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On the Rationality of the Period Mapping.

ANDREW JOHN SOMMESE (*)

Summary. – *Let S be a Zariski open set of a connected projective manifold $\bar{S}$, where $\bar{S} - S$ has only normal crossing singularities. Let $q: X \to S$ be a proper maximal rank holomorphic surjection with connected fibres such that $R^2_{q_*}(X, \mathbf{Z})$ has a section whose restriction to each fibre of q is represented by a Kaehler form. Let $p: S \to \Gamma \backslash D$ be the associated period mapping. By a theorem of Griffiths there exists a Zariski open set $\tilde{S}$ of $\bar{S}$ containing S to which p extends as an holomorphic proper map $\tilde{p}: \tilde{S} \to \Gamma \backslash D$. Let $\mathcal{N}(\tilde{p}(\tilde{S}))$ denote the normalization of $\tilde{p}(\tilde{S})$ and let $\mathcal{N}(\tilde{p})$: $\tilde{S} \to \mathcal{N}(\tilde{p}(\tilde{S}))$ denote the associated map. It is shown that* there exists a normal quasi-projective variety M and a holomorphic proper map $\Phi: M \to \mathcal{N}(\tilde{p}(\tilde{S}))$ such that:

1) Φ is a biholomorphism from $M - \Phi^{-1}(\mathfrak{D})$ to $\mathcal{N}(\tilde{p}(\tilde{S})) - \mathfrak{D}$ where $\mathfrak{D}$ is the degeneracy set of $\mathcal{N}(\tilde{p})$, i.e. the set of those points $y \mu \mathcal{N}(\tilde{p}(\tilde{S}))$ such that $\dim_{\mathbf{C}} \mathcal{N}(\tilde{p})^{-1}(y) > \dim_{\mathbf{C}} \tilde{S} - \dim_{\mathbf{C}} \mathcal{N}(\tilde{p}(\tilde{S}))$;
2) $\Phi^{-1}(\mathfrak{D})$ is algebraic and $\mathfrak{D}$ is the degeneracy set of Φ;
3) $\Phi^{-1} \circ \mathcal{N}(\tilde{p}): \tilde{S} \to M$ is a meromorphic rational map, and given any bimeromorphic map $\Psi: \mathcal{N}(\tilde{p}(\tilde{S})) \to M'$, where M' is a normal quasi-projective variety such that $\Psi \circ \mathcal{N}(\tilde{p})$ is a rational meromorphic map, then $\Psi \circ \Phi$ gives a birational equivalence of M and M';
4) let $\mathcal{U} \subseteq \mathcal{N}(\tilde{p}(\tilde{S}))$ be the image under $\mathcal{N}(\tilde{p})$ of those points of S at which dp is of maximal rank; then $\mathcal{U}$ is open and possesses a quasi-projective structure compatible with Φ and M;
5) if Γ is torsion free then given any holomorphic map $A: Z \to \mathcal{N}(\tilde{p}(\tilde{S}))$ with Z a normal quasi-projective variety and $A(Z)$ containing an open set of $\mathcal{N}(\tilde{p}(\tilde{S}))$, then $\Phi^{-1} \circ A$ is rational.

One corollary of the above is that if $\dim_{\mathbf{C}} \tilde{p}(\tilde{S}) \leqslant 2$ *then* $\mathcal{N}(\tilde{p}(\tilde{S}))$ *is Zariski open in a compact Moisezon space.*

To each proper holomorphic maximal rank surjection $q: X \to S$ of connected complex manifolds with connected fibres, such that $R^2_{q_*}(X, \mathbf{Z})$

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has a section whose restriction to each fibre of q is represented by a Kaehler form, one has associated an holomorphic map $p: S \to \Gamma\backslash D$, called the period mapping [cf. 10, 11, 12, 13, 24], where $\Gamma\backslash D$ is a complex space that is generally not quasi-projective in any way compatible with its analytic structure. If S is Zariski open in a projective manifold $\bar{S}$ and $\bar{S} - S$ has only normal crossing singularities, then by a theorem of Griffiths there is a Zariski open set $\tilde{S} \subseteq \bar{S}$ that contains S and to which p extends as a proper holomorphic map $\tilde{p}$. It is a basic question [cf. 11, pg. 259 for a discussion] whether if S is quasi-projective then $\tilde{p}(\tilde{S})$ possesses a functorial algebraic structure such that $\tilde{p}$ is rational.

This is known [7] if the fibres of q are either one-dimensional, abelian varieties, or $K - 3$ surfaces, since then D is a bounded Hermitian symmetric domain. Griffiths [10, III.9.7] showed it if $\tilde{p}(\tilde{S})$ is compact and the author [29, 30] showed it if p is proper, and the image has at most isolated singularities.

In this article I use the methods of [29, 30] to prove the above mentioned conjecture on the function field level. More precisely I show (Proposition IV) that when S is a quasi-projective manifold and $p: S \to \Gamma\backslash D$ arises from geometry as mentioned in the first paragraph, then $\tilde{p}(\tilde{S})$ possesses a quasi-projective desingularization $\delta: Y \to \tilde{p}(\tilde{S})$ such that the meromorphic map $\delta^{-1} \circ p$ is rational. Furthermore when Γ is torsion free then the negative curvature of D in horizontal directions forces on $\mathcal{N}(\tilde{p}(\tilde{S}))$ the GAGA property [25] that given a second quasi-projective desingularization $\delta': Y' \to \tilde{p}(\tilde{S})$, then $(\delta')^{-1} \circ \delta: Y \to Y'$ is a birational equivalence.

In fact it is shown that if $\mathcal{N}(\tilde{p}): \tilde{S} \to \mathcal{N}(\tilde{p}(\tilde{S}))$ denotes the map from $\tilde{S}$ to the normalization, $\mathcal{N}(\tilde{p}(\tilde{S}))$, of $\tilde{p}(\tilde{S})$ associated to $\tilde{p}$, then $\mathcal{N}(\tilde{p}(\tilde{S})) - \mathfrak{D}$ is quasi-projective where $\mathfrak{D}$ is the degeneracy set of $\mathcal{N}(\tilde{p})$, i.e. the set of points y of $\mathcal{N}(\tilde{p}(\tilde{S}))$ with $\dim_{\mathbf{C}} \mathcal{N}(\tilde{p})^{-1}(y)$ greater than $\dim_{\mathbf{C}} \tilde{S} - \dim_{\mathbf{C}} \mathcal{N}(\tilde{p}(\tilde{S}))$.

§ I is devoted to background material and especially the theory of the generalized canonical sheaf of Grauert and Riemenschneider [9].

In § II some criteria, based on § I, the L^2 estimates for $\bar{\partial}$, and Kodaira's proof of his embedding theorem, are given for the image of a proper holomorphic map to be bimeromorphic to a quasi-projective variety in a way compatible with the proper map.

In § III some results about proper maps are given that allow us to sharpen the results of § II.

In § IV I prove my results about the period mapping.

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§ I. – I essentially follow the notation of [29]. If Y is a complex space and E is a holomorphic vector bundle on Y, then $\mathcal{O}(E)$ denotes the sheaf of germs of holomorphic sections of E, and $\mathcal{O}_Y$, the holomorphic structure sheaf. If Y is a complex manifold, then K_Y, $\Lambda^q T^*_Y$, and Ω^*_Y denote the canonical bundle of Y, the q-th exterior power of the holomorphic cotangent bundle of Y, and $\mathcal{O}(\Lambda^q T^*_Y)$ respectively. If F and G are holomorphic vector bundles on a complex space Y, then $F \otimes G$ denotes the tensor product over $\boldsymbol{C}$; if $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on H, then $\mathcal{F} \otimes \mathcal{G}$ denotes the tensor product over $\mathcal{O}_Y$. Denote the manifold points of a reduced analytic space Y by Reg Y.

If Y is a reduced and irreducible analytic space, recall $\mathcal{K}_Y$, the generalized canonical sheaf of Grauert and Riemenschneider [9]. Given a desingularization $A\colon \tilde{Y} \to Y$, then $\mathcal{K}_Y = A_* \mathcal{O}(K_{\tilde{Y}})$. This definition is local and doesn't depend on the given desingularization. To see this one first notes that by Hironaka's desingularization theorem [17], if one has a second desingularization $B\colon W \to Y$, one can find another complex manifold $\tilde{W}$ and proper modifications $C\colon \tilde{W} \to \tilde{Y}$ and $D\colon \tilde{W} \to W$ such that $A \circ C = B \circ D$. Now we would be done if one know that $C_* \mathcal{O}(K_{\tilde{W}}) = \mathcal{O}(K_{\tilde{Y}}) = D_* \mathcal{O}(K_{\tilde{W}})$, or in other words that $\mathcal{K}_Y$ is functorially isomorphic with the canonical sheaf at manifold points of Y. This follows immediately from the following [9]:

LEMMA I-A. *Let V be a normal connected analytic space with $S \subset V$ an analytic set containing the singular set of V. Let ω be a holomorphic n form on $V - S$ with $n = \dim_{\boldsymbol{C}} V$. Assume $(\sqrt{-1})^{n^2} \int_{V-S} \omega \wedge \bar{\omega} < \infty$. Then given any desingularization $q\colon \tilde{V} \to V$, $q^* \omega$ extends to an holomorphic n form on $\tilde{V}$.*

PROOF. Clearly

$$(\sqrt{-1})^{m2} \int_{\tilde{V} - q^{-1}(S)} (q^* \omega) \wedge (\overline{q^* \omega}) < \infty .$$

One now uses Fubini's and Hartog's theorems to reduce to the punctured disc where it is obvious. Q.E.D.

This gives an important characterization [9] of $\mathcal{K}_Y$ for a normal, irreducible analytic space with $\dim_{\boldsymbol{C}} Y = n$. At a point $y \in Y$, $\mathcal{K}_{Y|y}$ consists of germs of holomorphic n forms, ω, defined on the intersection of some neighborhood U of y and $\mathrm{Reg}(Y)$ and such that

$$(\sqrt{-1})^{n2} \int_{n \cap \mathrm{Reg}(Y)} \omega \wedge \bar{\omega} < \infty .$$

The following is needed.

LEMMA I-*B*. *If $\varphi: X \to Y$ is a proper holomorphic surjection from a connected complex manifold X onto a normal analytic space Y, then $\varphi^* \mathcal{K}_Y$ is functorially isomorphic to a subsheaf of Ω_X^n where $n = \dim_C Y$.*

PROOF. To see this, Remmert-Stein factorize φ as $\varphi_2 \circ \varphi_1$ where $\varphi_1: X \to \tilde{X}$ is a proper surjection with connected fibres onto a normal analytic space $\tilde{X}$ and $\varphi_2: \tilde{X} \to Y$ is a proper finite to one holomorphic map. Using the above characterization of $\mathcal{K}_Y$ by means of integration one immediately sees that $\varphi_2^* \mathcal{K}_Y$ is functorially isomorphic to a subsheaf of $\mathcal{K}_{\tilde{X}}$. Thus one may assume φ has connected fibres.

Now what must be shown is that given a point $y \in Y$ and any neighborhood U of y and a section ω of $\mathcal{K}_U$, then $p^* \omega|_{\varphi^{-1}(\mathrm{Reg} U)}$ has a unique extension to $\varphi^{-1}(U)$ as an holomorphic n form. Let $p: \tilde{Y} \to Y$ be a desingularization of Y and X_p, the irreducible component of the fibre product of p and φ that surjects on X. One has the commutative diagram

$$\begin{array}{ccc} X_p & \xrightarrow{\;q\;} & X \\ {\scriptstyle\tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \tilde{Y} & \longrightarrow & Y \end{array}$$

where $\tilde{p}$ and $\tilde{\varphi}$ are the holomorphic maps induced by p and φ, respectively. Since φ has connected fibres, $\tilde{p}$ is a modification. Let $p': X' \to X$- be a desingularization and note that $\tilde{p} \circ p': X' \to X$ is a modification since $\tilde{p}$ is a modification. Now given y, ω, and U as above, one can assume that Y is U since the desired result depends only on $\varphi: \varphi^{-1}(U) \to U$. Now $p^* \omega|_{p^{-1}(\mathrm{Reg}(Y))}$ extends to be an holomorphic n form γ on Y by lemma I-*A*. Thus $(\tilde{\varphi} \circ p')^* \gamma$ is an holomorphic n form on X', and in particular $(\varphi \circ \tilde{p} \circ p')^* \omega|_{(\varphi \cdot \tilde{p} \cdot p')^{-1}(\mathrm{Reg}(Y))}$ extends to be an holomorphic n form on X'. Now $\tilde{p} \circ p'$ is a biholomorphism of a Zariski open set of X' with $X - S$ where the codimension of S is at least two. Thus $\varphi^* \omega|_{\varphi^{-1}(\mathrm{Reg}(Y))}$ extends to an holomorphic n form on $X - S$. Since S is of codimension at least two, $\varphi^* \omega$ extends to an holomorphic n form on X. Q.E.D.

Now recall some basic facts about positivity. I refer to the good account of Siu [27, p. 58 ff.]. If V is a complex vector space and $\{e_1, \ldots, e_n\}$

is a basis of the dual V^*, then a (k, k) covector $u \in (\Lambda^k V^*) \otimes (\Lambda^k \overline{V}^*)$ is said to be positive if for any $\{w_{k+1}, \dots, w_n\} \subseteq V^*$ the complex number α in:

$$u \wedge \left(\frac{\sqrt{-1}}{2} w_{k+1} \wedge \overline{w}_{k+1}\right) \wedge \dots \wedge \left(\frac{\sqrt{-1}}{2} w_n \wedge \overline{w}_n\right) = \\ = \alpha \left(\frac{\sqrt{-1}}{2} e_1 \wedge \overline{e}_1\right) \wedge \dots \wedge \left(\frac{\sqrt{-1}}{2} e_n \wedge \overline{e}_n\right)$$

is non-negative. A continuous (k, k) form u on a complex manifold Y is said to be positive if at each point y of Y, u is a positive (k, k) covector of $T_Y|_y$.

