assume that* $\mathcal{H}(\Omega)$ *is an algebra* i.e. $q = 1$ and $u, v \in \mathcal{H}(\Omega)$ implies that $u \cdot v \in \mathcal{H}(\Omega)$. This is for instance the case if $q = 1$ and all operators in the matrix A_0 are homogeneous of the first order. *Then* $\forall c \geqslant 1$ *we have* $\hat{K}_\Omega(c) = \hat{K}_\Omega(1)$.

Indeed $\hat{K}_\Omega(1) \subset \hat{K}_\Omega(c)$ $\forall c \geqslant 1$. On the other hand if $x \in \hat{K}_\Omega(c)$, then $\forall u \in \mathcal{H}(\Omega)$

$$|u(x)| \leqslant c\|u\|_K$$

thus $\forall l > 0$ integer,

$$|u^l(x)| \leqslant c\|u^l\|_K,$$

and therefore

$$|u(x)| \leqslant c^{1/l}\|u\|_K.$$

Letting $l \to +\infty$ we get then, as $\lim\limits_{l \to +\infty} c^{1/l} = 1$, that

$$|u(x)| \leqslant \|u\|_K$$

and therefore $\hat{K}_\Omega(c) \subset \hat{K}_\Omega(1)$.

(*b*) Let A_0 be an operator with constant coefficients, Ω an open set in $\mathbb{R}^n$. Let us consider the following two conditions:

$(K)_\Omega$: $\forall$ *compact* $K \subset \Omega$, $\forall c \geqslant 1$ *the set* $\hat{K}_\Omega(c)$ *is also compact*

$(D)_\Omega$: $\forall$ *divergent* (7) *sequence* $\{x_\nu\} \subset \Omega$ *there exists* $u \in \mathcal{H}(\Omega)$ *such that*

$$\sup_\nu |u(x_\nu)| = \infty.$$

Clearly $(D)_\Omega \Rightarrow (K)_\Omega$.

Indeed if it is not so there exist a compact K in Ω and a constant $c \geqslant 1$ such that $\hat{K}_\Omega(c)$ is not compact. Then we can find a divergent sequence $\{x_\nu\}$ in $\hat{K}_\Omega(c)$. On that sequence we have, for some $u \in \mathcal{H}(\Omega)$, $\sup |u(x_\nu)| = \infty$. But on the other hand $|u(x_\nu)| \leqslant c\|u\|_K < \infty$: this gives a contradiction.

THEOREM 3. *If* A_0 *is an elliptic operator then the two conditions* $(K)_\Omega$ *and* $(D)_\Omega$ *are equivalent.*

PROOF. Since the opposite inclusion has already been proved above, we have only to show that $(K)_\Omega \Rightarrow (D)_\Omega$. If this is not the case there exists a divergent sequence $\{x_\nu\} \subset \Omega$ such that $\forall u \in \mathcal{H}(\Omega)$

$$\sup_\nu |u(x_\nu)| < \infty.$$

(7) Divergent means without point of accumulation in Ω.

Set

$$A = \left\{ u \in \mathcal{H}(\Omega) \mid \sup_{\nu} |u(x_\nu)| \leqslant 1 \right\}.$$

The subset $A \subset \mathcal{H}(\Omega)$ is convex, closed, and symmetric with respect to the origin $A = - A$. The only property that needs a proof is the fact that A is closed.

Now if $u_0 = \lim_{\mu} u_\mu$ with $u_\mu \in A$ and if $\alpha = \sup |u_0(x_\nu)| > 1$ then for $0 < \varepsilon < \frac{1}{2}(\alpha - 1)$ we can find ν_0 such that

$$|u_0(x_{\nu_0})| > \alpha - \varepsilon .$$

Since, for μ sufficiently large, we have

$$|u_\mu(x_{\nu_0}) - u_0(x_{\nu_0})| < \varepsilon ,$$

it follows that

$$\begin{aligned} |u_\mu(x_{\nu_0})| &\geqslant |u_0(x_{\nu_0})| - |u_0(x_{\nu_0}) - u_\mu(x_{\nu_0})| \\ &\geqslant \alpha - 2\varepsilon > 1 . \end{aligned}$$

This is impossible as $u_\mu \in A$. Hence $\alpha = \sup |u_0(x_\nu)| \leqslant 1$ and $u_0 \in A$.

Now, we have

$$\mathcal{H}(\Omega) = \bigcup_{m=1}^{\infty} mA$$

and thus one of the sets mA (and therefore A which is homeomorphic to mA) contains an interior point. Because A is convex and balanced, it must contain also a neighborhood of the origin. There exist therefore a compact set $K \subset \Omega$ and an $\varepsilon > 0$ such that (as A_0 is hypoelliptic)

$$V(K, \varepsilon) = \{u \in \mathcal{H}(\Omega) \mid \|u\|_K < \varepsilon\} \subset A .$$

We claim then that $\forall u \in \mathcal{H}(\Omega)$

$$(*) \qquad \sup |u(x_\nu)| \leqslant \frac{2}{\varepsilon} \|u\|_K .$$

Indeed, given $u \in \mathcal{H}(\Omega)$, we can find $\lambda > 0$ such that

$$\lambda u \in V(K, \varepsilon) .$$

If $\|u\|_K = 0$, then $\lambda u \in V(K, \varepsilon)$ $\forall \lambda > 0$. Thus

$$(**) \qquad \lambda \sup |u(x_\nu)| \leqslant 1$$

and hence $u(x_\nu) = 0$ for every ν.

If $\|u\|_K \neq 0$ we can take $\lambda = \varepsilon/(2\|u\|_K)$ and we get the same conclusion $(*)$.

But the condition $(*)$ says that $\{x_\nu\} \subset \hat{K}(2/\varepsilon)$, which is by assumption compact. Thus $\{x_\nu\}$ cannot be divergent. This establishes the implication $(K)_\Omega \Rightarrow (D)_\Omega$.

DEFINITION. *We will say that the open set Ω is A_0-convex if property $(K)_\Omega$ is satisfied.*

COROLLARY. *If Ω is a domain A_0-convex (and A_0 is elliptic) then Ω is a regularity domain.*

Indeed this follows from proposition 4: property $(D)_\Omega$ shows that the condition of that proposition is satisfied.

(*c*) The converse of the last corollary is not true. For instance if A_0 is the operator of exterior differentiation:

$$A_0 u = {}^t\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$$

then $\mathbb{R}^n$ is a regularity domain which is not A_0-convex as for any non empty compact $K \subset R^n$ we have $\hat{K}(c) = \mathbb{R}^n$ $\forall c \geqslant 1$.

Let Ω be an open set in $\mathbb{R}^n$, and ϱ_Ω a pseudodistance from the boundary of Ω. For any compact $K \subset \Omega$ we set

$$\varrho_\Omega(K) = \inf_{x \in K} \varrho_\Omega(x) .$$

This is also a minimum, as ϱ_Ω is lower semicontinuous. Thus $\varrho_\Omega(K) > 0$. We have in any case the following

PROPOSITION 5. *Let Ω be an open set in $\mathbb{R}^n$, K a compact subset of Ω and $c \geqslant 1$. For any $\xi \in \hat{K}(c)$ and any $u \in \mathcal{K}(\Omega)$ the Taylor series $\mathcal{T}_\xi u$ of u at ξ is convergent in the (complex) polycylinder*

$$|z_i - \xi_i| < \varrho_\Omega(K) \qquad 1 \leqslant i \leqslant n .$$

PROOF. We have

$$\mathcal{T}_\xi u = \sum \frac{D^\alpha u(\xi)}{\alpha!} (z - \xi)^\alpha .$$

Since A_0 has constant coefficients, $D^\alpha u \in \mathcal{H}(\Omega)$. Therefore

$$|D^\alpha u(\xi)| \leqslant c \|D^\alpha u\|_K .$$

For any $\lambda > 1$, the functions $u \in \mathcal{H}(\Omega)$ satisfy an estimate of the form

$$\|D^\alpha u\|_K \leqslant \alpha!\, c_1 \lambda^{|\alpha|} \frac{\|u\|_{K'(\lambda)}}{\varrho_\Omega(K)^{|\alpha|}}$$

where $K'(\lambda)$ is a convenient compact subset of Ω depending only on K and λ (this is a consequence of Corollary 1 of theorem 2).

Therefore $\mathcal{C}_\xi u$ is majorized by the series

$$\sum c c_1 \lambda^{|\alpha|} \frac{\|u\|_{K'(\lambda)}}{\varrho_\Omega(K)^{|\alpha|}} |z-\xi|^{|\alpha|} \equiv c c_1 \|u\|_{K'(\lambda)} \sum \lambda^{|\alpha|} \frac{|z-\xi|^{|\alpha|}}{\varrho_\Omega(K)^{|\alpha|}} .$$

The last series converges for

$$|z_i - \xi_i| < \frac{\varrho_\Omega(K)}{\lambda}$$

and thus the same is true for $\mathcal{C}_\xi u$. As $\lambda > 1$ is arbitrary we get the statement of the proposition.

COROLLARY. *Let Ω be a regularity domain for A_0: Then for any compact $K \subset \Omega$, and for any $c \geqslant 1$ we have*

$$\varrho_\Omega(K) = \varrho_\Omega\big(\hat{K}(c)\big)$$

(*where* $\varrho_\Omega(C) = \inf_{x \in C} \varrho_\Omega(x)$, *for C closed in Ω*).

PROOF. We have $K \subset \hat{K}(c)$ and thus

$$\varrho_\Omega(K) \geqslant \varrho_\Omega\big(\hat{K}(c)\big) .$$

From the previous proposition we must have also the opposite inequality

$$\varrho_\Omega(K) \leqslant \varrho_\Omega\big(\hat{K}(c)\big)$$

as Ω is a domain of regularity and therefore, $\forall \xi \in \hat{K}(c)$, we have $\varrho_\Omega(\xi) \geqslant \varrho_\Omega(K)$.

In particular, if $\mathbb{R}^n$ is A_0-convex we deduce the *analogue of Cartan-Thullen* *theorem*:

THEOREM 4. *If* $\mathbb{R}^n$ *is* A_0*-convex* (8), *the necessary and sufficient condition for* Ω *to be a regularity domain is that* Ω *is* A_0*-convex.*

PROOF. If Ω is A_0-convex then Ω is a domain of regularity by the Corollary to theorem 3. If conversely Ω is a domain of regularity, we have for any compact $K \subset \Omega$ and any $c \geqslant 1$, that

$$\varrho_\Omega(K) = \varrho_\Omega(\hat{K}(c)) .$$

On the other hand, as $\mathbb{R}^n$ is A_0-convex, $\hat{K}(c)$ is a bounded set. Therefore it is a compact subset of Ω and thus Ω is A_0-convex.

(*d*) A function $f \in \mathcal{H}(\Omega)$ will be called a multiplicator for $\mathcal{H}(\Omega)$ if

$$A_0 f u = 0 \qquad \forall u \in \mathcal{H}(\Omega) .$$

The multiplicators form a ring $\mathcal{R}_{A_0}(\Omega) \supset \mathbb{C}$. If $\mathcal{H}(\Omega)$ is an algebra, then $\mathcal{R}_{A_0}(\Omega) = \mathcal{H}(\Omega)$.

One can improve the statement of Proposition 5, as follows: *Let* $f \in \mathcal{R}_{A_0}(\Omega)$ *and assume that*

$$|f(z)| \leqslant \varrho_\Omega(z) \qquad \forall z \in K .$$

Then for any $c \geqslant 1$, *any* $\xi \in \hat{K}(c)$, *and any* $u \in \mathcal{H}(\Omega)$, *the Taylor series* $\mathcal{T}_\xi u$ *of* u *at* ξ *is convergent in the polycylinder*

$$|z_i - \xi_i| < |f(\xi)| \qquad 1 \leqslant i \leqslant n .$$

PROOF. Set in $\mathbb{C}^n$, for K compact in Ω and $0 < t < 1$, $\|z\| = \sup_{1 \leqslant i \leqslant n} |z_i|$

$$K_{tf} = \{z \in \mathbb{C}^n \mid \|z - w\| \leqslant t|f(w)| \text{ for some } w \in K\} .$$

If

$$U = \{z \in \mathbb{C}^n \mid \|z - w\| < \varrho_\Omega(w) \text{ for some } w \in \Omega\}$$

we have that

i) $\mathcal{H}_A(U) \overset{\rightarrow}{\sim} \mathcal{H}_{A_0}(\Omega)$;

ii) K_{tf} is compact in U.

(8) Cf. Remark after Proposition 17 in Section 18.

There exist a compact set $K_1(t) \subset \Omega$ and a constant $c_1 > 0$ such that

$$\sup_{K_{tf}} |\tilde{u}| \leqslant c_1 \sup_{K_1(t)} |u|$$

where $\tilde{u}$ is the extension of u to U.

We have $\forall \alpha \in \mathbf{N}^n$ and $\xi \in \hat{K}(c)$ that, for some $z \in K$

$$\left| f(\xi)^{|\alpha|} \frac{D^\alpha u(\xi)}{\alpha!} \right| \leqslant c \left| f(z)^{|\alpha|} \frac{D^\alpha u(z)}{\alpha!} \right|$$

because $f(z)^{|\alpha|}\big(D^\alpha u(z)/\alpha!\big) \in \mathcal{H}(\Omega)$ as A_0 has constant coefficients and $f \in \mathcal{R}_{A_0}(\Omega)$.

Also by Cauchy formula

$$\frac{D^\alpha u(z)}{\alpha!} = \frac{1}{(2\pi i)^n} \underset{\substack{|\xi_j - z_j| = tf(z) \\ 1 \leqslant j \leqslant n}}{\int \cdots \int} \frac{\tilde{u}(\xi)\, d\xi_1, \ldots, d\xi_n}{\prod_{j=1}^{n} (\xi_j - z_j)^{\alpha_j + 1}} .$$

From this we deduce

$$\left| \frac{D^\alpha u(z)}{\alpha!} \right| \leqslant \frac{\sup_{K_{tf}} |\tilde{u}|}{t^{|\alpha|} |f(z)|^{|\alpha|}} .$$

Therefore

$$\left| f(z)^{|\alpha|} \frac{D^\alpha u(z)}{\alpha!} \right| \leqslant c_1 \frac{\sup_{K_1(t)} |u|}{t^{|\alpha|}} .$$

Consequently

$$\left| \frac{D^\alpha u(\xi)}{\alpha!} \right| \leqslant c c_1 \frac{\sup_{K_1(t)} |u|}{t^{|\alpha|} |f(\xi)|^{|\alpha|}} .$$

From this we derive the convergence of $\mathcal{T}_\xi u$ in the polycylinder

$$|z_i - \xi_i| < t|f(\xi)| \qquad 1 \leqslant i \leqslant n .$$

As this is true for any $t < 1$ we get then convergence of $\mathcal{T}_\xi u$ in the polycylinder

$$|z_i - \xi_i| < |f(\xi)| \qquad 1 \leqslant i \leqslant n .$$

7. – We have obtained a characterization of domains of regularity under the assumption that $\mathbf{R}^n$ is A_0-convex. To remove this restriction one can proceed as follows

LEMMA. *The following two conditions are equivalent (for Ω open in $\mathbb{R}^n$)*

i) $\forall$ *compact* $K \subset \Omega$, $\forall c \geqslant 1$ *we have*

$$\varrho_\Omega(\hat{K}(c)) > 0$$

ii) $\forall$ *sequence* $\{x_\nu\} \subset \Omega$ *such that* $\lim_\nu \varrho_\Omega(x_\nu) = 0$ *there exists an* $u \in \mathcal{H}(\Omega)$ *with*

$$\sup_\nu |u(x_\nu)| = \infty .$$

PROOF. ii) $\Rightarrow$ i) If not, there exist K compact in Ω and a constant $c \geqslant 1$ such that $\varrho_\Omega(\hat{K}(c)) = 0$. Thus we can select $\{x_\nu\} \subset \hat{K}(c)$ with $\varrho_\Omega(x_\nu) \to 0$. Now $\forall u \in \mathcal{H}(\Omega)$, $|u(x_\nu)| \leqslant c\|u\|_K$ which contradicts ii).

Conversely i) $\Rightarrow$ ii). If not, there exists a sequence $\{x_\nu\} \subset \Omega$ with $\varrho_\Omega(x_\nu) \to 0$ such that $\forall u \in \mathcal{H}(\Omega)$

$$\sup_\nu |u(x_\nu)| < \infty .$$

Set

$$A = \left\{u \in \mathcal{H}(\Omega) \,|\, \sup_\nu |u(x_\nu)| \leqslant 1\right\}.$$

Then A is closed convex and symmetric with respect to the origin and therefore (by the same argument as in the proof of theorem 3) must contain a neighborhood of the origin in $\mathcal{H}(\Omega)$. Thus there exist K compact and $\varepsilon > 0$ such that

$$V(K, \varepsilon) = \left\{u \in \mathcal{H}(\Omega) \,|\, \sup_K |u| < \varepsilon\right\} \subset A .$$

Then

$$\sup |u(x_\nu)| \leqslant \frac{2}{\varepsilon} \|u\|_K$$

as in the proof of theorem 3. Therefore $\{x_\nu\} \subset \hat{K}(2/\varepsilon)$. This contradicts the assumption that $\varrho_\Omega(x_\nu) \to 0$.

THEOREM 5. *Necessary and sufficient condition for $\Omega \subset \mathbb{R}^n$ to be a domain of regularity is that Ω satisfies the equivalent conditions* i) *or* ii) *of the above Lemma.*

PROOF. Assume Ω verifies i) and ii). Then, $\forall \{x_\nu\} \subset \Omega$ with dist $(x_\nu, \partial\Omega) \to 0$, we have $\varrho_\Omega(x_\nu) \to 0$. Then there exists an $u \in \mathcal{H}(\Omega)$ with

$$\sup_\nu |u(x_\nu)| = \infty .$$

This shows that Ω is a domain of regularity by virtue of proposition 4.

Conversely if Ω is a domain of regularity we have $\forall K$ compact in Ω and $\forall c \geqslant 1$

$$\varrho_\Omega(\mathrm{K}) = \varrho_\Omega\big(\hat{K}(c)\big) .$$

Therefore condition i) (and thus ii)) is satisfied.

8. – Convexity theory. Operators with analytic coefficients.

(a) We will assume now that $A_0(x, D)$ is an operator with real analytic coefficients (complex valued) on $\mathbb{R}^n$ and elliptic. We want to characterize its domains of regularity.

We gather here some preliminary lemmas.

