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The Obstacle Problem for the Biharmonic Operator (*).

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Summary. – *In this work we consider the obstacle problem for a plate. Thus, we study the variational inequality*

$$\Delta^2 u \geqslant 0\,, \quad u \geqslant \varphi\,, \quad \Delta^2 u \cdot (u - \varphi) = 0$$

in a domain, subject to boundary conditions; φ is the given obstacle. We prove regularity theorems for the solution u and obtain some results for the free boundary.

1 – Introduction.

Let Ω be a bounded domain in R^n with $C^{2+\alpha}$ boundary $\partial\Omega$, where $0 < \alpha < 1$. Let $\varphi(x)$ be a function in $C^2(\bar{\Omega})$ such that

$$\varphi \leqslant 0 \quad \text{on } \partial\Omega\,.$$

We introduce the closed convex set in $H_0^2(\Omega)$:

$$K = \{v \in H_0^2(\Omega);\ v \geqslant \varphi \text{ a.e. in } \Omega\}\,. \tag{1.1}$$

Consider the following variational inequality for Δ^2: find a minimum u of the functional

$$\int_\Omega |\Delta v|^2 dx\,, \qquad v \in K\,.$$

By standard results [7] [9], this problem has a unique solution u.

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Frehse [3] has proved that

$$u \in H^3_{\text{loc}}(\Omega)\,. \tag{1.2}$$

He also proved, in [4], that

$$\Delta u \in L^\infty_{\text{loc}}(\Omega) \tag{1.3}$$

and, moreover,

$$D^2 u \in L^\infty_{\text{loc}}(\Omega)\,. \tag{1.4}$$

Thus u is continuously differentiable in Ω and its first derivatives are locally Lipschitz continuous.

In this paper we establish several results regarding the regularity of u. To describe them, let us introduce the sets

$$C = \{x \in \Omega;\, u(x) = \varphi(x)\}$$
$$N = \{x \in \Omega;\, u(x) > \varphi(x)\}\,.$$

The set C is called the *coincidence set* and the set N is called the *noncoincidence set*. The set

$$F = \partial C \cap \Omega$$

is called the *free boundary*.

In Section 2 we prove that

$$\text{if } x^0 \in C \quad \text{then } \Delta u(x^0) \geqslant \Delta\varphi(x^0)\,, \tag{1.5}$$

where Δu is the upper semicontinuous version of the subharmonic distribution Δu.

In Section 3 we give a new proof of (1.3). This proof uses (1.5) and the maximum principle for upper semicontinuous subharmonic functions [2] [6].

The distribution derivative $\Delta^2 u$ is a nonnegative measure μ. Thus it has a finite mass on every compact subset of Ω. Assuming that $\varphi < 0$ on $\partial\Omega$ we prove, in Section 4, that the total mass $\mu(\Omega)$ is finite.

In Section 5 we establish the smoothness of u up to the boundary in case $n \leqslant 4$, provided $\varphi < 0$ on $\partial\Omega$.

In Section 6 it is proved that, for $n = 2$, u is in $C^2(\Omega)$; this is probably the most important result of this paper. In Section 7 it is shown, by a counterexample, that an a priori estimate on the modulus of continuity of $D^2 u$, in a compact subset K of Ω, cannot hold. Therefore also an a priori estimate on the $W^{2,p}(K)$ norm of u is not possible if $p > n$.

In Sections 8, 9 we study the free boundary. In Section 8 we assume that $\Delta^2\varphi > 0$ in Ω and prove that the open set N is connected. In Section 9 we study the behaviour of the free boundary in a neighborhood of a point $x^0 \in F$, in case $n = 2$. We show, for example, that if $\Delta u(x^0) > \Delta\varphi(x^0)$ then the free boundary in a neighborhood of x^0 is contained in a continuously differentiable curve.

The results of the preceding sections extend to the case where the space $H_0^2(\Omega)$ in (1.1) is replaced by $H^2(\Omega) \cap H_0^1(\Omega)$. That means that u satisfies (in a generalized sense) the boundary conditions $u = 0$, $\Delta u = 0$. We obtain in this case (Section 10) some additional global inequalities on u and Δu.

Variational inequalities for Δ^2 with the convex set K defined by

$$v \in H_0^2(\Omega)\,, \qquad \alpha \leqslant \Delta v \leqslant \beta$$

were studied by Brézis and Stampacchia [1], and, in case $n = 1$, by Cimatti [12] and by Stampacchia [11].

2. – Proof of (1.5).

From the definition of u it follows that

$$\int_\Omega |\Delta(u + \varepsilon\zeta)|^2\,dx \geqslant \int_\Omega |\Delta u|^2\,dx$$

for any $\varepsilon > 0$, $\zeta \in H_0^2(\Omega)$, $\zeta \geqslant 0$. Hence

$$\int_\Omega \Delta u \cdot \Delta\zeta\,dx \geqslant 0\,.$$

This implies that

$$\mu \equiv \Delta^2 u \geqslant 0 \tag{2.1}$$

where $\Delta^2 u$ is taken in the distribution sense. Hence [10] μ is a measure in Ω. It follows that for any compact subset $K \subset \Omega$, $\mu(K) < \infty$. (In Section 4 we shall prove that $\mu(\Omega) < \infty$ if $\varphi < 0$ on $\partial\Omega$.)

LEMMA 2.1. *There exists a function w satisfying*:

(*a*) $w = \Delta u$ *a.e. in* Ω;

(*b*) w *is upper semicontinuous in* Ω;

(c) *for any* $x^0 \in \Omega$ *and for any sequence of balls* $B_\varrho(x^0)$ *with center* x^0 *and radius* ϱ,

$$\frac{1}{|B_\varrho(x^0)|}\int_{B_\varrho(x^0)} w\,dx \downarrow w(x^0) \qquad \text{if } \varrho\downarrow 0\,,$$

where $|B_\varrho(x^0)| =$ *volume of* $B_\varrho(x^0)$.

Thus, w is upper semicontinuous and satisfies the mean value property. (That is, the mean value taken over $B(x^0)$ is $\geqslant w(x^0)$.).

Any other version of Δu which is upper semicontinuous and satisfies the mean value property must coincide with w everywhere [6].

PROOF. Let

$$w_\varrho(x) = \frac{1}{|B_\varrho(x)|}\int_{B_\varrho(x)} \Delta u(y)\,dy\,. \tag{2.2}$$

We claim: for any $x^0 \in \Omega$,

$$w_\varrho(x^0) \quad \text{is decreasing in } \varrho\,. \tag{2.3}$$

Indeed, if $u \in C^\infty$ then we can write

$$\Delta u(x^0) = \frac{1}{|S_\varrho|}\int_{S_\varrho} (\Delta u)\,dS - \int_{B_\varrho} (\Delta^2 u)\,G_\varrho\,dx$$

where $B_\varrho = B_\varrho(x^0)$, $S_\varrho = \partial B_\varrho$, $|S_\varrho| =$ area of S_ϱ, and where

$$G_\varrho = \gamma(r^{2-n} - \varrho^{2-n}) \qquad (\gamma > 0)$$

is Green's function (if $n = 2$, $G_\varrho = \gamma \log(\varrho/r)$). Similarly, if $\varrho' > \varrho$,

$$\Delta u(x^0) = \frac{1}{|S_{\varrho'}|}\int_{S_{\varrho'}} (\Delta u)\,dS - \int_{B_{\varrho'}} (\Delta^2 u)\,G_\varrho\,dx\,.$$

Since $G_\varrho > G_{\varrho'}$ and $\Delta^2 u \geqslant 0$, we get

$$\frac{1}{|S_\varrho|}\int_{S_\varrho} \Delta u \leqslant \frac{1}{|S_{\varrho'}|}\int_{S_{\varrho'}} \Delta u$$

and, by integration,

$$\frac{1}{|B_\varrho|}\int_{B_\varrho} \Delta u \leqslant \frac{1}{|B_{\varrho'}|}\int_{B_{\varrho'}} \Delta u\,. \tag{2.4}$$

For general $u \in H^2(\Omega)$ with $\Delta^2 u \geqslant 0$, we introduce the C^∞ functions

$$w_m = J_{1/m}(\Delta u)$$

where J_ε is the mollifier defined by

$$(J_\varepsilon f)(x) = \int j_\varepsilon(x-y) f(y)\, dy\,, \qquad j_\varepsilon(x-y) = \varepsilon^{-n} j\left(\frac{x-y}{\varepsilon}\right),$$

where $j(x) = j_0(|x|)$ and $j_0(t)$ is a C^∞ function, $j_0(t) = 0$ if $|t| > 1$, $j_0(t) \geqslant 0$, $\int j_0(t)\, dt = 1$. Then $\Delta w_m \geqslant 0$ and, therefore, (2.4) holds with Δu replaced by w_m. Taking $m \to \infty$, the inequality (2.4) follows, *i.e.*, (2.3) is proved. We conclude that

$$w_\varrho(x) \downarrow w(x) \quad \text{as } \varrho \downarrow 0\,. \tag{2.5}$$

Since each w_ϱ is continuous,

$$w(x) \quad \text{is upper semicontinuous}\,. \tag{2.6}$$

Since $\Delta u \in L^2_{\text{loc}}(\Omega)$, w is locally integrable. Hence, for a.e. $x^0 \in \Omega$,

$$w_\varrho(x^0) \to \Delta u(x^0)\,.$$

It follows that

$$w = \Delta u \quad \text{a.e.}\,. \tag{2.7}$$

From (2.5)-(2.7) follow all the assertions of Lemma 2.1.

REMARK. For any ball B, if z is harmonic in B and $z = w$ on ∂B then $w \leqslant z$ everywhere in B. Indeed, Green's third formula is valid for w_ϱ; letting $\varrho \to 0$ and recalling that $w_\varrho \downarrow w$ everywhere and that $\Delta w_\varrho \geqslant 0$, the assertion follows.