If u is a positive (k, k) form and w is a positive $(1, 1)$ form, then $u \wedge w$ is a positive $(k+1, k+1)$ form.

One has a partial order on the (k, k) forms by $u \leqslant v$ if $v - u$ is positive. Note further that if w and $\mathcal{E}$ are two positive $(1, 1)$ forms on a pure n dimensional complex manifold with $\mathcal{E} \leqslant w$, then $u \wedge \mathcal{E} \leqslant u \wedge w^r$ for any positive (k, k) form u. To see this note that $u \wedge w^r - u \wedge \mathcal{E}^r = u \wedge w^{r-1} \wedge (w - \mathcal{E}) + u \wedge w^{r-2} \wedge (w - \mathcal{E}) \wedge \mathcal{E} + \dots + u \wedge (w - \mathcal{E}) \wedge \mathcal{E}^{r-1}$.

If η is a $(k, 0)$ form on a complex manifold X, then $(\sqrt{-1})^{k^2} \eta \wedge \bar{\eta}$ is positive. One consequence of this that will be needed below is that if η is a $(k, 0)$ form on a polydisc Δ^n, and w is an Hermitian $(1, 1)$ form on Δ^n which is greater than $\mathcal{E}$, the Euclidean Kaehler form, then

$$(\sqrt{-1})^{k^2} \eta \wedge \bar{\eta} \wedge w^{n-k} \geqslant (\sqrt{-1})^{k^2} \eta \wedge \bar{\eta} \wedge \mathcal{E}^{n-k}.$$

Now given a continuous Hermitian structure on an holomorphic line bundle L on a normal analytic space Y, then $L^2(L \otimes \mathcal{K}_Y)$ will denote the space of L^2 sections of $L|_{\mathrm{Reg}(Y)} \otimes K_{\mathrm{Reg}(Y)}$. Any element of $L^2(L \otimes \mathcal{K}_Y)$ which is holomorphic on $\mathrm{Reg}(Y)$ extends, by the integration characterization of $\mathcal{K}_Y$, to a section over Y of $\mathcal{O}(L) \otimes \mathcal{K}_Y$.

Now to globalize lemma I-A and I-B we need the concept of an Hermitian structure with L^2 poles at infinity. This is basically the same as the « croissance modérée » of [8, II.2.17] except that the L^2 version is better adapted to our needs. Let X be a Zariski open set in a connected complex manifold $\overline{X}$. Given a manifold point $p \in \overline{X} - X$ one can find a neighborhood U of p in $\overline{X}$ biholomorphic to a polydisc Δ^n and $U \cap X$ biholomorphic to $(\Delta^*)^a \times \Delta^{n-a}$ where a is the codimension of the irreducible component of $\overline{X} - X$ to which p belongs. Let the coordinates be denoted $(z_1, \dots, z_n)$ with $\{z_i = 0 | 1 \leqslant i \leqslant a\} = \Delta^n \cap (\overline{X} - X)$.

Now let L be an holomorphic line bundle on X. *A continuous Hermitian structure,* $\|\ \|$, *on* $L|_X$, *is said to have* L^2 *poles at infinity relative to* L *if given* *any irreducible component Z of $\bar{X}-X$, of codimension one in $\bar{X}$, one can find a point $p \in Z$ and a neighborhood U of p in $\bar{X}$ such that:*

1) *p is a manifold point of $\bar{X}-X$,*

2) *U is as in the last paragraph*

3) *there is a trivializing holomorphic section, s, of L over U and an integer N such that $\int_{\Delta^* \times \Delta^{n-1}} |z_1|^N \|s\|^{-2} dl < \infty$ where dl is Lebesgue measure on $\Delta^* \times \Delta^{n-1}$.*

LEMMA I-C. *Let X be a Zariski open set of a connected Kaehler manifold $\bar{X}$. Let $p: X \to Y$ be a proper holomorphic surjection onto a normal analytic space Y. Let L be an holomorphic line bundle on Y such that p^*L extends to an holomorphic line bundle $\mathfrak{L}$ on $\bar{X}$. Let L have a continuous Hermitian structure on Y such that the induced Hermitian structure on p^*L has L^2 poles at infinity relative to $\mathfrak{L}$. Then given a section s of $\mathcal{O}(L) \otimes \mathcal{K}_Y$ with $s \in L^2(L \otimes \mathcal{K}_Y)$, p^*s extends to $\bar{X}$ as a meromorphic section of $\mathfrak{L} \otimes \Lambda^k T^*_{\bar{X}}$ where $\dim_{\mathbf{C}} Y = k$.*

PROOF. Let ω denote the Kaehler form on $\bar{X}$. Let ν be the (k, k) form on $\operatorname{Reg}(Y)$ constructed from s. In local coordinates $\{w_1, \ldots, w_k\}$ at some point $y \in \operatorname{Reg}(Y)$, one has $s = e \otimes dw_1 \wedge \ldots \wedge dw_k$ where e is an holomorphic section of L in a neighborhood of y and

$$\nu = \|e\|^2 \left(\frac{\sqrt{-1}}{2} dw_1 \wedge d\bar{w}_1\right) \wedge \ldots \wedge \left(\frac{\sqrt{-1}}{2} dw_k \wedge d\bar{w}_k\right).$$

One has $\int_{\operatorname{Reg}(Y)} \nu < \infty$. Note p^*s gives rise to an holomorphic section of $p^*L \otimes \wedge^k T^*_X$ by lemma I-*B*. Thus $p^*\nu$ gives rise to a well defined continuous (k, k) form on X denoted by γ.

Note that $\int_X \gamma \wedge \omega^{n-k} < \infty$ where $\dim_{\mathbf{C}} X = n$. To see this note that there is a dense open set V of $\operatorname{Reg} Y$ and hence of Y such that $p: p^{-1}(U) \to U$ is a C^∞ fibre bundle. Now by Fubini's theorem one has: $\int_X \gamma \wedge \omega^{n-k} = \int_V \left(\int_{p^{-1}(y)} \omega^{n-k}\right) \nu$ but since X is Kaehler $\int_{p^{-1}(y)} \omega^{n-k}$ is independent of $y \in V$, and thus the integral over X is finite.

Now by Hartog's extension theorem, to show extension it suffices to get p^*s to extend over a neighborhood of one point of each irreducible codimension one component of $\bar{X}-X$. Given a codimension one component Z of $\bar{X}-X$, let p, U, e, and z_1 be as given in the definition of L^2 poles at infinity. Letting $s = e \otimes \eta$ where η is an holomorphic k form on $U \cap X$, one has $(\sqrt{-1})^{k^2}\int\limits_{U\cap X}\|e\|^2\eta\wedge\bar{\eta}\wedge\omega^{n-k}<\infty$ and by the definition of L^2 poles at infinity $\int\limits_{U\cap X}\|e\|^{-2}|z_1|^N\omega^n<\infty$ for some integer N.

It may be assumed from the discussion of positivity above that ω is the Euclidean Kaehler form after possibly shrinking U, since $(\sqrt{-1})^{k^2}\eta\wedge\bar{\eta}$ is positive. Letting $\eta=\sum\limits_{|J|=k}\eta_J\,dz_J$ where J runs over multiindices $(j_1,\ldots,j_k)$, one sees that the former integral implies for each J, that:

$$\int\limits_{U\cap X}\|e\|^2|\eta_J|^2\omega^n<\infty.$$

Thus one has up to a positive constant:

$$\int\limits_{U\cap X}|z_1|^{N/2}|\eta_J|\omega^n\leqslant\Big(\int\limits_{U\cap X}|z_1|^N\|e\|^{-2}\omega^n\Big)^{\frac12}\Big(\int\limits_{U\cap X}|\eta_J|^2\,\|e\|^2\omega^n\Big)^{\frac12}<\infty\,.$$

Thus η_J extends meromorphically to U and thus so does η, and hence p^*s extends meromorphically as a section of $L\otimes\Lambda^kT^*_{\bar{X}}$ to $\bar{X}$. Q.E.D.

REMARK I-*A*. It seems worth noting that the above lets one define generalized holomorphic differential k forms for any k on an irreducible reduced complex space Y. Let U be a neighborhood of a point $y\in Y$ that is embedded in $\mathbf{C}^N$. Now let ω be an Hermitian form on $\mathbf{C}^N$ associated to an Hermitian metric. If η is an holomorphic k form on Reg U, $(\sqrt{-1})^{k^2}\int\limits_{\text{Reg }U}\eta\wedge\bar{\eta}\omega^{n-k}<\infty$ where $\dim_{\mathbf{C}}U=n$, and if $p:\tilde{U}\to U$ is a desingularization of U, then $p^*\eta$ extends to a meromorphic k-form on $\tilde{U}$ with poles of bounded degree along $\tilde{U}-p^{-1}(\text{Reg}(U))$. This should be compared with [14, II(a)].

§ II. – In this section I collect some material pertaining to the L^2 estimates for $\bar{\partial}$ and prove Proposition I and Proposition II which give criteria for the image of a proper map to be bimeromorphic to a quasi-projective variety.

Let L be an holomorphic line bundle on a reduced and irreducible complex space Y. *L is said to be semi-ample* (*almost-positive* [9, 22], *ample*) *if it possesses an Hermitian structure that has positive semi-definite curvature on Y, and that has positive curvature at least one point* (*on a dense open set, at every point*).

The relevant lemma is:

LEMMA II-*A*. *Let $p:X\to Y$ be a generically finite to one proper holomorphic surjection from a connected complete Kaehler manifold X to a normal analytic space Y. Let L be an holomorphic line bundle on Y with a continuous Hermitian structure $\|\ \|$. Assume $p^*\|\ \|$ is C^∞ on p^*L and that p^*L is semi-*

ample with respect to $p^*\| \ \|$. Let U be the intersection of the complement of the image under p of the set where p isn't a covering and the set of those points $y \in Y$ such that the curvature form of $p^*\| \ \|$ is positive definite in a neighborhood of $p^{-1}(y)$.

Then for any finite set $\{p_m\} \subseteq U$ and any finite set $\{\lambda_m\}$ of non-negative integers corresponding to $\{p_m\}$, there exists an N_0 such that for $N \geqslant N_0$, there exists a section e of $\mathcal{O}(L^N) \otimes \mathcal{K}_Y$ with $e \in (L^N \otimes \mathcal{K}_Y)$ and prescribed λ_m jet at p_m for each m.

PROOF. For each m choose a neighborhood U_m of p_m such that the $\{U_m\}$ are disjoint, $U_m \subseteq U$, and $p|_{p^{-1}(U_m)}: p^{-1}(U_m) \to U_m$ is a covering. Let $\mathcal{N} = \bigcup_m U_m$.

Now one can use lemma B of [30] to find an $N_0 > 0$ such that for each $N \geqslant N_0$, there exists a section $\tilde{e}$ of $\mathcal{O}(p^* L^N \otimes K_X)$ with $\tilde{e} \in L^2(p^* L^N \otimes K_X)$ and such that $\operatorname{tr} \tilde{e}$, the trace of $\tilde{e}$ over $p^{-1}(\mathcal{U})$ with respect to p, is an holomorphic section of $L^N|_{\mathcal{U}} \otimes K_{\mathcal{U}}$ with the desired λ_m jets for each p_m.

The proof will be finished if one shows $\operatorname{tr} \tilde{e} \in L^2(L^N \otimes \mathcal{K}_Y)$, since then by Lemma I-$A$, it gives rise to a section over Y of $\mathcal{O}(L^N) \otimes \mathcal{K}_Y$.

Therefore I will show that $\operatorname{tr} \tilde{e} \in L^2(L^N \otimes \mathcal{K}_Y)$. Let S equal the union of the singular set of Y and the image under p of the set where p is not a covering. S is an analytic variety of Y by Remmert's proper mapping theorem and $p: X - p^{-1}(S) \to Y - S$ is a covering. It suffices to show $\operatorname{tr} \tilde{e} \in L^2(L^N|_{Y-S} \otimes K_{Y-S})$ since S is a set of measure zero. The inequality

$$\|\operatorname{tr} \tilde{e}\|^2 \leqslant \frac{(\deg p)^2}{2} \|\tilde{e}\|^2$$

where $\deg p$ is the sheet number of p is a trivial consequence of

$$\Big|\sum_{1 \leqslant i \leqslant r} x_i\Big|^2 \leqslant \frac{r^2}{2}\Big(\sum_{1 \leqslant i \leqslant r} |x_i|^2\Big)$$

where $\{x_i\}$ is a set of r complex numbers. Q.E.D.

For a further discussion of the trace operator see [14, II(b)].

The following lemma will be used to prove Proposition II. The assumption that X is quasi-projective could be dropped with extra work.

LEMMA II-B. *Let $r: X \to Y$ be a proper holomorphic surjection from a quasi-projective manifold X onto a normal irreducible analytic space Y. Assume that X is complete Kaehler and that r is biholomorphic on a dense open set of X. Let L be an holomorphic line bundle on a reduced analytic space Z;*

assume there is an Hermitian structure $\| \;\|$ *on* L *with respect to which* L *is semi-ample. Let* $q\colon Y \to Z$ *be an holomorphic map that expresses* Y *as the normalization of* Z. *Let* U *be an open set of* Y *belonging to the inverse image under* q *of the set on* Z *where the curvature of* $\| \;\|$ *is positive definite. Then given any finite set* $\{p_m\} \subseteq U$ *and any finite set* $\{\lambda_m\}$ *of non-negative integers corresponding to* $\{p_m\}$, *then there exists an* N_0 *such that for any* $N \geqslant N_0$, *there exists a section* e *of* $\mathcal{O}(q^* L^N) \otimes \mathcal{K}_Y$ *with* $e \in L^2(q^* L^N \otimes \mathcal{K}_Y)$ *and with prescribed image in* $(\mathcal{O}_Y/\mathcal{I}_{pm}^{\lambda m}) \otimes \mathcal{K}_Y \otimes \mathcal{O}(q^* L^N)$ *for each* m.

PROOF. The proof is a modification of Kodaira's basic proof [18].