(α) LEMMA 1. *Let Ω be open in $\mathbb{R}^n$ and let K be a compact subset of Ω. For every $\lambda > 1$ we can find a compact set $K_1(\lambda)$ and a constant $c(\lambda) > 0$ such that*

$$\sup_{x\in K}\left|\frac{D^\alpha u(x)}{\alpha!}\right| \leqslant \lambda^{|\alpha|} c(\lambda)\, \frac{\sup_{K_1(\lambda)} |u|}{\varrho_\Omega(K)^{|\alpha|}}$$

where $\varrho_\Omega(K) = \inf_{x\in K} \varrho_\Omega(x)$.

This follows from Corollary 1 to theorem 2 in section 5.

(β) For every positive integer m we set

$$|u(x)|_m = \sum_{|\alpha|\leqslant m}\left|\frac{D^\alpha u(x)}{\alpha!}\right|$$

and for every compact $K \subset \Omega$ we set

$$\|u\|_{K,m} = \sup_{x\in K} |u(x)|_m .$$

Given Ω open in $\mathbb{R}^n$, $u \in \mathcal{H}(\Omega)$ we set

$$U(u, x) = \sum_0^\infty |u(x)|_m \xi^m$$

$$\lambda(u, x) = \max \lim \sqrt[m]{|u(x)|_m}$$

so that

$$r(u, x) = \inf\{\lambda(u, x)^{-1}, 1\}$$

is a radius of convergence of the power series $U(u, x)$ in ξ.

As usual we set, for

$$z \in \mathbf{C}^n, \quad \|z\| = \sup_{1 \leq i \leq n} |z_i|.$$

LEMMA 2. *Let u be an analytic function defined in Ω. Let $\mathcal{T}_x u$, for $x \in \Omega$, denote the Taylor series of u at x. Let*

$$\sigma(x) = \sup\{\sigma \in \mathbf{R} | 0 < \sigma < 1, \ \mathcal{T}_x u \text{ converges in the polycylinder } \|z - x\| < \sigma\}.$$

Then $\sigma(x) = r(u, x)$.

PROOF. We have: $\sigma(x) \geq r(u, x)$. Indeed if $U(u, x)$ converges in the disc $|\xi| < r(u, x)$, for $0 < \lambda < 1$ we have an estimate

$$|u(x)|_m \leq \frac{c}{(\lambda r(u, x))^m}.$$

Therefore

$$\left|\frac{D^\alpha u(x)}{\alpha!}\right| \leq \frac{c}{(\lambda r(u, x))^{|\alpha|}}$$

and thus $\mathcal{T}_x u$ is convergent in the polycylinder

$$\|z - x\| < \lambda r(u, x)$$

or, letting $\lambda \to 1$ in $\|z - x\| < r(u, x)$. Hence $\sigma(x) \geq r(u, x)$.

We have: $\sigma(x) \leq r(u, x)$. Since $\mathcal{T}_x u$ is convergent in $\|z - x\| < \sigma(x)$, for $0 < \lambda < 1$ we do have estimates

$$\left|\frac{D^\alpha u(x)}{\alpha!}\right| \leq \frac{c}{(\lambda \sigma(x))^{|\alpha|}}.$$

The set of $\alpha \in N^n$ with $|\alpha| \leq m$ contains less than $(m+1)^n$ elements. Therefore

$$|u(x)|_m \leq (m+1)^n \frac{c}{(\lambda \sigma(x))^m}.$$

Then $U(u, x)$ is majorized by the series

$$c \sum (m+1)^n \frac{|\xi|^m}{(\lambda \sigma(x))^m}$$

and therefore $U(u, x)$ is convergent for $|\xi| < \sigma(x)$. Hence $\sigma(x) \leqslant r(u, x)$, and the proof is complete.

(b) Let Ω be an open set in $\mathbb{R}^n$ and let K be a compact subset of Ω. For any $c \geqslant 1$ and $L \geqslant 1$ we define

$$\hat{K}_\infty(L, c) = \{x \in \Omega \,\big|\, |u(x)|_m \leqslant L c^m \|u\|_{K,m}\ \forall u \in \mathcal{H}_{A_0}(\Omega),\ \forall m \in \mathbb{N}\}\,.$$

We will say that Ω is A_0-convex iff the following condition is satisfied

$(K_\infty)_\Omega$ *For every K compact in Ω, $\forall c \geqslant 1$, $\forall L \geqslant 1$ the set $\hat{K}_\infty(L, c)$ is also compact.*

If the operator A_0 has constant coefficients this notion reduces to the notion of A_0-convexity already introduced. We have indeed the following

PROPOSITION 6. *Suppose that A_0 has constant coefficients. Then for every $c \geqslant 1$, every $L \geqslant 1$ and every compact $K \subset \Omega$ we have*

$$\hat{K}_\infty(L, c) = \hat{K}_\infty(L, 1) = \hat{K}(L)\,.$$

PROOF. We have: $\hat{K}(L) \subset \hat{K}_\infty(L, 1) \subset \hat{K}_\infty(L, c)$. Indeed $\forall u \in \mathcal{H}_{A_0}(\Omega)$ we have if $x \in \hat{K}(L)$

$$\left|\frac{D^\alpha u(x)}{\alpha!}\right| \leqslant L \sup_K \left|\frac{D^\alpha u(x)}{\alpha!}\right|.$$

Hence

$$|u(x)|_m \leqslant L\|u\|_{K,m}\,.$$

Therefore $x \in \hat{K}_\infty(L, 1) \subset \hat{K}_\infty(L, c)$.

Conversely $\hat{K}_\infty(L, c) \subset \hat{K}(L)$. Indeed if $x \in \hat{K}_\infty(L, c)$ we have $\forall u \in \mathcal{H}_{A_0}(\Omega)$ (taking $m = 0$ in the conditions defining $\hat{K}_\infty(L, c)$)

$$|u(x)| \leqslant L\|u\|_K\,.$$

Therefore, $x \in \hat{K}(L)$.

(c) We set

$$\delta_\Omega(x) = \inf\big(\varrho_\Omega(x), 1\big)$$

and for any compact $K \subset \Omega$

$$\delta_\Omega(K) = \inf_{x \in K} \delta_\Omega(x)\,.$$

Clearly $\delta_\Omega(K) > 0$ for any compact subset K of Ω.

PROPOSITION 7. *Let $A_0(x, D)$ be elliptic with C^∞ coefficients. Let Ω be open in $\mathbb{R}^n$, K compact in Ω and let $L \geqslant 1$, $c \geqslant 1$. Then for any $\xi \in \hat{K}_\infty(L, c)$ and for any $u \in \mathcal{H}(\Omega)$ the Taylor series $\mathcal{T}_\xi u$ of u at ξ is convergent in the complex polycylinder*

$$|z_i - \xi_i| < \frac{1}{c}\,\delta_\Omega(K) \qquad 1 \leqslant i \leqslant n .$$

PROOF. Since $\xi \in \hat{K}_\infty(L, c)$ we have $\forall \alpha \in N^n$, $\forall u \in \mathcal{H}(\Omega)$

$$\left|\frac{D^\alpha u(\xi)}{\alpha!}\right| \leqslant L c^{|\alpha|} \|u\|_{K,|\alpha|} .$$

Let $\lambda > 1$; by lemma 1, we can find $K_1(\lambda)$ compact in Ω such that

$$\sup_{x \in K}\left|\frac{D^\beta u(x)}{\beta!}\right| \leqslant c(\lambda)\,\lambda^{|\beta|}\,\frac{\sup_{K_1(\lambda)} |u|}{\delta_\Omega(K)^{|\beta|}} .$$

Therefore

$$\|u\|_{K,m} \leqslant c(\lambda)(m+1)^n \lambda^m \frac{\sup_{K_1(\lambda)} |u|}{\delta_\Omega(K)^m} .$$

Consequently

$$\left|\frac{D^\alpha u(\xi)}{\alpha!}\right| \leqslant L c(\lambda) \sup_{K_1(\lambda)} |u| (|\alpha| + 1)^n \frac{\lambda^{|\alpha|} c^{|\alpha|}}{\delta_\Omega(K)^{|\alpha|}} .$$

This shows that the Taylor series of u at ξ is convergent in the polycylinder

$$|z_i - \xi_i| < \frac{\delta_\Omega(K)}{\lambda c} \qquad 1 \leqslant i \leqslant n$$

Letting $\lambda \to 1$ we deduce the statement of this proposition.

COROLLARY 1. *If Ω is a domain of regularity for $A(x, D)$ then for any compact set $K \subset \Omega$, for any $L \geqslant 1$, any $c \geqslant 1$ we have*

$$\delta_\Omega\big(\hat{K}_\infty(L, c)\big) \geqslant \frac{1}{c}\,\delta_\Omega(K) .$$

In particular for any K compact in Ω, for any $L \geqslant 1$, any $c \geqslant 1$ we have

$$\delta_\Omega\big(\hat{K}_\infty(L, c)\big) > 0 .$$

COROLLARY 2. *If* $\mathbb{R}^n$ *itself is* A_0*-convex, then any domain of regularity* $\Omega \subset \mathbb{R}^n$ *is also* A_0*-convex.*

PROOF. For any K compact in Ω, any $L \geqslant 1$, any $c \geqslant 1$, $\hat{K}_\infty(L, c)$ is a bounded subset of Ω (as $\mathbb{R}^n$ is A_0-convex) closed in Ω and such that $\delta_\Omega(\hat{K}_\infty(L, c)) > 0$. Therefore $\hat{K}_\infty(L, c)$ is closed and bounded in $\mathbb{R}^n$ and thus is a compact set.

(d) Given a positive constant c and a function u in $\mathcal{H}(\Omega)$, we define for $x \in \Omega$

$$\tau(c, x, u) = \sup_\alpha c^{|\alpha|} \frac{|D^\alpha u(x)|}{\alpha!}.$$

This is a positive function of x on Ω, with values in the extended real line $\mathbb{R} \cup \{+\infty\}$.

We have the following:

PROPOSITION 8. *For any open set* Ω *in* $\mathbb{R}^n$ *the following two conditions are equivalent*:

(i) *For every compact subset* K *of* Ω *and every constants* $c \geqslant 1$, $L \geqslant 1$, *we have*

$$\varrho_\Omega(\hat{K}_\infty(L, c)) > 0.$$

(ii) *For every sequence* $\{x_\nu\}$ *in* Ω *such that* $\varrho_\Omega(x_\nu) \to 0$ *and for every constant* c *with* $0 < c < 1$, *there is a function* $u \in \mathcal{H}(\Omega)$ *such that*

$$\sup_\nu \tau(c, x_\nu, u) = +\infty.$$

PROOF. i) $\Rightarrow$ ii) By contradiction: assume there is a sequence $\{x_\nu\}$ in Ω with $\varrho_\Omega(x_\nu) \to 0$ and a constant c with $0 < c < 1$ such that

$$\sup_\nu \tau(c, x_\nu, u) < \infty \quad \text{for every } u \in \mathcal{H}(\Omega).$$

Set

$$\begin{aligned} \mathbb{A} &= \left\{u \in \mathcal{H}(\Omega) \mid \sup_\nu \tau(c, x_\nu, u) \leqslant 1\right\} \\ &= \left\{u \in \mathcal{H}(\Omega) \,\Big|\, \left|\frac{D^\alpha u(x_\nu)}{\alpha!}\right| \leqslant \frac{1}{c^{|\alpha|}} \quad \forall \alpha \in \mathbb{N}^n, \quad \forall \nu \in \mathbb{N}\right\}. \end{aligned}$$

We have that

i) $\mathbb{A} = -\mathbb{A}$;

ii) $\mathbf{A}$ is convex;

iii) $\mathbf{A}$ is closed.

The properties i) and ii) are immediate. If $u_n \to u_0$ in $\mathcal{H}(\Omega)$ and $u_n \in \mathbf{A}$, then also $u_0 \in \mathbf{A}$ because the conditions

$$\left|\frac{D^\alpha u(x_\nu)}{\alpha!}\right| \leqslant \frac{1}{c^{|\alpha|}}$$

are closed conditions (under the Schwartz topology of $\mathcal{H}(\Omega)$ which is the same than the « sup » topology on compact sets) for any $\alpha \in \mathbf{N}^n$ and any $\nu \in \mathbf{N}$.

As $\mathcal{H}(\Omega) = \bigcup_1^\infty m\mathbf{A}$ then one of the sets $m\mathbf{A}$ (and therefore $\mathbf{A}$ itself) must contain an interior point and thus a neighborhood of the origin.

There exist therefore a compact $K \subset \Omega$ and an $\varepsilon > 0$ such that

$$V(K, \varepsilon) = \left\{u \in \mathcal{H}(\Omega) \mid \sup_K |u| < \varepsilon\right\} \subset \mathbf{A}\,.$$

Therefore $\forall \alpha \in \mathbf{N}^n$, $\forall \nu \in \mathbf{N}$, $\forall u \in \mathcal{H}(\Omega)$

$$\left|\frac{D^\alpha u(x_\nu)}{\alpha!}\right| \leqslant \frac{\|u\|_{K,0}}{\varepsilon}\,\frac{1}{c^{|\alpha|}}\,.$$

If $c_2 > 0$ is such that $\forall m \in \mathbf{N}$

$$c_2^m \geqslant (m+1)^n \frac{1}{\varepsilon}$$

we get $\forall u \in \mathcal{H}(\Omega)$

$$|u(x_\nu)|_m \leqslant \left(\frac{c_2}{c}\right)^m \|u\|_{K,m}\,.$$

This shows that $\{x_\nu\} \subset \hat{K}_\infty(1, c_2/c)$ and thus $\varrho_\Omega(x_\nu) \geqslant \varrho_\Omega\big(\hat{K}_\infty(1, c_2/c)\big) > 0$ contrary to the assumption that $\varrho_\Omega(x_\nu) \to 0$.

(ii) $\Rightarrow$ (i). If not, there exist a compact set K and constants $L_0 \geqslant 1$, $c_0 \geqslant 1$ such that $\varrho_\Omega\big(\hat{K}_\infty(L_0, c_0)\big) = 0$, so that we can find a sequence $\{x_\nu\} \subset \subset \hat{K}_\infty(L_0, c_0)$ with $\varrho_\Omega(x_\nu) \to 0$.

For any $u \in \mathcal{H}(\Omega)$ we thus have, for every ν,

$$|u(x_\nu)|_m \leqslant L_0 c_0^m \|u\|_{K,m}\,.$$

By lemma 1, we can find a compact set K_1 and positive constants $c_1 \geqslant 1$, L_1, such that

$$\sup_{x\in K}\left|\frac{D^\alpha u(x)}{\alpha!}\right| \leqslant L_1 c_1^{|\alpha|}\|u\|_{K_1,0}\,.$$

If $c_2 > 0$ is so chosen that $\forall m \in \mathbb{N}$

$$c_2^m > (m+1)^n$$

then we have

$$\|u\|_{K,m} \leqslant L_1(c_1c_2)^m\|u\|_{K_1,0}\,.$$

In particular, $\forall u \in \mathcal{H}(\Omega)$ we have, for every ν,

$$(*)\qquad |u(x_\nu)|_m \leqslant L_0L_1(c_0c_1c_2)^m\|u\|_{K_1,0}\,.$$

Let $0 < c < 1$ be such that $c < (c_0c_1c_2)^{-1}$. Then by assumption (ii) there exists $u \in \mathcal{H}(\Omega)$ such that

$$\sup_\alpha c^{|\alpha|}\left|\frac{D^\alpha u(x_\nu)}{\alpha!}\right| > L_0L_1\|u\|_{K_1,0}$$

for infinitely many ν's i.e. for infinitely many ν's and $\alpha = \alpha(\nu)$

$$\left|\frac{D^\alpha u(x_\nu)}{\alpha!}\right| > L_0L_1\left(\frac{1}{c}\right)^{|\alpha|}\|u\|_{K_1,0}\,.$$

Now

$$\left(\frac{1}{c}\right)^{|\alpha|} \geqslant (c_0c_1c_2)^{|\alpha|}$$

as $cc_0c_1c_2 < 1$. Thus for infinitely many ν's and $\alpha = \alpha(\nu)$ we have

$$\left|\frac{D^\alpha u(x_\nu)}{\alpha!}\right| > L_0L_1(c_0c_1c_2)^{|\alpha|}\|u\|_{K_1,0}\,.$$

This inequality contradicts inequality $(*)$ established above.

PROPOSITION 9. *Let Ω be a domain in $\mathbb{R}^n$ satisfying condition* (ii) *of the previous proposition. Then Ω is a domain of regularity for A_0.*

PROOF. Assume, if possible, that Ω is not a domain of regularity. Then there exist two domains $\Delta \subset \tilde{\Delta}$ with $\Delta \subset \Omega$, $\tilde{\Delta} \not\subset \Omega$ and with $\tilde{\Delta}$ an $\mathcal{H}(\Omega)|\Delta$-

completion of Δ. Let $a \in \Delta$, $b \in \tilde{\Delta} - \tilde{\Delta} \cap \Omega$ and let $\gamma\colon [0,1] \to \tilde{\Delta}$ be a continuous path from a to b, with $\gamma(0) = a$, $\gamma(1) = b$. Let $x_0 = \gamma(t_0)$ be the first point of this path which is not in Ω. We must have $0 < t_0 \leqslant 1$. Set $x_\nu = \gamma(t_0(1 - 1/\nu))$, $\nu = 2, 3, \ldots$; then $\{x_\nu\} \subset \Omega$ and $x_\nu \to x_0$.

We can select $0 < r < \frac{1}{2}$ so that $\forall u \in \mathcal{H}(\Omega)$ the extension to $\tilde{\Delta}$ of $u|\Delta$ has a Taylor series at x_0 which is convergent in the closed polycylinder

$$P_0 = \{\|z - x_0\| \leqslant 2r\}\,.$$

If $u \in \mathcal{H}(\Omega)$ is given and ν is sufficiently large we have

$$\left|\frac{D^\alpha u(x_\nu)}{\alpha!}\right| \leqslant \frac{\sup\limits_{P_0} |u|}{r^{|\alpha|}} \qquad \forall \alpha \in \mathbf{N}^n\,.$$

Therefore

$$\sup_\nu \sup_\alpha r^{|\alpha|} \left|\frac{D^\alpha u(x_\nu)}{\alpha!}\right| < \infty\,.$$

This is in contradiction with property (ii) of proposition 8.

COROLLARY 1. (*Generalized Cartan-Thullen theorem*) *The necessary and sufficient condition for Ω to be a domain of regularity for A_0 is that Ω satisfies condition* (i) *or* (ii) *of proposition* 8.

COROLLARY 2. *If $\mathbf{R}^n$ is A_0-convex the necessary and sufficient condition for a domain Ω to be a domain of regularity for A_0 is that Ω is A_0-convex.*

9. – Some remarks on non elliptic operators.