THEOREM 2.2. *For any point $x^0 \in \Omega$ which belongs to the support μ,*

$$w(x^0) \geqslant \Delta\varphi(x^0)\,. \tag{2.8}$$

PROOF. Extend the definition of u into $R^n \setminus \Omega$ so that it remains in H^2_{loc}. Let u_ε be the mollifier of u (defined throughout Ω) and let $x^0 \in \Omega$. Suppose there exists a neighborhood W of x^0 and a $\delta > 0$ such that

$$u_\varepsilon(x) - \varphi(x) > \delta \text{ for all } x \in W\,. \tag{2.9}$$

Then $u_\varepsilon \pm \zeta$ belongs to the convex set K for any $\zeta \in C_0^\infty(W)$, $|\zeta| < \delta$. By the variational principle applied to $v = u_\varepsilon \pm \zeta$ we get

$$\int_\Omega |\Delta u|^2 \leqslant \int_\Omega |\Delta u_\varepsilon \pm \Delta\zeta|^2 .$$

Since

$$\int_\Omega |\Delta u_\varepsilon|^2 \to \int_\Omega |\Delta u|^2 ,$$

we obtain

$$\int \Delta u \cdot \Delta\zeta = 0 ,$$

so that $\Delta^2 u = 0$ in W. We thus conclude that the support of μ is contained in the set of points where (2.9) is not satisfied. Thus, it remains to prove (2.8) at a point x^0 for which the following is true:

There exists a sequence of points $\{x_m\}$, $x_m \to x^0$, and a sequence of positive numbers $\{\varepsilon_m\}$, $\varepsilon_m \to 0$, such that

$$u_{\varepsilon_m}(x_m) - \varphi(x_m) \to 0 . \tag{2.10}$$

By Green's formula

$$u_\varepsilon(x_m) = \frac{1}{|S_{\varrho,m}|} \int_{S_{\varrho,m}} u_\varepsilon(y)\, dS_y - \int_{B_{\varrho,m}} \Delta u_\varepsilon(y) \cdot G(x_m - y)\, dy \tag{2.11}$$

where $B_{\varrho,m} = \{y;\ |y - x_m| < \varrho\}$, $S_{\varrho,m} = \partial B_{\varrho,m}$, $|S_{\varrho,m}| =$ area of $S_{\varrho,m}$ and

$$G(z) = \gamma |z|^{2-n} \quad (\gamma > 0) , \quad \left(G(z) = \gamma \log \frac{1}{|z|} \text{ if } n = 2 \right)$$

is the fundamental solution. Similarly,

$$\varphi_\varepsilon(x_m) = \frac{1}{|S_{\varrho,m}|} \int_{S_{\varrho,m}} \varphi_\varepsilon(y)\, dS_y - \int_{B_{\varrho,m}} \Delta\varphi_\varepsilon(y) \cdot G(x_m - y)\, \mathrm{d}y . \tag{2.12}$$

Since $u \geqslant \varphi$, also $u_\varepsilon \geqslant \varphi_\varepsilon$. Hence

$$\int_{S_{\varrho,m}} u_\varepsilon \geqslant \int_{S_{\varrho,m}} \varphi_\varepsilon .$$

Using this inequality and (2.10), we obtain by comparing (2.11) with (2.12), that

$$\lim_{m\to\infty}\Big[\int_{B_{\varrho,m}} \Delta u_{\varepsilon_m}(y)\cdot G(x_m-y)\,dy-\int_{B_{\varrho,m}}\Delta\varphi_{\varepsilon_m}(y)\cdot G(x_m-y)\,dy\Big]\geqslant 0\,. \tag{2.13}$$

We can write

$$\begin{aligned}\int_{B_{\varrho,m}}\Delta u_\varepsilon(y)\cdot G(x_m-y)\,dy &=\int_{B_{\varrho,m}} dy G(x_m-y)\int_{|y-z|<\varepsilon} j_\varepsilon(y-z)\,\Delta u(z)\,dz \\ &= \int_{B_{\varrho,m}}\Big(\int_{|x_m-y-z|<\varepsilon} j_\varepsilon(x_m-y-z)\,G(z)\,dz\Big)\Delta u(y)\,dy+\lambda_{\varepsilon m} \\ &=\int_{B_{\varrho,m}}(J_\varepsilon G)w+\lambda_{\varepsilon,m}\end{aligned} \tag{2.14}$$

where $\lambda_{\varepsilon,m}\to 0$ if $\varepsilon\to 0$ (independently of m). A similar relation holds for the second integral in (2.13). Hence,

$$\lim_{m\to\infty}\int_{B_{\varrho,m}}(J_{\varepsilon_m}G)(w-\Delta\varphi)\,dy\geqslant 0\,.$$

By the mean value theorem there are then points $x_{m,\varrho}$ such that

$$x_{m,\varrho}\in B_{\varrho,m}$$

and

$$w(x_{m,\varrho})-\Delta\varphi(x_{m,\varrho})\geqslant-\delta_m\,,\qquad \delta_m\to 0 \text{ if } m\to\infty\,.$$

Taking a subsequence of $\{x_{m,\varrho}\}$ which converges to some point x_ϱ and using the upper semicontinuity of w, we obtain $w(x_\varrho)-\Delta\varphi(x_\varrho)\geqslant 0$. As $\varrho\to 0$, $x_\varrho\to x^0$ and, by the upper semicontinuity of w, $w(x^0)-\Delta\varphi(x^0)\geqslant 0$.

3. – Δu is locally bounded.

THEOREM 3.1. *Δu is in $L^\infty_{\mathrm{loc}}(\Omega)$.*

PROOF. Take a point $x^0\in\Omega$ and denote by B_ϱ the ball with center x^0 and radius ϱ. Fix R so that $\overline{B_R}\subset\Omega$ and let $\zeta\in C_0^\infty(B_R)$, $\zeta=1$ in $B_{2R/3}$, $\zeta\geqslant 0$ elsewhere. Let $u_\varepsilon=J_\varepsilon u$ be the mollifier of u. For any $x\in B_{2R/3}$,

$$\Delta u_\varepsilon(x)=\Delta u_\varepsilon(x)\cdot\zeta(x)=-\int_{B_R} V\,\Delta(\Delta u_\varepsilon\cdot\zeta)\,dy$$

where $V = V(x-y) = \gamma|x-y|^{2-n}$ $(\gamma > 0)$ is the fundamental solution for Δ. Hence

$$(3.1)\qquad \Delta u_\varepsilon(x) = -\int_{B_{R/2}} V\Delta^2 u_\varepsilon - \int_{B_R/B_{R/2}} V[\Delta^2 u_\varepsilon\cdot\zeta + 2\nabla(\Delta u_\varepsilon)\cdot\nabla\zeta + \Delta u_\varepsilon\cdot\Delta\zeta]\,.$$

Since $\Delta u \in L^2(\Omega)$,

$$(3.2)\qquad \int_K |\Delta u_\varepsilon|^2 \leqslant C \quad \text{for any compact subset } K\subset\Omega\,,$$

where C is a generic constant independent of ε. Let $G_R = B_R\setminus B_{2R/3}$. Notice that the support of $\nabla\zeta$ is contained in G_R. Hence, by integration by parts,

$$\int_{B_R/B_{R/2}} V\nabla(\Delta u_\varepsilon)\cdot\nabla\zeta = -\int_{G_R}\Delta u_\varepsilon\nabla(V\nabla\zeta)\,.$$

Using this and (3.2), we obtain from (3.1)

$$(3.3)\qquad \Delta u_\varepsilon(x) = -\int_{B_{R/2}} V\Delta^2 u_\varepsilon - \int_{B_R/B_{R/2}} V\Delta^2 u_\varepsilon\cdot\zeta + \alpha_\varepsilon(x)$$

where $|\alpha_\varepsilon(x)| \leqslant C$ if $x\in B_{R/2}$.

Since $\mu = \Delta^2 u \geqslant 0$, also $\Delta^2 u_\varepsilon \geqslant 0$. By integration by parts (cf. (2.14)),

$$(3.4)\qquad \int_{B_{R/2}} V(x-y)\,\Delta^2 u_\varepsilon(y)\,dy = \int_{B_{R/2}} V_\varepsilon(x-y)\,\Delta^2 u(y)\,dy + \beta_\varepsilon(x)$$

where $V_\varepsilon(z)$ is the mollifier of $V(z)$ and $\beta_\varepsilon(x)\to 0$ uniformly in $x\in B_{R/2}$, if $\varepsilon\to 0$.

Consider now the integral

$$(3.5)\qquad \hat{V}(x) = \int_{B_{R/2}} V(x-y)\,d\mu(y)\,.$$

It exists in the sense of improper integrals, i.e., as

$$(3.6)\qquad \lim_{\delta\to 0} \int_{|x-y|>\delta,\ y\in B_{R/2}} V(x-y)\,d\mu(y)\,,$$

for almost all x. In fact, since μ is a measure and $\int V(x-y)\,dx$ is a bounded function, the limit in (3.6) exists in the $L^1_{\text{loc}}(\Omega)$ sense.

Observe that $V_\varepsilon(z) = V(z)$ if $|z| > \varepsilon$ (since V is harmonic and the mollifier is obtained by taking averages on spheres) and $V_\varepsilon(z) \leqslant V(z)$ if $|z| < \varepsilon$. Therefore

$$\lim_{\varepsilon\to 0} \int\limits_{B_{R/2}} V_\varepsilon(x-y)\,d\mu(y) \text{ exists a.e. and is equal to } \hat{V}(x)\,. \tag{3.7}$$

Next, as $\varepsilon \to 0$,

$$\begin{aligned}\Delta u_\varepsilon(x) &= \iint u(z)\,\Delta j_\varepsilon(x-z)\,dz = \iint \Delta u(z)\cdot j_\varepsilon(x-z)\,dz\\ &= \iint w(z)\,j_\varepsilon(x-z)\,dz = \int_0^\varepsilon\!\!\int \lambda_\varepsilon(\varrho)\,w(\varrho,\theta)\,dS_\theta\,d\varrho\end{aligned}$$

where $(\varrho, \theta) = (\varrho, \theta_1, \ldots, \theta_{n-1})$ are the spherical coordinates about x and $\lambda_\varepsilon(\varrho)$ is a smooth nonnegative function. Since

$$\frac{1}{\omega_n}\int w(\varrho,\theta)\,dS_\theta \downarrow w(x) \qquad \text{as } \varrho\downarrow 0$$

where ω_n is the area of the unit sphere, the mean value theorem gives

$$\Delta u_\varepsilon(x) \to w(x) \qquad \text{as } \varepsilon \to 0\,.$$

Combining this with (3.4), (3.7) we deduce from (3.3), upon taking $\varepsilon \to 0$, that

$$w(x) = -\hat{V}(x) - \int\limits_{B_R/B_{R/2}} \zeta(y)\,V(x-y)\,\Delta^2 u(y)\,dy + \delta(x)\,, \tag{3.8}$$

$$\delta(x) \text{ is bounded in } B_{R/2}\,.$$

We shall need the following maximum principle for superharmonic functions [2] [6]: Let Z be a superharmonic function in R^n for which the measure $\nu = -\Delta Z$ is supported on a bounded set S. If $Z \leqslant M$ on S then $Z \leqslant M$ in all of R^n.

