ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

KLAUS HULEK

A remark on runge approximation of meromorphic functions

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 6, nº 1 (1979), p. 185-191
<http://www.numdam.org/item?id=ASNSP_1979_4_6_1_185_0>

A Remark on Runge Approximation of Meromorphic Functions (*).

KLAUS HULEK (**)

0. – Introduction.

Let Ω_1 be an open subset of the complex manifold Ω_2. In [2] Hirschowitz calls the pair (Ω_1, Ω_2) meromorphic-convex if every function holomorphic in Ω_1 may be uniformly approximated by functions meromorphic in Ω_2. He calls the pair (Ω_1, Ω_2) μ-convex if even every function meromorphic in Ω_1 may be uniformly approximated by functions meromorphic in Ω_2. In [2, Theorem 5.1] it is claimed that a meromorphic-convex pair of Stein manifolds is μ-convex if and only if the natural homomorphism $H_2(\Omega_1, \boldsymbol{R}) \to \to H_2(\Omega_2, \boldsymbol{R})$ is injective. In this paper I shall prove by means of a counter-example that this condition is not necessary.

I am particularly indebted to Professor Karl Stein for his suggestions and for many helpful discussions.

1. – An approximation theorem.

PROPOSITION 1. *Let Ω_1 be an open and Stein subset of Ω_2 such that the pair (Ω_1, Ω_2) is meromorphic convex. Assume for each hypersurface $h \subset \Omega_1$ and for each $\alpha \in H_2(\Omega_1, \boldsymbol{Z}_n)$, $n \in \boldsymbol{N}_0$, that the intersection number $S(h, \alpha)$ vanishes. Then (Ω_1, Ω_2) is μ-convex.*

PROOF. Let m be an in Ω_1 meromorphic function which is holomorphic in a neighbourhood of the compact set K. For a given $\varepsilon > 0$ we shall have to construct a meromorphic function $\tilde{m}$ which is also holomorphic in a

(*) The results presented in this paper are part of the author's Diplomarbeit written at the University of München.

(**) Mathematisches Institut der Universität, Erlangen.

Pervenuto alla Redazione il 10 Novembre 1977.

neighbourhood of K, such that $\|m-\tilde{m}\|_K<\varepsilon$. Since Ω_1 is Stein we can exhaust it by special analytic polyhedra. Therefore we can choose such polyhedra P_1, P_2 with

$$K\subset P_1\subset \overline{P}_1\subset P_2\subset \Omega_1 .$$

Let h be the set of poles of m. According to our hypothesis we have $S(h,\alpha)=0$ for all $\alpha\in H_2(\Omega_1, Z_n)$ where n is an arbitrary non-negative integer. It then follows from [7] that the Poincarè-problem has a solution for m on P_2, i.e. there are functions $f, g\in \mathcal{O}(P_2)$ which are relatively prime, s. th. $m|_{P_2}=f/g$. In particular we have $M_g:=\inf_{x\in K}|g(x)|>0$. According to [9] f and g can be uniformly approximated by functions holomorphic in Ω_1. We can choose $f_1, g_1\in\mathcal{O}(\Omega_1)$ with $\|f-f_1\|_K<\varepsilon$ and $\|g-g_1\|_K<\varepsilon$.

Since (Ω_1, Ω_2) is meromorphic-convex, there are functions m_f and m_g meromorphic in Ω_2, which are holomorphic in a neighbourhood of K, s. th.

$$\|f_1-m_f\|_K<\varepsilon \quad \text{and} \quad \|g_1-m_g\|_K<\varepsilon .$$

Put $\tilde{m}:=m_f/m_g$. For sufficiently small ε, $\tilde{m}$ is holomorphic in a neighbourhood of K and for $\varepsilon<\operatorname{Min}\{M_g/4, \|f\|_K, \|g\|_K\}$ we have

$$\begin{aligned}\|m-\tilde{m}\|_K&=\left\|m-\frac{m_f}{m_g}\right\|_K\leqslant\left\|\frac{f}{g}-\frac{f_1}{g_1}\right\|_K+\left\|\frac{f_1}{g_1}-\frac{m_f}{m_g}\right\|_K\leqslant\\ &\leqslant\frac{1}{M_g(M_g-\varepsilon)}\|fg_1-gf_1\|_K+\frac{1}{(M_g-\varepsilon)(M_g-2\varepsilon)}\|f_1m_g-g_1m_f\|_K\leqslant\\ &\leqslant\frac{4}{M_g^2}\left(\|fg_1-fg\|_K+\|fg-f_1g\|_K+\|f_1m_g-f_1g_1\|_K+\|f_1g_1-g_1m_f\|_K\right)\leqslant\\ &\leqslant\varepsilon\frac{12}{M_g^2}\left(\|f\|_K+\|g\|_K\right) . \qquad ***\end{aligned}$$