(*a*) The considerations of the preceeding sections are strongly based on the assumption that the operator A_0 is an elliptic operator. If we drop this assumption then in general we loose the following properties:

(i) the topology of the sheaf $\mathcal{O}_{A_0}$ of germs of C^∞ *solutions* of $A_0 u = 0$ is a Hausdorff topology;

(ii) for every open set $\Omega \subset R^n$ the Schwartz topology of the space $\mathcal{H}(\Omega)$ of C^∞ solutions of $A_0 u = 0$ on Ω coincides with the topology defined by the seminorms $\|u\|_K = \sup\limits_{x \in K} |u(x)|$ for K compact in Ω.

Here is some indications of how to cope with these difficulties or at least of how to formulate the corresponding problems.

(*b*) Given a topological space X with a topology not necessarily Hausdorff we can consider on $X \times X$ the diagonal $\Delta_X = \{(x, y) \in X \times X | x = y\}$ and the natural identification $i: X \simeq \Delta_X$.

For every point $(x_0 \times x_0) \in \Delta_X$ we can consider the following property

(α) there exists an open neighborhood $U(x_0 \times x_0)$ of $(x_0 \times x_0)$ in $X \times X$ such that $\Delta_X \cap U(x_0 \times x_0)$ is a closed subset of $U(x_0 \times x_0)$.

This means that Δ_X is locally closed at $(x_0 \times x_0)$. The set of points $(x_0 \times x_0)$ where Δ_X is locally closed form an open subset $\mathcal{H}(\Delta) \subset \Delta_X$.

Let $\mathcal{H}(X) = i^{-1}(\mathcal{H}(\Delta))$. Then $\mathcal{H}(X)$ *is an open subset of* X *and the topology induced by* X *on* $\mathcal{H}(X)$ *is a Hausdorff topology.* We will call $\mathcal{H}(X)$ the *Hausdorff part of* X; *it is the maximal open set on which the induced topology is Hausdorff.*

We can then repeat the considerations of sections 3 and 4 replacing the notion of « connected component of ... » with the notion of « the Hausdorff part of the connected component of ... ».

In this way we can talk about the S-envelope with respect to a part S of $\mathcal{H}(\Omega)$ (and the envelope of regularity) of an open set Ω.

(*c*) As an *example*, consider the equation on $\mathbb{R}^2$ (x and y are cartesian coordinates in $\mathbb{R}^2$)

$$A_0 u \equiv \frac{\partial^2 u}{\partial x\, \partial y} = 0\,, \qquad \text{on the open set } \Omega \equiv \{x^2 + y^2 < 1\}\,.$$

Let $\mathcal{O}_{A_0}$ be the sheaf of germs of C^∞ solutions of $A_0 u = 0$. Then the envelope of regularity of Ω is the open set

$$\tilde{\Omega} = \{(x, y) \in \mathbb{R}^2 \big| |x| < 1,\ |y| < 1\}$$

which is larger then Ω.

(*d*) Let Ω be an open set in $\mathbb{R}^n$ and $\gamma: [0, 1] \to \mathbb{R}^n$ a continuous path with $\gamma(0) \in \Omega$. We will say that $g \in \mathcal{H}(\Omega)$ has a unique continuation along γ if

i) there exists a section $G \in \Gamma([0, 1], \gamma^* \mathcal{O}_{A_0})$ with $G_0 = g_{\gamma(0)}$;

ii) every section $S \in \Gamma([0, 1], \gamma^* \mathcal{O}_{A_0})$ with $S_0 = g_{\gamma(0)}$ must coincide with the section G considered above.

Given two paths $\gamma_i: [0, 1] \to \mathbb{R}^n$ $(i = 1, 2)$ with $\gamma_i(0) = a \in \Omega$ $(i = 1, 2)$ and $\gamma_i(1) = b \in \mathbb{R}^n$ $(i = 1, 2)$, assume that every $g \in \mathcal{H}(\Omega)$ admits a unique continuation on γ_i $(i = 1, 2)$; we will call the two paths equivalent if, denoting by $G_g(i)$ the continuation of g along γ_i $(i = 1, 2)$, we have

$$\forall g \in \mathcal{H}(\Omega)\,, \qquad G_g(1)_1 = G_g(2)_1$$

i.e. the germs defined by analytic continuation to b are the same for every $g \in \mathcal{H}(\Omega)$.

One can then verify that, if Ω is connected, given $a \in \Omega$ the envelope of regularity $\tilde{\Omega} \xrightarrow{\pi} \mathbf{R}^n$ of Ω is in one to one correspondence with the equivalence classes of paths $\gamma\colon [0, 1] \to \mathbf{R}^n$ with $\gamma(0) = a$, along which every $g \in \mathcal{H}(\Omega)$ admits a unique continuation.

(*e*) We end up this section by the following

REMARK. *Assume that* A_0 *has constant coefficients.*

Let $\gamma\colon [0, 1] \to \mathbf{R}^n$ *be a path along which a germ* $g_{\gamma(0)} \in \mathcal{O}_{A_0\gamma(0)}$ *has a non unique continuation. Then we can find* $t_0 \in \,]0, 1]$ *and a continuation* G *of* $g_{\gamma(0)}$ *along* $\gamma_t = \gamma|_{[0,t_0[}\, [0, t_0[\to \mathbf{R}^n$ *such that* (9)

$$\sup_{t<t_0} |G(t)| = +\infty\,.$$

PROOF. Let G_1, G_2 be two continuations of $g_{\gamma(0)}$ along γ, i.e.

(i) $G_i \in \Gamma([0, 1], \gamma^* \mathcal{O}_{A_0})$ $i = 1, 2$;

(ii) $(G_1)_0 = (G_2)_0 = g_{\gamma(0)}$.

We assume that $G_1 \neq G_2$. If $A = [0, t_0[$ is the connected component of 0 in the open set $B = \{t \in [0, 1] | (G_1)_t = (G_2)_t\}$, we have $0 < t_0 \leqslant 1$.

A non difficult argument shows that without loss of generality we may assume that γ is a simple path. Then, denoting by I the set $\gamma([0, 1])$, we can find a neighborhood U of I in $\mathbf{R}^n$ and a section $G \in \Gamma(U, \mathcal{O}_{A_0})$ such that $\gamma^* G = G_1 - G_2$ on $[0, 1]$. We assume that $\gamma(t_0) = 0$, the origin of $\mathbf{R}^n$. There is a sequence $\{x^{(n)}\}$ converging to 0 in U with the properties that

$$G_n = G(x + x^{(n)}) \quad \text{is defined in } W_n = U - x^{(n)} \supset I$$

and $G_n(0) = G(x^{(n)}) \neq 0$.

As

$$\frac{G(x^{(n)})}{|G(x^{(n)})|} \to \frac{G(0)}{|G(0)|}\,,$$

(9) For $G \in \mathbf{C}^p$ we set $|G| = ({}^t\bar{G}G)^{\frac{1}{2}}$.

we can also assume, by deleting sufficiently many terms at the beginning of the sequence $\{x^{(n)}\}$, that for any choice of $n_1, \ldots, n_\nu$ we have:

$$\left| \frac{G(x^{(n_1)})}{|G(x^{(n_1)})|} + \ldots + \frac{G(x^{(n_\nu)})}{|G(x^{(n_\nu)})|} \right| > \frac{\nu}{2}.$$

Moreover we can assume that every G_n is defined in a fixed neighborhood W of I independent of n.

Since A_0 has constant coefficients we have

$$A_0 G_n = 0 \quad \text{on } W.$$

The set $A = \{x \in W | G_x = 0\}$ is open and contains $\gamma([0, t_0[)$, but not the point 0 $(= \gamma(t_0))$.

We claim that there is a subsequence $\{x^{(n_\nu)}\}$ of $\{x^{(n)}\}$, such that, setting

$$H(x) = \sum \frac{G_{n_\nu}(x)}{|G(x^{(n_\nu)})|}$$

one has:

(i) $H(x) \in C^\infty(A)$ and $A_0 H = 0$ on A;

(ii) we can find a sequence $\{t_\nu\}$ converging to t_0, with $0 < t_\nu < t_0$, such that

$$\lim_{\nu\to\infty} H(\gamma(t_\nu)) = +\infty.$$

To prove the claim (i), we note that, for $x_0 \in A$, we can find a compact neighborhood $U(x_0) = \{|x - x_0| \leqslant \varepsilon\}$ of x_0 in A and correspondingly an integer $n_0 = n_0(x_0, \varepsilon)$ such that, for $n > n_0$,

$$U(x_0) + x^{(n)} \subset A.$$

Therefore, for $n > n_0$, $G_n(x)$ vanishes in $\{|x - x_0| < \varepsilon\}$. Thus, in a neighborhood of each point of A, the series defining $H(x)$ contains only finitely many terms which are different from 0 and hence the first claim is proved.

To prove ii), we first choose a decreasing sequence $\{\varepsilon_n\}$ of positive numbers such that $U_n = \{|x| < \varepsilon_n\} \subset U$, $\varepsilon_n \to 0$ and

$$|G(x + x^{(n)}) - G(x^{(n)})| < \tfrac{1}{4}|G(x^{(n)})| \quad \text{on } U_n.$$

We fix $x^{(1)}$ and select t_1 with $0 < t_1 < t_0$ such that

$$\gamma(t_1) \in U_1 .$$

Deleting from the sequence $\{x^{(n)}\}$ sufficiently many terms after $x^{(1)}$ at the beginning, and renumbering, we may satisfy also the condition:

$$\gamma(t_1) \in A \bigcap_{n \geqslant 2} (A - x^{(n)}) .$$

Let $x^{(2)}$ be the second element of the sequence thus obtained. We can select t_2 with $0 < t_2 < t_0$ and

$$\gamma(t_2) \in U_2 .$$

Deleting sufficiently many terms after $x^{(2)}$ in the previous sequence, we can fulfill also the condition

$$\gamma(t_2) \in A \bigcap_{n \geqslant 3} (A - x^{(n)}) .$$

Repeating this argument, we obtain a subsequence of $\{x^{(n)}\}$, that we still denote by $\{x^{(n)}\}$, such that the function $H(x) \in C^\infty(A)$ constructed from it has the properties:

(α) the germ $H_{\gamma(0)} = 0$ (this is the case if enough elements are deleted at the beginning of the sequence $\{x^{(n)}\}$).

$$(\beta) \qquad H(\gamma(t_\nu)) = \sum_{j=1}^{\nu} \frac{G(\gamma(t_\nu) + x^{(j)})}{|G(x^{(j)})|}$$

(for $G(\gamma(t_\nu) + x^{(j)}) = 0$ if $j > \nu$ as $\gamma(t_\nu) + x^{(j)} \in A$).

Now we have:

$$H(\gamma(t_\nu)) = \left| \sum_{j=1}^{\nu} \frac{G(x^{(j)}) + \big(G(\gamma(t_\nu) + x^{(j)}) - G(x^{(j)})\big)}{|G(x^{(j)})|} \right| \geqslant$$

$$\geqslant \frac{\nu}{2} - \sum_{j=1}^{\nu} \frac{|G(\gamma(t_\nu) + x^{(j)}) - G(x^{(j)}))|}{|G(x^{(j)})|} > \frac{\nu}{4} .$$

Then, if we set $G = G_1 + \gamma^* H$, the proof is completed.

§ 2. - Envelopes of regularity. Operators with constant coefficients in one unknown function. First examples.

10. – Reduction to $\bar\partial$-suspended operators.

a) Let $A_0(D)\colon \mathcal{E}^q(\Omega) \to \mathcal{E}^p(\Omega)$ be an *elliptic* operator with constant coefficients defined on $\mathbb{R}^n$ by a $p \times q$ matrix of differential operators with constant coefficients; $(D = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ is the symbol of differentiation and Ω denotes an open set in $\mathbb{R}^n)$.

In $\mathbb{C}^n$, where $z_1, \dots, z_n$ are the holomorphic coordinates, $z = x + iy$ i.e. $z_j = x_j + iy_j$, $1 \leqslant j \leqslant n$, and where $\mathbb{R}^n$ is imbedded as $\mathbb{R}^n = \{z \in \mathbb{C}^n | y = 0\}$, we consider the set $U = \{z \in \mathbb{C}^n | z = x + iy,\ x \in \Omega,\ y \in \mathbb{R}^n \text{ and } \|z\| < \varrho_\Omega(x)\}$ where $\|z\| = \sup_{1 \leqslant j \leqslant n} |z_j|$. This is an open set, $\Omega \subset U$ and by corollary 2 to theorem 2 the natural restriction map

$$r^U{}_\Omega\colon \mathcal{H}_A(U) \xrightarrow{\sim} \mathcal{H}_{A_0}(\Omega)$$

is an isomorphism (here $A = A_0 \oplus \bar\partial$ is the $\bar\partial$ suspension of the operator A_0). We will assume Ω and thus U connected.

Let $\tilde U \xrightarrow{\tilde\pi} \mathbb{C}^n$ be the envelope of regularity of U with respect to the operator A and let $F_{\mathcal{H}_A(U)}\colon U \to \tilde U$ be the natural imbedding of U into its envelope. Similarly let $\tilde\Omega \xrightarrow{\omega} \mathbb{R}^n$ be the envelope of regularity of Ω with respect to A_0 and let $F_{\mathcal{H}_{A_0}(\Omega)}\colon \Omega \to \tilde\Omega$ the natural imbedding of Ω into its envelope of regularity.

We have a natural isomorphism

$$\mathcal{O}_A|_{\mathbb{R}^n} \xrightarrow{\sim} \mathcal{O}_{A_0}.$$

Thus one deduces the following statement:

PROPOSITION 10. *We have*

i) $F_{\mathcal{H}_A(U)}|_\Omega = F_{\mathcal{H}_{A_0}(\Omega)}$;

ii) $\tilde\Omega$ *is the connected component of* $F_{\mathcal{H}_{A_0}(\Omega)}(\Omega)$ *in* $\tilde\pi^{-1}(\mathbb{R}^n)$, i.e. *the part of* $\tilde U$ *lying above* $\mathbb{R}^n$.

REMARK. We do not know whether $\tilde\pi^{-1}(\mathbb{R}^n)$, the part of $\tilde U$ above $\mathbb{R}^n$, can actually have more than one connected component.

From the previous proposition it follows that the knowledge of the domain of regularity $\tilde U \xrightarrow{\tilde\pi} \mathbb{C}^n$ for U and the suspended operator $A = A_0 \oplus \bar\partial$ entails the knowledge of the domain of regularity $\tilde{\Omega} \xrightarrow{\omega} \mathbf{R}^n$ for the original operator A_0.

For this reason we will mainly study suspended operators $A = A_0 \oplus \bar{\partial}$, for which the assumption of ellipticity of the operator A_0 can be dropped, as a $\bar{\partial}$-suspended operator is always elliptic.

We may remark that, given a $\bar{\partial}$-suspended operator A (with constant coefficients), a connected open set $\tilde{\Omega} \subset \mathbf{C}^n$ will be a domain of regularity for A if and only if for any sequence $\{z_\nu\} \subset \tilde{\Omega}$ such that $\varrho_{\tilde{\Omega}}(z_\nu) \to 0$ there exists $u \in \Gamma(\tilde{\Omega}, \mathcal{O}_A) = \mathcal{H}_A(\tilde{\Omega})$ such that

$$\sup_\nu |u(z_\nu)| = \infty .$$

This by virtue of the generalization of Cartan Thullen theorem. In particular for any sequence $\{z_\nu\} \subset \Omega$ with $z_\nu \to z_0 \in \partial\Omega$ there exists $u \in \mathcal{H}_A(\Omega)$ with $\sup |u(z_\nu)| = \infty$.

This shows that *a domain of regularity for a $\bar{\partial}$-suspended operator in $\mathbf{C}^n$ is necessarily a domain of holomorphy.*

The purpose of this paragraph 2 is to investigate the particularities that a domain of holomorphy of $\mathbf{C}^n$ has to have in order to be also a domain of regularity for a given suspended operator $A = A_0 \oplus \bar{\partial}$.

b) We will restrict at the beginning at least our investigation to a $\bar{\partial}$-suspended operator $A = A_0 \oplus \bar{\partial}$ in one unknown function ($q = 1$). The system of equations $Au = 0$ will then be of the form

$$(*) \qquad \left\{\begin{array}{l} \psi_1\left(\dfrac{\partial}{\partial z}\right) u = 0\,, \\ \cdots\cdots \\ \psi_l\left(\dfrac{\partial}{\partial z}\right) u = 0\,, \\ \bar{\partial} u = 0\,, \end{array}\right.$$

where $\psi_i(\xi) = \psi_i(\xi_1, \ldots, \xi_n)$ are polynomials in $\xi = (\xi_1, \ldots, \xi_n)$, and $\partial/\partial z = (\partial/\partial z_1, \ldots, \partial/\partial z_n)$. Such a system is completely defined by the ideal

$$\mathfrak{b} = \mathbf{C}[\xi_1, \ldots, \xi_n]\big(\psi_1(\xi), \ldots, \psi_l(\xi)\big)$$

generated by the polynomials $\psi_j(\xi)$ $1 \leqslant j \leqslant l$. For this reason we *will call the system* (*) *the system of the suspended differential ideal* $\mathfrak{b}$.

Given an ideal of polynomials

$$\mathfrak{b} \subset \mathbb{C}[\xi_1, \ldots, \xi_n]$$

we can associate to it the algebraic (affine) variety

$$V(\mathfrak{b}) = \{z \in \mathbb{C}^n | g(z) = 0 \;\; \forall g \in \mathfrak{b}\} .$$

It will be called *the characteristic variety* of the given system (∗).

Besides the ideal $\mathfrak{b}$ one can consider the homogeneous ideal $\mathfrak{a} \subset C_0[\xi_1, \ldots, \xi_n]$ (by $C_0[\xi_1, \ldots, \xi_n]$ we denote the graded ring of homogeneous polynomials) of the homogeneous parts of maximal degree (principal parts) of the elements of $\mathfrak{b}$. We will call $\mathfrak{a}$ the *asymptotic ideal* and its variety of zeros

$$V(\mathfrak{a}) = \{z \in \mathbb{C}^n | g(z) = 0, \;\; \forall g \in \mathfrak{a}\}$$

the *asymptotic variety* (of the characteristic variety). It is the cone of complex lines joining the origin to the points of the variety $W(\mathfrak{a})$, in the projective space $\mathbb{P}_{n-1}(\mathbb{C})$ at ∞, of the points at infinity of the characteristic variety $V(\mathfrak{b})$.