I will do the proof for a single point $p \in U \subseteq Y$ and a single integer λ; the general case is a trivial modification of this.

By definition one can choose a neighborhood $W \subseteq Z$ of $q(p)$ such that:

a) there is an embedding of W into a neighborhood $\tilde{W}$ of the origin O in $\boldsymbol{C}^N$ for some N,

b) $q(p)$ is identified with O by this embedding,

c) $L|_W$ is the restriction of an holomorphic line bundle $\tilde{L}$ on $\tilde{W}$ and $\| \;\|$ is the restriction of an Hermitian metric $\widetilde{\| \;\|}$ on $\tilde{L}$ that has a positive definite curvature form on $\tilde{W}$.

By shrinking W and $\tilde{W}$ if necessary, it can be assumed that there is a neighborhood $V \subseteq Y$ of p such that $q\colon V \to W$ is proper and $q^{-1}(q(p)) \cap V = p$.

Now let $\tilde{W}'$ denote $\tilde{W}$ with the origin blown up and let $\delta\colon \tilde{W}' \to \tilde{W}$ be the blowing down map. Let $\mathcal{Q}$ denote $\delta^{-1}(0)$. Let V' be the normalized irreducible component of the fibre product $\tilde{W}' \times_{\tilde{W}} V$ of δ and q that surjects onto V under the map induced from the projection of $\tilde{W}' \times_{\tilde{W}} V$ onto V. Let $\delta'\colon V' \to V$ and $q'\colon V' \to \tilde{W}'$ be the induced maps. One has the commutative diagram:

$$\begin{array}{ccc} V' & \xrightarrow{\;q'\;} & \tilde{W}' \\ {\scriptstyle \delta'}\downarrow & & \downarrow{\scriptstyle \delta} \\ V & \xrightarrow{\;q\;} & \tilde{W} \end{array}$$

Note that:

1) $\delta'_*\left(q'^* \mathcal{O}([\mathcal{Q}]^{-\lambda})\right) \subseteq \mathcal{I}_p^\lambda$.

To see this, note that $\delta'_*(\delta'^*(\mathcal{S})) = \mathcal{S}$ for any coherent sheaf $\mathcal{S}$ on V since V' and V are normal and δ' is proper with connected fibres. Thus:

$$\delta'_* q'^* \mathcal{O}([\mathcal{Q}]^{-\lambda}) = \delta'_*((\delta \circ q')^* \mathcal{I}_0^\lambda) = \delta'_*((q \circ \delta')^*(\mathcal{I}_0^\lambda)) = q^*(\mathcal{I}_0^\lambda) \subseteq (q^* \mathcal{I}_0)^\lambda \subseteq \mathcal{I}_p^\lambda .$$

Let $\| \ \|_N$ and $\widetilde{\| \ \|}_N$ denote the Hermitian structures induced by $\| \ \|$ and $\widetilde{\| \ \|}$ respectively on q^*L^N and $\tilde{L}^N$.

By the basic argument of Kodaira [18] one can find an $N_0 > 0$ such that for all $N \geqslant N_0$ one can modify $\delta'^*\widetilde{\| \ \|}_N$ to get an Hermitian metric $\widetilde{\| \ \|}_{\lambda,N}$ on $\tilde{L}^N \otimes [\mathfrak{Q}]^{-\lambda}$ such that:

a) $\widetilde{\| \ \|}_{\lambda,N}$ agrees with $\delta'^*\widetilde{\| \ \|}_N$ outside some compact neighborhood of $\mathfrak{Q}$.

b) $\widetilde{\| \ \|}_{\lambda,N}$ has positive curvature on $\tilde{W}'$.

Let Y' denote Y with V replaced by V'. This makes sense because $\delta' \colon V' \to V$ is a biholomorphism from $V' - \delta'^{-1}(p)$ to $V - p$. No confusion will result from letting $\delta' \colon Y' \to Y$ denote the map induced by $\delta' \colon V' \to V$.

Since $q'^*[\mathfrak{Q}]$ is the trivial line bundle outside of the compact set $\delta'^{-1}(p)$ of V, it is easy to see it has a unique extension as an holomorphic line bundle Q on Y that is the trivial holomorphic bundle outside of $\delta'^{-1}(p)$. Note that $\delta'^*(q^*L^N) \otimes Q^{-\lambda}$ is semi-ample for all $N \geqslant N_0$. To see this, simply note that δ' is a biholomorphism outside of $\delta'^{-1}(p)$, $(q \circ \delta')^* L^N$ agrees with $(q \circ \delta')^*(L^N) \otimes Q^{-\lambda}$ outside $\delta'^{-1}(p)$, and $q'^*\widetilde{\| \ \|}_{\lambda,N}$ agrees with $\delta'^* q^*(\| \ \|_N)$ outside a compact set of V'.

Let X' be the normalization of the irreducible component of the fibre product $X \times_Y Y'$ of r and δ' that surjects onto Y' under the map induced by the projection of $X \times_Y Y'$ onto Y'. Let $a \colon X' \to X$ and $b \colon X' \to Y'$ denote the induced maps. Note that a is a biholomorphism outside of a compact set, i.e. a gives a biholomorphism between $X' - (r \circ a)^{-1}(p)$ and $X - a^{-1}(p)$. This implies that X' is Zariski open in a Moisezon space. To see this, note that X is Zariski open in some projective manifold $\bar{X}$ and that $(r \circ a)^{-1}(p)$ is disjoint from $\bar{X} - X$.

Therefore by Hironaka's theorem, there exists a quasi-projective manifold $\tilde{X}$ and an holomorphic proper surjection $h \colon \tilde{X} \to X'$ such that h is a biholomorphism on a dense open set of $\tilde{X}$.

Now note that $\tilde{X}$ possesses a complete Kaehler metric. To see this simply add the Kaehler metric of $\tilde{X}$ pulled back by a map into projective space to the pullback of the Kaehler metric on X by $a \circ h$.

Note that there is the commutative diagram:

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\ h\ } & X' & \xrightarrow{\ a\ } & X \\ & & {\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ & & Y' & \xrightarrow{\ \delta\ } & Y \end{array}$$

Also, by the above for any $N \geqslant N_0$, $\mathcal{L}_{N,\lambda} = (q\circ r\circ a\circ h)^* L^N \otimes (b\circ h)^* Q^{-\lambda}$ is semiample. Note that as a consequence of 1) above, one has:

$$1')\ (r\circ a\circ h)_*(b\circ h)^*\, \mathcal{O}(Q^{-\lambda}) = (\delta'\circ b)_* b^*\, \mathcal{O}(Q^{-\lambda}) = \delta'_*\, \mathcal{O}(Q^{-\lambda}) \subseteq \mathcal{J}_p^{\lambda}.$$

Also any holomorphic section of $(q\circ r\circ a\circ h)^* L^N \otimes K_{\tilde{X}}$ that is L^2 gives rise to a section of $\mathcal{K}_Y \otimes \mathcal{O}(q^* L^N)$ that belongs to $L^2(\mathcal{K}_Y \otimes q^* L^N)$. This follows since:

$$2')\ [(\delta'\circ b\circ h)_*(q\circ r\circ a\circ h)^*\, \mathcal{O}(L^N)] \otimes \mathcal{K}_Y = q^*\, \mathcal{O}(L^N) \otimes \mathcal{K}_Y,$$

and since the L^2 norm depends only on the metrics on $(q\circ r\circ a\circ h)^* L^N$ and $q^* L^N$, and these agree outside compact sets.

Now after possibly shrinking V one can find a section $\bar{e}$ of $\mathcal{K}_V \otimes \mathcal{O}(q^* L^N|_V)$ over V where $N \geqslant N_0$ and the image in $[\mathcal{O}_Y/\mathcal{J}_p^{\lambda}) \otimes \mathcal{K}_Y \otimes \mathcal{O}(q^* L^N)$ is prescribed. If V is small enough one can factor $\bar{e}$ as $s\cdot\beta$ where β is a non-vanishing section of $\mathcal{O}(q^* L^N|_V)$ and s is a section of $\mathcal{K}_V$. Let β' denote $\beta\cdot\varrho$ where ϱ is a C^∞ function with compact support in V such that ϱ is 1 in a neighborhood of p. By the definition of $\mathcal{K}_Y$, it follows that $(r\circ a\circ h)^* s$ is an holomorphic section $\tilde{s}$ of $K_{\tilde{V}}$ where $\tilde{V} = (r\circ a\circ h)^{-1}(V)$. Further, $\tilde{\beta} = (r\circ a\circ h)^* \beta'$ is a compactly supported C^∞ section of $(q\circ r\circ a\circ h)^* L^N|_{\tilde{V}}$. Thus $\tilde{s}\,\tilde{\beta}$ is a C^∞ section of $[q\circ r\circ a\circ h)^* L^N|_{\tilde{V}} \otimes K_{\tilde{V}}$.

Thus, since $\bar{\partial}(\tilde{s}\tilde{\beta})$ is 0 in neighborhood of $(r\circ a\circ h)^{-1}(p)$, it is a compactly supported C^∞, $K_{\tilde{X}} \otimes \mathcal{L}_{N,\lambda}$ valued $(0, 1)$ form. Thus by [30, Lemma A] and the fact that $\mathcal{L}_{N,\lambda}$ is semi-ample, there exists a C^∞ section $A \in L^2(K_{\tilde{X}} \otimes \mathcal{L}_{N,\lambda})$ such that $\bar{\partial}A = \bar{\partial}\tilde{s}\tilde{\beta}$. Now $\tilde{s}\tilde{\beta} - A$ is an holomorphic section of $K_{\tilde{X}} \otimes (q\circ r\circ a\circ h)^* L^N$ and gives rise to an holomorphic section e of $\mathcal{K}_Y \otimes q^*\, \mathcal{O}(L^N)$, by 2'). Now clearly $e \in L^2(\mathcal{K}_Y \otimes q^* L^N)$. Thus the lemma will be proved if it is shown that e gives rise to the prescribed section of $(\mathcal{O}_Y/\mathcal{J}_p^{\lambda}) \otimes \mathcal{K}_Y \otimes \mathcal{O}(q^* L^N)$. Now by construction $\tilde{s}\tilde{\beta}$ is holomorphic in a neighborhood of $(r\circ a\circ h)^{-1}(p)$ and gives rise to the prescribed section. Thus it suffices to note that in a neighborhood of $(r\circ a\circ h)^{-1}(p)$, A is an holomorphic section of $\mathcal{L}_{N,\lambda}$ and by 1') gives rise in a neighborhood of p to a section of $q^*\, \mathcal{O}(L^N) \otimes \mathcal{K}_Y \otimes \mathcal{J}_p^{\lambda}$.

Q.E.D.

Meromorphic maps will be in the sense of Remmert [21] with a few exceptions for the sake of convenience. Given two normal analytic spaces X and Y, a bimeromorphic map $A: X \to Y$ will be a meromorphic map that gives a biholomorphism between a dense open set of X and a dense open set of Y. I am not requiring that A^{-1} be meromorphic; the property that is allowed to fail is that $A^{-1}(y)$ with $y \in Y$ need not be compact. Also if X and Y are irreducible quasi-projective varieties and one has an irreducible algebraic set Φ in $X \times Y$ that gives a biholomorphism between dense open sets, one will say that Φ is birational. This is useful and conforms to popular usage though neither Φ nor its inverse is meromorphic.

Though in general one can not compose meromorphic maps, there is one situation where one can which covers the needs of this paper. Namely, let X, Y, and Z be normal irreducible analytic spaces. Let $A: X \to Y$ and $B: Y \to Z$ be meromorphic maps. If A is surjective then there is a dense open set U of X such that $A|_U$ is holomorphic, $A(U)$ is dense in Y, and $B|_{A(U)}$ is holomorphic. Thus $B \circ A|_U$ is a well defined holomorphic function. Now one can define $B \circ A$ as the closure of the graph of $B \circ A|_U$ in $X \times Z$.

The somewhat involved form of the following is dictated by my application in IV to the period mapping.

PROPOSITION I. *Let $p: X \to Y$ be a proper holomorphic surjection from a Zariski open set X of a connected compact Kaehler manifold $\bar{X}$ onto a normal analytic space Y. Let L be an holomorphic line bundle on Y with a continuous Hermitian structure $\| \ \|$. Assume that p^*L extends to an holomorphic line bundle $\mathfrak{L}$ on $\bar{X}$ and $p^*\| \ \|$ has L^2 poles at infinity relative to $\mathfrak{L}$. Let $q: Z \to Y$ be a proper generically finite to one holomorphic surjection of a complete Kaehler manifold Z onto Y. Assume that $q^*\| \ \|$ is a C^∞ Hermitian structure of q^*L and that q^*L is almost positive with respect to $q^*\| \ \|$. Then there exists a Zariski open set M of a projective manifold $\bar{M}$ and a bimeromorphic map $\Phi: Y \to M$ such that $\Phi \circ p$ extends to a meromorphic map from $\bar{X}$ to $\bar{M}$. If $\Phi': Y \to M'$ is a bimeromorphic mapping of Y to a Zariski open set M of a compact analytic space $\bar{M}'$ such that $\Phi' \circ p$ extends to a meromorphic map from $\bar{X}$ to $\bar{M}'$, then $\Phi \circ \Phi'^{-1}$ extends to a bimeromorphic map from $\bar{M}'$ to $\bar{M}$. The function field of $\bar{M}$ is characterized as the set of those meromorphic functions f such that $(f/M) \circ \Phi \circ p$ extends to a meromorphic function on $\bar{X}$.*

PROOF. Using Lemma I-C it follows that a section s of $\mathcal{O}(L^N) \otimes \mathcal{K}_Y$ with $s \in L^2(L^N \otimes \mathcal{K}_Y)$ is such that p^*s extends to be a meromorphic section of $\mathfrak{L}^N \otimes \Omega^k_{\bar{X}}$ where $\dim_C Y = k$. Now using this and Lemma II-A instead of Lemma I-B of [29] one can follow the argument of Proposition I of [29]. What one gets is a finite dimensional family S, of sections of $\mathcal{O}(L^t) \otimes (\mathcal{K}_Y)^r$ for some t and r such that

1) p^*s, with $s \in S$, extends to be a meromorphic section of $(\mathfrak{L})^t \otimes (\Omega^k_{\bar{X}})^{\otimes r}$ on $\bar{X}$.