COROLLARY. *Again let Ω_1 be open and Stein in Ω_2, such that (Ω_1, Ω_2) is meromorphic-convex. If $H_2(\Omega_1, Z)$ is divisible and $H_1(\Omega_1, Z)$ is torsion free, then (Ω_1, Ω_2) is μ-convex.*

PROOF. Because of Proposition 1 it suffices to prove that for each hypersurface $h\subset\Omega_1$ and each $\alpha\in H_2(\Omega_1, Z_n)$ we have $S(h,\alpha)=0$. For $n=0$ this is an immediate consequence of the divisibility of $H_2(\Omega_1, Z)$. For $n\neq 0$ the universal coefficient theorem and our hypothesis yield

$$H_2(\Omega_1, Z_n)\simeq H_2(\Omega_1, Z)\otimes Z_n\oplus \operatorname{Tor}\left(H_1(\Omega_1, Z), Z_n\right)=0 . \qquad ***$$

2. – Construction of a counterexample.

Let $D := \{(z_1, z_2) \in \boldsymbol{C}^2;\ |z_1| < 1,\ |z_2| < 1\}$ be the standard dicylinder in $\boldsymbol{C}^2$.

PROPOSITION 2. *There exists a domain of holomorphy $G \subset D$ with the following properties*:

(i) (G, D) *is meromorphic-convex.*

(ii) $H_1(G, \boldsymbol{Z}) \cong \boldsymbol{Q}$.

(iii) $H_2(G, \boldsymbol{Z}) = 0$.

PROOF. In carrying out this construction we follow ideas of Pontrjagin, Stein and Ramspott (see [4], [8] and [5]). We shall construct a sequence of biholomorphic mappings $f_n\colon \boldsymbol{C}^2 \to \boldsymbol{C}^2$ with $f_n(0) = 0$, of smooth analytic sets $B_n \subset D_n := f_n(D)$ and of neighbourhoods V_n of B_n such that with $A_n := f_n^{-1}(B_n)$ and $U_n := f_n^{-1}(V_n)$ the following conditions are fulfilled:

(1) $B_n = \{(z_1, z_2) \in D_n;\ z_2^n - c_n z_1 = 0\}$ for some $c_n \in \boldsymbol{R}_+$. There is a smooth neighbourhood of $\{z_2^n - c_n z_1 = 0\} \cap \partial D_n$ in ∂D_n and the two manifolds intersect transversally.

(2) 0 is deformation retract of B_n.

(3) $D_n - V_n$ and $D_n - B_n$ have the same homotopy type.

(4) $\overline{U}_n \subset U_{n-1}$ where $\overline{U}_n$ is the closure of U_n in D.

(5) $d(D - U_n, A_n) > 0$ where d is the Euclidean distance.

(6) $\tilde{d}(A_n, A_{n+1}) < 1/2^n$ where $\tilde{d}$ denotes the Hausdorff metric (see [1], [2]).

(7) $\tilde{d}(\overline{U}_n, A_n) < 1/2^n$.

The conditions (6) and (7) imply that the sequences $(A_n)_{n\in\boldsymbol{N}}$ and $(U_n)_{n\in\boldsymbol{N}}$ converge to a common limit A. (2), (3) and (4) will enable us to compute the homology of $G := D - A$, the other conditions are necessary for the induction.

To start the induction we choose

$$f_1 := id\,,$$

$$A_1 := B_1 := \{(z_1, z_2) \in D;\ z_2 - \tfrac{1}{2} z_1 = 0\}\,.$$

$$U_1 := V_1 := \{(z_1, z_2) \in D;\ |z_2 - \tfrac{1}{2} z_1| < \tfrac{1}{4}\}\,.$$

Now we assume that f_n, B_n and V_n are given.

Put $g_n\colon \boldsymbol{C}^2 \to \boldsymbol{C}^2;\ (z_1, z_2) \mapsto (z_2, z_2^n - c_n z_1)$.