If we consider $\mathbb{C}^n \subset \mathbb{P}_n(\mathbb{C})$ and $\mathbb{P}_{n-1}(\mathbb{C}) = \mathbb{P}_n(\mathbb{C}) - \mathbb{C}^n$ then

$$W(\mathfrak{a}) = \{\text{closure in } \mathbb{P}_n(\mathbb{C}) \text{ of } V(\mathfrak{b})\} \cap \mathbb{P}_{n-1}(\mathbb{C}) .$$

We have the relations between complex dimension, *provided* $\dim_{\mathbb{C}} V(\mathfrak{b}) \geqslant 1$ or assuming that the dimension of the empty set can be any integer $\gtreqless 0$:

$$\dim_{\mathbb{C}} V(\mathfrak{b}) = \dim_{\mathbb{C}} V(\mathfrak{a}) = \dim_{\mathbb{C}} W(\mathfrak{a}) + 1 .$$

This dimension is also called the dimension of the ideal $\mathfrak{b}$.

We remark that, if $V(\mathfrak{b}) = \emptyset$, then by Hilbert's « Nullstellensatz »

$$1 = \sum_{s=1} A_s(\xi)\, \psi_s(\xi)$$

for some polynomials A_s, i.e. $\mathfrak{b} = C[\xi_1, \ldots, \xi_n]$. In this case both for the operator A_0 and for its suspension A we will have $\mathcal{O}_{A_0} = 0 = \mathcal{O}_A$: the only solution (holomorphic or not) of the equations $\psi_i(\partial/\partial x)\, u = 0$, $1 \leqslant i \leqslant l$, on any open set Ω is $u = 0$.

We will therefore in the sequel assume that the characteristic variety $V(\mathfrak{b})$ is non void.

11. – Differential ideals of dimension zero.

(*a*) For the sake of simplicity we will first restrict our considerations to the case of an ideal $\mathfrak{b} \subset \mathbf{C}[\xi_1, \dots, \xi_n]$ of dimension zero coinciding with its radical: $\mathfrak{b} = \sqrt{\mathfrak{b}}$.

Let $a^{(1)} \cup a^{(2)} \cup \dots \cup a^{(\mu)} = V(\mathfrak{b})$. *The condition* $\mathfrak{b} = \sqrt{\mathfrak{b}}$ *is equivalent with the condition*

$$\operatorname{rank} \left\{ \frac{\partial(\psi_1, \dots, \psi_l)}{\partial(\xi_1, \dots, \xi_n)} \right\}_{a^{(i)}} = n \qquad \text{for } 1 \leqslant i \leqslant \mu .$$

Indeed, if $\mathfrak{b} = \sqrt{\mathfrak{b}}$, $\mathfrak{b}$ consists of all polynomials vanishing at the points $a^{(i)}$, $1 \leqslant i \leqslant \mu$, and therefore the condition on the rank of the jacobian is satisfied.

Conversely, assuming that such condition is satisfied, one has to show that if $p \in \mathbf{C}[\xi_1, \dots, \xi_n]$ vanishes at the points $a^{(i)}$, $1 \leqslant i \leqslant \mu$, then $p \in \mathfrak{b}$.

The rank condition entails that $\forall i$ with $1 \leqslant i \leqslant \mu$ we can find formal power series $q_j^{(i)}(\xi)$, $1 \leqslant j \leqslant l$ centered at the point $a^{(i)}$ such that

$$p = \sum_{j=1}^{l} q_j^{(i)} \psi_j .$$

By a theorem of M. Noether (cfr. Gröbner [6] p. 151) there exist polynomials g_i, $1 \leqslant i \leqslant \mu$, such that

i) $g_i(a^{(i)}) \neq 0$;

ii) $g_i p = \sum_{j=1}^{l} b_j^{(i)} \psi_j$ where $b_j^{(i)}$ are polynomials.

The ideal generated by $\psi_1, \dots \psi_l, g_1, \dots, g_\mu$ is then the trivial ideal so that we do have an identity of the form

$$\sum_{s=1}^{\mu} b_s g_s \equiv 1 \bmod \mathfrak{b}$$

with b_s polynomials.

From this and condition (ii) we then derive

$$p \equiv \left(\sum_{s=1}^{\mu} b_s g_s \right) p \equiv 0 \bmod \mathfrak{b} .$$

(*b*) We have the following

PROPOSITION 11. *Let* $\mathfrak{b} = \sqrt{\mathfrak{b}}$ *be* 0*-dimensional and let*

$$V(\mathfrak{b}) = a^{(1)} \cup \ldots \cup a^{(\mu)}\,; \qquad a^{(i)} = (a_1^{(i)}, \ldots, a_n^{(i)}) \in \mathbf{C}^n\,, \qquad 1 \leqslant i \leqslant \mu\,.$$

For any open connected set $\Omega \subset \mathbf{C}^n$ *we have the direct sum decomposition*

$$\mathcal{H}(\Omega) = \sum_{s=1}^{\mu} \mathbf{C} \exp\left(\sum a_j^{(s)} z_j\right).$$

Setting $\mathcal{H}_s = \mathbf{C} \exp\left(\sum a_j^{(s)} z_j\right)$, *the projection operator*

$$\mathfrak{D}_s : \mathcal{H}(\Omega) \to \mathcal{H}_s$$

is a differential operator and $1 = \sum_{s=1}^{\mu} \mathfrak{D}_s$ *on* $\mathcal{H}(\Omega)$.

PROOF. Let $a = (a_1, \ldots, a_n) \in V(\mathfrak{b})$. We can assume that

$$\det \left\{\frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\xi_1, \ldots, \xi_n)}\right\}_{\xi = a} \neq 0\,.$$

Therefore

$$(+) \qquad \psi_j = \sum_1^n (\xi_i - a_i) H_{ji}(\xi)$$

with polynomials H_{ji} such that

$$H_{ji}(a) = \frac{\partial \psi_j}{\partial \xi_i}(a)\,.$$

Thus, setting $D_a = \det\left(H_{ji}(\xi)\right)$ we get $D_a(a) \neq 0$ and from equations $(+)$ we deduce

$$(\xi_i - a_i) D_a(\xi) \equiv 0 \bmod \mathfrak{b} \qquad 1 \leqslant i \leqslant n\,.$$

So we have:

for every zero $a = (a_1, \ldots, a_n) \in V(\mathfrak{b})$ *we can find a polynomial* $D_a(\xi)$ *such that*

(i) $D_a(a) \neq 0$;

(ii) $(\xi_i - a_i) D_a(\xi) \equiv 0 \bmod \mathfrak{b}$, $1 \leqslant i \leqslant n$.

Let $\Phi = \sum_{a \in V(\mathfrak{b})} D_a(\xi)$. Then the ideal generated by $\psi_1, \ldots, \psi_l$ and Φ has no zeros. Therefore it is a trivial ideal and we have an identity of the form:

$$1 \equiv B \sum_{a \in V(\mathfrak{b})} D_a(\xi) \bmod \mathfrak{b}$$

with B a convenient polynomial.

Set now $\mathfrak{D}_a = BD_a$. We have

$$\mathfrak{D}_a(a) = 1 \quad \text{and} \quad \mathfrak{D}_a(b) = 0 \text{ if } b \in V(\mathfrak{b}) \text{ and } b \neq a\,.$$

Set

$$\mathcal{H}_a = \left\{u \in \mathcal{E}(\Omega) \,|\, \bar{\partial} u = 0, \left(\frac{\partial}{\partial z_i} - a_i\right) u = 0 \text{ for } 1 \leqslant i \leqslant n\right\} = \mathbf{C} \exp\left[\sum a_i z_i\right].$$

From $1 = \sum_{a \in V(\mathfrak{b})} \mathfrak{D}_a(\partial/\partial z)$ on $\mathcal{H}(\Omega)$, i.e.

$$u = \sum_{a \in V(\mathfrak{b})} \mathfrak{D}_a\left(\frac{\partial}{\partial z}\right) u \qquad \forall u \in \mathcal{H}(\Omega)$$

we derive the direct sum decomposition

$$\mathcal{H}(\Omega) = \sum_{a \in V(\mathfrak{b})} \mathcal{H}_a$$

where the projection map of $\mathcal{H}(\Omega)$ onto $\mathcal{H}_a$ is given by the differential operator

$$\mathfrak{D}_a\left(\frac{\partial}{\partial z}\right) : \mathcal{H}(\Omega) \to \mathcal{H}_a\,.$$

REMARK 1. The same theorem, with the same proof, holds for the non suspended system on $\mathbf{R}^n$:

$$\psi_j\left(\frac{\partial}{\partial x}\right) u = 0 \qquad 1 \leqslant j \leqslant l\,.$$

REMARK 2. If $\mathcal{O}$ is either the sheaf $\mathcal{O}_A$ of germs of solutions of the suspended system (*) or the sheaf $\mathcal{O}_{A_0}$ of germs of solutions of the non suspended one, we have an isomorphism

$$\mathcal{O} \simeq \mathbf{C} \oplus \ldots \oplus \mathbf{C} \qquad \mu \text{ times}$$

for some integer μ.

COROLLARY. *For an open set Ω in $\mathbf{C}^n$ (or in $\mathbf{R}^n$) the necessary and sufficient condition that, for any given $j > 0$,*

$$H^j(\Omega, \mathcal{O}) = 0$$

is that

$$H^j(\Omega, \mathbb{C}) = 0 .$$

(*c*) In general we will have $\sqrt{\mathfrak{b}} \supsetneq \mathfrak{b}$. For a sufficiently large integer ϱ we will have

$$(\sqrt{\mathfrak{b}})^\varrho \subset \mathfrak{b} \subset \sqrt{\mathfrak{b}} .$$

Correspondingly, if we denote by $\mathcal{H}_{\mathfrak{b}}(\Omega)$ the space of solution of the system of equations corresponding to the ideal $\mathfrak{b}$ on the open connected set Ω, we have

$$\mathcal{H}_{\sqrt{\mathfrak{b}}}(\Omega) \subset \mathcal{H}_{\mathfrak{b}}(\Omega) \subset \mathcal{H}_{(\sqrt{\mathfrak{b}})^\varrho}(\Omega) .$$

Proposition 11 extends in general with the following

PROPOSITION 12. *Let* $\mathfrak{b} = \mathbb{C}[\xi_1, \ldots, \xi_n](\psi_1(\xi), \ldots, \psi_l(\xi))$ *be any* 0*-dimensional ideal.*

There exists a finite set of holomorphic solutions w_j $1 \leqslant j \leqslant k$ *on* $\mathbb{C}^n$ *of the system of equations*

$$(*) \qquad \psi_j\left(\frac{\partial}{\partial z}\right)u = 0 \qquad 1 \leqslant j \leqslant l$$

linearly independent over $\mathbb{C}$ *such that for every open connected set* $\Omega \subset C^n$ *we have a direct sum decomposition*

$$\mathcal{H}(\Omega) = \sum_{j=1}^{k} \mathbb{C} w_j .$$

PROOF. Because of the inclusion $\mathcal{H}_{\mathfrak{b}}(\Omega) \subset \mathcal{H}_{(\sqrt{\mathfrak{b}})^\varrho}(\Omega)$ it will be enough to prove the statement for $\mathfrak{b} = (\sqrt{\mathfrak{b}})^\varrho$. Here we can proceed by induction on ϱ, as for $\varrho = 1$ we are reduced to the previous proposition. Let $\sqrt{\mathfrak{b}}$ be generated by $\varphi_1(\xi), \ldots, \varphi_t(\xi)$. We set $\varphi(\xi) = (\varphi_1(\xi), \ldots, \varphi_t(\xi))$, so that, denoting by $\omega_\alpha^{(\varrho)}(\eta) = \eta^\alpha$ $(\alpha \in N^t,\ |\alpha| = \sum \alpha_i = \varrho)$ the monomials of degree ϱ in $\eta_1, \ldots, \eta_t$, the ideal $(\sqrt{\mathfrak{b}})^\varrho$ is generated by the polynomials $\omega_\alpha^{(\varrho)}(\varphi(\xi))$. The system $(*)$ now reduces to

$$(*) \qquad \omega_\alpha^{(\varrho)}\left(\varphi\left(\frac{\partial}{\partial z}\right)\right)u = 0 \qquad \alpha \in N^t,\ |\alpha| = \varrho ,$$

that can be written also as:

$$\begin{cases} \varphi_i\left(\dfrac{\partial}{\partial z}\right)\omega_\beta^{(\varrho-1)}\left(\varphi\left(\dfrac{\partial}{\partial z}\right)\right)u = 0 . \qquad 1 \leqslant i \leqslant t , \\ \beta \in N^t,\ |\beta| = \varrho - 1 . \end{cases}$$

Therefore

$$(**)\qquad v_\beta = \omega_\beta^{(\varrho-1)}\left(\varphi\left(\frac{\partial}{\partial z}\right)\right)u \qquad |\beta| = \varrho - 1$$

is for every β a solution of the system

$$(***)\qquad \varphi_i\left(\frac{\partial}{\partial z}\right)v = 0 \qquad 1 \leqslant i \leqslant t\,.$$

Then, by the previous proposition, we have

$$v_\beta = \sum_{a\in V(\mathfrak{b})} c_{\beta a} \exp\left(\sum a_i z_i\right).$$

Then there is a finite set of vectors $(f_\beta^{(j)})_{|\beta|=\varrho-1}$ $(1 \leqslant j \leqslant k)$, whose components are in $\mathcal{H}_{\sqrt{\mathfrak{b}}}$ and therefore are entire functions on $\mathbb{C}^n$, such that, for any solution u of $(*)$ on Ω, we have:

$$(****)\qquad \omega_\beta^{(\varrho-1)}\left(\varphi\left(\frac{\partial}{\partial z}\right)\right)u = \sum_1^k \lambda_j f_\beta^{(j)} \qquad |\beta| = \varrho - 1$$

for some $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$, while each $(f_\beta^{(j)})_{|\beta|=\varrho-1}$ satisfies the integrability conditions of $(**)$ (cf. [1]). Then for each $j = 1, \ldots, k$ we can find an entire function w_j on $\mathbb{C}^n$ with $w_j = u$ satisfying $(****)$ with $\lambda_j = 1$ and $\lambda_i = 0$ for $i \neq j$.

Set now $u_0 = u - \sum \lambda_j w_j$. Then u_0 is a solution on Ω of the homogeneous system

$$\omega_\beta^{(\varrho-1)}\left(\varphi\left(\frac{\partial}{\partial z}\right)\right)u_0 = 0 \qquad |\beta| = \varrho - 1\,.$$

By the inductive hypothesis there is a finite set of entire solutions of this system, say $\{\theta_s | s = 1, \ldots, h\}$, whose linear combinations span all its solutions in Ω. Thus, for some $\mu_1, \ldots, \mu_h \in \mathbb{C}$,

$$u = \lambda_1 w_1 + \ldots + \lambda_k w_k + \mu_1\theta_1 + \ldots + \mu_h\theta_h\,.$$

As we can assume that $w_1, \ldots, w_k$, $\theta_1, \ldots, \theta_h$ are linearly independent, the proof is complete.

REMARK 1. The same argument and conclusion apply to the non suspended system on $\mathbb{R}^n$:

$$\psi_j(D)u = 0 \qquad 1 \leqslant j \leqslant l\,.$$

REMARK 2. If $\mathcal{O}$ is either the sheaf of germs of solutions of the system $(*)$ or the sheaf of germs of solutions of the non suspended one, then

$$\mathcal{O} \simeq \mathbf{C} \oplus \dots \oplus \mathbf{C} \quad \text{(a direct sum of finitely many copies of the constant sheaf)} .$$

Therefore for any open connected set Ω we have

$$H^j(\Omega, \mathcal{O}) = 0 \Leftrightarrow H^j(\Omega, C) = 0 .$$

REMARK 3. A more precise statement about the nature of the generators of $\mathcal{H}(\Omega)$ could be given but for our purposes the qualitative statement of proposition 12 is sufficient information. In particular *the only domain of regularity for the system $(*)$ relative to a 0-dimensional ideal is the entire space $\mathbf{C}^n$ itself.*

12. – Differential homogeneous ideals of dimension one.

(a) Let us consider the case where $\mathfrak{b} = C_0[\xi_1, \dots, \xi_n]\,\big(\psi_1(\xi), \dots, \psi_l(\xi)\big)$ is a homogeneous ideal in the graded ring $C_0[\xi_1, \dots, \xi_n]$ of homogeneous polynomials. This amounts to suppose that generators $\psi_1, \dots, \psi_l$ of $\mathfrak{b}$ can be found that are homogeneous polynomials. In this case $\mathfrak{b}$ coincides with its asymptotic ideal $\mathfrak{a}$ and $V = V(\mathfrak{b}) = V(\mathfrak{a})$ is a cone with vertex at the origin of $\mathbf{C}^n$.

We *will assume* that

$$\dim_{\mathbf{C}} V(\mathfrak{b}) = 1 .$$

Therefore the projective variety $W(\mathfrak{a}) \subset \mathbf{P}_{n-1}(\mathbf{C})$ consists of finitely many points $a^{(s)} = (a_1^{(s)}, \dots, a_n^{(s)})$ $1 \leqslant s \leqslant \mu$.

We will make the *further assumption* that each point $a^{(s)} \in W(\mathfrak{a})$ is a simple point i.e. *that*

$$\operatorname{rank} \left\{ \frac{\partial\big(\psi_1(\xi), \dots, \psi_l(\xi)\big)}{\partial(\xi_1, \dots, \xi_n)} \right\}_{\xi = a^{(s)}} = n - 1 \qquad 1 \leqslant s \leqslant \mu .$$

REMARK. *If $\mathfrak{b} = \sqrt{\mathfrak{b}}$ the last assumption is automatically satisfied.*

Indeed, let s be fixed $(1 \leqslant s \leqslant \mu)$, and let $l_1 = 0, \dots, l_{n-1} = 0$ be $n-1$ hyperplanes in $\mathbf{P}_{n-1}(\mathbf{C})$ passing through $a^{(s)}$ and linearly independent

$$\Big(\text{i.e.: } l_j(\xi) \equiv \sum^n c_{ji}\,\xi_i,\ \sum c_{ji}\, a_i^{(s)} = 0,\ 1 \leqslant j \leqslant n-1,\ \operatorname{rank}\,(c_{ji}) = n-1\Big).$$

Let φ be a homogeneous polynomial having the property

$$\varphi(a^{(s)}) \neq 0 \quad \text{and} \quad \varphi(a^{(j)}) = 0 \quad \text{if } j \neq s .$$

Then $l_1\varphi, l_2\varphi, \ldots, l_{n-1}\varphi \in \mathfrak{b}$ as $\mathfrak{b} = \sqrt{\mathfrak{b}}$ and

$$\operatorname{rank}\left(\frac{\partial(l_1\varphi, l_2\varphi, \ldots, l_{n-1}\varphi)}{\partial(\xi_1, \ldots, \xi_n)}\right)_{a^{(s)}} = n-1 .$$

Therefore the above rank condition on the generators ψ must be also satisfied.