2) the meromorphic map $\varphi: \bar{Y} \to \boldsymbol{CP}^R$ for some R associated to S is an embedding on a dense open set.

Now 1) implies that $\varphi \circ p$ extends meromorphically to $\bar{X}$ and thus that $\varphi(Y)$ is Zariski open in its closure which is a projective variety. Using Hironaka's desingularization theorem one can assume that $\varphi(Y)$ is a manifold.

The remaining assertions are immediate consequences of the following minor variant of Lemma I-*E* of [29]; it is a straightforward consequence of Remmert's proper mapping theorem applied to graphs.

LEMMA II-*C*. *Let X, Y, and Z be analytic spaces and let $\bar{X}$ and $\bar{Y}$ be compact normal spaces in which X and Y are respectively Zariski open. Let $p: X \to Y$ be a meromorphic surjection that extends to a meromorphic map from $\bar{X}$ to $\bar{Y}$. Let $f: Y \to Z$ be a meromorphic map. The composition $f \circ p$ extends to a meromorphic map from $\bar{X}$ to Z if and only if f extends as a meromorphic map to $\bar{Y}$.*

The following corollary will be important.

COROLLARY II-*A*. *Let $Y, L, \| \ \|$, $M, \bar{M}$, Φ, Z and q be as above. If $f: X' \to M$ is a proper holomorphic surjection from a Zariski open set X' of a compact complex manifold $\bar{X}'$ and if:*

A) f^*L extends to an holomorphic line bundle $\mathcal{L}'$ on $\bar{X}'$,

B) $f^*\| \ \|$ has L^2 poles at infinity relative to $\mathcal{L}'$, then $\Phi \circ f$ extends to a meromorphic map from $\bar{X}'$ to $\bar{M}$. Φ can be chosen to be a biholomorphism on a set which includes the intersection of the complement of the image under q of the set where q isn't a covering and the set of those points $y \in Y$ such that the curvature form of $q^*\| \ \|$ is positive definite in a neighborhood of $q^{-1}(y)$.

PROOF. The first assertion will follow if it is shown that the meromorphic $\varphi \circ f: X' \to \boldsymbol{CP}^R$ extends meromorphically to X where $\varphi: Y \to \boldsymbol{CP}^R$ is as in the proof of Proposition I. Since φ is defined by a space S of sections of $\mathcal{O}(L)^a \otimes (\mathcal{K}_Y)^b$, and since these sections are sums of products of L^2 sections of $\mathcal{O}(L^a) \otimes \mathcal{K}_Y$ for various $a > 0$, it suffices to show that given an holomorphic $s \in L^2(L^a \otimes \mathcal{K}_Y)$, then f^*s extends to be a meromorphic section of $\mathcal{L}'^a \otimes \Lambda^k T^*_{\bar{X}}$, where $k = \dim_{\boldsymbol{C}} Y$. This follows from the L^2 pole condition by Lemma I-*C*.

The latter statement follows immediately from Lemma II-*A* and the proof of Proposition II. Q.E.D.

Finally there is:

PROPOSITION II. *Let $r: M \to Y$ be a proper generically one to one holomorphic surjection of M, a Zariski open set of a connected projective manifold M onto a normal analytic space. Let $q: Y \to Z$ be an holomorphic surjection of Y onto a reduced analytic space Z, that expresses Y as the normalization of Z. Let L be an holomorphic line bundle on Z with an almost positive*

Hermitian structure $\| \|$. Assume that $(q \circ r)^* L$ extends to an holomorphic line bundle $\mathfrak{L}$ on M and that $(q \circ r)^* \| \|$ has L^2 poles at infinity with respect to $\mathfrak{L}$. Then there exists a meromorphic map $\Phi Y \to \boldsymbol{CP}^N$, for some N, such that $\Phi \circ r$ is rational and Φ is an embedding on the set of manifold points in the inverse image under q of the set of points of Z where the curvature of $\| \|$ is positive definite.

PROOF. As in the last proposition, one uses the proof of the main proposition of [29] combined with Lemma II-B. Q.E.D.

Appendix to § II.

It seems worthwhile to record the following cute consequence of Lemma II-A:

COROLLARY TO LEMMA II-A. *Let be Y a Zariski open set in a compact connected complex manifold $\overline{Y}$ such that the codimension of $\overline{Y} - Y$ is at least two in $\overline{Y}$. Let $p: X \to Y$ be a generically finite to one surjection where X is a complete Kaehler manifold. If there is a closed positive semi-definite* (1, 1) *form ω on Y that is positive definite at least one point and if ω has rational periods, then $\overline{Y}$ is Moisezon.*

PROOF. If one produced a holomorphic line bundle L on Y with curvature form equal to some multiple of ω and if L extended to a holomorphic line bundle on $\overline{Y}$, one would be done. The extension follows from a result of Shiffman [27]. The existence of L is a consequence of both the following lemmas, and also the fact that Y being Zariski open in $\overline{Y}$ implies $H^2(Y, \boldsymbol{Q})$ is finite dimensional and thus that there exists an integer N such that $N \cdot \omega$ has integral periods.

LEMMA A-II-A. *Let Y be a complex manifold and let ω be closed, real,* (1, 1) *form with integral periods, on Y. There exists a holomorphic line bundle L on Y with curvature form equal to ω.*

PROOF. This is a very easy consequence of Weil's proof of the equivalence of de Rham and Čěch cohomology [31]. Using a Riemannian metric on Y, one can choose a cover $\{U_\alpha\}$ of Y by open sets with each U_α convex. Thus $U_{\alpha\beta} = U_\alpha \cap U_\beta$ is convex and thus contractible whenever $U_{\alpha\beta}$ is non empty. Now it can be assumed after shrinking if necessary that each U_α is relatively compact in an open set V_α that is biholomorphic to a polydisc. Now it can be further assumed that the $\{V_\alpha\}$ and hence the $\{U_\alpha\}$ are locally finite. Choose a set of points $\{P_{\alpha\beta}\} \subseteq Y$ for each pair (α, β) such that $U_{\alpha\beta}$ is non-empty, $p_{\alpha\beta} \in U_{\alpha\beta}$, and $p_{\alpha\beta} = p_{\beta\alpha}$.

Now there exist a set of real valued C^∞ functions $\{\sqrt{-1}\tilde{\varphi}_\alpha\}$ with $\partial\bar{\partial}\tilde{\varphi}_\alpha = -\omega|_{U_\alpha}$. Let $\varphi_\alpha = \tilde{\varphi}_\alpha|_{U_\alpha}$. It can be assumed with out loss of generality that $\varphi_\alpha(p_{\alpha\beta}) = 0$ for each β such that $U_{\alpha\beta}$ is non-empty, since the $\{\varphi_\alpha\}$ are only unique up to pluriharmonic functions and $U_{\alpha\beta}$ is non-empty for only finitely many β. For each $U_{\alpha\beta}$, $\varphi_{\alpha\beta} = \varphi_\alpha - \varphi_\beta$ is pluriharmonic. Thus $\{\partial\varphi_{\alpha\beta}\}$ is a one cocycle of holomorphic one forms.

Since $U_{\alpha\beta}$ is convex and hence contractible, the function $\int_{p_{\alpha\beta}}^{p} \partial\varphi_{\alpha\beta} = \psi_{\alpha\beta}(p)$ is well defined. Noting that $d\partial\varphi_\alpha = \omega|_{U_\alpha}$, one sees that the set of constants $\{\psi_{\alpha\beta\gamma}\}$ where $\psi_{\alpha\beta\gamma} = \psi_{\alpha\beta} + \psi_{\beta\gamma} + \psi_{\gamma\alpha}$ is the Cěch representative for ω. Thus there are a set of real constants $\{C_{\alpha\beta}\}$ such that $\psi_{\alpha\beta\gamma} - (C_{\alpha\beta} + C_{\beta\gamma} + C_{\gamma\alpha})$ is an integer. Thus the set of non-vanishing holomorphic functions $\{\exp[2\pi\sqrt{-1}(\psi_{\alpha\beta} - C_{\alpha\beta})]\}$ is a multiplicative cocycle and defines an holomorphic line bundle L.

Now L is trivialized by the cover $\{U_\alpha\}$ and a section f is given by functions $\{f_\alpha\}$ with:

$$f_\beta|_{U_{\alpha\beta}} = (f_\alpha|_{U_{\alpha\beta}})\left(\exp[-2\pi\sqrt{-1}(\psi_{\alpha\beta} - C_{\alpha\beta})]\right).$$

I claim one has an Hermitian structure given by

$$\|f\|^2 = \{|f_\alpha|^2 \exp[-2\pi\sqrt{-1}\varphi_\beta]\}.$$

To check this note that on $U_{\alpha\beta}$ one has

$$|f_\beta|^2 \exp[-2\pi\sqrt{-1}\varphi_\beta] = |f_\alpha|^2 \exp[-2\pi\sqrt{-1}(\psi_{\alpha\beta} - \bar{\psi}_{\alpha\beta} + \varphi_\beta)].$$

Now note

$$\psi_{\alpha\beta} - \bar{\psi}_{\alpha\beta} = \int_{p_{\alpha\beta}}^{p} \partial\varphi_{\alpha\beta} - \bar{\partial}\bar{\varphi}_{\alpha\beta} = \int_{p_{\alpha\beta}}^{p} d\varphi_{\alpha\beta}$$

since $\{\sqrt{-1}\varphi_\alpha\}$ consists of real functions. Thus noting that $\varphi_{\alpha\beta}(p_{\alpha\beta}) = 0$ one has $\psi_{\alpha\beta} - \bar{\psi}_{\alpha\beta} = \varphi_{\alpha\beta}$. Thus $\psi_{\alpha\beta} - \bar{\psi}_{\alpha\beta} + \varphi_\beta = \varphi_\alpha$ on $U_{\alpha\beta}$.

Now note that the curvature form of this Hermitian structure is given by

$$\left\{\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial ln \exp[2\pi\sqrt{-1}\varphi_\alpha]\right\} = \{\partial\bar{\partial}\varphi_\alpha\} = \omega. \qquad \text{Q.E.D.}$$

REMARK A-II-A. Note that the above lemma implies that a complete Kaehler manifold Y with $\dim_{\mathbf{Q}} H^2(Y, \mathbf{Q}) \leq 1$ has enough global meromorphic functions to give holomorphic coordinates in a neighborhood of any point.

It is not difficult to figure out a uniformity condition that will let one construct enough holomorphic sections of $K \otimes L^N$ for a fixed N to give an holomorphic embedding of Y into $\mathbf{CP}^N$. Indeed in [4; this was written in 1961!] such a condition is derived and used to show that one can embed the universal cover of any projective manifold into $\mathbf{CP}^M$ for some M.

§ III. – In this section I collect a few concepts and results connected with proper maps. These allow us to form a bridge from the conclusions of Proposition I to the hypotheses of Proposition II; this will be useful in § IV in the study of the period mapping. They, especially Proposition III, could be somewhat generalized, using [16] for example, but I will for simplicity work only in the degree of generality needed.

Let $p: X \to Y$ be a proper map from an irreducible normal analytic space X to a complex manifold Y. The singularity locus of p, $J(p)$ is the union of the singular locus S, of X with those regular points of X where p is not of maximal rank. To see that it is an analytic set note first this is trivial if X is a complex manifold. If X isn't, let $\delta: \tilde{X} \to X$ be a desingularization that exists by Hironaka's theorem, with δ proper and $\delta: \tilde{X} - \delta^{-1}(S) \to$ $\to X - S$, a biholomorphism. Now note that $\delta J(p \circ \delta) = J(p)$, and thus by Remmert's proper mapping theorem $J(p)$ is an analytic set. If $\tilde{X}$ were only a reduced analytic space, one could still define $J(p)$ by locally embedding Y in $\mathbf{C}^N$. If X is not irreducible, let $\{X_\alpha | \alpha \in \mathcal{A}\}$ be the irreducible components of X. Define $J(p)$ in this case as $\bigcup_\alpha J\{p|_{X_\alpha}\}$.

Let $p: X \to Y$ be an holomorphic map from a reduced analytic space X into a reduced analytic space Y. Define $\mathcal{S}_r(p)$ as those points $x \in X$ such that the irreducible component of the fibre $p^{-1}(p(x))$ that contains x is of dimension greater than or equal to r. Clearly $\mathcal{S}_0(p) \supseteq \mathcal{S}_1(p) \supseteq \ldots$, and the sequence terminates if X is finite dimensional. Define $\mathfrak{D}_r(p)$ as $p(\mathcal{S}_r(p))$; if X is irreducible and $k = \dim_{\mathbf{C}} X - \dim_{\mathbf{C}} p(X)$, then $\mathfrak{D}_{k+1}(p)$ is called the degeneracy locus of p. Note that if $q: Y \to Z$ is a finite to one holomorphic map to a reduced analytic space Z, then $q(\mathfrak{D}_r(p)) = \mathfrak{D}_r(q \circ p)$ and $\mathcal{S}_r(q \circ p) = \mathcal{S}_r(p)$. For this reason it suffices to assume Y is normal and p surjective in the following lemmas.

If $q: A \to B$ is a holomorphic map between reduced analytic spaces, then $\mathcal{N}(q): \mathcal{N}(A) \to \mathcal{N}(B)$ denotes the associated holomorphic mapping between the normalizations of A and B.

If $p: X \to Y$ is an holomorphic map between normal analytic spaces, let $p_0 = p$ and define inductively $p_{j+1} = \mathcal{N}(p_j|_{J(p_j)})$ from $\mathcal{N}(J(p_j))$ to Y. Let $i_0: J(p) \to X$ be the inclusion mapping. Define inductively $i_{j+1} =$ $= \mathcal{N}(i_j|_{J(p_j)})$ from $\mathcal{N}(J(p_j))$ to X.

