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Uniqueness Theorems for Some Open and Closed Surfaces in Three-Space (*).

J. J. STOKER (**)

dedicated to Hans Lewy

1. – Introduction.

The problems considered are the following:

1) *Christoffel's problem.* The sum $R_1 + R_2$ of the principal radii of curvature of a surface is prescribed as a function of the direction of its normals.

2) *Minkowski's problem.* This the same as 1) except that the Gauss curvature $K = 1/R_1R_2$ is prescribed as a function of the direction of the normals.

3) *Weyl's problem.* The line element is prescribed as a function of Gaussian parameters on the surface.

4) *Liebmann's problem.* This is an infinitesimal version of 3) in which a surface $S(0)$ is embedded in a set of neighboring surfaces $S(\varepsilon)$, in one-to-one correspondence with $S(0)$, with a few continuous derivatives in ε near $\varepsilon = 0$. It is assumed that the surfaces $S(\varepsilon)$ are approximately isometric in the correspondence set up near $\varepsilon = 0$, in the sense that corresponding curves differ in length by a quantity of order ε^2.

(*) This paper is dedicated to Hans Lewy; at the end of the introduction a few remarks about that are made. The author wishes to express thanks to the Guggenheim Foundation for an award. The ideas that resulted in this paper came about in the course of using it.

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In all of these problems it is to be shown that any solutions for a surface $S(u, v)$, with continuous third derivatives, say, is uniquely determined within orthogonal transformations, or, put differently, that any two surfaces satisfying the conditions of the problems (which must, of course, include boundary conditions in case the surfaces are not closed surfaces) can differ only by a rigid motion plus a possible reflection in a plane. In all cases the Gauss curvature K is assumed not to vanish, except possibly at boundary points. In the case of problem 4) uniqueness refers naturally to infinitesimal rigid motions, which in effect means that for $\varepsilon \simeq 0$, the displacement field of surface points is that of a rigid motion within second order terms in ε.

These problems have been treated at length by many writers. A good deal of the relevant literature up to 1950 is cited in the author's paper [9], in which the closed convex surfaces were treated, together with some having boundaries of a very special character. More recently the books of Pogorelov [7], Efimov [1], and a paper by Voss [14], give discussions of these and various other problems concerning surfaces with boundaries, and with the use of a considerable variety of methods. Only references to papers cited here are given in the bibliography.

In the paper of the author cited above, the object was to give uniqueness proofs in all four problems by making use of Minkowski's support function, which means that the problems are treated using homogeneous functions in three-space. At the same time it was desired to give proofs with a certain unity since the problems seemed to be in some sense closely related. This was partially successful, but not as completely as was desired. The purpose of this paper is in part to bring this out more clearly by procedures using only the maximum principle in quite simple ways which avoid, for example, the need for the symmetry relation between three arbitrary homogeneous functions of degree one, followed by an integration over the unit sphere in three-space to obtain an invariant integral with an integrand of one sign. All four of the uniqueness problems for closed surfaces can be treated with the aid of a properly chosen covariant differential form α of first degree. Its exterior derivative $d\alpha$ vanishes when it is integrated over the surface, and α is selected so that the integrand does not change sign and hence is everywhere zero, and in addition, is such that this leads to the desired uniqueness theorems. In the author's book [11] uniqueness proofs for all four problems are treated in this way. They are quite concise, but somewhat mysterious, since the choice of the differential α has little motivation, and the conciseness of the proofs results from the compactness of the notation, which conceals a great deal of differential geometry.

The present paper also deals with various problems for open surfaces when boundaries on the surfaces occur, or they extend to infinity. All boundaries are assumed to be simple disjoint closed curves on the surfaces. These problems are also solved in quite elementary ways using the maximum principle. Some problems for complete open surfaces with $K > 0$ are treated (these are boundaries of unbounded three-dimensional convex bodies).