For any $a^{(s)} \in W(\mathfrak{a})$ let us consider the projection map

$$\pi_s : \mathbf{C}^n \to \mathbf{C}$$

defined by

$$(z_1, \ldots, z_n) \to \sum_1^n a_i^{(s)} z_i .$$

This projection map is defined up to multiplication by an element $\varrho \in \mathbf{C}^*$, as the coordinates of $a^{(s)} \in \mathbf{P}_{n-1}(\mathbf{C})$ are so defined.

Let us denote by $\mathcal{O}_{\mathbf{C}}$ the sheaf of germs of holomorphic functions on $\mathbf{C}$ and by $\mathcal{O}_s$ the sheaf

$$\mathcal{O}_s = \pi_s^* \mathcal{O}_{\mathbf{C}}$$

reciprocal image of the sheaf $\mathcal{O}_{\mathbf{C}}$ via the holomorphic map π_s. In other words $\mathcal{O}_s$ is the sheaf of germs of holomorphic function on $\mathbf{C}^n$ depending only upon the variable $\sum a_i^{(s)} z_i$.

Let $\mathcal{O}_A$ denote, as usual, the sheaf of germs of holomorphic solution of the system

$$\psi_j\left(\frac{\partial}{\partial z}\right)u = 0 \qquad 1 \leqslant j \leqslant l .$$

We remark that we have a natural inclusion map

$$i_s : \mathcal{O}_s \to \mathcal{O}_A ,$$

because for each j the operator $\psi_j(\partial/\partial z)$ is homogeneous of a certain degree n_j, and for any holomorphic function $F(\xi)$ of the variable $\xi = \sum a_i^{(s)} z_i$, we have

$$\psi_j\left(\frac{\partial}{\partial z}\right) F\left(\sum a_i^{(s)} z_i\right) = F^{(n_j)}\left(\sum a_i^{(s)} z_i\right) \psi_j(a^{(s)}) .$$

We obtain therefore a natural homomorphism of sheaves

$$\coprod_{s=1}^{\mu} \mathcal{O}_s \xrightarrow{\alpha} \mathcal{O}_A$$

where $\alpha = \coprod_{s=1}^{\mu} i_s$. We want to investigate kernel and cokernel of α.

(*b*) Let again $\mu = \# W(\mathfrak{a})$ be the cardinality of $W(\mathfrak{a})$ and let $\mathbf{C}_{\mu-2}[t]$ denote the space of polynomials in t with coefficients in $\mathbf{C}$ and degree $\leqslant \mu - 2$; let $\mathcal{S} \subset \mathbf{C}_{\mu-2}[t]^{\mu}$ be the subspace of μ-tuples of polynomials of degree $\leqslant \mu - 2$, $s = \begin{pmatrix} s_1(t) \\ \vdots \\ s_\mu(t) \end{pmatrix}$ such that

$$\sum s_i\left(\sum a_j^{(i)} z_j\right) \equiv 0 .$$

Then $\mathcal{S}$ is a finite dimensional vector space and the elements of $\mathcal{S}$ can be viewed as sections of a sheaf $\underset{\sim}{\mathcal{S}}$ on $\mathbf{C}^n$ isomorphic to a finite sum $\mathbf{C} \oplus \ldots \oplus \mathbf{C}$, the number of components being equal to the dimension of $\mathcal{S}$.

We define a natural map

$$\underset{\sim}{\mathcal{S}} \xrightarrow{\beta} \coprod \mathcal{O}_s$$

by

$$\begin{pmatrix} s_1(t) \\ \vdots \\ s_\mu(t) \end{pmatrix} \longrightarrow \begin{pmatrix} s_1\left(\sum_1^n a_j^{(1)} z_j\right) \\ \cdots\cdots \\ s_\mu\left(\sum_1^n a_j^{(\mu)} z_j\right) \end{pmatrix}.$$

PROPOSITION 13. *Let $\mathfrak{b} = \mathfrak{a}$ be homogeneous of dimension one and let*

$$W(\mathfrak{a}) = a^{(1)} \cup \ldots \cup a^{(\mu)} \subset \mathbf{P}_{n-1}(\mathbf{C})$$

consist of μ distinct points. Then, with the above notations, the sequence

$$0 \longrightarrow \underset{\sim}{\mathcal{S}} \xrightarrow{\beta} \coprod_1^{\mu} \mathcal{O}_s \xrightarrow{\alpha} \mathcal{O}_A$$

is an exact sequence and $\underset{\sim}{\mathcal{S}} \simeq \mathbf{C} \oplus \ldots \oplus \mathbf{C}$, (a finite sum).

PROOF. We first establish the following

LEMMA. *Let $(a, b, \ldots, c)$ be a set of μ distinct points in $\mathbf{P}_{n-1}(\mathbf{C})$.*

There exists an integer $\nu \leqslant \mu - 1$ such that, denoting by the column vector $\{\omega_\sigma(z)\}_{|\sigma|=\nu}$ the set of dissimilar monomials of degree ν in $z_1, \ldots, z_n$, the rank of the matrix $(\omega_\sigma(a), \omega_\sigma(b), \ldots, \omega_\sigma(c))_{|\sigma|=\nu}$ equals μ.

There exist homogeneous polynomials of degree $\mu - 1$ (product of $\mu - 1$ linear forms) vanishing on $\mu - 1$ of the given μ points but not vanishing on the last of them.

This proves that, if $\nu \geqslant \mu - 1$; then

$$\operatorname{rank} (\omega_\sigma(a), \omega_\sigma(b), \ldots, \omega_\sigma(c))_{|\sigma|=\nu} = \mu .$$

Therefore there exists a smallest integer $\nu \leqslant \mu - 1$ for which the lemma is verified.

We go back now to the proof of Proposition 13.

Let $p \in \mathbf{C}^n$ and $f = \begin{pmatrix} f_1 \\ \vdots \\ f_\mu \end{pmatrix} \in (\coprod \mathcal{O}_s)_p$. If $B(p)$ is a sufficiently small open ball centered at p then f_i will be holomorphic on $\pi_i(B) \subset \mathbf{C}$ as a function of $t = \sum_1^n a_j^{(i)} z_j$.

Assume that $\alpha(f) = 0$. Thus we have

$$\sum_{s=1}^{\mu} f_s\left(\sum a_j^{(s)} z_j\right) \equiv 0 \quad \text{on } B .$$

Taking all ν-th partial derivatives

$$\frac{\partial^{\alpha_1 + \ldots + \alpha_n}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}, \qquad \sum \alpha_i = \nu ,$$

we get the relation:

$$(\omega_\sigma(a^{(1)}), \omega_\sigma(a^{(2)}), \ldots, \omega_\sigma(a^{(\mu)}))_{|\sigma|=\nu} \frac{d^\nu f}{dt^\nu} \equiv 0 .$$

By virtue of the lemma, as the matrix in parentheses is of rank μ, we derive that

$$\frac{d^\nu f_s}{dt^\nu}(t) = 0 \qquad \text{on } \pi_s(B) .$$

This shows that $f_s | \pi_s(B)$ is a polynomial in $t = \sum_{j=1}^n a_j^{(s)} z_j$ of degree $\leqslant \nu - 1 \leqslant \mu - 2$.

This completes the proof as the injectivity of β and the fact that $\alpha \circ \beta = 0$ are immediate from the definitions.

(*c*) Now we make use of the assumption that the μ points $a^{(s)} \in W(\mathfrak{a})$ are simple. Modulo a linear (real) transformation of variables we may assume that $a_n^{(s)} \neq 0$ for all s $(1 \leqslant s \leqslant \mu)$, so that

$$X_1^{(s)} = z_n a_1^{(s)} - z_1 a_n^{(s)}, \ldots, X_{n-1}^{(s)} = z_n a_{n-1}^{(s)} - z_{n-1} a_n^{(s)}$$

is a basis of linear forms vanishing on $a^{(s)}$.

Then we can write

$$\psi_\alpha(z) = \sum_{j=1}^{n-1} X_j^{(s)} \varphi_{j\alpha}^{(s)} \qquad 1 \leqslant \alpha \leqslant l$$

so that

$$d\psi_\alpha(a^{(s)}) = \sum_{j=1}^{n-1} \varphi_{j\alpha}^{(s)}(a^{(s)})\, dX_j^{(s)} .$$

Now the assumption of the simplicity of $a^{(s)}$ states that for $n-1$ distinct indices between 1 and l, $\alpha_1, \ldots, \alpha_{n-1}$, we do have

$$d\psi_{\alpha_1}(a^{(s)}) \wedge \ldots \wedge d\psi_{\alpha_{n-1}}(a^{(s)}) \neq 0 .$$

From the equations

$$\begin{cases} \psi_{\alpha_1}(z) = \sum_{j=1}^{n-1} X_j^{(s)}(z) \varphi_{j\alpha_1}^{(s)}(z) \\ \cdots\cdots\cdots\cdots \\ \psi_{\alpha_{n-1}}(z) = \sum_{j=1}^{n-1} X_j^{(s)}(z) \varphi_{j\alpha_{n-1}}^{(s)}(z) , \end{cases}$$

setting

$$\mathfrak{D}_s(z) = \det \left(\varphi_{j\alpha_k}^{(s)}(z)\right)_{\substack{1 \leqslant j \leqslant n-1 \\ 1 \leqslant k \leqslant n-1}}$$

we deduce that:

i) $X_j^{(s)}(z)\, \mathfrak{D}_s(z) \equiv 0 \bmod \mathfrak{b}$;

ii) $\mathfrak{D}_s(a^{(s)}) \neq 0$ and $\mathfrak{D}_s(a^{(r)}) = 0$ for $r \neq s$, so that for $r \neq s$ $\mathfrak{D}_s(z) = \sum X_j^r(z)\, g_j(z)$ with $g_j(z)$ homogeneous polynomials.

Therefore in the graded ring $C_0[z_1, \ldots, z_n]$ of homogeneous polynomials the ideal

$$C_0[z_1, \ldots, z_n]\big(\psi_1(z), \ldots, \psi_l(z), \mathfrak{D}_1(z), \ldots, \mathfrak{D}_\mu(z)\big)$$

has no non trivial zero and thus there exists an integer $\varrho \geqslant 1$ such that for any multiindex $\alpha \in \mathbf{N}^n$ with $|\alpha| \geqslant \varrho$ we have

iii) $$z^\alpha \equiv \sum B_{\alpha s}(z)\, \mathfrak{D}_s(z) \bmod \mathfrak{b}$$

where $B_{\alpha s}(z)$ are homogeneous polynomials.

d) For every s, $1 \leqslant s \leqslant \mu$, the homogeneous ideal generated by

$$X_1^{(s)}, \ldots, X_{n-1}^{(s)}, \mathfrak{D}_s \text{ is the trivial ideal}.$$

Therefore if $\alpha = X_1^{(s)} dt_1 + \ldots + X_{n-1}^{(s)} dt_{n-1} + \mathfrak{D}_s dt_n$ and if $\mathcal{A}^r$ denotes the space of exterior forms of degree r in $dt_1, \ldots, dt_n$ and with coefficients polynomials in $z_1, \ldots, z_n$ (homogeneous if we want so), the sequence (of graded free modules)

$$0 \longrightarrow \mathcal{A}^0 \xrightarrow{\Lambda\alpha} \mathcal{A}^1 \xrightarrow{\Lambda\alpha} \mathcal{A}^2 \xrightarrow{\Lambda\alpha} \ldots$$

is an exact sequence ([1]).

We derive therefore the following

STATEMENT. *Let Ω be an open convex subset of $\mathbf{C}^n$. Consider the system of equations*

$$\begin{cases} \mathfrak{D}_s\left(\dfrac{\partial}{\partial z}\right) u = f_n \\ X_j^{(s)}\left(\dfrac{\partial}{\partial z}\right) u = f_j \qquad 1 \leqslant j \leqslant n-1 \end{cases}$$

with $f_j \in \Gamma(\Omega, \mathcal{O})$ (i.e. holomorphic on Ω). The necessary and sufficient condition in order that the given system should be solvable with $u \in \Gamma(\Omega, \mathcal{O})$ (i.e. *holomorphic on Ω) is that*

$$\begin{cases} X_j^{(s)}\left(\dfrac{\partial}{\partial z}\right) f_k - X_k^{(s)}\left(\dfrac{\partial}{\partial z}\right) f_j = 0 \quad 1 \leqslant j < k \leqslant n-1 \\ X_j^{(s)}\left(\dfrac{\partial}{\partial z}\right) f_n - \mathfrak{D}_s\left(\dfrac{\partial}{\partial z}\right) f_j = 0 \qquad 1 \leqslant j \leqslant n-1 . \end{cases}$$

In particular the system

$$\begin{cases} \mathfrak{D}_s\left(\dfrac{\partial}{\partial z}\right) u = f \qquad f \in \Gamma(\Omega, \mathcal{O}) \\ X_j^{(s)}\left(\dfrac{\partial}{\partial z}\right) u = 0 \end{cases}$$

is solvable with $u \in \Gamma(\Omega, \mathcal{O})$ if $X_j^{(s)}(\partial/\partial z) f = 0$ for $1 \leqslant j \leqslant n-1$.

e) Let Ω be an open convex set in $\mathbf{C}^n$ and let $u \in \mathcal{H}(\Omega) = \Gamma(\Omega, \mathcal{O}_A)$. Then because of i) above we have

$$X_j^{(1)}\left(\frac{\partial}{\partial z}\right)\mathcal{D}_1\left(\frac{\partial}{\partial z}\right)u = 0 .$$

Therefore we can find v_1 holomorphic on Ω and such that

$$\left\{\begin{array}{ll} \mathcal{D}_1\left(\dfrac{\partial}{\partial z}\right)v_1 = \mathcal{D}_1\left(\dfrac{\partial}{\partial z}\right)u & \\ X_j^{(1)}\left(\dfrac{\partial}{\partial z}\right)v_1 = 0 & 1 \leqslant j \leqslant n-1 . \end{array}\right.$$

Now remark that by the linear change of coordinates

$$\left\{\begin{array}{l} z_1 \quad\; = a_n^{(1)}\xi_1 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ z_{n-1} = a_n^{(1)}\xi_{n-1} \\ z_n \quad\; = -\,(a_1^{(1)}\xi_1 + \ldots + a_n^{(1)}\xi_n) \end{array}\right.$$

the last set of equations transforms into

$$\frac{\partial}{\partial \xi_1}v_1 = 0, \ldots, \frac{\partial}{\partial \xi_{n-1}}v_1 = 0 .$$

We deduce therefore that $v_1 \in \Gamma(\Omega, \mathcal{O}_1)$.

We can thus write

$$u = (u - v_1) + v_1 = u_1 + v_1$$

with $v_1 \in \Gamma(\Omega, \mathcal{O}_1)$ and $\mathcal{D}_1(\partial/\partial z)u_1 = 0$.

Moreover

$$\mathcal{D}_2\left(\frac{\partial}{\partial z}\right)u_1 = \mathcal{D}_2\left(\frac{\partial}{\partial z}\right)u$$

because $\mathcal{D}_2(\partial/\partial z)v_1 = 0$ (property ii)). Then we can find v_2 holomorphic on Ω and such that

$$\left\{\begin{array}{ll} \mathcal{D}_2\left(\dfrac{\partial}{\partial z}\right)v_2 = \mathcal{D}_2 u_1 & \\ X_j^{(2)}\left(\dfrac{\partial}{\partial z}\right)v_2 = 0 & 1 \leqslant j \leqslant n-1 . \end{array}\right.$$

We have $v_2 \in \Gamma(\Omega, \mathcal{O}_2)$ and $\mathcal{D}_2(\partial/\partial z)(u_1 - v_2) = 0$ so that

$$u = (u - v_1 - v_2) + v_1 + v_2 = u_2 + v_1 + v_2$$

with $v_1 \in \Gamma(\Omega, \mathcal{O}_1)$, $v_2 \in \Gamma(\Omega, \mathcal{O}_2)$, and

$$\mathcal{D}_1\left(\frac{\partial}{\partial z}\right)u_2 = 0 = \mathcal{D}_2\left(\frac{\partial}{\partial z}\right)u_2 .$$

Moreover

$$\mathcal{D}_3\left(\frac{\partial}{\partial z}\right)u_2 = \mathcal{D}_3\left(\frac{\partial}{\partial z}\right)u$$

and therefore we can find v_3 holomorphic in Ω and such that

$$\begin{cases} \mathcal{D}_3\left(\dfrac{\partial}{\partial z}\right)v_3 = \mathcal{D}_3\left(\dfrac{\partial}{\partial z}\right)u_2 & \\ X_j^{(3)}\left(\dfrac{\partial}{\partial z}\right)v_3 = 0 & 1 \leqslant j \leqslant n-1 \end{cases}$$

so that, setting $u = (u - v_1 - v_2 - v_3) + v_1 + v_2 + v_3 = u_3 + v_1 + v_2 + v_3$ we get

$$\begin{cases} v_j \in \Gamma(\Omega, \mathcal{O}_j) & 1 \leqslant j \leqslant 3 \\ \mathcal{D}_j u_j = 0 & 1 \leqslant j \leqslant 3 . \end{cases}$$

Proceeding in this way we see that

every $u \in \mathcal{H}(\Omega)$ can be written as a sum

$$u = w + v_1 + v_2 + \ldots + v_\mu$$

with $v_j \in \Gamma(\Omega, \mathcal{O}_j)$ and

$$\mathcal{D}_j\left(\frac{\partial}{\partial z}\right)w = 0 \qquad 1 \leqslant j \leqslant \mu .$$

Because of property iii) we deduce that w is a polynomial in $z_1, \ldots, z_n$ of degree $\leqslant \varrho - 1$.