We apply this result to $Z = \hat{V}$; the measure ν coincides with the restriction of μ to $B_{R/2}$. Hence on the support S we have, by Theorem 2.2,

$$w(x) \geqslant \Delta\varphi(x)\,.$$

Since the integral on the right hand side of (3.8) is $\geqslant 0$, we conclude that

$$\hat{V}(x) \leqslant -w(x) + \delta(x) \leqslant -\Delta\varphi(x) + \delta(x) \leqslant C$$

on the support of the measure of $-\Delta\hat{V}$. The maximum principle cited above yields

$$\hat{V}(x) \leqslant C \quad \text{in } R^n .$$

Since also $\hat{V}(x) \geqslant 0$, $\hat{V}$ is a bounded function in R^n.

Observing now that the integral on the right hand side of (3.8) is a bounded function in $B_{R/3}$, we conclude that w is a bounded function in $B_{R/3}$. This completes the proof.

4. – $\mu(\Omega) < \infty$.

THEOREM 4.1. *If $\varphi < 0$ on $\partial\Omega$ then $\mu(\Omega) < \infty$.*

That means that there is a constant C such that for any compact subset K of Ω,

$$\mu(K) < C .$$

PROOF. For any $\varepsilon > 0$, we introduce the functions

$$\gamma_\varepsilon(\lambda) = \begin{cases} \dfrac{\lambda^2}{\varepsilon} & \text{if } \lambda < 0 , \\ 0 & \text{if } \lambda > 0 , \end{cases} \tag{4.1}$$

$$\beta_\varepsilon(\lambda) = \gamma'_\varepsilon(\lambda) . \tag{4.2}$$

Consider the problem:

$$\underset{v \in H_0^2(\Omega)}{\text{minimize}} \int_\Omega [|\Delta v|^2 + \gamma_\varepsilon(v - \varphi)]\, dx . \tag{4.3}$$

By a standard argument one shows that this problem has a unique solution u_ε. By the variational principle,

$$\int_\Omega [\Delta u_\varepsilon \cdot \Delta v + \beta_\varepsilon(u - \varphi) v]\, dx = 0 \quad \text{for any } v \in H_0^2(\Omega) .$$

Hence

$$\Delta^2 u_\varepsilon + \beta_\varepsilon(u - \varphi) = 0 \quad \text{in } \Omega . \tag{4.4}$$

The standard elliptic theory shows that u_ε is a classical solution of (4.4).

Choosing $v \in H_0^2(\Omega)$ such that $v \geqslant \varphi$, we see that the minimum in (4.3) is bounded by a constant C, where C denotes a generic constant independent of ε.

Hence

$$\int_\Omega |\Delta u_\varepsilon|^2 \leqslant C\,, \tag{4.5}$$

and

$$\int_\Omega \gamma_\varepsilon(u_\varepsilon - \varphi) \leqslant C\,. \tag{4.6}$$

By (4.1), (4.2), $\beta_\varepsilon \leqslant 0$. Hence $\Delta^2 u_\varepsilon = -\beta_\varepsilon(u-\varphi) \geqslant 0$, i.e., $\mu_\varepsilon \equiv \Delta^2 u_\varepsilon$ is a measure. We claim that

(4.7) $\{\mu_\varepsilon(K)\}$ is bounded, for any compact subset K of Ω.

Indeed, if $\zeta \in C_0^\infty(\Omega)$, $\zeta = 1$ on K, $\zeta \geqslant 0$ elsewhere, then

$$\mu_\varepsilon(K) \leqslant \int_\Omega \zeta\, d\mu_\varepsilon = \int_\Omega \Delta\zeta \cdot \Delta u_\varepsilon \leqslant C$$

by (4.5).

We can now choose a sequence $\{\varepsilon'\}$ such that

$$u_{\varepsilon'} \to \bar{u} \quad \text{weakly in } H_0^2(\Omega)\,, \tag{4.8}$$

$$u_{\varepsilon'} \to \bar{u} \quad \text{strongly in } H^1(\Omega)\,, \tag{4.9}$$

$$\mu_{\varepsilon'} = \Delta^2 u_{\varepsilon'} \to \bar{\mu} \quad \text{weakly}\,. \tag{4.10}$$

The last convergence means that for any function f in $C_0^0(\Omega)$,

$$\int_\Omega f\, d\mu_{\varepsilon'} \to \int_\Omega f\, d\bar{\mu}\,. \tag{4.11}$$

From (4.6), (4.1) we have

$$\int_\Omega |(u_\varepsilon - \varphi)^-|^2 dx \leqslant C\varepsilon\,.$$

Using (4.9) and Chebychev's inequality we deduce that $(\bar{u} - \varphi)^- = 0$ a.e., that is, $\bar{u} \geqslant \varphi$. Thus $\bar{u} \in K$ where K is the set defined in (1.1). If we can show that $\bar{u}$ minimizes

$$\int_\Omega |\Delta v|^2, \qquad v \in K$$

then it would follow that $\bar{u}$ coincides with u.

To prove this, take any $v \in K$. Then

$$\int_\Omega |\Delta v|^2 = \int_\Omega [|\Delta v|^2 + \gamma_\varepsilon(v-\varphi)] \geqslant \int_\Omega [\Delta u|^2 + \gamma_\varepsilon(u-\varphi)] \geqslant \int_\Omega |\Delta u_\varepsilon|^2 .$$

Using (4.8) we get

$$\int_\Omega |\Delta v|^2 \geqslant \lim_{\varepsilon'\to 0} \int_\Omega |\Delta u_{\varepsilon'}{}^2 \geqslant \int_\Omega |\Delta \bar{u}|^2 .$$

We have thus completed the proof that $\bar{u} = u$. Taking $f \in C_0^2(\Omega)$ in (4.11) we get

$$\int f\, d\bar{\mu} = \lim_{\varepsilon'\to 0} \int \Delta f \cdot \Delta u_{\varepsilon'} = \int \Delta f \cdot \Delta u = \int f\, \Delta^2 u ,$$

so that $\bar{\mu} = \mu$.

Multiplying the inequality $\Delta^2 u_\varepsilon \geqslant 0$ by $u_\varepsilon - \varphi$ and integrating over Ω, we get

$$\int_\Omega \Delta^2 u_\varepsilon \cdot (u_\varepsilon - \varphi) \geqslant 0 . \tag{4.12}$$

Denote by Ω_δ the intersection of Ω with a δ-neighborhood of $\partial\Omega$. Since $\varphi < 0$ on $\partial\Omega$, there exists a positive number c such that

$$\varphi \leqslant -c \quad \text{in } \Omega_\delta$$

if δ is sufficiently small. It follows that

$$\int_\Omega \Delta^2 u_\varepsilon \cdot \varphi \leqslant -c \int_{\Omega_\delta} \Delta^2 u_\varepsilon + \int_{\Omega/\Omega_\delta} \Delta^2 u_\varepsilon \cdot \varphi .$$

Using this in (4.12) and noting that

$$\int \Delta^2 u_\varepsilon \cdot u_\varepsilon = \int |\Delta u_\varepsilon|^2 \geqslant 0 ,$$

we get

$$\int_{\Omega_\delta} \Delta^2 u_\varepsilon \leqslant C \int_{\Omega/\Omega_\delta} \Delta^2 u_\varepsilon .$$

Recalling (4.7), we conclude that

$$\int_\Omega \Delta^2 u_\varepsilon \leqslant C ,$$

i.e., $\mu_\varepsilon(\Omega) \leqslant C$. Since $\mu_{\varepsilon'} \to \mu$ weakly, we also have $\mu(\Omega) \leqslant C$, and the proof is complete.

5. – Smoothness near the boundary.

Since $u \in H^2(\Omega)$, Sobolev's inequality implies that u is continuous in $\bar{\Omega}$ if $n \leqslant 3$. If further $\varphi < 0$ on $\partial\Omega$ then $u > \varphi$ in some Ω neighborhood Ω_δ of $\partial\Omega$, so that

$$\Delta^2 u = 0 \quad \text{in} \quad \Omega_\delta, \quad u \in H_0^2(\Omega).$$

The standard theory of elliptic operators then implies that u is « as smooth » in $\bar{\Omega}_\delta$ as $\partial\Omega$; thus, if $\partial\Omega \in C^{m+\alpha}$ (m integer $\geqslant 4, 0 < \alpha < 1$) then $u \in C^{m+\alpha}(\bar{\Omega}_\delta)$.