LEMMA III-A. *Let $p: X \to Y$ be an holomorphic map between normal analytic spaces. $\mathcal{S}_r(p)$ and $\mathcal{D}_r(p)$ are analytic subsets of X and Y respectively for all r. If X is Zariski open in a normal compact analytic space, then $\mathcal{S}_r(p)$ extends to an analytic set in X for all r, if $i_j(J(p_j))$ extends to an analytic set in $\bar{X}$ for all j.*

PROOF. Clearly one can assume X is irreducible. Let $k = \dim_{\mathbf{C}} X - \dim_{\mathbf{C}} p(X)$. By the upper semicontinuity of the dimension of the fibres of an holomorphic map $\mathcal{S}_k(p) = X$.

Let Z be an irreducible component of $J(p_j)$. If $\dim_{\mathbf{C}} Z - \dim_{\mathbf{C}} p_j(Z) = k + r$ where $r > 0$ then $i_j(Z) \subseteq \mathcal{S}_{k+r}(p)$. Also $\mathcal{S}_{k+r}(p)$ for $r > 0$ is a union of such irreducible components as all j's are run over. To see this note that if $\dim_{\mathbf{C}} Z - \dim_{\mathbf{C}} p_j(Z)$ were less than or equal to k and $r > 0$ then $i_j(\mathcal{S}_{r+k}(p_j|_Z)) = i_{j+1}(\mathcal{S}_{r+k}(p_{j+1}|_{J(p_{j+1}|\mathcal{N}(Z))}))$; and that $J(p_{j+1}) \cap \mathcal{N}(Z) = J(p_{j+1}|_{\mathcal{N}(Z)})$.

Now the set of irreducible analytic components Z of $i_j(J(p_j))$ such that $\dim_{\mathbf{C}} Z - \dim_{\mathbf{C}} p(Z) = k + r$ is an analytic set of X. Thus $\mathcal{S}_{k+r}(p)$ is an analytic set if the sequence $J(p_j)$ eventually terminates. This is clear since $\dim_{\mathbf{C}} X < \infty$ and $\dim_{\mathbf{C}} J(p_{j+1}) < \dim_{\mathbf{C}} J(p_j)$.

The $\mathcal{D}_r(p)$ are analytic sets by Remmert's proper mapping theorem, since $p(\mathcal{S}_r(p)) = \mathcal{D}_r(p)$.

Finally note that if Z is an irreducible analytic component of $i_j(J(p_j))$ and $i_j(J(p_j))$ extends as an analytic set to $\bar{X}$, then Z extends as an analytic set to $\bar{X}$. Q.E.D.

The next lemma is quite well known:

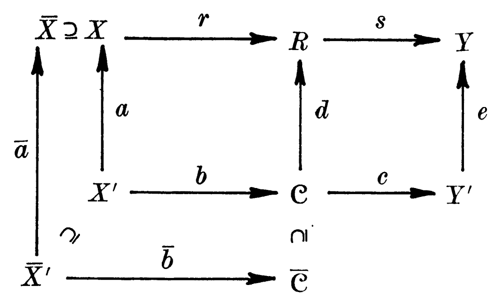
LEMMA III-B. *Let $p: A \to B$ be a proper, finite to one holomorphic map from a reduced and irreducible compact analytic space A to a projective variety B. Then A is a projective variety. In particular the normalization of a quasi-projective variety is a quasi-projective variety.*

PROOF. Let H be an ample line bundle on B; let us show $p^*(H)$ is ample on A. Using Serre's criterion let $\mathcal{S}$ be an analytic coherent sheaf in A. Since $R^i_{p^*}(\mathcal{S} \otimes \mathcal{O}_{p^*}(H^N)) = 0$ if $i > 0$ and $R^0_{p^*}(\mathcal{S}) \otimes \mathcal{O}(H^N)$ if $i = 0$, one is done, using the Leray spectral sequence.

If X is quasi-projective, i.e. Zariski open in $\bar{X}$ a projective variety, then $\mathcal{N}(X)$, the normalization of X is Zariski open in $\mathcal{N}(\bar{X})$. Thus since the map from $\mathcal{N}(\bar{X})$ to $\bar{X}$ is finite to one, $\mathcal{N}(\bar{X})$ is projective and hence $\mathcal{N}(X)$ is quasi-projective. Q.E.D.

PROPOSITION III. *Let $p: X \to Y$ be a proper holomorphic surjection of a Zariski open set X of a connected projective variety, $\bar{X}$, onto a normal* *analytic space Y. Let $p = s \circ r$ be the Remmert-Stein factorization of p with $r: X \to R$, a proper holomorphic map with connected fibres onto a normal space R and $s: R \to Y$ a proper finite to one map. Let $\mathfrak{D} = \mathfrak{D}_{k+1}(p)$ where $k = \dim_C X - \dim_C Y$ be the degeneracy set of p. Assume there exists a bimeromorphic map $\Phi: Y \to Z$ where Z is a normal quasi-projective variety and $\Phi \circ p$ is rational. Then there exist spaces X', $\bar{X}'$, $\mathfrak{C}$, $\bar{\mathfrak{C}}$ and Y' where X' and $\mathfrak{C}$ are Zariski open in the projective varieties $\bar{X}'$ and $\bar{\mathfrak{C}}$ respectively, and there exist proper holomorphic surjections $\bar{a}$, a, $\bar{b}$, b, c, d, and e such that:*

A) the diagram:



commutes, $\bar{a}$ is a birational morphism, and $\bar{b}$ has equidimensional fibres,

B) e, d, and a are biholomorphic when restricted to $Y' - e^{-1}(\mathfrak{D})$, $\mathfrak{C} - d^{-1}(s^{-1}(\mathfrak{D}))$ and $X' - a^{-1}(p^{-1}(\mathfrak{D}))$ respectively,

C) b has connected fibres and c is proper and generically finite to one,

D) Y' is quasi-projective and c is rational.

E) If p has connected fibres, then the above remains true without the assumption that Φ or Z exist.

Proof. It is convenient to first do the case when $R = Y$, i.e., when p has connected fibres. Now recall [cf. 1, 5] that the Chow space of $\bar{X}$ contains countably many irreducible components, $\mathfrak{C}_i$, and to each, one has associated an analytic space $Z_i \subseteq \bar{X} \times \mathfrak{C}_i$; denote by π_i and δ_i the projections of Z_i on the first and second factors. Now the $\mathfrak{C}_i$ and Z_i are projective and reduced and the fibres of δ_i are sums of subvarieties all of the same dimension. Since there are only countably many components, a category argument shows there exists one component $\bar{\mathfrak{C}}$ with associated Z and projections $\pi: Z \to \bar{X}$ and $\delta: Z \to \bar{\mathfrak{C}}$ such that $\pi(Z) = \bar{X}$ and containing an infinite number of the fibres of p. Now one can, by normalizing, assume that Z and $\bar{\mathfrak{C}}$ are normal. Now using the following well known rigidity theorem one sees that π is birational.

SUB-LEMMA A. *Let $p: X \to Y$ be a proper surjective map between normal analytic spaces. Assume p has connected fibres. Let F be a fibre. Then there exists a neighborhood U of F such that any connected analytic space $F' \subseteq U$ with $\dim_C F' = \dim_C F$ is a fibre of p if F' is compact.*

PROOF. By Remmert's proper mapping theorem one has $p(F')$ a subvariety of Y, but if U is small enough then $p(F')$ belongs to a Stein open set and is a point. By upper semi-continuity of dimension all fibres of p near enough to F have dimension at most $\dim_C F$. Finally use the connectedness of the fibres of p. Q.E.D.

Now let $\mathcal{C} = \delta(\pi^{-1}(X))$. Using the above lemma one sees that $\mathcal{C}$ is Zariski open in $\overline{\mathcal{C}}$. Now there is an holomorphic surjection d, from $\mathcal{C}$ to Y. Send $c \in \mathcal{C}$ to $p\big(\pi(\delta^{-1}(c))\big)$; by the rigidity theorem this would be well defined, and without difficulty holomorphic, if one shows $\delta^{-1}(\mathcal{C}) = \pi^{-1}(X)$. To see this assume there is a fibre F of $\delta: Z \to \mathcal{C}$ with $\pi(F) \not\subseteq X$. Let $\mathcal{E}$ be an algebraic curve in $\overline{\mathcal{C}}$, that contains c and such that there is an open set U of $\mathcal{E}$ such that $\pi(\delta^{-1}(U))$ are fibres of p. Now if for $t \in \mathcal{E}$, $\pi(\delta^{-1}(t)) \subseteq X$, then clearly by the rigidity lemma there exists an open set $U(t) \ni t$ such that $\pi\big(\delta^{-1}(U(t))\big) \subseteq X$. Let $\mathcal{U}$ be the largest open set of $\mathcal{E}$ such that $\pi(\delta^{-1}(\mathcal{U})) \subseteq X$. If $s \in \mathcal{C}$ is a boundary point of $\mathcal{U}$, then $\pi(\delta^{-1}(s)) \subseteq \overline{X} - X$. If not then $p\big(\pi(\delta^{-1}(s)) \cap X\big)$ would contain some interior point y of Y. By properness of p, given any point s' near enough to s, $p\big(\pi(\delta^{-1}(s')) \cap X\big)$ must intersect a fixed neighborhood of y. But by hypothesis there exists a sequence $\{x_n\} \subseteq \mathcal{U}$ with $x_n \to s$ where by the rigidity theorem, since $\pi(\delta^{-1}(s)) \not\subseteq X$, one has $p\big(\pi(\delta^{-1}(x_n))\big)$ diverging. It is trivial now that $\mathcal{U}$ must be Zariski open in $\mathcal{E}$, and thus if $\pi(F) \not\subseteq X$ then $\pi(F) \subseteq \overline{X} - X$; but this would contradict $c \in \delta(\pi^{-1}(X))$.

Note that $d: \mathcal{C} \to Y$ is generically one to one by the rigidity lemma. In fact the only time there can exist two points $\{c_1, c_2\} \subseteq \mathcal{C}$ such that $d(c_1) = d(c_2)$ is when the fibre $p^{-1}(d(c_1))$ is of dimension greater than $\dim_C X - \dim_C Y$, i.e., when $d(c_1) \in \mathfrak{D}$. Note the fibre of d must be positive dimensional in this case since otherwise $\pi: Z - \delta^{-1}(\overline{\mathcal{C}} - \mathcal{C}) \to X$ could not be onto. Further $d: \mathcal{C} - d^{-1}(\mathfrak{D}) \to Y - \mathfrak{D}$ is a biholomorphism since it is one to one and onto and $\mathcal{C}$ and Y are normal.

Now consider the case where $p: X \to Y$ is a proper generically finite to one surjection. Let λ be the sheet number of p. There is a well defined meromorphic map $\Lambda: Y \to X^{(\lambda)}$ where $X^{(\lambda)}$ is the λ-th symmetric power of X. To see this first consider $p|_{X-p^{-1}(\mathfrak{D})} X - p^{-1}(\mathfrak{D}) \to Y - \mathfrak{D}$ and let Λ send y to the λ tuple $p^{-1}(y)$ where $p^{-1}(\lambda)$ consists of λ distinct points; an easy argument with Riemann's extension theorem shows Λ extends holomorphically to $Y - \mathfrak{D}$. To go the rest of the way, note $\mathfrak{D}$ is of codimension two at least and $\bar{X}$ is projective. This guarantees that one gets a meromorphic map from Y to $\bar{X}^{(\lambda)}$. Now to show that $\Lambda(y) \subseteq X^{(\lambda)}$ for $y \in Y$ simply note that $\Lambda(y) \in p^{-1}(y)^{(\lambda)}$ for a dense set of $y \in Y$. This combined with the properness of p and a continuity argument shows that for all $y \in Y$ one has $\Lambda(y) \in p^{-1}(y)^{(\lambda)}$. Thus Λ is a meromorphic function from Y to $X^{(\lambda)}$.

Now if there is a bimeromorphic map from $\Phi\colon Y \to Z$ where Z is a normal quasi-projective variety and $\Phi \circ p$ is rational, then one can construct similarly with $\Phi \circ p$ a map $\Lambda'\colon Z \to \bar{X}^{(\lambda)}$ with an algebraic subset $\mathrm{Gr}\,(\Lambda') \subseteq Z \times \bar{X}^{(\lambda)}$ as graph. Now the projection of $\mathrm{Gr}\,(\Lambda')$ in $\bar{X}^{(\lambda)}$ is a locally closed algebraic set and has the image of Λ as one of its components. Thus $\Lambda(Y)$ is an algebraic set Y' in $X^{(\lambda)}$. Now using the natural map $p^{(\lambda)}\, X^{(\lambda)} \to Y^{(\lambda)}$ one sees by a continuity argument that $p^{(\lambda)}(Y')$ is in the diagonal and thus one gets an holomorphic map $e\colon Y' \to Y$ where Y' is quasi-projective. One can by normalizing assume that Y' is normal. Clearly $e\colon Y' - e^{-1}(\mathfrak{D}) \to Y - \mathfrak{D}$ is biholomorphic. Now one has a meromorphic and rational map from X to Y' by using the normalization of the graph $\mathcal{G}$ of the map $\Lambda \circ p$. Thus one has the commutative diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\;t\;} & X \\ {\scriptstyle q}\downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

with the projection $q\colon \mathcal{G} \to Y'$ a rational holomorphic map and t a birational morphism.

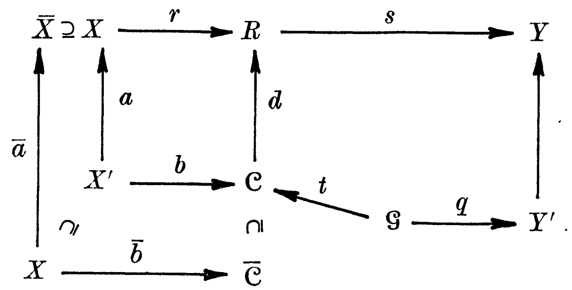
Now let us put the above together.