We can summarize the contents of the previous argument by the following

PROPOSITION 14. *We assume $\mathfrak{b} = \mathfrak{a}$ of dimension one and that the zeros of $\mathfrak{b}$ are simple.*

There exists an integer $\varrho \geqslant 1$ *such that, if* $W \subset \mathbf{C}_{\varrho-1}[z_1, \ldots, z_n]$ *is the vector space of polynomials* w *of degree* $\leqslant \varrho - 1$ *verifying the equations*

$$(*) \qquad \psi_j\left(\frac{\partial}{\partial z}\right) w = 0 \qquad 1 \leqslant 1 \leqslant l .$$

for any open convex set $\Omega \subset \mathbf{C}^n$ *we have*

$$\mathcal{H}_A(\Omega) = \sum_{j=1}^{\mu} \Gamma(\Omega, \mathcal{O}_j) + W .$$

COROLLARY 1. *If* $\mathfrak{b} = \mathfrak{a}$ *is of dimension one with simple zeros then* Ker α *and* Coker α *are sheaves locally isomorphic to a finite sum* $\mathbf{C} \oplus \ldots \oplus \mathbf{C}$ *i.e. we have an exact sequence of sheaves*

$$0 \to \mathcal{S} \to \coprod \mathcal{O}_s \xrightarrow{\alpha} \mathcal{O}_A \to W \to 0$$

with $\mathcal{S} \simeq \mathbf{C} \oplus \ldots \oplus \mathbf{C}$, $W \simeq \mathbf{C} \oplus \ldots \oplus \mathbf{C}$ *(finite sums).*

COROLLARY 2. *If* Ω *is an open subset of* $\mathbf{C}^n$ *such that* $H^l(\Omega, \mathbf{C}) = 0 \ \forall l > 0$ *then*

$$H^j(\Omega, \mathcal{O}_A) = \bigoplus_{s=1}^{\mu} H^j(\Omega, \mathcal{O}_s), (j > 0) .$$

The following is a sometimes useful remark

PROPOSITION 15. *Let* $\mathfrak{b} = \mathfrak{a}$ *be of dimension one with simple zeros. If* $\mathfrak{a} = \sqrt{\mathfrak{a}}$ *then we have for any open convex set* $\Omega \subset \mathbf{C}^n$

$$\mathcal{H}(\Omega) = \sum_{s=1}^{\mu} \Gamma(\Omega, \mathcal{O}_s) .$$

PROOF. With the notation of proposition 14. Let $\gamma \colon W \to \mathbf{C}$ be a linear function on W. We can write, δ_0 denoting the Dirac distribution at 0,

$$\gamma = \sum_{|\beta| \leqslant \varrho - 1} (-1)^{|\beta|} c_\beta D^\beta \delta_0$$

as W is a vector space of polynomials of degree $\leqslant \varrho - 1$. Let $\mathbf{C}_{\varrho-1}[t]$ be the space of polynomials of degree $\leqslant \varrho - 1$ in the variable t and let $p_s \in \mathbf{C}_{\varrho-1}[t]$, for $1 \leqslant s \leqslant \mu$. We will show that the vector space L described by

$$\sum_{s=1}^{\mu} p_s\left(\sum_{j=1}^{n} a_j^{(s)} z_j\right)$$

coincides with W. This will prove our contention. Now, if $L \subsetneq W$, we can find γ as above with $\gamma|L = 0$ but $\gamma \neq 0$ on W. From

$$\gamma\big(p(\sum a_j^{(s)} z_j)\big) = \sum_{k=0}^{\varrho-1} p^{(k)}(0) \Big(\sum_{|\beta|=k} c_\beta (a^{(s)})^\beta\Big) = 0$$

for any $p \in \mathbf{C}_{\varrho-1}[t]$, we derive that

$$\sum_{|\beta|=k} c_\beta (a^{(s)})^\beta = 0 \quad \text{for } 0 \leqslant k \leqslant \varrho - 1\,.$$

It follows that $\sum_{|\beta| \leqslant \varrho - 1} c_\beta z^\beta$ vanishes on all zeros of $\mathfrak{a} = \sqrt{\mathfrak{a}}$ and therefore is contained in $\mathfrak{a}$. We have thus

$$\sum_{|\beta|=k} c_\beta z^\beta = \sum_{j=1}^{l} A_j^{(k)}(z)\, \psi_j(z)$$

for $0 \leqslant k \leqslant \varrho - 1$.

But this shows that if $w \in W$ then $\gamma(w) = 0$ as $W \subset \mathcal{K}(\Omega)$, against the assumption that $\gamma \neq 0$ on W.

COROLLARY 3. *Let $\mathfrak{b} = \mathfrak{a}$ be of dimension one with simple zeros. Let Ω be an open convex subset of $\mathbf{C}^n$. The envelope of regularity of Ω is the set*

$$\tilde{\Omega} = \bigcap_{s=1}^{\mu} \pi_s^{-1} \pi_s(\Omega)$$

and every convex set $\tilde{\Omega}$ of this sort is a domain of regularity.

PROOF. For every Ω convex we have

$$\mathcal{K}(\Omega) = \sum_{s=1}^{\mu} \Gamma(\Omega, \mathcal{O}_s) + W$$

where W is a space of polynomials. It follows that every $w \in \mathcal{K}(\Omega)$ has an holomorphic extension to $\tilde{\Omega}$.

If $z_0 \in \partial\tilde{\Omega}$ then, for some s, $\pi_s(z_s) \in \partial\pi_s(\Omega)$ and we can construct an element $u \in \Gamma(\Omega, \mathcal{O}_s) \subset \mathcal{K}(\Omega)$ such that $\lim_{z \to z_0} |u(z)| = \infty$ (taking for instance $u = 1\big/\sum a_i^{(s)}(z_i - z_{0_i})\big)$. This shows that $\tilde{\Omega}$ is indeed the envelope of regularity of Ω and also proves the last part of the statement above.

13. – We keep the notations of the previous section. Let Ω be an open set in $\mathbf{C}^n$ and let $\pi_s^\Omega = \pi_s|_\Omega$.

PROPOSITION 16. *We assume that* $(\Omega, \pi_s^\Omega, \pi_s(\Omega))$ *is a differentiable fiber space with typical fiber* F *connected and such that*

$$H^l(F, \mathbf{C}) = 0 \qquad \text{for } 1 \leq l \leq j\,;$$

then

$$H^k(\Omega, \mathcal{O}_s) = 0 \qquad \text{for } 1 \leq k \leq j\,.$$

PROOF. We may assume that $\pi_s\colon \mathbf{C}^n \to \mathbf{C}$ is the map

$$(z_1, \ldots, z_n) \to z_n\,.$$

We set then $z = z_n = x_1 + ix_2$ and $y = (y_1, \ldots, y_{2n-2}) = (\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_{n-1}, \operatorname{Im} z_{n-1})$ and define

$C^{0,r}$ = the sheaf of germs of differential forms of degree r in the dy's and with C^∞ coefficients with respect to the variables x and y.

$C^{1,r}$ = the sheaf of germs of differential forms of degree r in the dy's and of degree 1 in $d\bar{z}$ with C^∞ coefficients with respect to x and y.

Thus $C^{1,r} = d\bar{z} \wedge C^{0,r}$. We have a soft resolution of $\mathcal{O}_s$

$$\begin{array}{ccccccccccc}
 & & & & C^{0,1} & \xrightarrow{d_y} & C^{0,2} & \xrightarrow{d_y} & C^{0,3} & \xrightarrow{d_y} & \\
 & & & {\scriptstyle d_y}\nearrow & & \searrow & & \searrow & & \searrow & \\
0 \to \mathcal{O}_s & \to & C^{0,0} & & \oplus & -\partial_{\bar{z}} & \oplus & \partial_{\bar{z}} & \oplus & -\partial_{\bar{z}} & \cdots \\
 & & & {\scriptstyle \partial_{\bar{z}}}\searrow & & \searrow & & \searrow & & \searrow & \\
 & & & & C^{1,0} & \xrightarrow[d_y]{} & C^{1,1} & \xrightarrow[d_y]{} & C^{1,2} & \xrightarrow[d_y]{} &
\end{array}$$

where d_y denotes exterior differentiation with respect to the variables y and where $\partial_{\bar{z}}$ denotes exterior differentiation with respect to the variable $\bar{z}$.

Let first $k \geq 2$, so that $j \geq 2$ and let

$$\beta^{0,k} \oplus \alpha^{1,k-1} \in \Gamma(\Omega, C^{0,k}) \oplus \Gamma(\Omega, C^{1,k-1})$$

be such that

$$d_y \beta^{0,k} = 0 \qquad \partial_{\bar{z}} \beta^{0,k} = d_y \alpha^{1,k-1}\,.$$

We have to show that there exist

$$u^{0,k-1} \oplus v^{1,k-2} \in \Gamma(\Omega, C^{0,k-1}) \oplus \Gamma(\Omega, C^{1,k-2})$$

such that

$$\beta^{0,k} = d_y u^{0,k-1} \qquad \alpha^{1,k-1} = d_y v^{1,k-2} - \partial_{\bar{z}} u^{0,k-1}\,.$$

Now because of the hypothesis we can find $u^{0,k-1} \in \Gamma(\Omega, C^{0,k-1})$ such that

$$\beta^{0,k} = d_y u^{0,k-1} .$$

Therefore

$$- d_y \partial_{\bar{z}} u^{0,k-1} = d_y \alpha^{1,k-1} .$$

From this we deduce that $d_y(\alpha^{1,k-1} + \partial_{\bar{z}} u^{0,k-1}) = 0$ and therefore, because of the assumption again, we can find $v^{1,k-2} \in \Gamma(\Omega, C^{1,k-2})$ such that

$$\alpha^{1,k-1} + \partial_{\bar{z}} u^{0,k-1} = d_y v^{1,k-2} .$$

This proves that if $j \geqslant 2$, $k \geqslant 2$, $j \geqslant k$, we have

$$H^k(\Omega, \mathcal{O}_s) = 0 .$$

Let now $k = 1$. Let $\beta^{0,1} \in \Gamma(\Omega, C^{0,1})$ and $\alpha^{1,0} \in \Gamma(\Omega, C^{1,0})$ be such that

$$d_y \beta^{0,1} = 0 \quad \text{and} \quad \partial_{\bar{z}} \beta^{0,1} = d_y \alpha^{1,0} .$$

First, as above, we see that there exists $u^{0,0} \in \Gamma(\Omega, C^{0,0})$ such that

$$\beta^{0,1} = d_y u^{0,0} .$$

Therefore

$$d_y(\alpha^{1,0} + \partial_{\bar{z}} u^{0,0}) = 0$$

i.e. $c^{1,0} = \alpha^{1,0} + \partial_{\bar{z}} u^{0,0}$ is an element of $\Gamma(\Omega, C^{0,1})$ which is independent of y: $c^{1,0} = v(z, \bar{z})\, d\bar{z}$, with v C^∞ on $\pi_s(\Omega)$.

Now we can find $w(z, \bar{z})$ defined and C^∞ on $\pi_s(\Omega)$ such that

$$\frac{\partial w(z, \bar{z})}{\partial \bar{z}} = v(z, \bar{z}) .$$

Let $U = u^{0,0} - w$. Then we have

$$\beta^{0,1} = d_y U \qquad \alpha^{1,0} = - \partial_{\bar{z}} U .$$

This completes the proof.

COROLLARY. *Assume that the open set $\Omega \subset \mathbf{C}^n$ has the following properties*

i) *For any s, $1 \leqslant s \leqslant \mu$, $(\Omega, \pi_s|\Omega, \pi_s(\Omega))$ is a differentiable fiber space with connected fiber F_s and with $H^j(F_s, \mathbf{C}) = 0$, $\forall j > 0$.*

ii) $H^j(\Omega, \mathbb{C}) = 0, \ \forall j > 0$;

then

$$H^j(\Omega, \mathcal{O}_A) = 0 \quad \forall j > 0 .$$

§ 3. - The envelope of regularity for a general system, constant coefficients, one unknown function.

14. – Cauchy-Kowalewska systems.

(*a*) Consider in $\mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}$, where $(z_1, \ldots, z_n, t)$ are holomorphic coordinates, the closed polycylinder

$$\bar{P} = \{(z, t) \in \mathbb{C}^n \times \mathbb{C} \mid \|z\| \leqslant R, \ |t| \leqslant r\}$$

where $\|z\| = \sup_{1 \leqslant i \leqslant n} |z_i|$. We set $\bar{P}_0 = \{(z, t) \in \bar{P} | t = 0\} = \{z \in \mathbb{C}^n | \|z\| \leqslant R\}$.

A Cauchy-Kowalewska system of order m is a system of partial differential equations in the unknown functions $u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$ of the form

$$D_t^m u = \sum_{\substack{\beta + |\alpha| \leqslant m \\ \beta < m}} c_{\alpha\beta}(z, t) D_z^\alpha D_t^\beta u + f(z, t) \tag{1}$$

where $c_{\alpha\beta}$ and f are holomorphic (respectively $N \times N$ and $N \times 1$ matrices) in a neighborhood of $\bar{P}$, together with initial conditions

$$u(z, 0) = v_0(z) , \qquad \frac{\partial u}{\partial t}(z, 0) = v_1(z), \ldots, \frac{\partial^{m-1} u}{\partial t^{m-1}}(z, 0) = v_{m-1}(z) \tag*{$(1)_0$}$$

where the $v_j(z)$'s are holomorphic ($N \times 1$ matrices) in a neighborhood of $\bar{P}_0$.

(*b*) Given a Cauchy-Kowalewska system (1), $(1)_0$ one can construct a Cauchy-Kowalewska system (1*), $(1^*)_0$ of first order and with coefficients holomorphic in a neighborhood of $\bar{P}$, having the following property:

If in an open connected set $G \supset P_0$ the Cauchy problem for (1*), $(1^*)_0$ admits a (resp. unique) solution then on G the Cauchy problem for (1), $(1)_0$ admits a (resp. unique) solution.

This is done by a standard procedure and therefore the argument is omitted.

(c) Let us thus consider a first order Cauchy-Kowalewska system

$$\frac{\partial u}{\partial t} = \sum_1^n c_j(z, t) \frac{\partial u}{\partial z_j} + c(z, t) u + f(z, t) \tag{1}$$

with initial conditions

$$u(z, 0) = v(z) .$$

Replacing $u(z, t)$ by $u(z, t) - v(z)$, we reduce to study the Cauchy problem for equation (1) $\big($where f is replaced by $\sum c_j(z, t)(\partial v(z)/\partial z_j) + c(z, t) v(z) + f(z, t)\big)$ with vanishing initial conditions:

$$u(z, 0) = 0 . \tag{1_0}$$

(d) Solve recursively the Cauchy problems

$$\begin{cases} \dfrac{\partial w_{\nu+1}}{\partial t} = \sum c_j(z, t) \dfrac{\partial w_\nu}{\partial z_j} + c(z, t) w_\nu + f(z, t) \\ w_{\nu+1}(z, 0) = 0 \qquad \nu = 0, 1, 2, \ldots \end{cases}$$

starting with $w_0 = 0$. Setting $v_\nu = w_{\nu+1} - w_\nu$, then

$$w(z, t) = \sum_{\nu=0}^{\infty} v_\nu(z, t)$$

is a solution of (1), $(1)_0$ within the region G where the series is convergent.

We have

$$v_0(z, t) = \int_0^t f(z, s)\, ds$$

and recursively

$$v_{\nu+1}(z, t) = \int_0^t \left\{ \sum_{j=1}^n c_j(z, s) \frac{\partial v_\nu(z, s)}{\partial z_j} + c(z, s) v_\nu(z, s) \right\} ds .$$

We now make use of the following two lemmas that we borrow from Hörmander's book ([7] p. 117).

Lemma 1. *If $v(t)$ is holomorphic on $|t| < R$ and if*

$$|v(t)| \leqslant \frac{A}{(R - |t|)^h}$$

then

$$|v'(t)|\leqslant\frac{Ae(1+h)}{(R-|t|)^{h+1}}.$$

LEMMA 2. *If $v(t)$ is holomorphic in $|t|<r$ and if*

$$|v'(t)|\leqslant B|t|^a \qquad v(0)=0$$

then

$$|v(t)|\leqslant\frac{B|t|^{a+1}}{a+1} \qquad \text{for } |t|<r.$$

Set

$$\phi=\sup_{\bar P}|f(z,t)|$$

$$c=\sup_{\bar P}\{|c_1(z,t)|,\dots,|c_n(z,t)|,|c(z,t)|\}\ (^{10}).$$

Then we obtain, assuming $R<1$,

$$|v_0(z,t)|\leqslant|t|\phi$$

$$\left|\frac{\partial v_1(z,t)}{\partial t}\right|\leqslant nc\sup_{1\leqslant j\leqslant n}\left|\frac{\partial v_0(z,t)}{\partial z_j}\right|+c|v_0(z,t)|$$

$$\leqslant nc\frac{|t|\phi}{R-\|z\|}e\cdot 1+c|t|\phi$$

$$\leqslant(n+1)c\frac{|t|\phi}{R-\|z\|}e\cdot 1, \qquad (\text{as } R<1)$$

hence

$$|v_1(z,t)|\leqslant\frac{|t|^2}{2}\frac{(n+1)ce\phi}{R-\|z\|}.$$

Similarly

$$\left|\frac{\partial v_2(z,t)}{\partial t}\right|\leqslant nc\frac{|t|^2}{2}\frac{(n+1)ce\phi}{(R-\|z\|)^2}e\cdot 2+c\frac{|t|^2}{2}\frac{(n+1)ce\phi}{R-\|z\|}\leqslant\frac{(n+1)^2c^2|t|^2e^2\Phi}{(R-\|z\|)^2}$$

Thus

$$|v_2(z,t)|\leqslant\frac{|t|^3}{3}\frac{(n+1)^2c^2e^2\Phi}{(R-\|z\|)^2}.$$

([10]) We denote by $|\cdot|$ any given norm on $\mathbf{C}^N$ or $\mathbf{C}^{N^2}$.

Proceeding in this way we get in general

$$|v_\nu(z,t)| \leqslant \frac{|t|^{\nu+1}}{\nu+1}\left(\frac{(n+1)ce}{R-\|z\|}\right)^\nu \phi$$

therefore we derive that the series defining $w(z,t)$ converges in the region

$$|t| < \frac{R-\|z\|}{(n+1)ce}, \quad \|z\| < R, \quad |t| < r.$$

Uniqueness of solution within the same region is proved by the usual argument.

We conclude this review of classical results by the following

STATEMENT. *Given a Cauchy-Kowalewska system* (1), $(1)_0$ *in* $\overline{P}$ (assuming $R<1$), *the corresponding Cauchy problem admits a unique holomorphic solution in a part of* $\overline{P}$ *defined by an inequality of the form*

$$|t| \leqslant c(R - \|z\|)$$

where $c>0$ *is a constant which depend only on* $\sup_P |c_{\alpha\beta}(z,t)|$. (*In particular* c *is independent of the Cauchy data and* f).

15. – Characteristic boundary points.