We shall obtain, in this section, the same result also for $n = 4$:

THEOREM 5.1. *If $\varphi < 0$ on $\partial\Omega$ and $n \leqslant 4$ then $u > \varphi$ in some Ω-neighborhood of $\partial\Omega$.*

PROOF. For any $x \in \Omega$, denote by $B_r(x)$ the ball with center x and radius r. We shall first prove that if $u \in H_0^2(\Omega)$ and $n \leqslant 4$ then

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |u| \to 0 \quad \text{if } r = \operatorname{dist}(x, \partial\Omega), \ r \to 0. \tag{5.1}$$

Let $x^0 \in \partial\Omega$, η a small positive number, and denote by V_η the η-neighborhood of x^0. Then $V_\eta \cap \partial\Omega$ can be represented parametrically, say in the form

$$x_n = f(x_1, \dots, x_{n-1})$$

with $x_n > f(x_1, \dots, x_{n-1})$ in $V_\eta \cap \Omega$. Set

$$\tilde{u}(y) = u(x)$$

where

$$y_i = x_i \quad (1 \leqslant i \leqslant n-1), \qquad y_n = x_n - f(x_1, \dots, x_{n-1}).$$

Denote by y^0 the image of x^0. If $\tilde{u}(y)$ is smooth up to the boundary $y_n = 0$ then $\tilde{u} = \tilde{u}_{y_n} = 0$ on $y_n = 0$ and, therefore,

$$\tilde{u}(y) = \int_{y_n}^{0}\int_{y_n}^{0} \tilde{u}_{y_n y_n}\, dy_n\, dy_n.$$

It follows that for any δ positive and sufficiently small,

$$\frac{1}{\delta^n}\int_{|v-v^0|<\delta}|\tilde{u}(y)|\,dy\leqslant C\frac{\delta^2}{\delta^n}\int_{|v-v^0|<\delta}|\tilde{u}_{v_nv_n}|\,dy\,.$$

By approximation, this inequality is valid for our present function $\tilde{u}$, since $u\in H_0^2(\Omega)$. We therefore get

$$\frac{1}{\delta^n}\int_{|v-v^0|<\delta}|\tilde{u}(y)|\,dy)\leqslant C\frac{\delta^2}{\delta^n}\,\delta^{n/2}\int_{|v-v^0|<\delta}|\tilde{u}_{v_nv_n}|^2\,dy\leqslant C\delta^{2-n/2}\int_{V_\eta\cap\Omega}|\Delta u|^2\,dx\,.$$

Since the integral on the right hand side converges to zero if $\eta\to 0$ and since $n\leqslant 4$, we obtain

$$\frac{1}{\delta^n}\int_{|v-v^0|<\delta}\tilde{u}(y)\leqslant\gamma(\eta)\,,\qquad \gamma(\eta)\to 0\ \text{ if }\ \eta\to 0\,.$$

This relation establishes (5.1).

We now introduce Green's function (with pole in x)

$$G_r=\gamma(\varrho^{2-n}-r^{2-n})\qquad(\gamma>0)$$

for Δ in $B_r(x)$, and the function

$$P_r=\gamma_1 r^{-n}(r^2-\varrho^2)\qquad(\gamma_1>0)$$

where γ_1 is chosen so that the normal derivative of the function

$$V_r=G_r-P_r$$

vanishes on $\partial B_r(x)$; here $\varrho=$ distance from x. Then

$$u(x)=\frac{1}{|B_r(x)|}\int_{B_r(x)}u\,dy-\int_{B_r}V_r w\,dy\,. \tag{5.2}$$

Since $\partial V_r/\partial\varrho<0$ if $\varrho<r$, we have $V_r>0$ if $\varrho<r$. We can therefore write, for a suitable function $\lambda(\varrho)\geqslant 0$,

$$\begin{aligned}\int_{B_r(x)}V_r w\,dy&=\int_0^r\lambda(\varrho)\left\{\frac{1}{|\partial B_\varrho(x)|}\int_{\partial B_\varrho(x)}w\,dS\right\}d\varrho\\&\geqslant\int_0^r\lambda(\varrho)w(x)\,d\varrho=w(x)\int_{B_r(x)}V_r\,dy\,.\end{aligned} \tag{5.3}$$

Hence,

$$\varphi(x) \leqslant u(x) \leqslant \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} u\, dy - w(x)\gamma_r \tag{5.4}$$

where

$$\gamma_r = \int\limits_{B_r(x)} V_r\, dy .$$

It follows that for $r = \operatorname{dist}(x, \partial\Omega)$,

$$w(x) \leqslant -\frac{\varphi(x)}{\gamma_r} + \frac{1}{\gamma_r}\frac{1}{|B_r(x)|}\int\limits_{B_r(x)} u\, dy \leqslant \frac{C}{\gamma_r}, \tag{5.5}$$

where (5.1) was used.

On the other hand, if $x = \bar{x}$ is a point of the coincidence set C, then

$$\varphi(\bar{x}) = u(\bar{x}) = \frac{1}{|B_r(\bar{x})|}\int\limits_{B_r(\bar{x})} u\, dy - \int\limits_{B_r(\bar{x})} V_r w\, dy .$$

Since w is subharmonic we can establish, by introducing spherical coordinates (as in (5.3)) and using (2.3), that

$$\int\limits_{B_r(\bar{x})} V_r w \leqslant \hat{w}(\bar{x}) \int\limits_{B_r(\bar{x})} V_r$$

where

$$\hat{w}(\bar{x}) = \frac{1}{|B_r(\bar{x})|}\int\limits_{B_r(\bar{x})} w$$

is the average of w over $B_r(\bar{x})$. It follows that

$$\varphi(\bar{x}) \geqslant \frac{1}{|B_r(\bar{x})|}\int\limits_{B_r(\bar{x})} u\, dy - \gamma_r \hat{w}(\bar{x}) .$$

Using (5.1) and the assumption that $\varphi < 0$ on $\partial\Omega$, we get

$$\hat{w}(\bar{x}) > \frac{c}{\gamma_r} \qquad (c > 0) \tag{5.6}$$

provided $r = \operatorname{dist}(\bar{x}, \partial\Omega)$ is sufficiently small.

Let Σ be the subset of $B_r(\bar{x})$ where $w > c_1/\gamma_r$, and let $|\Sigma| = \text{meas}\,\Sigma$. Then.

$$\hat{w}(\bar{x}) \leqslant \frac{|B_r(\bar{x})| - |\Sigma|}{|B_r(\bar{x})|}\frac{c_1}{\gamma_r} + \frac{|\Sigma|}{|B_r(\bar{x})|}\frac{C}{\gamma_r},$$

where (5.5) was used. Choosing $c_1 = c/2$ and using (5.6) we find that

$$|\Sigma| \geqslant c_2 |B_r(\bar{x})|$$

where c_2 is a positive constant independent of r. Thus

$$w \geqslant \frac{c_1}{\gamma_r} \quad \text{in } \Sigma, \ |\Sigma| \geqslant c_2 |B_r(\bar{x})|. \tag{5.7}$$

Consider the subset $\hat{\Sigma}$ of Σ consisting of all points $\hat{x}$ such that

$$\frac{\hat{r}}{r} > c_3 \qquad \text{where } \hat{r} = \text{dist}\,(\hat{x}, \partial\Omega),\ c_3 > 0\,. \tag{5.8}$$

Since $|\Sigma| > c_2 |B_r(\bar{x})|$, if c_3 is sufficiently small then $\text{meas}\,(\hat{\Sigma}) > c_4 |B_r(\bar{x})|$ where both c_3 and c_4 are independent of $\bar{x}$.

Applying (5.4) with $x = \hat{x} \in \hat{\Sigma}$ $r = \hat{r}$ and using (5.7), (5.8), we find that

$$u(\hat{x}) \leqslant \frac{1}{|B_{\hat{r}}(\hat{x})|} \int_{B_{\hat{r}}(\hat{x})} u\, dy - c_5\,,$$

where c_5 is a positive constant independent of $\hat{x}$, $\bar{x}$. In view of (5.1), the right hand side is $< - c_5/2$ if $\bar{x}$ is sufficiently close to $\partial\Omega$. But then,

$$\frac{1}{|B_r(\bar{x})|} \int_{B_r(\bar{x})} |u| \geqslant c_6 > 0$$

if $\bar{x}$ is sufficiently close to $\partial\Omega$, which contradicts (5.1). This shows that the coincidence set cannot have points arbitrarily close to $\partial\Omega$, and the proof is thereby completed.

6. – Further regularity of the solution.

THEOREM 6.1. $u \in W^{2,\infty}_{\text{loc}}(\Omega)$.

This result is due to Frehse [4]. We briefly describe his proof, since a part of it will be needed in the sequel.

Let $B_R = B_R(x^0)$ be a ball with center x^0 and radius R contained in Ω, and let $\zeta \in C_0^\infty(B_R)$, $\zeta = 1$ in $B_{2R/3}$, $\zeta \geqslant 0$ elsewhere. If $x \in B_{2R/3}$ then the mollifier u_ε of u can be represented in the form

$$u_\varepsilon(x) = \int_{B_R} V(x-y)\,\Delta^2(\zeta u_\varepsilon)(y)\,dy \geqslant \int_{B_{R/2}} V\nabla^2 u_\varepsilon + \beta_\varepsilon(x) \tag{6.1}$$

where V is the fundamental solution of Δ^2 and where (after writing β_ε explicitly and performing some integrations by parts) $\beta_\varepsilon(x)$ is a C^∞ function in $B_{R/3}$. The derivatives of $\beta_\varepsilon(x)$ are bounded independently of ε.

From the explicit form of V one deduces [4]:

$$\left(\frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\Delta\right) V \geqslant -c \qquad (c \text{ positive constant})\,. \tag{6.2}$$

In the case $n = 2$, $V(x) = |x|^2(\log|x| - 1)$, so that

$$\left(\frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\Delta\right) V(x) = \frac{2x_j^2}{|x|^2} \qquad \text{is bounded } (n = 2)\,. \tag{6.3}$$

Applying $\partial^2/\partial x_j^2 - \Delta/2$ to both sides of (6.1) and using (6.2), we get

$$\left(\frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\Delta\right) u_\varepsilon \geqslant -C \qquad \text{in } B_{R/3}\,,$$

where C is a generic positive constant independent of ε. Since Δu (and hence Δu_ε) is locally bounded, we conclude that

$$\frac{\partial^2}{\partial x_j^2} u_\varepsilon \geqslant -C\,.$$

Since also

$$\frac{\partial^2 u_\varepsilon}{\partial x_j^2} = \Delta u_\varepsilon - \sum_{i \neq j} \frac{\partial^2 u_\varepsilon}{\partial x_i^2} \leqslant \Delta u_\varepsilon + (n-1)C\,,$$

we deduce that

$$\left|\frac{\partial^2 u_\varepsilon}{\partial x_j^2}\right| \leqslant C \qquad \text{in } B_{R/3}\,.$$

Taking $\varepsilon \to 0$ the assertion of Theorem 6.1 follows.