Papers by Volkov and Oliker [13] and by Oliker [6], published in 1970 and 1971, deal with uniqueness theorems for problems 1) and 2) when formulated as boundary problems for relevant partial differential equations on the unit sphere. They are treated in the spherical image Ω of the surfaces in terms of Minkowski's support function h as dependent variable. The condition imposed for the uniqueness theorems is $h = 0$ on the boundary of Ω. Uniqueness theorems for such problems also lead, of course, to uniqueness theorems for the surfaces in three-space. The author feels that they can be obtained in a rather more elementary way by working in three-space. Also, though it is quite natural from the point of view of uniqueness theorems for boundary problems for a second order linear partial differential equation on the sphere to prescribe that solutions should take on prescribed values at the boundary, it is not the only reasonable way to formulate boundary conditions for the surfaces in three-space. For example, in the paper by Volkov and Oliker simple examples are given showing that the solution of Christoffel's problem for certain surfaces with boundaries is not always uniquely determined by prescribing the value of the support function on them. But in all cases the solution of these problems for the surfaces is uniquely determined if any one of the Cartesian coordinates at its boundary (rather than the values of the support function) has a prescribed value, since it then follows (as will be shown later) that the other two coordinates are uniquely determined within translations. The same remark applies to Minkowski's problem. It has long been known (apparently since Darboux) that the same thing holds for Weyl's problem (and also, within the infinitesimal approximation involved, for Liebmann's problem). The author has not seen this remark in the literature about the first two problems, although, as will be seen, it is very easy to verify it.

This paper is dedicated to Hans Lewy. He and I have been friends for more than thirty years. At the time it began we worked closely and fruitfully together for about a year on problems in fluid mechanics. However I, and students, have been much influenced by his two classic papers about Weyl's and Minkowski's problem, in which he dealt mostly with difficult existence questions; but he also gave the first uniqueness proof for the solution of Minkowski's problem for closed analytic surfaces. Thus it seems appropriate to me to write a paper for this occasion in the field of differential geometry in the large.

2. – Formulation of the problems using Minkowski's support function.

The formulas needed for this paper will be set down without derivations. These can be found in the paper [9] cited above or in the literature otherwise—in any case the theory is quite elementary in character.

The support function is defined as follows. A point p of a regular surface $S(u, v)$ in three-space is considered and a Cartesian coordinate system x^i, $i = 1, 2, 3$ is chosen with its origin not on S. The direction cosines of a normal to S at point p are denoted by α^i. The function $h(\alpha^1, \alpha^2, \alpha^3)$ is defined as the distance from the origin to the tangent plane of S at p. Since the Gaussian curvature K is assumed not to vanish, the points of S in a neighborhood of p are in one-to-one correspondence with the directions of the normals. The function $h(\alpha^1, \alpha^2, \alpha^3)$ is then extended into three-space as a homogeneous function $H(x^1, x^2, x^3)$ of degree one by setting

$$H(x^1, x^2, x^3) \equiv rh\left(\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}\right), \qquad r = [(x^1)^2 + (x^2)^2 + (x^3)^2]^{\frac{1}{2}} . \tag{2.1}$$

This is Minkowski's support function. It means that an infinite cone with vertex at the origin is mapped on the surface in such a way that all points on each ray from the origin (except the origin itself) map on a single point of S. This makes it possible to regard the surface as defined by H, with x^1, x^2, x^3 as independent variables defining normal directions of the surface. The Cartesian coordinates $\bar{x}^1$, $\bar{x}^2$, $\bar{x}^3$ of the *points* of S are then shown to be given by

$$\bar{x}^i = \frac{\partial H}{\partial x^i} = H_i . \tag{2.2}$$

In effect these are conditions that result because S is the envelope of its tangent planes. The coordinates $\bar{x}^i$ of the points of S are therefore determined by homogeneous functions of degree zero, since H is homogeneous of degree one, and that is as it should be because it means that every point of a given ray from the origin does indeed map on a uniquely determined point of S. It is vital to observe that (2.2) leads to the following conclusion: *if two support functions of surfaces differ by a linear function then the two surfaces are identical within a translation.*

Curvature properties of S are defined through consideration of the principal radii of curvature R_1, R_2, in connection with the formula of Rodrigues. It is found in a straightforward way that R_1 and R_2 are roots of the following algebraic equation:

$$\begin{vmatrix} H_{11}+R & H_{12} & H_{13} \\ H_{21} & H_{22}+R & H_{23} \\ H_{21} & H_{32} & H_{33}+R \end{vmatrix} = 0 \; . \tag{2.3}$$

Here H_{ik} is a symbol for second derivatives of H. This equation, seemingly cubic in R, is really quadratic, since the constant term in it, i.e. the determinant $|H_{ik}|$, vanishes simply because H is homogeneous of degree one.