(*a*) Let us consider a general $\bar\partial$-suspended system in one unknown function u

$$(*) \qquad \begin{cases} \psi_1\left(\dfrac{\partial}{\partial z}\right)u = 0\,, \\ \cdots\cdots\cdots \\ \psi_l\left(\dfrac{\partial}{\partial z}\right)u = 0\,, \\ \bar\partial u = 0\,. \end{cases}$$

Let $\mathfrak{b} = \mathbf{C}[\xi_1, \ldots, \xi_n]\big(\psi_1(\xi), \ldots, \psi_l(\xi)\big)$ be the corresponding polynomial ideal, $\mathfrak{a}$ its asymptotic ideal, $V(\mathfrak{a})$ the variety of zeros in $\mathbf{C}^n$ of the asymptotic ideal and $W(\mathfrak{a})$ its part at ∞ on $\mathbf{P}_{n-1}(\mathbf{C})$.

Let $\Omega \subset \mathbf{C}^n$ be an open set. We say that $a \in \partial\Omega = \overline{\Omega} - \Omega$ is a *C^k-boundary point* ($k \geqslant 1$) if we can find an open neighborhood $U(a)$ of a and a C^k function $\phi\colon U(a) \to R$ such that

$$\Omega \cap U(a) = \{z \in U(a) \,|\, \phi(z) < \phi(a)\} \quad d\phi(a) \neq 0\,.$$

We will call ϕ a defining function for the boundary of Ω near a. Let ψ be any other C^k defining function for the boundary of Ω near the same point a. Then in a sufficiently small open neighborhood $V(a)$ of a we will have, assuming for simplicity that $\psi(a) = 0 = \phi(a)$:

$$\psi = h\phi$$

where h is at least C^{k-1} in $V(a)$ and $h(a) > 0$. In particular if a is a C^1 boundary point the *complex gradient*

$$\operatorname{grad}_z \phi(a) = \left(\frac{\partial \phi}{\partial z_1}(a), \dots, \frac{\partial \phi}{\partial z_n}(a)\right)$$

is well defined up to multiplication by a positive constant.

Let $\Sigma \subset \partial\Omega$ be the part of the boundary of Ω which is of class C^1 at least. For every point $a \in \Sigma$, if ϕ is a defining function for $\partial\Omega$ near a we can consider $\operatorname{grad}_z \phi(a)$ as a point in the projective space $\mathbf{P}_{n-1}(\mathbf{C})$. We thus define a map

$$\tau\colon \Sigma \to \mathbf{P}_{n-1}(\mathbf{C})$$

which is independent of the choice of the defining local function ϕ.

We define the *characteristic set* of $\partial\Omega$ as the set $\tau^{-1}(W(\mathfrak{a}))$: it is the set of points $a \in \Sigma$ where $\operatorname{grad}_z \phi(a) \in V(\mathfrak{a})$. In other words a point $a \in \Sigma$ is not characteristic if and only if there exists a homogeneous polynomial $g \in \mathfrak{a}$ with $g(\operatorname{grad}_z \phi(a)) \neq 0$.

Choosing coordinates $z_1, \dots, z_{n-1}, t$ in $\mathbf{C}^n$ so that the t axis is parallel to the complex line $\mathbf{C} \operatorname{grad}_z \phi(a)$, then $a \in \Sigma$ in non characteristic if and only if there exists a polynomial $p \in \mathfrak{b}$ of degree $m > 0$ such that, in those coordinates

$$p(z, t) = t^m + c_1(z)t^{m-1} + c_2(z)t^{m-2} + \dots + c_n(z)$$

where, for $1 \leqslant j \leqslant m$, $c_j(z)$ is a polynomial in $z = (z_1, \dots, z_{n-1})$ of degree $\leqslant j$. In other words in the differential ideal corresponding to $\mathfrak{b}$ there is a Cauchy-Kowalewska equation (with constant coefficients) of the form

$$\frac{\partial^m u}{\partial t^m} = \sum_{\substack{|\alpha|+\beta \leqslant m \\ \beta < m}} c_{\alpha\beta} \frac{\partial^{|\alpha|+\beta} u}{\partial z^\alpha \, \partial t^\beta}.$$

16. – Extension lemma.

(*a*) As usual, if $\mathcal{O}_A$ is the sheaf of germs of solutions of the homogeneous system (∗), for Ω open in $\mathbb{C}^n$ we set

$$\mathcal{H}(\Omega) = \Gamma(\Omega, \mathcal{O}_A)\,.$$

EXTENSION LEMMA. *Let Ω be open and let $z_0 \in \partial\Omega$ be a C^2 boundary point. If z_0 is not characteristic, there exists an open neighborhood $V(z_0)$ of z_0 in $\mathbb{C}^n$ such that the restriction map*

$$\mathcal{H}(\Omega \cup V(z_0)) \to \mathcal{H}(\Omega)$$

is an isomorphism.

PROOF. α) We may assume that z_0 is at the origin of the coordinates; that $z_n = 0$ is the holomorphic tangent plane to $\partial\Omega$ at z_0; that in a sufficiently small neighborhood $U = U(\sigma, \eta)$ of the origin

$$U(\sigma, \eta) \equiv \left\{ \sum_1^{2n-1} |x_j|^2 < \eta,\ |x_{2n}| < \sigma \right\}$$

$\Omega \cap U$ is given by:

$$\Omega \cap U \equiv \left\{ x_{2n} < \sum_1^{2n-1} a_{ij} x_i x_j + o(\|x\|^2) \right\}$$

(we have set $z_j = x_{2j-1} + i x_{2j}$).

If η, σ are sufficiently small we can find $c > 0$ such that

i) $- c \sum_1^{2n-1} x_j^2 < \sum_1^{2n-1} a_{ij} x_i x_j$ on U

ii) $\Omega_1 = \left\{ x \in U(\sigma, \eta) \,|\, x_{2n} < - c \sum_1^{2n-1} x_j^2 \right\} \subset \Omega \cap U(\sigma, \eta)\,.$

It will be enough to prove the theorem with Ω_1 replacing Ω.

β) Consider for $\varepsilon > 0$, the analytic discs D_ε, defined by

$$D_\varepsilon = \{z_n = - i\varepsilon\} \cap \Omega_1\,.$$

These are given by $- \varepsilon < - c \sum_{j=1}^{2n-2} x_j^2$, i.e. by

$$\sum_{j=1}^{2n-2} x_j^2 < \frac{\varepsilon}{c}\,.$$

Therefore if $R(\varepsilon) = \dfrac{1}{\sqrt{2n-2}}\sqrt{\dfrac{\varepsilon}{c}}$ the polycylinder

$$\sup_{1\leq j\leq n-1} |z_j| < R(\varepsilon)$$

is contained in D_ε.

γ) Set $t = z_n + i\varepsilon$ (so that $t = 0$ corresponds to the complex hyperplane $\{z_n = -i\varepsilon\}$).

We can find in the differential ideal $\mathfrak{b}$ a Cauchy-Kowalewska equation

$$\frac{\partial^m u}{\partial t^m} = \sum_{\substack{|\alpha|+\beta\leq m\\ \beta<m}} c_{\alpha\beta}\frac{\partial^{|\alpha|+\beta}u}{\partial z^\alpha\,\partial t^\beta} \tag{1}$$

with constant coefficients $c_{\alpha\beta}$ independent of $\varepsilon < \sigma$.

For any $U \in \mathcal{H}(\Omega_1)$ we add to (1) the initial conditions

$$\left.\frac{\partial^s u}{\partial t^s}\right|_{t=0} = \left.\frac{\partial^s U}{\partial t^s}\right|_{t=0} \qquad (s = 0, 1, \dots, m-1) \tag*{$(1)_0$}$$

then the unique solution of (1), $(1)_0$ is defined and holomorphic in the region

$$|t| < c_1(R(\varepsilon) - \|\xi\|)\,, \qquad \text{where } \xi = (z_1, \dots, z_{n-1})\,,$$

the constant $c_1 > 0$ being independent of ε.

In particular the solution is holomorphic at $t = i\varepsilon$, $z_1 = \dots = z_{n-1} = 0$, i.e. at $z = 0$, if

$$\varepsilon < \frac{1}{2}c_1 R(\varepsilon) = \frac{1}{2}c_1\frac{1}{\sqrt{2n-2}}\sqrt{\frac{\varepsilon}{c}} = c_2\sqrt{\varepsilon}$$

i.e. for

$$\varepsilon < c_2^2\,.$$

But for these values of $\varepsilon > 0$ the region $|t| < c_1(R(\varepsilon) - \|\xi\|)$ covers a full neighborhood $V(z_0)$ of the origin z_0. This region is independent of U and therefore we have the isomorphism:

$$\mathcal{H}(\Omega_1 \cup V(z_0)) \xrightarrow{\sim} \mathcal{H}(\Omega_1)\,.$$

This completes the proof.

COROLLARY 1. *If Ω is a domain of regularity for* $(*)$, *at any point* $z_0 \in \partial\Omega$

of class C^2 we must have

$$\forall g \in \mathfrak{a}\,, \qquad g(\operatorname{grad}_z \phi(z_0)) = 0$$

ϕ being any function defining $\partial\Omega$ near z_0.

COROLLARY 2. *A bounded domain Ω of regularity for $(*)$ cannot have a boundary $\partial\Omega$ which is everywhere C^2 unless $\mathfrak{b} = 0$ i.e. unless $(*)$ reduces to the system of Cauchy-Riemann equations only.*

Indeed if $\partial\Omega$ is C^2 everywhere, given $a \in \mathbb{C}^n - \{0\}$ there exists a point $z_0 \in \partial\Omega$ where $\operatorname{grad}_z \phi(z_0) \in C^* a$, ϕ being a defining function for $\partial\Omega$ near z_0. This shows by Corollary 1 that $\mathfrak{a} = 0$. Thus $\mathfrak{b} = 0$.

b) An inspection of the previous proof shows that the assumption for $\partial\Omega$ to be of class C^2 at z_0 can be considerably relaxed:

It is enough that

$\partial\Omega$ be C^1 at z_0 with a defining function ϕ for $\partial\Omega$ near z_0 having the following property:

in a sufficiently small neighborhood U of z_0 we can find finitely many C^2 functions $\phi_i\colon U \to \mathbb{R}$, $1 \leqslant i \leqslant k$, so that $\phi(z_0) = \phi_1(z_0) = \ldots = \phi_k(z_0) = 0$ and $\phi = \sup(\phi_1, \ldots, \phi_k)$ on U.

Indeed we can assume, setting $z_j = x_{2j-1} + ix_{2j}$, that the tangent hyperplane to Ω at z_0 has the equation $x_{2n} = 0$, z_0 being at the origin 0 of the coordinates. Setting in U, $\Omega_h = \{\phi_h < 0\}$, we may assume Ω_h defined by an inequality of the form

$$x_{2n} < \sum_{1}^{2n-1} a_{ij}^{(h)} x_i x_j + o(|x|^2)\,.$$

We can then find a constant $c > 0$ so that

$$-c\sum_{1}^{2n-1} x_j^2 < \inf_h \left\{\sum a_{ij}^{(h)} x_i x_j\right\}$$

in a sufficiently small neighborhood of the origin. The argument then proceeds as before.

The same weakening of the assumptions applies to the previous corollaries.

17. – Envelope of regularity of convex sets.

With the same notations as before let us consider for any $a = (a_1, \ldots, a_n) \in$

$\in V(\mathfrak{a}) - \{0\}$ the projection map

$$\pi_a : \mathbf{C}^n \to \mathbf{C}$$

given by

$$\pi_a(z_1, \ldots, z_n) = \sum_1^n a_i z_i \,.$$

The following theorem is a generalization of Corollary 3 to proposition 14 of section 12.

THEOREM 6. *Let Ω be an open convex set in $\mathbf{C}^n$. The set*

$$\tilde{\Omega} = \overline{\bigcap_{a \in V(\mathfrak{a}) - \{0\}} \pi_a \pi_a^{-1}(\Omega)}^{\,0}$$

is the envelope of regularity of Ω.

PROOF. (α) We first assume $\partial\Omega$ to be of class C^2 and prove that we have, via the restriction map, an isomorphism

$$\mathcal{H}(\tilde{\Omega}) \to \mathcal{H}(\Omega) \,.$$

As Ω is convex and every projection π_a is linear we have that $\tilde{\Omega}$ is also convex.

Let $z_0 \in \tilde{\Omega} - \Omega$. Join z_0 to a point $p \in \Omega$ by a real line and let 0 be a point on the half line pz_0 after z_0 but in $\tilde{\Omega}$.

The convex envelope of 0 and Ω, that we denote by $\Gamma(0, \Omega)$, is contained in $\tilde{\Omega}$ and contains z_0 in its interior.

We will take the origin of the coordinates at 0 and define

$$\Omega(\mu) = \bigcup_{\mu < \lambda \leqslant 1} \lambda \Omega$$

for any μ with $0 \leqslant \mu \leqslant 1$. If μ is sufficiently small, then $\Omega(\mu)$ contains z_0 in its interior. Moreover

(i) the closure Σ of the part of $\partial\Omega$ contained in the interior of $\Gamma(0, \Omega)$ is all of non characteristic points.

Indeed each one of those points has an entire spherical neighborhood contained in $\tilde{\Omega}$, while at a characteristic point this is not possible by the very definition of $\tilde{\Omega}$.

(ii) For every μ_0 with $0 < \mu_0 < 1$ there exists an $\varepsilon > 0$ such that for every μ with $|\mu - \mu_0| < \varepsilon$ we have

$$\mathcal{H}(\Omega(\mu - \varepsilon)) \underset{\rightarrow}{\sim} \mathcal{H}(\Omega(\mu)) .$$

Indeed $\partial\Omega(\mu)$ for every $0 < \mu < 1$ satisfies the weakened assumptions under which we can apply the extension lemma. This is of pure geometric nature. Therefore we get an uniform extension for all μ near μ_0 ([11]).

It follows that the set

$$M = \{\mu | 0 \leqslant \mu \leqslant 1, \mathcal{H}(\Omega(\mu)) \underset{\rightarrow}{\sim} \mathcal{H}(\Omega)\}$$

is open and closed and thus $M = \{0 \leqslant \mu \leqslant 1\}$ so that

$$\mathcal{H}(\Omega(0)) \underset{\rightarrow}{\sim} \mathcal{H}(\Omega) .$$

This shows that (again because $\tilde{\Omega}$ is convex) we have

$$\mathcal{H}(\tilde{\Omega}) \underset{\rightarrow}{\sim} \mathcal{H}(\Omega) .$$

(β) If we drop the assumption that $\partial\Omega$ is of class C^2 we obtain the same conclusion

$$\mathcal{H}(\tilde{\Omega}) \underset{\rightarrow}{\sim} \mathcal{H}(\Omega) .$$

This by the use of the following *approximation lemma*: *Let Ω be an open convex set in $\mathbb{R}^N$. One can construct a sequence of open convex subsets $\Omega_\nu \subset \Omega$ $\nu = 1, 2, \ldots$ with the following properties*

i) $\Omega_\nu \subset \Omega_{\nu+1}$, $\cup\, \Omega_\nu = \Omega$

ii) $\Omega_\nu = \{x \in R^n | \phi_\nu(x) < 1\}$ *where* $\phi_\nu \colon \mathbb{R}^N \to \mathbb{R}$ *is real analytic and has*

([11]) Indeed there exists a finite number of open « triangular regions» Δ_ν, $1 \leqslant \nu \leqslant k$, as specified in the extension lemma, such that the closure of the boundary of $\partial\Omega(\mu_0)$ in $\Gamma(z_0, \Omega)$, say $\partial'\Omega(\mu_0)$, is contained in $\bigcup_1^k \Delta_\nu$:

$$\partial'\Omega(\mu_0) \subset \bigcup_1^k \Delta_\nu .$$

Therefore, if $|\mu - \mu_0| < \varepsilon$ and $\varepsilon > 0$ is sufficiently small, we do have also

$$\Omega(\mu) - \Omega(\mu_0) \subset \bigcup_1^k \Delta_\nu .$$

the properties:

(a) $d\phi_\nu \neq 0$ on $\partial\Omega_\nu$,

(b) $\sum \frac{\partial^2 \phi_\nu}{\partial x_i \partial x_j}(a)\, u_i u_j > 0 \qquad \forall a \in \mathbb{R}^N,\ u \in \mathbb{R}^N - \{0\}$.

This construction is obtained with unessential modification from an argument given in [4] p. 36, 37.

(γ) It remains to show that $\tilde{\Omega}$ is a domain of regularity, i.e. that, given any point z_0 in $\partial\tilde{\Omega}$, we can find a function u in $\mathcal{H}(\tilde{\Omega})$ which cannot be extended over z_0. Now every point z_0 of $\partial\tilde{\Omega}$ belongs to $\partial\pi_a^{-1}(\pi_a(\Omega))$ for some $a \in V(\mathfrak{a}) - \{0\}$ and $\tilde{\Omega}$ is convex; as we can assume that $z_0 = 0$, it is enough to show that, for $a \in V(\mathfrak{a}) - \{0\}$, the half space $\{\mathrm{Re}\,\langle a, z\rangle < 0\}$ is a regularity domain.

To this aim we prove the following

Lemma. *Let $U = \{z \in \mathbb{C}^n | \mathrm{Re}\ \langle a, z\rangle < 0\}$ be a half space in $\mathbb{C}^n$ $(a \neq 0)$. Then either $a \in V(\mathfrak{a})$ and U is a domain of regularity, or its envelope of regularity $\tilde{U}$ is the whole of $\mathbb{C}^n$.*

Proof. Note first that the envelope of regularity $\tilde{U}$ of U either coincides with U or is the whole of $\mathbb{C}^n$. Indeed, if every function $u \in \mathcal{H}(U)$ extends over a point $z_0 \in \partial U$, then there is a ball B of positive radius ε centered at z_0 such that the functions in $\mathcal{H}(U)$ extend (in a unique way) to functions in $\mathcal{H}(U \cup B)$.

But our equations, having constant coefficients, are translation invariant, thus the functions in $\mathcal{H}(U)$ extend to functions in $\mathcal{H}(U_\varepsilon)$, where $U_\varepsilon = \{z \in \mathbb{C}^n | \mathrm{Re}\,\langle a, z\rangle < \varepsilon |a|\}$ and therefore, iterating the argument, to functions in $\mathcal{H}(\mathbb{C}^n)$.