One has $p\colon X \to Y$ with Remmert-Stein factorization $X \xrightarrow{r} R \xrightarrow{s} Y$ and one has a bimeromorphic map $\Phi\colon Y \to Z$ such that $\Phi \circ p$ is rational.

First use the first half of the above argument applied to $r\colon X \to R$ to get the commutative diagram

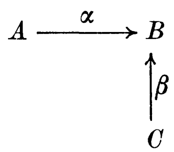
$$\begin{array}{ccccc} \bar{X} \supseteq X & \xrightarrow{\quad r \quad} & R & \xrightarrow{\quad s \quad} & Y \\ \uparrow \quad \uparrow {\scriptstyle a} & & \Big| {\scriptstyle d} & & \\ {\scriptstyle \bar{a}} \Big| \quad X' & \xrightarrow{\quad b \quad} & \mathcal{C} & & \\ \Big| \quad \cap & & \cap & & \\ \bar{X}' & \longrightarrow & \bar{\mathcal{C}} & & \end{array}$$

where $\bar{X}' = Z$, $\bar{a} = \pi$ and $b = \delta$. Now note that $\Phi \circ s \circ d$ is rational. To see this, simply note that the graph of $\Phi \circ s \circ d$ in $\mathcal{C} \times Z$ is a component of the algebraic set gotten by pulling the graph of $\Phi \circ p$ in $X \times Z$ to $X' \times Z$ using a, and then pushing it down to $\mathcal{C} \times Z$ by b. Thus one can use the second half of the argument applied to $s \circ d$ to get



Here $\mathcal{G}$ is the normalization of the graph in $\mathcal{C} \times Y'$ of the rational meromorphic map $\Lambda \circ s \circ d$. Let $\bar{\mathcal{G}}$ be a normal projective variety in which $\mathcal{G}$ is Zariski open. Note that q is a proper, generically finite to one map and that the birational morphism t extends to a birational mapping $\bar{t}: \bar{\mathcal{G}} \to \bar{\mathcal{C}}$. Replace $\mathcal{C}$ by $\mathcal{G}$. Replace $\bar{\mathcal{C}}$ by the normalization of the component Γ of the graph of $\bar{t}$ that surjects on both $\bar{\mathcal{C}}$ and $\bar{\mathcal{G}}$ under the maps induced from the projections $\bar{\mathcal{C}} \times \bar{\mathcal{G}} \to \bar{\mathcal{C}}$ and $\bar{\mathcal{C}} \times \bar{\mathcal{G}} \to \bar{\mathcal{G}}$. Replace X' by the normalization of the component of the fibre product $X' \times_{\mathcal{C}} \mathcal{G}$ of b and t that surjects onto X and $\mathcal{G}$ under the induced maps. Similarly replace $\bar{X}'$ by the normalization of the component of the fibre product $\bar{X}' \times_{\bar{\mathcal{C}}} \Gamma$ of $\bar{b}$ and the projection of Γ onto $\bar{\mathcal{C}}$, that surjects onto X and Γ under the induced projections.

Note that if:



is a commutative diagram of quasi-projective varieties and rational maps α and β, then the fibre product $A \times_{\beta} C$ of α and β is quasi-projective. To see this, simply consider the embedding of the fibre product in $A \times C$.

I leave it to the reader to check that the above satisfies the conclusion of the theorem. Note that Φ was used in an auxillary role to show quasi-projectivity of certain graphs and spaces but all maps above are independent of it. Q.E.D.

REMARK III-A. If $\dim_C X - \dim_C Y = k$ in the above theorem then if $\mathcal{S}_{k+1}(p)$ is algebraic it follows immediately from the above result that $e^{-1}(D)$ is algebraic where D is the degeneracy set of e.

REMARK III-B. One can use a combination of the rigidity lemma and the Chow space argument to show each component of $p^{-1}(\mathfrak{D}) \subseteq X$ is algebraic—of course there may be infinitely many components. The following question if answered affirmatively would, with the results of the next section, imply that the normalization of the image of the period mapping is algebraic.

QUESTION. *Let $p: X \to Y$ be a proper holomorphic surjection with X a Zariski open set of a normal irreducible projective variety $\bar{X}$ and Y a normal analytic space. Assume $p^{-1}(\mathfrak{D})$ is an algebraic set where $\mathfrak{D}$ is the degeneracy set and that p gives a biholomorphism between $X - p^{-1}(\mathfrak{D})$ and $Y - \mathfrak{D}$. Then does there exist a compact normal space $\bar{Y}$ in which Y is Zariski open such that p extends meromorphically to a map from $\bar{X}$ to $\bar{Y}$?*

REMARK III-C. If X is Zariski open in a normal compact irreducible analytic space $\bar{X}$ and $p: X \to Y$ is a proper holomorphic surjection with connected fibres onto a normal analytic space Y, then an analogue of the above proposition would hold if the Chow space of $\bar{X}$ had compact components; this is probably true if $\bar{X}$ is a Kaehler manifold. Precisely, there would exist a bimeromorphic map $\Phi: Y \to Z$ with Z a Zariski open set of a compact normal analytic space $\bar{Z}$ and such that $\Phi \circ p$ extends meromorphically to a map from $\bar{X}$ to $\bar{Z}$.

§ IV. – References for this section are [10, 11, 12, 13, 24, 29, 30]; I will not repeat all the usual definitions. D will denote a period domain and Γ an arithmetically defined discrete group that acts properly discontinuously on D. Thus $\Gamma \backslash D$ is normal complex space.

A smooth quasi-projective variation of Hodge structure is given by an holomorphic map $p: S \to \Gamma \backslash D$ of a Zariski open set of a connected smooth projective variety $\bar{S}$ in $\Gamma \backslash D$ such that:

A) there is an holomorphic map $p_u: S_u \to D$ of the universal cover S_u of S into D that covers S in the sense that

$$\begin{array}{ccc} S_u & \xrightarrow{\;p_u\;} & D \\ \downarrow & & \downarrow \\ S & \xrightarrow{\;p\;} & \Gamma \backslash D \end{array}$$

commutes,

B) fixing a basepoint s_0 of S and letting $\pi_1(S, s_0)$ act on S_u by covering transformations, then one has an homomorphism $\varphi: \pi_1(S, s_0) \to \Gamma$ with respect to which p_u is equivariant, *C*) p_u *is horizontal in the sense of* [24, *p.* 224],

D) $\bar{S} - S$ *has only normal crossing singularities.*

REMARK IV-*A*. All maps $p: S \to \Gamma\backslash D$ that arise in algebraic geometry, as sketched in the introduction, are smooth quasi-projective variations of Hodge structure. It will be convenient in a certain technical situation below to allow a variant of the above definition. A normal quasi-projective variation of Hodge structure is defined the same as above except $\bar{S}$ is only normal, p is proper, Γ is torsion free and *D*) doesn't apply.

Let $p: S \to \Gamma\backslash D$ be a smooth variation of Hodge structure where $\bar{S} - S$ has only normal crossings. Then, by a result of Griffiths [10; III p. 158] there exists a Zariski open set $\tilde{S}$ of $\bar{S}$ containing S and such that p extends to a proper holomorphic mapping $\tilde{p}: \tilde{S} \to \Gamma\backslash D$, called the *Griffiths extension of p*.

REMARK IV-*B*. In general $\tilde{p}: \tilde{S} \to \Gamma\backslash D$ is not a smooth quasi-projective variation of Hodge structure if Γ is not torsion free [cf. 10, III p. 172]. It is for this reason that the main result in [29] needs in its hypotheses either that p is proper or Γ is torsion free.

LEMMA IV-*A*. *Let* $p: S \to \Gamma\backslash D$ *be a smooth quasi-projective variation of Hodge structure with* S *Zariski open in a projective manifold* $\bar{S}$ *and let* $\tilde{p}: \tilde{S} \to \Gamma\backslash D$ *be the Griffiths extension. Then there exists a smooth quasi-projective variation of Hodge structure,* $p': S' \to \Gamma'\backslash D$ *with a Griffiths extension* $\tilde{p}': \tilde{S}' \to \Gamma\backslash X$ *where* Γ' *is torsion free and of finite index in* Γ *and*:

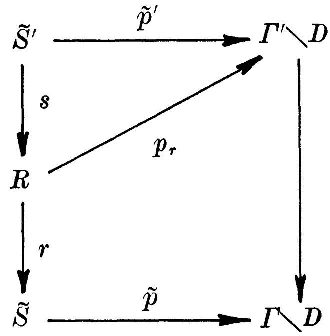
A) S' *and* $\tilde{S}'$ *are Zariski open in* $\bar{S}'$, *a smooth projective manifold, and there exists an holomorphic surjection* $\bar{q}: \bar{S}' \to \bar{S}$ *with restriction* $q = \bar{q}|_{S'}$: $S' \to S$ *a covering projection and restriction* $\tilde{q} = \bar{q}|_{\tilde{S}'}: \tilde{S}' \to \tilde{S}$ *a proper map,*

B) *the diagram*

$$\begin{array}{ccc} \tilde{S}' & \xrightarrow{\tilde{p}'} & \Gamma'\backslash D \\ {\scriptstyle\tilde{q}}\downarrow & & \downarrow \\ \tilde{S} & \xrightarrow{\tilde{p}} & \Gamma\backslash D \end{array}$$

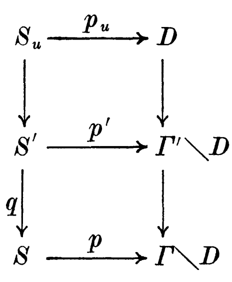
commutes.

Furthermore let $\bar{q} = \bar{r} \circ \bar{s}$ *denote the Remmert-Stein factorization of* $\bar{q}$ *where* $\bar{s}: \bar{S}' \to \bar{R}$ *is a proper surjection with connected fibres onto a normal variety* $\bar{R}$ *and* $\bar{r}: \bar{R} \to \bar{S}$ *is a proper finite to one surjection. Letting* $R = \bar{s}(\tilde{S})'$, *then* R *is Zariski open in* $\bar{R}$ *which is projective, and there is an holomorphic proper map* $p_r: R \to \Gamma'\backslash D$ *which is a normal quasi-projective variation of Hodge* *structure. Letting* $r = \bar{r}|_R$ *and* $s = \bar{s}|_{\tilde{S}'}$*, then* $\tilde{q} = r \circ s$ *is the Remmert-Stein factorization of* $\tilde{q}$ *and the diagram*



commutes.

PROOF. Recall Selberg's theorem [6, 17.4] that there exists a torsion free subgroup Γ' of finite index in Γ. Using B) of the definition of smooth quasi-projective variation of Hodge structure one has a commutative diagram:



where S' is a finite cover of S. Now S' is also quasi-projective in such a way that the covering projection $q: S' \to S$ is rational; this standard fact follows easily from § II.

Let $\bar{S}'$ and $\bar{S}$ be projective manifolds in which S' and S respectively are Zariski open. It can be assumed by Hironaka's theorem [17] that q extends to an holomorphic $\bar{q}$ from $\bar{S}'$ to $\bar{S}$, and further that $\bar{S}' - S'$ and $\bar{S} - S$ have only normal crossings. Let $\tilde{p}: \tilde{S} \to \Gamma\backslash D$ and $\tilde{p}': \tilde{S}' \to \Gamma'\backslash D$ be the Griffiths extensions of $\tilde{p}$ and $\tilde{p}'$ to proper maps [10, III p. 158]. Let $\tilde{q}$ denote the restriction of $\bar{q}$ to $\tilde{S}'$; it is easily checked from how the extensions $\tilde{p}$ and $\tilde{p}'$ are defined that $\tilde{q}$ gives an extension of q to a proper map from $\tilde{S}'$ to $\tilde{S}$. Let $\tilde{q}' = r \circ s$ be the Remmert-Stein factorization of $\tilde{q}'$ where $r: \tilde{S}' \to R$ has connected fibres and $s: R \to \tilde{S}$ is a finite to one proper map with R normal. Note that one can also Remmert-Stein factorize $\bar{q}'$ as $\bar{r} \circ \bar{s}$

where $\bar{r}\colon \bar{S}' \to \bar{R}$ has connected fibres and $\bar{s}\colon \bar{R} \to \bar{S}$ is a finite to one map. Since $\bar{s}$ is finite too it follows by Lemma III-B that $\bar{R}$ is projective and R, being Zariski open in $\bar{R}$, i.e. $\bar{r}|_R = r$ and $R = \bar{R} - \bar{r}(\bar{S}' - \tilde{S}')$, is quasi-projective. Thus the lemma will be proven if it can be shown that $\tilde{p}'$ factors as $p_r \circ r$ and the diagram:

$$\begin{array}{ccc} R & \xrightarrow{\;p_r\;} & \Gamma' \backslash D \\ \downarrow & & \downarrow \\ S & \xrightarrow{\;\tilde{p}\;} & \Gamma \backslash D \end{array}$$

commutes.

This follows immediately from the universal property of the Remmert-Stein factorization. To be precise, Remmert-Stein factorize $\tilde{p}' = s' \circ r'$ with $r'\colon \tilde{S}' \to R'$ a proper surjective map with connected fibres and R' normal and $s'\colon R' \to \Gamma \backslash D$ a finite to one proper map. Consider the commutative diagram:

$$\begin{array}{ccccc} \tilde{S}' & \xrightarrow{\;r'\;} & R' & \xrightarrow{\;s\;} & \Gamma' \backslash D \\ {\scriptstyle r}\downarrow & & & & \\ R & & & & \downarrow{\scriptstyle t} \\ {\scriptstyle s}\downarrow & & & & \\ \tilde{S} & & \xrightarrow{\;\tilde{p}\;} & & \Gamma \backslash D \end{array}$$

where $t\colon \Gamma' \backslash D \to \Gamma \backslash D$ is finite to one. Thus one has $(\tilde{p} \circ s) \circ r = (t \circ s) \circ r'$ where r and r' have connected fibres and $t \circ s$ is a finite to one map. Thus by the universal property (easily proved using the rigidity lemma of § III) of the Remmert-Stein factorization of $t \circ s \circ r'$ as $(t \circ s) \circ r'$, there exists a surjective proper map $u\colon R \to R'$ such that, letting $p_r = s \circ u$, one has the desired commutative diagram. Q.E.D.