It then follows that the support function satisfies two differential equations containing R_1 and R_2 in coefficients. One of them is

$$\nabla^2 H = H_{11} + H_{22} + H_{33} = -(R_1 + R_2) \; , \tag{2.4}$$

which results from consideration of the sum of the roots of (2.3). The second results because the product of the roots of (2.3) is $R_1 R_2 = 1/K$, and because H is homogeneous of degree one:

$$\begin{cases} H_{11}H_{22} - H_{12}^2 = (x^3)^2/r^2 K \; , \\ H_{22}H_{33} - H_{23}^2 = (x^1)^2/r^2 K \; , \\ H_{33}H_{11} - H_{13}^2 = (x^2)^2/r^2 K \, . \end{cases} \tag{2.5}$$

The following identities occur later; they hold when H and W are any homogeneous functions of degree one (not necessarily derived from a surface):

$$\begin{aligned} \frac{H_{11}W_{22} - 2H_{12}W_{12} + H_{22}W_{11}}{(x^3)^2} &= \frac{H_{22}W_{33} - 2H_{23}W_{23} + H_{33}W_{22}}{(x^1)^2} = \\ &= \frac{H_{11}W_{33} - 2H_{13}W_{13} + H_{33}W_{11}}{(x^2)^2} \, . \end{aligned} \tag{2.6}$$

If H and W were identical, the three ratios in (2.6) would all be equal to $1/r^2K$ in view of (2.5); thus each equation of (2.5) is equivalent to the others. For the sake of ready reference some of Euler's relations for a homogeneous function H of degree one are set down; they are used in deriving the above identities:

$$H_1 x^1 + H_2 x^2 + H_3 x^3 = H \; , \tag{2.7}$$

and

$$\sum_{k=1}^{3} H_{ik} x^k = 0 \; , \qquad i = 1, 2, 3 \; . \tag{2.8}$$

The second results because the first derivatives H_i are homogeneous of degree zero.

Suitable partial differential equations for problems 1) and 2) are, clearly, given by (2.4) and (2.5). For Weyl's and for Liebmann's problem the formulation in terms of a support function is less direct. If $\boldsymbol{X}(u, v)$, $\boldsymbol{Y}(u, v)$ are two surfaces S_1, S_2 which are isometric in the same parameters (u, v), it follows that $d\boldsymbol{X}\cdot d\boldsymbol{X} = d\boldsymbol{Y}\cdot d\boldsymbol{Y}$, the differential d to be taken with respect to (u, v). Introduction of $\boldsymbol{V} = \boldsymbol{X} - \boldsymbol{Y}$ and $\boldsymbol{Z} = \frac{1}{2}(\boldsymbol{X} + \boldsymbol{Y})$, the so-called mean surface, leads to the condition $d\boldsymbol{V}\cdot d\boldsymbol{Z} = 0$. It can then be shown that a uniquely determined rotation vector $\boldsymbol{\delta}(u, v)$ exists such that

$$d\boldsymbol{V} = \boldsymbol{\delta} \times d\boldsymbol{Z}, \tag{2.9}$$

in which $\times$ means the vector product; provided that the mean surface $\boldsymbol{Z}$ is a regular surface in the parameters (u, v). It is easily seen that this condition on $\boldsymbol{Z}$ will be satisfied if the two surfaces S_1 and S_2 are so placed in three-space that no pair of tangent vectors that correspond by the isometry are parallel in three-space, but oppositely directed. In the book of Pogorelov [7] the regularity of the mean surface is verified in a special case; in a recent letter to the author he states that this intriguing question remains unsettled in the general case. However, if the two surfaces are placed so as to be tangent to each other at a pair of corresponding points, it is clear that the mean surface will be regular even if not too large finite displacements of corresponding points in three-space occur. This, of course, means that an unwanted restriction is imposed if the mean surface is used, but nevertheless Weyl's problem rather than Liebmann's problem is involved: in the second problem the original surface $S(0)$ can be used legitimately in place of a mean surface.

The connection of this with a support function W results by defining W as the scalar product $\boldsymbol{\delta}\cdot\boldsymbol{n}$ of the rotation vector with the unit normal $\boldsymbol{n}$ of the mean surface, hence first as a point function on a unit sphere (the spherical image of the mean surface), which is then extended into three-space as a homogeneous function of degree one. An elementary discussion (carried out in [9] for Weyl's problem) shows that if $\boldsymbol{\delta}$ is a constant vector in the parameters (u, v) that then the isometric surfaces can differ only by a rigid motion. It turns out that the uniqueness theorems for problems 2), 3), and 4) then all lead to the same linear differential equations for a relevant homogeneous function (in spite of the fact that the differential equations for 2) and 3) are Monge-Ampère equations, hence not even quasilinear). This is basically the reason why a unified treatment of the uniqueness problems would seem feasible and worth looking for.