We take $f_{n+1} := g_n \circ f_n$ and get $D_{n+1} = f_{n+1}(D) = g_n(D_n)$. Furthermore

$$f_{n+1}(A_n) = g_n(B_n) = \{(z_1, z_2) \in D_{n+1};\ z_2 = 0\}\,.$$

The plane $\{z_2 = 0\}$ intersects ∂D_{n+1} transversally. We define

$$B_{n+1} := \{(z_1, z_2) \in D_{n+1};\ z_2^{n+1} - c_{n+1} z_1 = 0\}\,.$$

For sufficiently small $c_{n+1} \in \boldsymbol{R}_+$ the conditions (1) and (6) are clearly fulfilled. A retraction of B_n to 0 gives a retraction of $g_n(B_n)$ to 0 and out of this we can construct a retraction for B_{n+1}, hence (2) is valid. We also can choose c_{n+1} sufficiently small such that $B_{n+1} \subset g_n(V_n)$ and $d(B_{n+1}, D_{n+1} - g_n(V_n)) > 0$. We now have to find a suitable neighbourhood V_{n+1} of B_{n+1}. To do this we look at

$$g_{n+1}\colon \boldsymbol{C}^2 \to \boldsymbol{C}^2\,; \qquad (z_1, z_2) \mapsto (z_2, z_2^{n+1} - c_{n+1} z_1)\,.$$

Again we have
$g_{n+1}(B_{n+1}) = \{(z_1, z_2) \in D_{n+2};\ z_2 = 0\}$ where the plane $\{z_2 = 0\}$ intersects ∂D_{n+2} transversally. Put

$$W_{n+1} := \{(z_1, z_2) \in D_{n+2};\ |z_2| < \varrho_{n+1}\} \quad \text{for some } \varrho_{n+1} \in \boldsymbol{R}_+\,.$$

For sufficiently small ϱ_{n+1} we have according to the above $\overline{W_{n+1}} \subset g_{n+1}(g_n(V_n))$ and $d(D_{n+2} - g_{n+1}(B_{n+1}), D_{n+2} - W_{n+1}) > 0$.

Moreover we can acquire

$$\tilde{d}(\overline{f_{n+1}^{-1}(W_{n+1})}, A_{n+1}) < \frac{1}{2^{n+1}}\,.$$

The sets

$$D_{n+2} - g_{n+1}(B_{n+1}) = \{(z_1, z_2) \in D_{n+2};\ z_2 = 0\}$$

and

$$D_{n+2} - W_{n+1} = \{(z_1, z_2) \in D_{n+2};\ |z_2| < \varrho_{n+1}\}$$

have the same homotopy type. If we put $V_{n+1} := g_{n+1}^{-1}(W_{n+1})$ then the conditions (3), (4), (6) and (7) are fulfilled, i.e. V_{n+1} is a suitable neighbourhood of B_{n+1}.

Let A be the limit of the sequence $(A_n)_{n \in \boldsymbol{N}}$. A is non-empty. We claim that $G := D - A$ has the desired properties. We shall first prove that G is connected. Take two points $(z_1^{(1)}, z_2^{(1)}), (z_1^{(2)}, z_2^{(2)}) \in G$. For some big n_0 we have $(z_1^{(1)}, z_2^{(1)}) \notin U_{n_0} \not\ni (z_1^{(2)}, z_2^{(2)})$. Because of $A \subset \overline{U_{n_0+1}} \subset U_{n_0}$ it is sufficient to prove that $D - U_{n_0}$ is pathwise connected. But this is a consequence of the fact that $D - A_{n_0}$ is connected and that both sets have the same homotopy-type. G is a domain of holomorphy. To see this, consider

$$G_m := D - A_m \qquad \text{and} \qquad \hat{G}_n := \overset{\circ}{\widehat{\bigcap_{m \geq n} G_m}} = D - \overline{\bigcup_{m \geq n} A_m}\,.$$