LEMMA IV-B. *Let $p\colon S \to \Gamma \backslash D$ be a normal quasi-projective variation of Hodge structure. Let $\mathcal{N}(Z)$ be the normalization of an irreducible algebraic subvariety Z of S. Then $\mathcal{N}(p|_Z)\colon \mathcal{N}(Z) \to \Gamma \backslash D$ is a normal quasi-projective variation of Hodge structure.*

PROOF. The only thing that must be checked is the horizontality condition C). Since this is local, one can by choosing a neighborhood of a point of Z and lifting the situation, assume p is a map from S to D. Now p is horizontal means that dp maps the tangent space at each regular point of S to a certain holomorphic sub-bundle $\mathcal{H}$ of T_D. Clearly by continuity this condition must only be checked on a dense open set of the regular points. Now if Z is not contained in the singular set of S, then there is a dense open set of the regular points of Z that are regular points of S and everything is clear. Thus one can assume Z is contained in the singular set $\mathcal{S}$ of S. Use Hironaka's theorem to do a resolution $\delta: \tilde{S} \to S$ of S such that $\delta|_{\tilde{S}-\delta^{-1}(\mathcal{S})}: \tilde{S} - \delta^{-1}(\mathcal{S}) \to S - \mathcal{S}$ is a biholomorphic. By continuity $p\circ\delta$ is horizontal. Now consider $p\circ\delta|_{\delta^{-1}(Z)}$. Now $p\circ\delta$ is horizontal restricted to the regular points of $\delta^{-1}(Z)$. Further there is dense set U of the regular points of Z such that for each point y of U, there is at least one point x of $\delta^{-1}(y)$ such that $\delta|_{\delta^{-1}(U)}$ maps the tangent space of x in $\delta^{-1}(U)$ onto the tangent space of y in U. Thus considering $\delta^{-1}(U) \xrightarrow{\delta} U \xrightarrow{p} D$, one sees that p is horizontal on U. Q.E.D.

Now (§ III) consider the filtration $\mathcal{S}_0(\tilde{p}) \supseteq \mathcal{S}_1(\tilde{p}) \supseteq \ldots$ of $\tilde{S}$ where $\tilde{p}: \tilde{S} \to \Gamma\backslash D$ is the Griffiths extension of a smooth variation of Hodge structure $p: S \to \Gamma\backslash D$.

LEMMA IV-*C*. *With $\tilde{p}$ as above, the sets $\mathcal{S}_i(\tilde{p})$ are algebraic subvarieties of $\tilde{S}$.*

PROOF. Note that if $A: X \to Y$ and $B: Y \to Z$ are surjections of normal irreducible analytic spaces and A is finite to one, then $A(\mathcal{S}_i(B\circ A)) = \mathcal{S}_i(B)$. Thus by Lemma IV-*A* it suffices to show that $\mathcal{S}_0(p_r), \supseteq \mathcal{S}_1(p_r) \supseteq \ldots$ are algebraic where $p_r: R \to \Gamma\backslash D$ is a normal quasi-projective variation of Hodge structure.

Now consider the following lemma of Griffiths:

LEMMA (Griffiths). *Let $p: S \to \Gamma\backslash D$ be a normal quasi-projective variation of Hodge structure. The singular locus $J(p)$ is algebraic.*

PROOF. Using the remark preceding the statement of this lemma, one can assume using Hironaka's theorem that $p: S \to \Gamma\backslash D$ is a smooth quasi-projective variation of Hodge structure where p is proper and Γ is torsion free.

One notes that $T_{\Gamma\backslash D}$, the holomorphic tangent bundle of $\Gamma\backslash D$ is built out of the universal bundles on $\Gamma\backslash D$ by tensor products and dualizing [10, III p. 153]. Thus $p^*\mathcal{O}(T_{\Gamma\backslash D})$ extends as an analytic coherent sheaf $\mathcal{F}$ to $\bar{S}$, some projective manifold in which S is Zariski open [25; 4.13]. Now it can be shown [10, III p. 153 and 24, p. 225 ff] that the differential $dp \in \operatorname{Hom}(T_S, p^*(T_{\Gamma\backslash D}))$ is given by the induced action of the Gauss-Manin connection on $p^*(T_{\Gamma\backslash D})$. Thus by [24; 4.13] dp extends to a meromorphic sheaf map on $\bar{S}$ from $\mathcal{O}(T_S)$ to $\mathcal{F}$. The set where dp is not of maximal rank is thus algebraic. Q.E.D.

Now let

$$\begin{array}{ccc} \tilde{S}' & \xrightarrow{\ \tilde{p}'\ } & \Gamma'\backslash D \\ {\scriptstyle \tilde{q}}\big\downarrow & & \big\downarrow \\ \tilde{S} & \xrightarrow{\ \tilde{p}\ } & \Gamma\backslash D \end{array}$$

be as in lemma IV-A where $\tilde{p}: \tilde{S} \to \Gamma\backslash D$ and $\tilde{p}': \tilde{S}' \to \Gamma'\backslash D$ are the Griffiths extensions of the smooth quasi-projective variations of Hodge structure $p: S \to \Gamma\backslash D$ and $\tilde{p}': S' \to \Gamma'\backslash D$ respectively. Let A be a quasi-projective manifold of $\tilde{S}'$ such that $\tilde{p}'|_A$ is generically finite to one and proper. To construct such an A simply take $\bar{A}$ as a projective submanifold of $\bar{S}'$ that intersects a generic fibre of p' in a finite set of points and has dimension equal to that of $\tilde{p}'(\tilde{S}')$, and then let $A = \bar{A} \cap \tilde{S}'$. Letting $i_A: A \to \tilde{S}'$ denote the inclusion, let $\lambda: A \to \tilde{p}(\tilde{S})$ denote $\tilde{p}\circ\tilde{q}\circ i_A$ and let $\mu: A \to \Gamma'\backslash D$ denote $\tilde{p}'\circ i_A$.

One has the commutative diagram:

$$\begin{array}{ccc} & A & \xrightarrow{\ \mu\ } \Gamma'\backslash D \\ & {\scriptstyle\lambda}\big\downarrow & \\ \tilde{S} \xrightarrow{\ \tilde{p}\ } & \tilde{p}(\tilde{S}) & \end{array}$$

Note that K_D, the canonical bundle of D, since it is invariant under Γ, drops to a line bundle on $\Gamma\backslash D$ which by abuse of notation I denote $K_{\Gamma\backslash D}$; if Γ were actually torsion free this line bundle would be the canonical bundle of $\Gamma\backslash D$. K_D is built by taking tensor products, dual and determinant bundles out of the universal Hodge bundles on D. Therefore by a theorem of Griffiths [24; 4.13], the sheaf of germs of holomorphic sections of $\mathcal{K} = p^*(K_{\Gamma\backslash D})$ extends to $\bar{S}$ as a coherent analytic sheaf $\mathfrak{L}$. By a theorem of Rossi [23, cf. 22, § 2 for a good discussion] combined with Hironaka's theorem it can be assumed that $\mathfrak{L}$ is the locally free sheaf associated to an holomorphic line bundle L on $\bar{S}$.

D is of the form G/V where G is the connected component of the identity of all biholomorphic maps of D and V is a compact subgroup. Any G invariant metric on K_D (unique up to a constant multiple) drops to give a continuous metric on $K_{\Gamma\backslash D}$. In [29, p. 254] it is shown as a direct consequence of the results of [24] that the pullback of this Hermitian metric to $\mathcal{K}$ has L^2 poles at infinity relative to L. Further $\lambda^* K_{\Gamma\backslash D}|_{p(\tilde{S})} = \mu^*(K_{\Gamma'\backslash D})$ and the pullbacks of the Hermitian structures agree.

Now I claim that the curvature of this Hermitian structure on $K_{\Gamma'\backslash D}|_{p'(\tilde{S}')}$ is positive definite. Note that the G invariant metric on K_D is positive definite in the horizontal directions [12, p. 277]. Thus the assertion will follow from the following lemma.

LEMMA IV-*D*. *Let $\varphi: X \to D$ be an holomorphic horizontal embedding of an irreducible and reduced analytic space X into D. Assume that φ is a proper map from X to an open set $U \subseteq D$. Let ω be any real $C^\infty(1, 1)$ form defined in U and positive definite on horizontal directions. Then given a point $x \in X$, there exists a neighborhood V of $\varphi(x)$ in U and a plurisubharmonic function A on V such that:*

1) *$A \circ \varphi$ is identically zero,*
2) *$\sqrt{-1}\,\partial\bar{\partial} A + \omega|_V$ is positive on V.*

PROOF. Let $\{f_i | 1 \leqslant i \leqslant N\}$ be holomorphic functions defined in some neighborhood V' of $\varphi(x)$ that span the ideal sheaf of $\varphi(X) \cap V'$. Let $A = \sum_{i \leqslant N} |f_i|^2$.

Let $H \subseteq T_{V'}$ be the horizontal sub-bundle. If for a possibly smaller neighborhood V' of $\varphi(x)$ one had $\sqrt{-1}\,\partial\bar{\partial} A$ positive definite on some complementary sub-bundle to H in $T_{V'}$, the lemma would be done by restricting to some still smaller relatively compact neighborhood V of $\varphi(x)$, and multiplying A by a large enough positive constant.

Now let:

$$\mathfrak{Z}_{\varphi(x)} = \{e \in T_{V'}|_{\varphi(x)} | df_i|_x(e) = 0,\ 1 \leqslant i \leqslant N\}\,.$$

The desired conclusion about A will clearly occur if $\mathfrak{Z}_{\varphi(x)} \subseteq H|_{\varphi(x)}$. Now $\mathfrak{Z}_{\varphi(x)}$ is the Zariski tangent space of $\varphi(X)$ at $\varphi(x)$ and is well known to be the smallest vector subspace of $T_{V'}|_{\varphi(x)}$ spanned by the Zariski tangent cone $C_{\varphi(x)}$ of $\varphi(X)$ at x. Now let $\{y_r\}$ be a sequence of manifold points of $\varphi(X)$ with $y_r \to \varphi(x)$. Let $\{e_r\}$ be a sequence of tangent vectors $e_r \in T_{\varphi(X)}|_{y_r}$. Then by continuity and the horizontality of φ, it is clear that any limit $e' \in T_{V'}|_{\varphi(x)}$ must belong to $H|_{\varphi(x)}$. Thus by a theorem of Whitney [32, 22.1 on p. 547; cf. 15, p. 251 ff. for discussion] $H|_{\varphi(x)} \supseteq C_{\varphi(x)}$ and thus also $H|_{\varphi(x)} \supseteq \mathfrak{Z}_{\varphi(x)}$.
Q.E.D.

It is worth recording one consequence of this lemma; see [11, p. 258]:

COROLLARY IV-*A*. *Let $\varphi: X \to D$ be a proper horizontal holomorphic map where X is a reduced and irreducible analytic space. Then $\varphi(X)$ is a Stein space.*

PROOF. Let ϱ be the exhaustion function of D found in [12, p. 295]. The Leviform of ϱ is positive definite when restricted to the horizontal subbundle of H. Thus by the above, the restriction of ϱ to $\varphi(X)$ is a strongly plurisubharmonic exhaustion function. Now use Narasimhan's generalization of Grauert's solution of the Levi problem [19]. Q.E.D.

Next note that A is a complete Kaehler manifold. To see this, first note that since A is a quasi-projective manifold it possesses a Kaehler metric. Next add to this metric the pullback under μ of the complete metric d_G on $\Gamma'\backslash D$ that is induced by the essentially unique G invariant metric on G. It is easy to see [29, p. 254] that the Hermitian form associated to $\mu^* d_G$ is closed.

In the following $\mathcal{N}(Z)$ will denote the normalization of Z, for any reduced analytic space Z. If $f: Z \to W$ is an holomorphic map between reduced analytic spaces Z and W, then $\mathcal{N}(f)$ will denote the functorially associated map between $\mathcal{N}(Z)$ and $\mathcal{N}(W)$. As far as meromorphic mappings go, I will work with normal spaces. One can go to more general spaces by throwing everything back into the normalizations of the spaces in question, but this does not allow any better results below. It should be noted, though, that with the classical definition of meromorphic functions one has the result [1, 3.9] that $\mathcal{M}(\mathcal{N}(Z)) \approx \mathcal{M}(Z)$ where for a reduced and irreducible analytic space Z, $\mathcal{M}(Z)$ denotes the field of meromorphic functions.

Now the following is the major proposition of the paper.

PROPOSITION IV. *Let $p: S \to \Gamma\backslash D$ be a quasi-projective variation of Hodge structure and let $\tilde{p}: \tilde{S} \to \Gamma/D$ be the Griffiths extension of p. Let $\mathfrak{D}$ be the degeneracy set of $\mathcal{N}(\tilde{p})$. Then there exists a proper holomorphic surjection $\Phi: M \to \mathcal{N}(\tilde{p}(\tilde{S}))$ where M is a normal quasi-projective variety such that:*

A) $\Phi^{-1}(\mathfrak{D})$ is an algebraic subvariety of M and Φ gives a biholomorphism between $M - \Phi^{-1}(\mathfrak{D})$ and $\mathcal{N}(\tilde{p}(\tilde{S})) - \mathfrak{D}$,

B) the degeneracy set of Φ is $\mathfrak{D}$,

C) $\Phi^{-1}\circ\mathcal{N}(\tilde{p})$ is rational; if $\Psi: \mathcal{N}(\tilde{p}(\tilde{S})) \to Z$ is any bimeromorphic map onto a normal quasi-projective variety Z, such that $\Psi\circ\mathcal{N}(\tilde{p})$ is rational, then $\Psi\circ\Phi$ is a birational equivalence between M and Z,

D) Let $j: \mathcal{N}(\tilde{p}(\tilde{S})) \to \Gamma\backslash D$ denote the induced map. If $f: S' \to \mathcal{N}(\tilde{p}(\tilde{S}))$ is any holomorphic map where S' is Zariski open in a projective manifold $\bar{S}'$, $j\circ f$ is a smooth quasi-projective variation of Hodge structure, and $f(S')$ contains an open set of $\mathcal{N}(\tilde{p}(\tilde{S}))$, then $\Phi^{-1}\circ f$ is rational.