Consider now the case $n = 2$. By Green's formula

$$\Delta u_\varepsilon(x_0) = -\int_{\partial B_r(x^0)} \Delta u_\varepsilon \cdot \frac{\partial G}{\partial \varrho}\, dS - \int_{B_r(x^0)} G\, \Delta^2 u_\varepsilon\, dx$$

where $G = (1/2\pi)\log r/\varrho$ is Green's function with pole at x^0, $\varrho = |x - x^0|$, and u_ε is the mollifier of u. Since $|\Delta u_\varepsilon| \leqslant C$ in any compact subset of Ω, we deduce that

$$\int_{B_r(x^0)} \log \frac{1}{|x^0 - y|} \Delta^2 u_\varepsilon(y)\, dy \leqslant C\,.$$

Therefore the measure $\mu_\varepsilon = \Delta^2 u_\varepsilon$ satisfies:

$$\mu_\varepsilon(B_r(x^0)) \leqslant \frac{C}{\log(1/r)} \qquad (x^0 \in K) \tag{6.4}$$

where K is any compact subset of Ω and $r \leqslant r_0 = r_0(K)$.

Applying the operator

$$\square = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$$

to both sides of (6.1) we get

$$\square u_\varepsilon(x) \geqslant \int_{B_{R/2}} F(x, y)\, d\mu_\varepsilon(y) + \gamma_\varepsilon(x) \tag{6.5}$$

where $\gamma_\varepsilon(x)$ is continuous in x, uniformly with respect to ε, and $F(x, y) = \square V$. By (6.3),

(6.6) $F(x, y)$ is a bounded function, continuous in (x, y) if $x \neq y$.

By (6.4), the measure $\mu_\varepsilon(B_r(y))$ tends to zero as $r \to 0$, uniformly with respect to y, ε.

Using these remarks, it follows from (6.5), by a standard Potential Theory argument, that

$$\square u_\varepsilon(x) \quad \text{is uniformly continuous in } x,\ x \in K$$

uniformly with respect to ε, where K is any compact subset of Ω. By the Ascoli-Arzela theorem, there exists a sequence $\{\varepsilon'\}$ such that $\square u_{\varepsilon'}$ is convergent to a continuous function in compact subsets of Ω. Since also $\square u_{\varepsilon'} \to \square u$ in the distribution sense, there is a version of $u_{x_1x_1} - u_{x_2x_2}$ which is continuous in Ω. By change of coordinates

$$x_1 \to \frac{x_1 + x_2}{\sqrt{2}}\,, \qquad x_2 \to \frac{x_1 - x_2}{\sqrt{2}}$$

we find that also $u_{x_1x_2}$ has a continuous version in Ω. We have thus proved:

LEMMA 6.2. *The distribution derivatives $u_{x_1x_1} - u_{x_2x_2}$ and $u_{x_1x_2}$ are continuous functions in Ω.*

We shall next prove:

LEMMA 6.3. *If $n = 2$ then w is continuous in Ω.*

PROOF. Denote by S the support of $\mu = \Delta^2 u$ in Ω. By a continuity theorem for subharmonic functions [2], if w restricted to S is continuous then w is continuous on Ω. Thus it suffices to show:

$$\text{if } P_0 = (x_0, y_0) \in S\,, \quad \text{then } w|_S \text{ is continuous at } P_0\,. \tag{6.7}$$

Let $P_m = (x_m, y_m) \in S$, $P_m \to P_0$ be such that, if $\alpha_m =$ angle between $P_m \to P_0$ and the y-axis, then

$$\alpha_m \to 0 \quad \text{if } m \to \infty\,, \tag{6.8}$$

$$|P_{m+1} - P_0| < \tfrac{1}{5}|P_m - P_0|\,. \tag{6.9}$$

We shall prove that

$$w(P_m) \to w(P_0) = (u_{xx} - u_{yy})(P_0) + 2\varphi_{yy}(P_0)\,, \tag{6.10}$$

where $u_{xx} - u_{yy}$ is the continuous function asserted in Lemma 6.2.

Take for simplicity $(x_0, y_0) = (0, 0)$ and introduce the square

$$R_m\colon 0 < y < y_m\,, \qquad |x| < \tfrac{1}{2}y_m\,.$$

We can write a.e.

$$w = \Delta u = u_{xx} - u_{yy} + 2(u_{yy} - \varphi_{yy}) + 2\varphi_{yy}\,. \tag{6.11}$$

Since $u - \varphi = 0$, $\nabla(u - \varphi) = 0$ at P_m,

$$\begin{aligned}\iint_{R_m} (u_{yy} - \varphi_{yy})\,dx\,dy &= \int_{-\frac{1}{2}y_m}^{\frac{1}{2}y_m} (u_y - \varphi_y)(x, y_m)\,dx - \int_{-\frac{1}{2}y_m}^{\frac{1}{2}y_m} (u_y - \varphi_y)(x, 0)\,dx \\ &= \int_{-\frac{1}{2}y_m}^{\frac{1}{2}y_m} \int_{x_m}^{x} (u_{xy} - \varphi_{xy})(\xi, y_m)\,d\xi\,dx - \int_{-\frac{1}{2}y_m}^{\frac{1}{2}y_m} \int_{0}^{x} (u_{xy} - \varphi_{xy})(\xi, 0)\,d\xi\,dx\,.\end{aligned}$$

Using the continuity of u_{xy} and (6.8) we find that

$$\frac{1}{|R_m|}\iint_{R_m}(u_{yy}-\varphi_{yy})\,dx\,dy=\sigma_m\,,\qquad \sigma_m\to 0 \text{ if } m\to\infty\,. \tag{6.12}$$

The function

$$g=u_{xx}-u_{yy}+2\varphi_{yy}$$

is continuous, and the function $w-g$ is bounded, say by M, in a compact subset K of Ω which contains a neighborhood of P_0.

Let B_m be the ball with center P_0 and radius y_m and let $P\in B_m$. Denote by $B_{4y_m}(P)$ the ball with center P and radius $4y_m$. Then

$$R_m\subset B_{4y_m}(P)\,.$$

Using this and (6.11), (6.12) it follows that

$$\frac{1}{|B_{4y_m}(P)|}\int\int_{B_{4y_m}(P)}(w-g)\,dx\,dy\leqslant\lambda M+(1-\lambda)\sigma_m\qquad(0<\lambda<1)$$

where λ is independent of m and m is sufficiently large. Since w is subharmonic and g is continuous, the left hand side is

$$\geqslant w(P)-g(P)+\eta_m\,,\qquad \eta_m\to 0 \text{ if } m\to\infty\,.$$

Hence

$$w(P)-g(P)\leqslant\lambda M+(1-\lambda)\sigma_m+\eta_m\leqslant\lambda' M$$

where $\lambda<\lambda'<1$, provided m is sufficiently large. Thus

$$(w-g)(P)\leqslant\lambda' M\qquad\text{if } P\in B_m,\ m\geqslant m_0\,. \tag{6.13}$$

We can now repeat the process step by step, making use of (6.9). We obtain

$$w-g\leqslant(\lambda')^k M\qquad\text{in } B_{m_k}\,. \tag{6.14}$$

From (6.14) we get

$$\frac{1}{|B_{m_k}|}\int_{B_{m_k}}(w-g)\,dx\,dy\leqslant(\lambda')^k M\to 0\qquad k\to\infty\,.$$

Since w is subharmonic and g is continuous, the left hand side is

$$\geqslant w(P_0) - g(\tilde{P}_k), \qquad \tilde{P}_k \in B_{m_k}.$$

It follows that

$$w(P_0) \leqslant g(P_0). \tag{6.15}$$

For any small $h > 0$, we introduce the rectangle

$$T_h\colon x_m < x < x_m + h^2, \qquad y_m < y < y_m + h.$$

We have:

$$\iint_{T_h} (u-\varphi)_{yy}\,dx\,dy = \int_0^{h^2} (u-\varphi)(x_m + x, y_m + h)\,dx - \tag{6.16}$$

$$-\int_0^{h^2} (u-\varphi)(x_m + x, y_m)\,dx - \int_0^{h^2} h\,(u-\varphi)_y(x_m + x, y_m)\,dx \equiv J_1 - J_2 - J_3.$$

Clearly $J_1 \geqslant 0$. Since

$$|(u-\varphi)(x_m + x, y_m)| \leqslant C|x|^2,$$

$$|(u-\varphi)_y(x_m + x, y_m)| \leqslant C|x|,$$

we also have

$$|J_2| \leqslant Ch^6, \qquad |J_3| \leqslant Ch^5.$$

Hence (6.16) yields

$$\operatorname*{ess\,sup}_{T_h}\,(u-\varphi)_{yy} \geqslant -Ch.$$

Therefore there exists a point $Q_h \in T_h$ such that

$$(u-\varphi)_{yy}(Q_h) > -2Ch.$$

Recalling (6.11), we deduce that

$$w(Q_h) - g(Q_h) \geqslant -4Ch.$$

Taking $h \to 0$ and using the fact that w is upper semicontinuous, it follows that

$$w(P_m) \geqslant \overline{\lim}\, w(Q_h) \geqslant \lim g(Q_h) = g(P_m).$$

Thus

$$w(P_m) \geqslant g(P_m) \quad \text{for any } P_m. \tag{6.17}$$

The proof of (6.17) is valid also for P_0; thus $w(P_0) \geqslant g(P_0)$. Recalling (6.15) we then have

$$w(P_0) = g(P_0).$$

From (6.17) it follows that

$$\varliminf_{m\to\infty} w(P_m) \geqslant g(P_0) = w(P_0).$$

Since w is also upper semicontinuous, the assertion (6.10) follows.