As above, one sees that $\hat{G}_n$ is connected. Being the open kernel of an intersection of domains of holomorphy $\hat{G}_n$ is a domain of holomorphy itself. Moreover $\hat{G}_n \subset \hat{G}_{n+1}$ and $G = \bigcup_{n \in N} \hat{G}_n$. Hence G is a domain of holomorphy. (See [3, p. 38]). A is a limit of hypersurfaces, hence it is a limace in the terminology of Hirschowitz. It follows from [2; Theorem 3.5] that (G, D) is meromorphic conex. The next step will be to prove $H_2(G, \boldsymbol{Z}) = 0$. Let β be a 2-cycle in G. For sufficiently big n_0, β is contained in $D - U_{n_0}$. Thus it suffices to prove $H_2(D - U_{n_0}, \boldsymbol{Z}) \cong H_2(D - A_{n_0}, \boldsymbol{Z}) = 0$. The exact homology sequence of the pair $(D, D - A_{n_0})$ yields

$$\ldots \to H_3(D, D - A_{n_0}, \boldsymbol{Z}) \to H_2(D - A_{n_0}, \boldsymbol{Z}) \to \underset{\substack{\parallel \\ 0}}{H_2(D, \boldsymbol{Z})} \to \ldots\,.$$

On the other hand Alexander-Pontrjagin duality implies $H_3(D, D - A_{n_0}, \boldsymbol{Z}) \cong$ $\cong H^1_*(A_{n_0}, \boldsymbol{Z})$, where the star denotes cohomology with compact support. Since A_{n_0} has no singularities Poincarè duality gives $H^1_*(A_{n_0}, \boldsymbol{Z}) \cong$ $\cong H_1(A_{n_0}, \boldsymbol{Z}) = 0$, since A_{n_0} is contractible. Hence $H_3(D, D - A_{n_0}, \boldsymbol{Z}) = 0$, and this clearly implies $H_2(D - A_{n_0}, \boldsymbol{Z}) = 0$.

It remains to prove $H_1(G, \boldsymbol{Z}) \cong \boldsymbol{Q}$. According to [5, Satz 2] we have $H_1(D - U_n, \boldsymbol{Z}) \cong H_1(D - A_n, \boldsymbol{Z}) \cong \boldsymbol{Z}$. We want to construct a generating cycle for these homology groups. Therefore consider

$$f_{n+1}(A_n) = g_n(B_n) = \{(z_1, z_2) \in D_{n+1};\, z_2 = 0\}$$

and

$$W_n = f_{n+1}(U_n) = \{(z_1, z_2) \in D_{n+1};\, |z_2| < \varrho_n\}\,.$$

As a generating cycle for $H_1(D_{n+1} - f_{n+1}(A_n), \boldsymbol{Z}) \cong H_1(D_{n+1} - W_n, \boldsymbol{Z})$ we can choose

$$\alpha_n := \{(0, \varrho_n \exp[2\pi i t]);\, 0 \leq t \leq 1\}\,.$$

Put $t_n := f_{n+1}^{-1}(\alpha_n)$, denote by $\overline{t_n}$ the homology class in $H_1(D - U_n, \boldsymbol{Z})$ and by $\overline{\overline{t_n}}$ the homology class in $H_1(G, \boldsymbol{Z})$. The classes $\overline{\overline{t_n}}$ generate $H_1(G, \boldsymbol{Z})$.

To see this take a 1-cycle α with homology class $\bar{\bar{\alpha}}$. Then for some n_0, α is contained in $D - U_{n_0}$ and there it is homologous to some $m \cdot t_{n_0}$. In particular $\bar{\bar{\alpha}} = m \cdot \overline{\overline{t_{n_0}}}$. Now we have to find the relations between the $\overline{\overline{t_n}}$. Therefore consider

$$g_{n+2}(\alpha_n) = \{(\varrho_n \exp[2\pi i t], \varrho_n^{(n+1)} \exp[2\pi i(n+1)t]); 0 \leqslant t \leqslant 1\}.$$

In $D_{n+2} - W_{n+1}$ the cycle $g_{n+2}(\alpha_n)$ is homologous to $(n+1)\cdot\alpha_{n+1}$. This implies $\overline{\overline{t_n}} = (n+1)\cdot\overline{\overline{t_{n+1}}}$. Moreover $m\cdot\overline{\overline{t_n}} \neq 0$ for all $m \neq 0$. Because, if we assumed $m\cdot\overline{\overline{t_n}} = 0$ this would imply that $m\cdot t_n$ was homologous to 0 in some set $D - U_{n_0}$, $n_0 \geqslant n$. But this would mean $m\cdot(n+1)\dots n_0\cdot\overline{t_{n_0}} = 0$, a contradiction to $H_1(D - U_{n_0}, \boldsymbol{Z}) \cong \boldsymbol{Z}$. This also means that apart from the relations $\overline{\overline{t_n}} = (n+1)\cdot\overline{\overline{t_{n+1}}}$ there are no other relations between the $\overline{\overline{t_n}}$. The map $\overline{\overline{t_n}} \mapsto 1/n!$ gives an isomorphism $H_1(G, \boldsymbol{Z}) \cong \boldsymbol{Q}$. ***