3. – Christoffel's problem.

Consider two surfaces S_1, S_2, possibly with boundaries, such that $K \neq 0$ holds, but $K < 0$ as well as $K > 0$ may occur. For both, the support functions $H^{(1)}$, $H^{(2)}$ are then sometimes assumed to have the same values at any boundaries of their spherical images. This is the assumption made by Volkov and Oliker [13], and by Oliker [6]. In effect, $H^{(1)}$ and $H^{(2)}$ are assumed to be prescribed at boundaries of the spherical image. In the papers quoted the significance of this for the surfaces S_1 and S_2 is mentioned only in passing, probably because the problem is dealt with as a boundary problem on the sphere, with differential equations also defined on it. It seems to the author easier to deal with the problem in three-space since the differential equation is simpler and the uniqueness theorems are proved in a straighforward way that is quite elementary.

THEOREM C_1: It has some point to deal with the classical problem first, i.e. the case in which S_1 and S_2 are closed convex sufaces with the origin in their interiors, and thus the spherical image covers the whole sphere. It is to be shown that S_1 and S_2 differ only by a translation. Their support functions $H^{(1)}$, $H^{(2)}$ both satisfy (2.4) with the same right hand sides. Thus $H = H^1 - H^2$ is evidently a harmonic function in three-space (it would be whether S_1 and S_2 are closed or not, and also whether $K < 0$ or $K > 0$ holds):

$$\nabla^2 H = H_{11} + H_{22} + H_{33} = 0 . \tag{3.1}$$

The harmonic function H is defined in the entire x^1, x^2, x^3-space, except at the origin. However, the origin is an isolated point with H defined and single-valued in its neighborhood, and $H \to 0$ as $r \to 0$ because H is homogeneous of degree one, hence H is bounded near the origin. Consequently the origin is a removable singularity and H is harmonic there if its value is defined to be zero. The function behaves linearly in r when $r \to \infty$; it follows that H is a linear function of the variables x^i (from Liouville's theorem for harmonic functions in three dimensions), and because of (2.2) the coordinates $\bar{x}^i$ of the points of the two surfaces S_1 and S_2 differ by a constant, and thus S_1 and S_2 differ by a translation. Thus an exceedingly simple proof of uniqueness of the solution of Christoffel's problem for closed convex surfaces results.

In the paper by Volkov and Oliker cited above examples are given (involving surfaces with $K < 0$) for which the solution of the problem with boundaries is not uniquely determined when the support function H is prescribed at boundaries. However, the solutions are uniquely determined if any one of the Cartesian coordinates $\bar{x}^i = H_i$ (cf. (2.2)) is prescribed, as will be shown next.

THEOREM C_2: For surfaces with $K>0$ or $K<0$ and any number of boundaries, the uniqueness proof is hardly more complicated than it was for closed surfaces if the boundary condition imposed is that a Cartesian coordinate, $\bar{x}^3$ say, is prescribed at all boundaries. Since H in (2.4), hence also in (3.1), is rather naturally assumed to have at least continuous second derivatives, it follows that solutions of (3.1) would have continuous derivatives of all orders since they are harmonic functions. Thus the derivative $H_3 = H_3^{(1)} - H_3^{(2)}$ would be a solution of

$$H_{3,11} + H_{3,22} + H_{3,33} = 0 \,. \tag{3.2}$$

Consider the values of H_3 on the spherical image Ω, regarded as an open set, which lies on the unit sphere centered at the origin. At the points of $\partial\Omega$ corresponding to boundary curves of S_1 and S_2, the boundary condition assumed means that the difference $H_3^{(1)} - H_3^{(2)}$ has the value zero at all boundaries. Thus unless $H_3 \equiv 0$ holds in Ω, a positive maximum of H_3 could be assumed to exist at a point p in Ω. H_3 is a homogeneous function of degree zero, hence is constant along each ray from the origin to points of Ω; it therefore takes on a maximum in three-space at p. But that is impossible since H_3 is a harmonic function. Thus $H_3 \equiv 0$ holds, and $H = H^{(1)} - H^{(2)}$ is independent of x^3.