We can now easily complete the proof of (6.7). Indeed, if (6.7) is not true then there exist two sequences $\{P_m\}$ and $\{\tilde{P}_m\}$ in S such that

$$w(P_m) \to A, \qquad w(\tilde{P}_m) \to \tilde{A}, \qquad A \neq \tilde{A}.$$

By extracting a subsequence we may assume that P_m satisfies (6.8), (6.9) with respect to a ray which we take to be the y-axis. Hence, by (6.10),

$$A = \lim w(P_m) = w(P_0).$$

Similarly, $\tilde{A} = w(P_0) = A$; a contradiction.

From Lemmas 6.2, 6.3 it follows that also $u_{x_1x_1}$, $u_{x_2x_2}$ are continuous. We have thus established.

THEOREM 6.4. *If $n = 2$ then $u \in C^2(\Omega)$.*

7. – Counter-example for higher regularity.

We shall show, by a counter-example, that

(7.1) *for any compact subdomain $K \subset \Omega$, there cannot exist an a priori estimate on the modulus of continuity of D^2u in K.*

PROOF. Let Ω be the unit ball and $\varphi = \varepsilon - r^2$ $(r = |x|)$. The set $N = \{u > \varphi\}$ is open. We claim that

$$N = \{x; r > \delta\}. \tag{7.2}$$

Indeed, N is nonempty and if (7.2) is not true then there is a shell $\alpha < r < \beta$ such that

$$u > \varphi \quad \text{if } \alpha < r < \beta,$$
$$u - \varphi = 0, \quad \frac{\partial}{\partial r}(u - \varphi) = 0 \quad \text{on } r = \alpha, \ r = \beta.$$

Since

$$\Delta^2(u - \varphi) = \Delta^2 u - \Delta^2 \varphi = 0 \quad \text{if } \alpha < r < \beta,$$

it follows that $u - \varphi \equiv 0$ if $\alpha < r < \beta$, which is absurd.

Notice that $\delta > 0$. Indeed, if $\delta = 0$ then $\Delta^2 u = 0$ if $r > 0$ and therefore (see the beginning of Section 9) $\Delta^2 u = 0$ also at $r = 0$. Therefore $u \equiv 0$ in Ω. But this is impossible since $u \geqslant \varphi > 0$ if $r^2 < \varepsilon$.

We claim that

$$w(1) \geqslant 0 . \tag{7.3}$$

Indeed, since $\Delta w \geqslant 0$, if $w(1) < 0$ then, by the maximum principle, $w \leqslant 0$ in Ω. But this is impossible since

$$\int_\Omega w = \int_{\partial\Omega} \frac{\partial u}{\partial r} = 0 .$$

The solution u is in $C^2(\Omega)$. Indeed, the variational inequality for u is actually one-dimensional, and since (by Frehse [3]) $u \in H^3_{\text{loc}}(\Omega)$, it follows that $u \in C^2(\Omega)$. Recalling that $u = \varphi$ if $r < \delta$, we therefore deduce that

$$w(\delta) = \Delta\varphi(\delta) = -2n .$$

We can now write the harmonic function w in $\delta < r < 1$ in the form:

$$\begin{aligned} w &= -\frac{2n + w(1)}{\delta^{2-n} - 1} r^{2-n} + \left\{ w(1) + \frac{2n + w(1)}{\delta^{2-n} - 1} \right\} && \text{if } n > 2 , \\ w &= -\frac{4 + w(1)}{\log(1/\delta)} \log\frac{1}{r} + w(1) && \text{if } n = 2 . \end{aligned} \tag{7.4}$$

As $\varepsilon \to 0$, $\varphi \to 0$ and, from the variational definition of $u = u_\varepsilon$ we find that

$$\int_\Omega w^2 \to 0 .$$

It follows that $u_\varepsilon \to 0$ and, therefore, $\delta = \delta_\varepsilon \to 0$.

Since $w(1)\geqslant 0$, the functions in (7.4), with $\delta \to 0$, do not have a uniform modulus of continuity in any compact neighborhood of the origin. This establishes the assertion (7.1).

Frehse [3] has proved that the solution u is in $W^{3,2}_{\mathrm{loc}}(\Omega)$. The assertion (7.1), in conjunction with the Sobolev inequality, shows that

(7.5) it is not possible to have an a priori $W^{3,p}(K)$ estimate on u, $p > n$,

where K is any compact subdomain of Ω.

8. – The non-coincidence set is connected.

We shall now assume that $\varphi \in C^4(\bar{\Omega})$ and study the behavior of the non-coincidence set N.

If $\Delta^2\varphi < 0$ then the coincident set C has no interior points; indeed, if such interior points exist then

$$\Delta^2 u = \Delta^2 \varphi < 0$$

at such points, which is impossible (since $\Delta^2 u \geqslant 0$ in Ω).

THEOREM 8.1. *Let Ω_0 be a subdomain of Ω such that $\Delta^2\varphi \geqslant 0$ in Ω_0, and let K be any component of $N \cap \Omega_0$. Then ∂K must intersect $\partial\Omega_0$.*

COROLLARY 8.2. *If $\Delta^2\varphi\geqslant 0$ in Ω, $\varphi < 0$ on $\partial\Omega$ and $n \leqslant 4$, then N is a connected open set.*

Indeed, by Theorem 5.1, N contains an Ω neighborhood of $\partial\Omega$ and, by Theorem 8.1, the boundary of any component of N intersects $\partial\Omega$; hence the result.

PROOF OF THEOREM 8.1. Suppose the assertion is not true. Then any point of ∂K lies in the support of μ. Therefore, by Lemma 2.2,

$$w - \Delta\varphi \geqslant 0 \quad \text{on } \partial K. \tag{8.1}$$

We also have

$$\Delta(w - \Delta\varphi) = 0 - \Delta^2\varphi \leqslant 0 \quad \text{in } K. \tag{8.2}$$

Using (8.1), (8.2) we shall prove that

$$w - \Delta\varphi \geqslant 0 \quad \text{in } K. \tag{8.3}$$

It would then follow that $\Delta(u-\varphi)\geqslant 0$ in K. Since $u-\varphi=0$ on ∂K, the maximum principle gives $u-\varphi\leqslant 0$ in K, which is impossible. Thus it remains to prove (8.3). (Notice that if we knew that w is continuous in $\bar{K}$ then (8.3) would simply follow from the maximum principle.)

Let Ω_1, Ω_2, Ω_3 be open sets such that

$$\bar{K}\subset\Omega_1, \quad \bar{\Omega}_1\subset\Omega_2, \quad \bar{\Omega}_2\subset\Omega_3, \quad \bar{\Omega}_3\subset\Omega,$$

and let $\zeta\in C_0^\infty(\Omega_3)$, $\zeta\geqslant 1$ in Ω_2, $\zeta\geqslant 0$ elsewhere. For any $x\in\Omega_2$,

$$\begin{aligned} w_\varepsilon(x) = (\zeta w_\varepsilon)(x) &= -\int_{\Omega_3} G(x-y)\,\Delta(\zeta w_\varepsilon)\,dy \\ &= -\int_{\Omega_1} G(x-y)\,\Delta w_\varepsilon(y)\,dy - \int_{\Omega_3/\Omega_1} G(x-y)[\Delta w_\varepsilon\cdot\zeta + 2\nabla w_\varepsilon\cdot\nabla\zeta + w_\varepsilon\Delta\zeta]\,dy \end{aligned}$$

where G is the fundamental solution of Δ and w_ε is the mollifier of w. Integrating by parts,

$$\int_{\Omega_3/\Omega_1} G\nabla w_\varepsilon\cdot\nabla\zeta = -\int_{\Omega_3/\Omega_2} w_\varepsilon\nabla(G\nabla\zeta)$$

and taking $\varepsilon\to 0$, we obtain the relation

$$w(x) = -\int_{\Omega_1} G(x-y)\,d\mu(y) + \beta(x) \tag{8.4}$$

where $\beta(x)$ is continuous in Ω_1.

Let

$$\begin{aligned} \Phi(x) &= -\int_{\Omega_1} G(x-y)\,d\mu(y)\,, \qquad x\in\Omega_1\,, \\ \Phi_\varepsilon(x) &= -\int_{\Omega_1\cap\{|x-y|>\varepsilon\}} G(x-y)\,d\mu(y)\,, \quad x\in\Omega_1\,. \end{aligned}$$

Note that $\Phi_\varepsilon\downarrow\Phi$ a.e. in Ω_1. Hence, by Egoroff's theorem, for any $\delta>0$ there is a closed subset F_δ of Ω_1 such that

$$\begin{aligned} &\operatorname{meas}(\Omega_1\setminus F_\delta)<\delta\,, \\ &\Phi_\varepsilon\downarrow\Phi \text{ uniformly on } F_\delta\,. \end{aligned}$$

Denote by μ_δ the restriction of μ to F_δ, and define

$$\begin{aligned} \Phi_{\varepsilon,\delta}(x) &= -\int_{\Omega_1\cap\{|y-x|>\varepsilon} G(x-y)\,d\mu_\delta(y)\,. \\ \Psi_\delta(x) &= -\int_{\Omega_1} G(x-y)\,d\mu_\delta(y)\,. \end{aligned}$$

Then

$$0 \leqslant \Phi_{\varepsilon,\delta}(x) - \Psi_\delta(x) = \int\limits_{\Omega_1 \cap \{|y-x|<\varepsilon\}} G(x-y)\, d\mu_\delta(y)$$
$$\leqslant \int\limits_{\Omega_1 \cap \{|y-x|<\varepsilon\}} G(x-y)\, d\mu(y) \to 0$$

uniformly on F_δ. Therefore

$$\Phi_{\varepsilon,\delta} \to \Psi_\delta \quad \text{uniformly on } F_\delta .$$

Hence Ψ_δ is continuous on F_δ, which contains the support of μ_δ. By a continuity theorem for superharmonic functions [2; p. 16, Theorem 2] it follows that Ψ_δ is continuous on Ω_1. Since also $\Psi_\delta \geqslant \Phi$, we have

$$v_\delta \equiv \Psi_\delta + \beta \geqslant w \quad \text{in } \Omega_1 .$$

Hence $v_\delta \geqslant \Delta\varphi$ on ∂K. But since also

$$\Delta v_\delta = 0 \quad \text{in } K$$

and since v_δ is continuous in $\bar{K}$, we can apply the maximum principle to $v_\delta - \Delta\varphi$ and conclude that $v_\delta \geqslant \Delta\varphi$ in K.