PROOF. A), B), and C) follow directly from Proposition I, Proposition III, Lemma IV-C, Lemma IV-D and the discussion preceding the statement of the above Proposition IV.

D) Follows from Corollary II-A and the discussion preceding the above Proposition IV. Q.E.D.

COROLLARY IV-*B*. *Let $\tilde{p}: \tilde{S} \to \Gamma\backslash D$ be the Griffiths extension of a smooth quasi-projective variation of Hodge structure $p: S \to \Gamma\backslash D$; S and $\tilde{S}$ are Zariski open in $\bar{S}$, a projective manifold. Assume that the degeneracy set $\mathfrak{D} \subseteq \mathcal{N}(\tilde{p}(\tilde{S}))$ of $\mathcal{N}(\tilde{p})$ is a discrete set, e.g.* $\dim_{\mathbf{C}} \tilde{p}(\tilde{S}) \leqslant 2$. *Then there is a compact Moisezon space Y in which $\mathcal{N}(\tilde{p}(\tilde{S}))$ is Zariski open and such that $\mathcal{N}(\tilde{p})$ extends to a meromorphic map from $\bar{S}$ to Y.*

Now let me record an amazing GAGA [25] like corollary of the above proposition that amplifies *D*) of Proposition IV.

COROLLARY IV-*C*. *Let $p: \tilde{S} \to \Gamma\backslash D$ be the Griffiths extension of a smooth quasi-projective variation of Hodge structure. Assume Γ is torsion free. Let U_1 and U_2 be two open sets of $\mathcal{N}(\tilde{p}(\tilde{S}))$. If one can put a quasi-projective structure on U_1 and U_2, then $U_1 \cap U_2$ possesses a quasi-projective structure compatible with both. The union $\mathcal{U}$ of all points that possess a quasi-projective neighborhood is a scheme of finite type ove* **C**.

PROOF. Let $\Phi: M \to \mathcal{N}(\tilde{p}(\tilde{S}))$ be as in the above proposition. Using Hironaka's theorem, let $\delta_i: U_i' \to U_i$ for $i = 1$ and $i = 2$ be smooth desingularizations with δ_i proper and U_i' Zariski open in a projective manifold $\bar{U}_i'$ such that $\bar{U}_i' - U_i'$ has only normal crossing singularities. Let $j: \mathcal{N}(\tilde{p}(\tilde{S})) \to \Gamma\backslash D$ denote the induced map. Since Γ is torsion free, $j \circ \delta_i$ is a smooth quasi-projective variation of Hodge structure. Thus by *D*) of the above Proposition IV, $\Phi^{-1} \circ \delta_i$ is rational. In particular each $\Phi^{-1}(U_i)$ is a constructible set by Chevalley's theorem. Since the $\Phi^{-1}(U_i)$ are also open in the complex topology they are open in the algebraic Zariski topology on M and the maps $\Phi: \Phi^{-1}(U_i) \to U_i$ are rational. Thus $Z = M - \Phi^{-1}(U_2)$ is algebraic and therefore $\Phi(\Phi^{-1}(U_1) - Z) = U_1 - \Phi(Z) = U_1 \cap U_2$ is open in U_1 in the algebraic Zariski topology on U_1. Thus the quasi-projective structures on U_1 and U_2 are compatible.

Now let $\mathcal{U} = \bigcup_\alpha U_\alpha$ where each U_α has a quasi-projective structure. Note that $\Phi^{-1}(\mathcal{U}) = \bigcup_\alpha \Phi^{-1}(U_\alpha)$. Since a quasi-projective variety is compact in the algebraic Zariski topology, it follows that $\mathcal{U}$ is a finite union. By the first paragrah, everything is compatible. Q.E.D.

REMARK IV-*C*. The above is a reflection of the negative curvature and the resemblance of *D* to a bounded domain in horizontal directions.

PROPOSITION V. *Let $\tilde{p}: \tilde{S} \to \Gamma\backslash D$ be the Griffiths extension of a smooth quasi-projective variation of Hodge structure $p: S \to \Gamma\backslash D$. Let U denote the image in $\mathcal{N}(p(S))$ of the set of points in S where dp, the differential of p, is of maximal rank. Then U is an open set possessing a quasi-projective structure. Further U has the GAGA* [25] *property; it possesses only one quasi-projective* *structure compatible with the underlying analytic structure in the sense that if $\{U_i | i = 1$ and $i = 2\}$ denote U with each of two quasi-projective structures, then the identity map on U gives a birational equivalence between U_1 and U_2.*

PROOF. By Proposition IV there exists a normal irreducible quasi-projective variety M and a proper, generically one to one, holomorphic surjection $\Phi: M \to \mathcal{N}(\tilde{p}(\tilde{S}))$. Using Hironaka's theorem [17] to desingularize M, one can assume that M is Zariski open in a projective manifold $\bar{M}$ and $\bar{M} - M$ has only normal crossing singularities. Let $j: \mathcal{N}(\tilde{p}(\tilde{S})) \to \Gamma \backslash D$ be the induced map.

Assume first that Γ is torsion free. Then $j \circ \Phi$ is a smooth quasi-projective variation of Hodge structure. Thus one can apply Lemma IV-C and the discussion preceding Proposition IV to $j \circ \Phi$, in order to conclude that the hypotheses of Proposition II are satisfied. Thus there exists a meromorphic map $\Psi: \mathcal{N}(\tilde{p}(\tilde{S})) \to \boldsymbol{CP}^N$ for some N, such that:

A) $\Psi \circ \Phi$ is rational,

B) Ψ is an embedding on the set of manifold points of $\mathcal{N}(\tilde{p}(\tilde{S}))$.

By A) and Chevalley's theorem, the image of $\Psi \circ \Phi$ is constructible. Thus by normalizing the closure of the image of $\Psi \circ \Phi$ one can assume that there exists a normal irreducible projective variety Z and a meromorphic map $\Psi': \mathcal{N}(\tilde{p}(\tilde{S})) \to Z$ with dense image such that:

A) $\Psi' \circ \Phi$ is rational,

B) Ψ' is an embedding on the set of manifold points of $\mathcal{N}(\tilde{p}(\tilde{S}))$.

Since by Griffiths' lemma stated during the proof of Lemma IV-D, the singular set of p is algebraic, it follows that U is open in $\bar{S}$ in the Zariski topology. Thus, since $\Psi' \circ \Phi$ is rational, $\Psi' \circ \Phi(U)$ is constructible and since $\Psi' \circ \mathcal{N}(\tilde{p}(U))$ is open in the complex topology, it is quasi-projective. Thus Ψ' gives a biholomorphism of $\mathcal{N}(\tilde{p}(U))$ with a quasi-projective variety.

Now let Γ be arbitrary. By Lemma IV-A there is a smooth quasi-projective variation of Hodge structure $p': S' \to \Gamma' \backslash D$ with Griffiths extension $\tilde{p}': \tilde{S}' \to \Gamma' \backslash D$ where Γ' is a torsion free subgroup of finite index in Γ and where the diagram:

$$\begin{array}{ccc} S' & \xrightarrow{p'} & \Gamma' \backslash D \\ {\scriptstyle q}\downarrow & & \downarrow \\ S & \xrightarrow{p} & \Gamma \backslash D \end{array}$$

commutes with q, a covering map. Now let U' be the image in $\mathcal{N}(\tilde{p}'(\tilde{S}'))$ under $\mathcal{N}(p')$ of the open set of points in S where dp is of maximal rank;

it is a manifold by the implicit function theorem. The map from $\Gamma' \backslash D$ to $\Gamma \backslash D$ induces a holomorphic generically finite to one map $r\colon \mathcal{N}(\tilde{p}'(\tilde{S}')) \to \to \mathcal{N}(\tilde{p}(\tilde{S}))$.

Note that $r|_{U'}$ is a finite to cover from U' onto U. There is a map φ from U to $U'^{(\lambda)}$ where $r|_{U'}$ is λ to 1 and φ sends $a \in U$ to the unordered λ tuple $r|_{U'}^{-1}(a)$.

Now U' possesses a quasi-projective structure by the earlier part of the proof. Using Hironaka's theorem [17] one can assume U' is Zariski open in a projective manifold $\bar{U}'$ and $\bar{U}' - \bar{U}'$ has only normal crossing singularities. Since $j \circ r|_{U'}\colon U' \to \Gamma \backslash D$ is then a quasi-projective variation of Hodge structure and $r|_{U'}(U')$ contains the open set U of $\mathcal{N}(\tilde{p}(\tilde{S}))$, one can conclude by D) of Proposition IV that $\Phi^{-1} \circ r|_{U'}$ is rational with image $A \subseteq M$. Now A is Zariski open in M in the algebraic Zariski topology, since $A = \Phi^{-1}(U)$ is open in the complex topology and $A = \Phi^{-1} \circ r|_{U'}(U')$ is constructible by Chevalley's theorem.

Thus one can define a rational meromorphic map $\varphi'\colon A \to U'^{(\lambda)}$ by sending a generic point $a \in A$ to $(\Phi^{-1} \circ r|_{U'})^{-1}(a)$. Now $\varphi'(A)$ and $\varphi(U)$ have the same image. Since φ is an embedding of U into $U'^{(\lambda)}$ and since φ is rational, it follows that U possesses a quasi-projective structure.

Now I will show U has only one quasi-projective structure in the strong sense that if there were two, the identity map would be birational. Assume U possesses two quasi-projective structures; denote U with these two structures U_1 and U_2. Now U' is a finite cover of U and thus each U_i induces a quasi-projective structure on U', as is easily seen. But by Corollary IV-C, U' has a unique quasi-projective structure. It is easy to see that this forces U_1 and U_2 to be equivalent. Q.E.D.

REMARK IV-D. The above has as one consequence the fact that in certain situations, when the local Torelli theorem holds and a moduli space exists, e.g. [20], the moduli space has a unique quasi-projective structure supported by the underlying analytic structure.

Note that since the degeneracy locus $\mathfrak{D}$ of $\tilde{p}$ is the image of an algebraic set, one can conclude that there is a Zariski open set of $\mathcal{N}(\mathfrak{D})$ with a quasi-projective structure and so on.

The above results can be somewhat improved. By using Lemma II-B at all points, not only at the smooth ones, and by using the proof of Proposition I of [29], it can be shown that if Γ is torsion free then the normalization X_k of the monoidal transform of $\mathcal{N}(\tilde{p}(\tilde{S}))$ with respect to $\mathcal{K}^N_{\mathcal{N}(\tilde{p}(\tilde{S}))}$ for some N is quasi-projective. This fact, combined with some further arguments, can be used to show that if Γ is torsion free then the various bimeromorphic embeddings of $\mathcal{N}(\tilde{p}(\tilde{S}))$ used above can be gotten by using holomorphic sections of the pullback L^M of $K^M_{\Gamma^p}$ to $\mathcal{N}(\tilde{p}(\tilde{S}))$ for some integer $M > 0$. Roughly one chooses a quasi-projective desingularization $\tilde{X}_k$ of X_k. Letting R be the invertible sheaf on $\tilde{X}_k$ which is the pullback of $\mathcal{K}^N_{\mathcal{N}(\tilde{p}(\tilde{S}))}$ modulo torsion, one notes that the direct image under the surjection of $\tilde{X}_k$ onto $\mathcal{N}(\tilde{p}(\tilde{S}))$ of $\mathcal{K}_{\tilde{X}k} \otimes R^{-1}$ is the trivial sheaf at the non-singular points of $\mathcal{N}(\tilde{p}(\tilde{S}))$. Letting L' be the pullback of L to $\tilde{X}_k$ one does a curvature calculation on D that implies $\mathcal{O}(L'^r) \otimes R^{-1}$ is almost positive for $r \gg 0$. One now constructs sections of $\mathcal{O}(K_{\tilde{X}k} \otimes L'^b) \otimes R^{-a}$ for various a and b, including $a = 0$. By tensoring appropriate induced sections of the direct images on $\mathcal{N}(\tilde{p}(\tilde{S}))$, one gets holomorphic sections of L^s for various s on $\mathcal{N}(\tilde{p}(\tilde{S}))$ minus the singular set of $\mathcal{N}(\tilde{p}(\tilde{S}))$; these sections then extend to $\mathcal{N}(\tilde{p}(\tilde{S}))$. I will not give full proofs of these extra results because their importance does not seem to me commensurate with the extra space the proofs would require.

It is clear from the above that to say whether the image of $\tilde{p}$ itself has an algebraic structure, one must study the behavior of $\tilde{p}$ at infinity. E.g. if X is an algebraic curve, $\tilde{p}: \tilde{X} \to \tilde{p}(\tilde{X})$ assumed generically one to one would be algebraic unless there were two sequences of points $\{x_n\}$ and $\{y_n\}$ of X with $x_n \neq y_n$, $p(x_n)$ diverging, and $p(x_n) = p(y_n)$. This type of question is considered in detail in [28]. One positive result that the author proves [28, Prop. V] in this direction is that if $\varphi: \Delta^* \to \Gamma \backslash D$ is a horizontal, holomorphic, locally liftable map, then there is a smaller subdisc on which $\varphi: \Delta^* \to \varphi(\Delta^*)$ is a finite to one cover.

Note added in proof.

In Remark III-C we mention an improvement possible if the Chow space of a compact Kaehler manifold had compact components. This has been shown independently by A. FUJIKI, Publ. RIMS, Kyoto Univ., **14** (1978), pp. 1-52 and D. LIEBERMAN, Sem. Fran. Norguet (1976), pp. 140-186.

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