We can now deliver our counterexample. Take $\dot{E} := \{z \in \boldsymbol{C}; 0 < |z| < 1\}$ to be the punctured unit-disc in $\boldsymbol{C}$. The pair $(\dot{E}, \boldsymbol{C})$ is meromorphic-convex. $(G \times \dot{E}, D \times \boldsymbol{C})$ is meromorphic-convex since it is the product of meromorphic-convex pairs. The Künneth formula yields

$$H_2(G \times \dot{E}, \boldsymbol{Z}) \cong \boldsymbol{Q}.$$

$$H_1(G \times \dot{E}, \boldsymbol{Z}) \cong \boldsymbol{Z} \oplus \boldsymbol{Q}.$$

By virtue of our corollary $(G \times \dot{E}, D \times \boldsymbol{C})$ is μ-convex. On the other hand it follows from the universal coefficient theorem that

$$H_2(G \times \dot{E}, \boldsymbol{R}) \cong \boldsymbol{R}.$$

Since

$$H_2(D \times \boldsymbol{C}. \boldsymbol{R}) = 0$$

the canonical homomorphism $H_2(G \times \dot{E}, \boldsymbol{R}) \to H_2(D \times \boldsymbol{C}, \boldsymbol{R})$ cannot be injective.

3. – Remarks

As A. Hirschowitz has pointed out in a discussion, it is the first sentence that contains the mistake in the proof of [2; Theorem 5.1]. There it is assumed that the mapping $H_2(\Omega_1, \boldsymbol{R}) \to \operatorname{Hom}(H^2(\Omega_1, \boldsymbol{Z}), \boldsymbol{R})$ is injective. This is not true in general. If however the homology of Ω_1 is of finite type there is an exact sequence

$$0 \to \operatorname{Ext}(H^3(\Omega_1, \boldsymbol{Z}), \boldsymbol{R}) \to H_2(\Omega_1, \boldsymbol{R}) \to \operatorname{Hom}(H^2(\Omega_1, \boldsymbol{Z}), \boldsymbol{R}) \to 0.$$

(Cf. [6, p. 248]). Since $\boldsymbol{R}$ is divisible $\mathrm{Ext}\,(H^3(\Omega_1, \boldsymbol{Z}), \boldsymbol{R}) = 0$. Under this condition as well as under other conditions which imply the injectivity of $H_2(\Omega_1, \boldsymbol{R}) \to \mathrm{Hom}\,(H^2(\Omega_1, \boldsymbol{Z}), \boldsymbol{R})$ the arguments given in [2] remain true.

REFERENCES

[1] A. Hirschowitz, *Sur l'approximation des hypersurfaces*, Ann. Scuola Norm. Sup. Pisa, **25** (1971), pp. 47-58.

[2] A. Hirschowitz, *Entre les hypersurfaces et les ensembles pseudoconcaves*, Ann. Scuola Norm. Sup. Pisa, **27** (1973), pp. 873-887.

[3] R. P. Plug, *Holomorphiegebiete, pseudokonvexe Gebiete und das Levi-Problem*, Springer Lecture Notes, vol. **432** (1975).

[4] L. Pontrjagin, *Über den Inhalt topologischer Dualitätssätze*, Math. Ann., **105** (1931), pp. 165-205.

[5] K. J. Ramspott, *Existenz von Holomorphiegebieten zu vorgegebener erster Bettischer Gruppe*, Math. Ann., **123** (1959), 342-355.

[6] E. H. Spanier, *Algebraic Topology*, MacGraw Hill (1966).

[7] K. Stein, *Topologische Bedingungen für die Existenz analytischer Funktionen komplexer Veränderlichen zu vorgegebenen Nullstellenflächen*, Math. Ann., **117** (1941), pp. 727-757.

[8] K. Stein: *Analytische Funktionen mehrerer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem*, Math. Ann., **123** (1951), pp. 201-222.

[9] K. Stein: *Überlagerungen holomorph-vollständiger komplexer Räume*, Arch. Math., **7**, (1961), pp. 354-361.