Noting that

$$\left|\Psi_\delta(x) - \int\limits_{\Omega_1} G(x-y)\, d\mu(y)\right| \leqslant \int\limits_{\Omega_1} G(x-y)\, d(\mu - \mu_\delta) \to 0 \qquad (\delta \to 0)$$

if $x \in K$ (since $\mu = 0$ on K), we conclude that

$$w(x) = \lim v_\delta(x) \geqslant \Delta\varphi \quad \text{in } K .$$

This completes the proof.

Remark. Consider the obstacles

$$\varphi_\varepsilon(r) = 1 - \frac{r^2}{2n} + \varepsilon r^4 \qquad (\varepsilon \leqslant 0)$$

in the ball with center 0 and radius $\varrho > (2n)^{\frac{1}{2}}$. If $\varepsilon = 0$ then $\Delta^2\varphi \geqslant 0$ and, since $\varphi_0(\varrho) < 0$, Corollary 8.2 shows that the non-coincidence set is connected. It follows that the coincidence set C_0 consists of a ball $r < \alpha_0$; since $\varphi_0(0) = 1 > 0$, we must have $\alpha_0 > 0$. Thus

$$C_0 = \{x; r < \alpha_0\}, \quad \alpha_0 > 0 . \tag{8.5}$$

On the other hand, if $\varepsilon < 0$ then $\Delta^2\varphi_\varepsilon \leqslant - c_\varepsilon$ $(c_\varepsilon > 0)$ so that $\Delta(\Delta u - \Delta\varphi_\varepsilon) = \Delta^2 u - \Delta^2\varphi_\varepsilon \geqslant c_\varepsilon$. It follows that

$$\text{(8.6)} \qquad \int_{B_R} \Delta(u - \varphi_\varepsilon)\,dx \quad \text{is strictly increasing in } R$$

where B_R is a ball with center 0 and radius R. If the free boundary contains two spheres ∂B_{R_1}, ∂B_{R_2} then

$$\int_{B_{R_i}} \Delta(u - \varphi_i)\,dx = \int_{\partial B_{R_i}} \frac{\partial}{\partial \nu}(u - \varphi)\,dS = 0\,,$$

which contradicts (8.6). It follows that

$$\text{(8.7)} \qquad C_\varepsilon \text{ consists of just one sphere } \partial B_{R_\varepsilon}$$

where C_ε is the coincidence set. This example illustrates the unstable behavior of the non-coincidence set as a function of the obstacle.

9. – The behavior of the free boundary.

In this section we shall study the free boundary F. Suppose F_0 is a subset of F and N_0 is an open subset of the non-coindicence set N such that

$$\text{(9.1)} \qquad N_0 \cup F_0 \quad \text{is a domain},$$

$$\text{(9.2)} \qquad F_0 \quad \text{has zero capacity}.$$

The last condition holds if F_0 is contained in a smooth $(n-2)$-dimensional manifold.

Since Δu is harmonic in N_0 and is bounded in $N_0 \cup F_0$, it has a removable singularity at all the points of F_0 (see [5]). Thus F_0 is an « incidental » coincidence set; if we modify the obstacle by lowering the values of φ on F_0, the solution u does not change.

In what follows we shall assume that $n = 2$. Recall that in this case u is in $C^2(\Omega)$; in particular, Δu is continuous.

THEOREM 9.1. *Let $P_0 = (x_0, y_0)$ belong to F. If $\Delta u(P_0) > \Delta\varphi(P_0)$ then there exists a neighborhood W of P_0 such that $F \cap W$ is contained in a C^1 curve.*

PROOF. We choose W so that $\Delta u > \Delta\varphi$ in W. For any $P \in F \cap W$, the free boundary points Q can approach P in at most one direction l_P. Indeed. if there are two directions l_P and l'_P then

$$\frac{\partial^2}{\partial l_P}(u-\varphi) = 0\,, \qquad \frac{\partial^2}{\partial l'_P}(u-\varphi) = 0 \quad \text{at } P\,.$$

Since however $u - \varphi \geqslant 0$ and $u - \varphi = 0$, $\nabla(u - \varphi) = 0$ at P, it follows that $\partial^2(u - \varphi)/\partial l^2 = 0$ at P for any direction l. In particular $\Delta(u - \varphi)(P) = 0$ which contradicts the choice of W.

Suppose P is not an isolated point of $F \cap W$. Introducing Cartesian coordinates (ξ, η) in which l_P is in the η-axis and $P = (0, 0)$, we then have

$$\frac{\partial^2}{\partial \eta^2}(u-\varphi) = 0\,, \qquad \frac{\partial^2}{\partial \xi^2}(u-\varphi) > 0 \qquad \text{at } (0,0)\,.$$

Consequently,

$$u - \varphi > \alpha\xi^2 + o(\xi^2 + \eta^2) \qquad (\alpha > 0)\,.$$

It follows that the intersection of the coincidence set C with a small neighborhood of P lies inside a set consisting of two cusp-like regions about the l_P axis.

If P is an isolated point of F, we denote by l_P a direction l along which $\partial^2(u - \varphi)(P)/\partial l^2$ is minimum. Since

$$\frac{\partial^2}{\partial m^2}(u-\varphi)(P) \geqslant \frac{1}{2}\Delta(u-\varphi)(P) > 0$$

if m is orthogonal direction to l_P, we again conclude (by expanding $u - \varphi$ about P by Taylor's formula) that the intersection of the set C with a small neighborhood of P is contained in a set consisting of two cusp-like regions about the axis l_P.

To be more precise we introduce the set

$$D_{\alpha,h,\eta} = \{(x, y)\,;\, x^2 + y^2 < h^2,\ \alpha x^2 + \eta(x^2 + y^2) \leqslant 0\}$$

where $\alpha > 0$, $h > 0$, $\eta(t) \to 0$ if $t \to 0$, $\eta(t) > 0$ if $t > 0$. Let $D_{\alpha,h,\eta}(P)$ be the set obtained from $D_{\alpha,h,\eta}$ by performing a translation $(0, 0) \to P$ and a rotation of the y-axis into l_P. Then, there exist constants α, h and a function $\eta(t)$ such that

(9.3) for any $P \in F \cap W$ the set $C \cap \{h\text{-neighborhood of } P\}$ is contained in $D_{\alpha,h,\eta}(P)$.

This follows from the previous remarks and the fact that D^2u is uniformly continuous in compact subsets of Ω. From the latter fact and the definition of l_P it also follows that

(9.4) for any P, Q in $F \cap W$, if $\theta =$ angle between l_P, l_Q, then

$$\theta < \sigma(|P-Q|), \text{ where } \sigma(t) \to 0 \text{ if } t \to 0 .$$

Take for simplicity $P_0 = (0, 0)$ and $l_{P_0} = y$-axis. For any β with $|\beta|$ sufficiently small, consider the line $y = \beta$. It intersects $F \cap W$ in at most one point (if W is chosen sufficiently small). Indeed, suppose that it intersects $F \cap W$ in two points $P_1 = (x_1, \beta)$, $P_2 = (x_2, \beta)$. Then, by (9.3), (9.4), l_{P_0} forms a small angle with l_{P_1}, which is the y-axis, and also a small angle with P_1P_2, which is the x-axis; this is of course impossible.

We have thus proved that the points of $F \cap W$ coincide with a graph $x = \psi_0(y)$ where y varies in a closed subset of an interval $(-\gamma, \gamma)$. We can complete it linearly into a graph $x = \psi(y)$ and, in view of (9.3), (9.4), $\psi(y)$ is a Lipschitz continuous function. Thus

(9.5) $F \cap W$ lies on a Lipschitz curve $x = \psi(y)$.

In order to reconstruct a C^1 curve $x = \tilde{\psi}(y)$ which extends $x = \psi_0(y)$, we take any partition of $(-\gamma, \gamma)$ into m intervals and in each interval choose a point of $F \cap W$, if such a point exists. We connect two adjacent points P_1, P_2 by a C^1 parabolic curve $x = \lambda(y)$ such that the tangents at P_1, P_2 coincide with l_{P_1}, l_{P_2} respectively. Denote this curve by $x = \psi_m(y)$ and the modulus of continuity of the derivative $\psi_m'(y)$ by $\sigma_m(t)$. Using (9.3), (9.4) it follows that $\sigma_m(t)$ is bounded by a modulus of continuity $\sigma(t)$ independent of m, but depending on the σ in (9.4).

By the Ascoli-Arzelà theorem, there exists a subsequence of ψ_m which is convergent to a C^1 function $\tilde{\psi}$. Since $\tilde{\psi}(y) = \psi(y)$ on a set of y's for which the points $(y, \psi(y))$ form a dense subset of $F \cap W$, we conclude that $\tilde{\psi}(y)$ is an extension of $\psi_0(y)$. This completes the proof.

In Theorem 9.1 we have assumed that $\Delta u > \Delta \varphi$ at P_0. We shall now consider the case where $\Delta u = \Delta \varphi$ in $F \cap W$, where W is a neighborhood of P_0. First we establish two lemmas.

LEMMA 9.2. *Let $P_0 \in F$ and let N_0 be a component of N such that $P_0 \in \partial N_0$. Then there exists a sequence of points Q_m in N_0 such that*

(9.6) $$\Delta(u-\varphi)(Q_m) > 0 , \quad Q_m \to P_0 .$$

PROOF. Otherwise there is a ball B_ε with center P_0 and radius $\varepsilon > 0$

such that

$$\Delta(u-\varphi)\leqslant 0 \quad \text{in } B_\varepsilon \cap N_0 .$$

Since $u-\varphi=0$ on ∂N_0 and $u-\varphi\geqslant 0$ elsewhere in $B_\varepsilon \cap N_0$, the maximum principle gives

$$\frac{\partial}{\partial\nu}(u-\varphi)\neq 0 \quad (\nu \text{ normal to } \partial N_0) \tag{9.7}$$

at any point $Q\in B_\varepsilon \cap \partial N_0$ which has the inside disc property (that is, there is an open disc D such that $D\subset N_0$, $\bar{D}\cap \partial N_0=\{Q\}$). Since such points Q clearly exist, the inequality (9.7) must hold at some points of $B_\varepsilon\cap \partial N_0$; but this is impossible since $\nabla(u-\varphi)=0$ on the free boundary.