Thus we need only to prove that, if $\tilde{U} = \mathbb{C}^n$, $a \notin V(\mathfrak{a}) - \{0\}$. Indeed, if $\tilde{U} = \mathbb{C}^n$, the restriction map $\mathcal{H}(\mathbb{C}^n) \to \mathcal{H}(U)$ is a topological isomorphism. In particular, we can find a compact subset K of U and a constant $c > 0$ such that

$$\sup_{|z| \leqslant 1} |u(z)| \leqslant c \sup_{z \in K} |u(z)| \qquad \text{for } u \in \mathcal{H}(\mathbb{C}^n).$$

Notice that

$$\sup_{z \in K} \mathrm{Re}\,\langle a, z\rangle = -\varepsilon < 0 .$$

Assume that $a \in V(\mathfrak{a}) - \{0\}$. Then we can find a sequence $\{\tau_n\}$ of positive real numbers and a sequence $\{\eta_n\}$ in $\mathbb{C}^n$ such that:

(i) $\tau_n(a + \eta_n) \in V(\mathfrak{b})$, $\forall n$;

(ii) $\tau_n \to +\infty$ as $n \to \infty$;

(iii) $\eta_n \to 0$ as $n \to \infty$.

For every n the function $u_n(z) = \exp\left(\tau_n \langle a + \eta_n, z\rangle\right)$ belongs to $\mathcal{H}(\mathbf{C}^n)$. Therefore, evaluating u_n for $z = \bar{a}/|a|$, we have from the above estimates:

$$\exp\left(\tau_n|a| + \tau_n \operatorname{Re}\left\langle\eta_n, \frac{\bar{a}}{|a|}\right\rangle\right) \leqslant c \exp(-\varepsilon\tau_n) \sup_{z\in K} \exp\left(\tau_n \operatorname{Re}\langle\eta_n, z\rangle\right).$$

This implies the estimate

$$\exp\left(\frac{1}{2}\tau_n|a|\right) \leqslant c \exp\left(-\frac{\varepsilon\tau_n}{2}\right).$$

By (ii) this gives a contradiction.

Example. Let $\Omega = \{x_1^2 + x_2^2 < 2\}$ be the disc of center 0 and radius $\sqrt{2}$ in $\mathbf{R}^2$. On $\mathbf{R}^2$ we consider the Laplace equation:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0.$$

Every harmonic function on Ω extends to a holomorphic function on the region $\tilde{\Omega} \subset \mathbf{C}^2$ given by

$$\tilde{\Omega} = \left\{x_1^2 + x_2^2 + y_1^2 + y_2^2 < \inf\{2(1 - x_1 y_2 + y_1 x_2), \right.$$
$$\left. 2(1 + x_1 y_2 - y_1 x_2)\}\right\} \quad \text{(Lie's lens)}$$

(here $x_1 + iy_1$, $x_2 + iy_2$ are holomorphic coordinates in $\mathbf{C}^2$).

18. – The distance from the boundary on a regularity domain.

We set $|z| = \left(\sum |z_i|^2\right)^{\frac{1}{2}}$ for the euclidean norm in $\mathbf{C}^n$.

Let $B = \{z \in \mathbf{C}^n | |z| < 1\}$ be the unit ball in $\mathbf{C}^n$.

We consider its envelope of regularity $\tilde{B}$ with respect to the system $Au = 0$ (A being the system $(*)$ of differential equations associated to a $\bar{\partial}$-suspended differential ideal).

We have

$$\tilde{B} = \overline{\bigcap_{a\in V(\mathfrak{a})-\{0\}} \pi_a^{-1}(\pi_a(B))}^{\,0} = \operatorname{Interior}\{z \in \mathbf{C}^n | |\textstyle\sum a_i z_i| < |a| \ \forall a \in V(\mathfrak{a}) - \{0\}\}.$$

The following criterion tells when $\tilde{B}$ is relatively compact:

PROPOSITION 17. *The necessary and sufficient condition for $\tilde{B}$ to be relatively compact is that $V(\mathfrak{a})$ generates $\mathbf{C}^n$.*

PROOF. The sufficiency follows from the characterization of $\tilde{B}$ given above. To prove the necessity, assume that $V(\mathfrak{a})$ is contained in a hyperplane $\left\{\sum_1 a_i z_i\right\} = 0$. By a holomorphic linear change of coordinates we can assume it is the hyperplane $\{z_n = 0\}$. Then $\tilde{B}$ must contain the cylinder $\{|z_1|^2 + \ldots + |z_{n-1}|^2 < 1\}$ and therefore is not bounded.

REMARK. The condition that $V(\mathfrak{a})$ generates $\mathbf{C}^n$ is therefore sufficient for $\mathbf{C}^n$ to be A-convex.

We give here another sufficient criterion for $\mathbf{C}^n$ to be A-convex, involving the characteristic variety $V(\mathfrak{b})$.

We denote by Λ the closed convex cone with vertex at 0 generated by $V(\mathfrak{b})$: Λ is the closure in $\mathbf{C}^n$ of the convex cone

$$C = \{\lambda_1 a^{(1)} + \ldots + \lambda_k a^{(k)} | a^{(i)} \in V(\mathfrak{b}), \lambda_i \in \mathbf{R}, \lambda_i \geq 0\} .$$

The criterion reads as follows:

If $\Lambda = \mathbf{C}^n$, then $\mathbf{C}^n$ is A-convex.

Note that this condition is fulfilled if $V(\mathfrak{a})$ generates $\mathbf{C}^n$.

PROOF. First notice that, if $\Lambda = \mathbf{C}^n$, then $\mathbf{C}^n$ is the convex cone with vertex at 0 generated by finitely many points $a^{(1)}, \ldots, a^{(N)}$ of $V(\mathfrak{b})$.

Indeed $V(\mathfrak{b})$ must contain a basis $a^{(1)}, \ldots, a^{(2n)}$ of $\mathbf{C}^n$ considered as a real vector space. Then

$$a = a^{(1)} + \ldots + a^{(2n)}$$

is an interior point of C.

Since $\Lambda = \mathbf{C}^n$, the point $-a$ belongs to the closure of C.

Therefore, since the interior points of the line joining an interior point of a convex set with one of its boundary points are interior to the convex set, 0 is an interior point of C, and thus $C = \mathbf{C}^n$.

Hence we have

$$-a = \lambda_1 b^{(1)} + \ldots + \lambda_k b^{(k)}$$

with $b^{(1)}, \ldots, b^{(k)} \in V(\mathfrak{b})$ and $\lambda_1, \ldots, \lambda_k \geq 0$.

Therefore C is the convex cone with vertex at 0 generated by $a^{(1)}, \ldots, a^{(2n)}, b^{(1)}, \ldots, b^{(k)}$.

Let K be a compact set in $\mathbf{C}^n$, and $c \geqslant 1$.

If K is contained in the ball $\{|z| \leqslant R\}$, then

$$\hat{K}_{\mathbf{C}^n}(c) \subset \{z \in \mathbf{C}^n | \operatorname{Re} \sum a_i z_i \leqslant R|a| + \ln c \ \ \forall a \in V(\mathfrak{b})\} .$$

Let $a \in \mathbf{C}^n$. We have

$$a = \lambda_1 a^{(1)} + \ldots + \lambda_N a^{(N)}$$

with $\lambda_1, \ldots, \lambda_N \geqslant 0$. If $z \in \hat{K}_{\mathbf{C}^n}(c)$ we have then

$$\operatorname{Re} \langle a, z \rangle \leqslant \sum \lambda_i R |a^{(i)}| + N \ln c .$$

Therefore $\hat{K}_{\mathbf{C}^n}(c)$ is contained in a cube with center at 0 and sufficiently large side. The proof is complete.

In the case $\mathfrak{b} = \sqrt{\mathfrak{b}}$, we have the following necessary condition:

If $\mathbf{C}^n$ *is A-convex and* $\mathfrak{b} = \sqrt{\mathfrak{b}}$, *then* $V(\mathfrak{b})$ *generates* $\mathbf{C}^n$.

Indeed, assume $V(\mathfrak{b})$ is contained in a complex hyperplane. We may as well assume it is the hyperplane $\{z_n = 0\}$. Then $\partial/\partial z_n$ belongs to the differential ideal of $\mathfrak{b}$ and therefore the functions in $\mathcal{K}(\mathbf{C}^n)$ are independent of z_n.

This condition, however, is not sufficient, as the example of the equation $\partial u/\partial z - u = 0$ (case $n = 1$) shows.

The following is a partial converse of the sufficient criterion given above:

Assume that $\mathfrak{a} = \sqrt{\mathfrak{a}}$. *Then, if* $\mathbf{C}^n$ *is A-convex, it follows that* $\Lambda = \mathbf{C}^n$.

Proof. If $\Lambda \neq C^n$, then Λ is contained in a closed half-space of $\mathbf{C}^n$. By a holomorphic linear change of coordinates we can assume that

$$\Lambda \subset \{z \in \mathbf{C}^n | \operatorname{Re} z_n \geqslant 0\} .$$

Then $V(\mathfrak{a}) \subset \{z \in \mathbf{C}^n | z_n = 0\}$ and therefore the differential ideal of $\mathfrak{b}$ contains a differential polynomial of the form

$$p(D) = \frac{\partial}{\partial z_n} - \lambda$$

with $\operatorname{Re}\lambda \geqslant 0$. This implies that the functions u in $\mathcal{H}(\mathbb{C}^n)$ are of the form

$$u(z) = \varphi(z_1, \dots, z_{n-1}) \exp(\lambda z_n)$$

with holomorphic functions φ. If $K = \{z \in \mathbb{C}^n \,|\, |z|^2 \leqslant 1\}$, we have

$$\hat{K}_{\mathbb{C}^n}(1) \supset \{z \in \mathbb{C}^n \,|\, |z_1|^2 + \dots + |z_{n-1}|^2 \leqslant 1,\ \operatorname{Re} z_n < 0\}$$

and therefore $\mathbb{C}^n$ is not A-convex. Then we get a contradiction and the proof is complete.

We obtain analogous results for the non suspended system A_0 if we substitute

$$\operatorname{Re} V(\mathfrak{a}) = \{\operatorname{Re} a \,|\, a \in V(\mathfrak{a})\} \text{ and } \operatorname{Re} V(\mathfrak{b}) = \{\operatorname{Re} a \,|\, a \in V(\mathfrak{b})\}$$

for $V(\mathfrak{a})$ and $V(\mathfrak{b})$ and denote by Λ the closed convex cone in $\mathbb{R}^n$ generated by $\operatorname{Re} V(\mathfrak{b})$. We have, repeating the same proofs given in the suspended case:

If $\Lambda = \mathbb{R}^n$, then $\mathbb{R}^n$ is A_0-convex.

If $\mathfrak{a} = \sqrt{\mathfrak{a}}$ and $\mathbb{R}^n$ is A_0-convex, then $\Lambda = \mathbb{R}^n$.

If $\mathfrak{b} = \sqrt{\mathfrak{b}}$ and $\mathbb{R}^n$ is A_0-convex, then $V(\mathfrak{b})$ generates $\mathbb{C}^n$.

(*b*) We will need the following easy lemmas:

LEMMA 1. *Let $z_0 \in \partial B \cap \partial\tilde{B}$. Then $\bar{z}_0 \in V(\mathfrak{a})$.*

PROOF. If $z_0 \in \partial B \cap \partial\tilde{B}$ then $|z_0| = 1$ and for some $a \in V(\mathfrak{a}) - \{0\} \cdot$ $\cdot |\sum a_i z_{0i}| = |a|$. This implies that $\bar{z}_0 = \mu a$ for some $\mu \in C^*$ and hence that $\bar{z}_0 \in V(\mathfrak{a})$.

LEMMA 2. *Let $f(x)$ be defined on $\mathbb{R}^N$ and let $x_0 \in \mathbb{R}^N$. If f is differentiable at x_0 and if for $\xi \in \mathbb{R}^N$ we have*

$$f(x) - f(x_0) \leqslant \langle \xi, x - x_0 \rangle + o(|x - x_0|)$$

(*where $\langle \cdot, \cdot \rangle$ is the euclidean scalar product*) *then* $\xi = \operatorname{grad} f(x_0)$.

PROOF. For $y \neq 0$, $y \in \mathbb{R}^N$ we have

$$\lim_{t \to 0^+} \frac{1}{t} \{f(x_0 + ty) - f(x_0) - \langle \xi, ty \rangle\} \leqslant 0 .$$

Thus

$$\langle \operatorname{grad} f(x_0), y \rangle - \langle \xi, y \rangle \leqslant 0 .$$

As this is possible only if $\langle \operatorname{grad} f(x_0) - \xi, y\rangle = 0, \forall y$, we have $\operatorname{grad} f(x_0) = \xi$.

Now we remark that, given any open set $\Omega \subset \mathbb{C}^n = \mathbb{R}^N$ $(N = 2n)$, the function

$$d(z, \partial\Omega) = \inf_{w\in C(\Omega)} |z - w|$$

is a continuous and Lipschitz function with Lipschitz constant $= 1$.

By a theorem of H. Rademaker ([12]) $d(z, \partial\Omega)$, as a function of z, is differentiable almost everywhere.

Let us now assume that Ω is a domain of regularity in $\mathbb{C}^n$ for A. We identify $\mathbb{C}^n$ with the underlying real space $\mathbb{R}^{2n}$, whose cartesian coordinates we denote by $x = (x_1, \ldots, x_{2n})$ $(z_j = x_{2j-1} + ix_{2j})$ and set

$$\langle x, y\rangle_{\mathbb{R}^{2n}} = \sum_1^{2n} x_i y_i$$

for the scalar product in $\mathbb{R}^{2n}$.

If $z_0 \in \Omega$ and $r = d(z_0, \partial\Omega)$, then Ω contains the ball $z_0 + rB$ and thus, Ω being a domain of regularity, also the set $z_0 + r\tilde{B}$. Then we can find $w \in \partial\Omega \cap \partial(z_0 + r\tilde{B}) \cap \partial(z_0 + rB)$. We denote by x_0 the cartesian coordinates of z_0 and by y the cartesian coordinates of w.

We claim that, if $d(x, \partial\Omega)$ is differentiable at z_0, then

$$\operatorname{grad}_x d(z, \partial\Omega)\big|_{z_0} = \frac{y - x_0}{|y - x_0|}$$

where $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^{2n}$.

Indeed, for $z \in z_0 + rB$, if we denote by x the cartesian coordinates of z, we have

$$d(z, \partial\Omega)^2 \leqslant d(z, w)^2 = |y - x_0|^2 + |x - x_0|^2 + 2\langle y - x_0, x - x_0\rangle_{\mathbb{R}^{2n}}$$

Thus:

$$d(z, \partial\Omega)^2 - d(z_0, \partial\Omega)^2 \leqslant 2\langle y - x_0, x - x_0\rangle + |x - x_0|^2 .$$

As $|x - x_0|^2 = o(|x - x_0|)$, we deduce from lemma 2 that

$$2(y - x_0) = \operatorname{grad}_x d(z, \partial\Omega)^2\big|_{z_0} = 2d(z_0, \partial\Omega) \operatorname{grad}_x d(z, \partial\Omega)\big|_{z_0}$$

([12]) Cf. H. Rademaker: *Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variablen, und über die Transformation der Doppelintegrale*, Math. Ann., **79** (1918), pp. 340-359.

and thus

$$\operatorname{grad}_x d(x, \partial\Omega)|_{z_0} = \frac{y - x_0}{|y - x_0|}.$$

From this it follows that

$$\operatorname{grad}_z d(x, \partial\Omega)|_{z_0} = \frac{1}{2}\,\frac{\overline{w - z_0}}{|w - z_0|}.$$

Since by lemma 1 we have $\overline{w - z_0} \in V(\mathfrak{a})$, it follows that

$$\operatorname{grad}_z d(x, \partial\Omega)|_{z_0} \in V(\mathfrak{a}).$$

We have therefore proved the following

THEOREM 7. *Let Ω be a domain of regularity for the $\bar\partial$-suspended differential ideal $\mathfrak{b}$ and let $d(z) = d(z, \partial\Omega)$ be the euclidean distance of $z \in \Omega$ from $\partial\Omega$.*

Then $d(z)$ is Lipschitz continuous (with Lipschitz constant 1) and therefore is differentiable almost everywhere in Ω.

For every $g \in \sqrt{\mathfrak{a}}$ ($\mathfrak{a}$ = the asymptotic ideal) we then have

$$g(\operatorname{grad}_z d(z, \partial\Omega)) = 0 \qquad a.e.\ in\ \Omega.$$

If for instance Ω has a piecewise smooth boundary then $d(z)$ is also C^2 almost everywhere. Note that the function $\delta(z) = -\log d(z)$ has the same type of regularity than $d(z)$.

From theorem 7 we deduce the following useful

COROLLARY. *Let Ω be a domain of regularity as in theorem 7.*

Let

$$\delta(z) = -\log d(z).$$

Set $\forall g \in \sqrt{\mathfrak{a}}$ and $\mu_1, \ldots, \mu_n \in \mathbf{C}$

$$L = \sum_{\alpha=1}^{n} \frac{\partial g}{\partial z_\alpha}(\operatorname{grad}_z \delta(z_0))\frac{\partial}{\partial z_\alpha} + \sum_{\beta=1}^{n} \mu_\beta \frac{\partial}{\partial \bar z_\beta}.$$

Then $L\bar L$ is a second order operator with constant coefficients.

If $\delta(z)$ is of class C^2 at z_0 then we have

$$(L\bar L\delta)(z_0) \geqslant 0.$$

PROOF. We have

$$g\left(\frac{\partial\delta(z)}{\partial z_1}, \dots, \frac{\partial\delta(z)}{\partial z_n}\right) = 0 \qquad \forall g \in \sqrt{\mathfrak{a}}$$

in a neighborhood of z_0.

By differentiation we deduce then

$$\sum_{\alpha=1}^{n} \frac{\partial g}{\partial z_\alpha}\left(\frac{\partial\delta}{\partial z}(z_0)\right)\frac{\partial^2\delta(z_0)}{\partial z_\alpha\,\partial z_\gamma} = 0 \quad 1 \leqslant \gamma \leqslant n$$

$$\sum_{\alpha=1}^{n} \frac{\partial g}{\partial z_\alpha}\left(\frac{\partial\delta}{\partial z}(z_0)\right)\frac{\partial^2\delta(z_0)}{\partial z_\alpha\,\partial \bar{z}_\gamma} = 0 \quad 1 \leqslant \gamma \leqslant n.$$

We have therefore

$$(L\bar{L}\delta)(z_0) = \sum \mu_\alpha \mu_{\bar{\beta}} \frac{\partial^2\delta(z_0)}{\partial z_\alpha \partial \bar{z}_\beta}.$$

This expression is $\geqslant 0$ because Ω, being a domain of regularity is also a domain of holomorphy and therefore the « logarithmic distance from the boundary » is plurisubharmonic.

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