LEMMA 9.3. *Let $P_0\in F$ and denote by B_ϱ the disc with center P_0 and radius ϱ. If $\Delta^2\varphi(P_0)<0$ and $\Delta(u-\varphi)=0$ in $F\cap B_R$ for some $R>0$, then, for all sufficiently small ϱ,*

$$\sup_{\partial B_\varrho \cap N_0} \Delta(u-\varphi)\geqslant c\varrho^2$$

where N_0 is any component of N with $P_0\in\partial N_0$ and c is a positive constant.

PROOF. Let $Q_m\in B_\varrho\cap N_0$ be such that (9.6) is satisfied and let $r_m=$ the distance function from Q_m. Consider the function

$$v=\Delta(u-\varphi)-cr_m^2 \quad \text{in } B_\varrho\cap N_0 .$$

It satisfies

$$\Delta v=-\Delta^2\varphi-4c>0 \quad \text{in } B_\varrho\cap N_0 \ (c>0) \tag{9.8}$$

if $\varrho\leqslant\varrho_0$ and ϱ_0, c are sufficiently small. Also

$$\begin{aligned} &v(Q_m)>0 , \\ &v(Q)\;=-cr_m^2<0 \quad \text{on } B_\varrho\cap F . \end{aligned} \tag{9.9}$$

From (9.8) it follows that v takes its maximum on the boundary. From (9.9) it follows that the maximum is positive and is attained on $\partial B_\varrho\cap N_0$. Since $r_m\geqslant\varrho-|Q_m-P_0|$ on $\partial B_\varrho\cap N_0$, we get, after taking $m\to\infty$,

$$\sup_{B_\varrho\cap N_0}\Delta(u-\varphi)=\sup_{\partial B_\varrho\cap N_0}\Delta(u-\varphi)\geqslant c\varrho^2 .$$

THEOREM 9.4. *Let $P_0\in F$, $\Delta^2\varphi(P_0)<0$, and assume that $\Delta(u-\varphi)=0$ in $F\cap W$, where W is a neighborhood of P_0. Let N_0 be a component of N and* *let Γ_1, Γ_2 be two curves lying in ∂N_0, initiating at P_0, and forming an angle α (with respect to N_0) at P_0. Then $\alpha \geqslant \pi/2$.*

PROOF. Let R_δ denote the region in N_0 bounded by Γ_1, Γ_2 and a circle S_δ with center P_0, radius δ; δ is sufficiently small. If $\alpha < \pi/2$, then choose β such that $\alpha < \beta < \pi/2$ and assume for simplicity that $P_0 = (0, 0)$ and that the bisector of Γ_1, Γ_2 at P_0 is the positive x-axis. Consider the function

$$v = \Delta(u - \varphi) - C\varrho^{\pi/\beta} \cos\frac{\pi\theta}{\beta} \qquad \text{in } R_\delta \ (C > 0),$$

where (ϱ, θ) are the polar coordinates. Clearly

$$v \leqslant 0 \qquad \text{on } (\Gamma_1 \cup \Gamma_2) \cap \partial R_\delta$$

if δ is sufficiently small. Choosing $C = C(\delta)$ sufficiently large we also have

$$v \leqslant 0 \qquad \text{on } S_\delta \cap \partial R_\delta .$$

Since $\Delta v = \Delta^2(u - \varphi) = -\Delta^2\varphi > 0$ in R_δ, the maximum principle gives $v \leqslant 0$ in R_δ. Thus

$$\Delta(u - \varphi) \leqslant C\varrho^{\pi/\beta} \cos\frac{\pi\theta}{\beta} .$$

Taking $\theta = 0$ we get $\Delta(u - \varphi) \leqslant C\varrho^{\pi/\beta}$, which contradicts Lemma 9.3 since $(\pi/\beta) > 2$. Thus α must be $\geqslant \pi/2$.

10. – The obstacle problem when $\Delta u = 0$ on $\partial\Omega$.

Consider the variational inequality of minimizing

$$\int_\Omega |\Delta v|^2 dx, \qquad v \in K$$

where K is given by

$$K = \{v \in H^2(\Omega) \cap H_0^1(\Omega),\ v \geqslant \varphi\} . \tag{10.1}$$

We denote the unique solution by u. From the variational principle we deduce that, in a generalized sense,

$$\Delta u = 0 \qquad \text{on } \partial\Omega . \tag{10.2}$$

All the local results of Sections 2-9 renain valid for the present problem. We shall now establish some additional results for this case. We shall assume that $\partial\Omega$ is in C^4.

THEOREM 10.1. *If w is the upper semicontinuous version of Δu, then*

$$\inf_{\Omega} \Delta\varphi \leqslant w \leqslant 0\,, \tag{10.3}$$

$$u \geqslant 0\,. \tag{10.4}$$

PROOF. We have

$$\int_{\Omega} \Delta u \cdot \Delta(v - u) \geqslant 0 \qquad \text{for any } v \in K\,.$$

Let B be a ball with center x^0 and with closure in Ω and let ψ be the solution of

$$\Delta\psi = -\chi_B \quad \text{in } \Omega\,,$$

$$\psi \in H^2(\Omega) \cap H^1_0(\Omega)\,.$$

Then $\psi \geqslant 0$ in Ω and therefore $v = u + \varphi$ belongs to K. It follows that

$$\int_{\Omega} \Delta u \cdot \Delta\varphi \geqslant 0\,,$$

that is,

$$\int_{\Omega} \Delta u \leqslant 0\,.$$

Hence $w(x^0) \leqslant 0$, and the second inequality in (10.3) follows.

To prove the first inequality in (10.3), consider first the obstacle $\varphi - \varepsilon$ and denote the corresponding solution by u_ε. Since $\Delta u_\varepsilon \leqslant 0$ and $u_\varepsilon \in H^2(\Omega) \cap$ $\cap\, H^1_0(\Omega)$, the maximum principle gives $u_\varepsilon \geqslant 0$ in Ω. Since $\varphi_\varepsilon < 0$ in Ω_δ (= the intersection of Ω with a δ-neighborhood of $\partial\Omega$) if δ is sufficiently small, we have $u_\varepsilon > \varphi_\varepsilon$ in Ω_δ. Consequently,

$$\Delta(\Delta u_\varepsilon) = 0 \quad \text{in } \Omega_\delta\,.$$

Since $\Delta u_\varepsilon = 0$ on $\partial\Omega$ in a generalized sense, it follows (by [8]) that Δu_ε is actually smooth up to $\partial\Omega$ and vanishes on $\partial\Omega$.

Now, Δu_ε is subharmonic function in Ω, and it is continuous and vanishes on $\partial\Omega$. By the maximum principle for subharmonic functions,

$$\Delta u_\varepsilon \geqslant \inf\{\Delta u_\varepsilon(y);\ y \text{ varies on the support of } \Delta^2 u_\varepsilon\}\,.$$

For such points y, $\Delta u_\varepsilon(y) \geqslant \Delta\varphi_\varepsilon(y)$ (by Theorem 2.2). Hence

$$\Delta u_\varepsilon \geqslant \inf_{\Omega} \Delta\varphi_\varepsilon .$$

Taking $\varepsilon \to 0$ so that $u_\varepsilon \to u$ weakly in $H^2(\Omega)$, we find that $\Delta u \geqslant \inf \Delta\varphi$ a.e. Appealing to the upper semicontinuity of w, the first inequality in (10.3) follows.

The inequality (10.4) is a consequence of the maximum principle, since $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\Delta u \leqslant 0$ (by (10.3)).

From (10.3) we see that $\Delta u \in L^\infty(\Omega)$. Hence:

COROLLARY 10.2. *$u \in C^{1+\beta}(\bar{\Omega})$ for any $0 < \beta < 1$.*

We now make the assumptions:

$$(10.5) \qquad \varphi < 0 , \quad \Delta\varphi \leqslant 0 \quad \text{on } \partial\Omega ,$$

$$(10.6) \qquad \Delta^2\varphi \geqslant 0 \quad \text{in } \Omega .$$

THEOREM 10.3. *If (10.5), (10.6) hold then $w \geqslant \Delta\varphi$ in Ω.*

PROOF. Consider the function

$$v = w - \Delta\varphi .$$

On the support of $\mu = \Delta^2 u$, $v \geqslant 0$ by Lemma 2.2, and on $\partial\Omega$, $v = 0 - \Delta\varphi \geqslant 0$. Further, on $\Omega \setminus (\operatorname{supp} \mu)$,

$$\Delta v = \Delta w - \Delta^2\varphi = -\Delta^2\varphi \leqslant 0 .$$

Thus, if w is continuous in Ω then we can apply the maximum principle to v and deduce that $v \geqslant 0$ in $\Omega \setminus (\operatorname{supp} \mu)$, i.e., $w \geqslant \Delta\varphi$.

Since we do not know that w is continuous, we have to proceed more carefully, analogously to the proof of Theorem 8.1. We first represent w in the form

$$\begin{aligned} w(x) &= -\int V(x-y)\, d\mu(y) + \int W(x, y)\, d\mu(y) \\ &\equiv -\zeta(x) + \eta(x) \end{aligned}$$

where V is the fundamental solution for Δ and $V - W$ is Green's function for Δ in Ω. We then approximate ζ by $\zeta^\delta \geqslant \zeta$ such that ζ^δ is continuous in $\bar{\Omega}$, $w^\delta = -\zeta^\delta + \eta$ is subharmonic and $\Delta\zeta^\delta$ is supported on a closed subset F_δ of Ω (with meas $(\Omega \setminus F_\delta) < \delta$). We apply the maximum principle to the continuous function $v^\delta - \Delta\varphi$ and deduce that $v^\delta \geqslant \Delta\varphi$. Taking $\delta \to 0$, the assertion $w \geqslant \Delta\varphi$ follows.

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