LEMMA C: From this it follows readily that the other Cartesian coordinates $\bar{x}^1$ and $\bar{x}^2$ of S_1 and S_2 differ by constants at all corresponding points. The proof is simple. Since H is independent of x^3, it follows that $H_{11} + H_{22} = 0$ holds. At the same time it follows from (2.8) that

$$H_{11}x^1 + H_{12}x^2 = 0 \,,$$

$$H_{21}x^1 + H_{22}x^2 = 0 \,,$$

and since these are identities for arbitrary values of x^1 and x^2, the determinant $H_{11}H_{22} - H_{12}^2 = 0$; because $H_{11} = -H_{22}$ it follows that $H_{11} = H_{22} = H_{12} = 0$ holds everywhere. Thus H_1 and H_2 are constants and this means that the coordinates $\bar{x}^1$ and $\bar{x}^2$ at all corresponding points of S_1 and S_2 differ everywhere by a constant, and the desired lemma is obtained. With it the proof of Theorem C_2 is completed.

THEOREM C_3: Examples of surfaces with boundaries for which this uniqueness theorem holds for $K < 0$ as well as $K > 0$ are readily given. For example, any closed convex surface with holes cut out anywhere, for example, a band in the form of a ring with two boundaries, and thus not simply connected, would serve for the case $K > 0$. For $K < 0$ such bands also exist. In addition the bands might be such that the spherical image could have a number of sheets, and the surfaces could have self-intersections. The maximum principle would again serve to establish the uniqueness theorems, since that involves a purely local statement, and the fact that H would be defined as a harmonic function in a multi-sheeted domain in three-space would cause no difficulty.

Other uniqueness problems have been considered by M. Tsuji [12], including particular kinds of closed surfaces having arbitrary genus. Such surfaces have of necessity regions in which K must be positive, as well as negative. To construct such examples, the surfaces were assumed to be built up of pieces of negative curvature joined to pieces with positive curvature across special curves where $K = 0$ holds. On the parts with $K < 0$, the Christoffel condition is assumed to hold, while on a part with $K > 0$ either that condition or Minkowski's condition is assumed.

THEOREM C_4: The books of Pogorelov and Efimov cited earlier deal extensively with the case of bounded surfaces S in which $K_T = 2\pi$ and the spherical image is a hemisphere. For Christoffel's problem such cases are readily treated in three-space not only when a coordinate of boundary curves is fixed, but also if the support function is prescribed. In case H is prescribed at the boundary suppose that $x^3 = 0$ contains the great circle that bounds the spherical image. It follows for two surfaces S_1 and S_2 that $H = H^{(1)} - H^{(2)} = 0$ on the unit circle centered at the origin in the plane $x^3 = 0$. The origin is located so that $H > 0$ holds generally. But $H \to 0$ when $r \to 0$ on any ray from the origin, and since $H = 0$ on each ray as it crosses the unit circle in $x^3 = 0$, it follows since H is homogeneous of degree one that it vanishes everywhere in the plane $x^3 = 0$. Since H is a harmonic function it can be continued analytically by reflection across the plane $x^3 = 0$, and consequently can be defined as a harmonic function everywhere in three-space except at the origin, where as in the earlier discussion it has a removable singularity. As before, H behaves linearly at ∞, hence is a linear function, so that the derivatives H_i are constants, which means that S_1 and S_2 are congruent. The uniqueness theorem C_4 is thus proved.

4. – Minkowski's problem.

This problem is to be attacked on the basis of the three equivalent equations (2.5), with a representative example of the form

$$H_{11}H_{22} - H_{12}^2 = (x^3)^2/r^2K\,, \quad x^3 \neq 0\,. \tag{4.1}$$

The two others are found by cyclic permutation of the subscripts.

The device used by the author in [9] will again be used here. It consists in introducing the sum $H = H^{(1)} + H^{(2)}$, and the difference $W = H^{(1)} - H^{(2)}$ of the support functions of two convex surfaces S_1 and S_2 ($K > 0$ holds throughout this section, in other words) such that (4.1) is satisfied when K is the same for both. The result is readily found to be the following linear differential equation for W:

$$H_{22} W_{11} - 2H_{12} W_{12} + H_{11} W_{22} = 0 \; . \tag{4.2}$$

The functions W and H are both, evidently, homogeneous functions of degree one, and H is the support function of a closed surface with positive Gauss curvature since it is the sum of support functions for two such surfaces, and the surface is then the boundary of a bounded convex set (Hadamard's theorem). It follows that

$$H_{11} H_{22} - H_{12}^2 > 0 \; , \tag{4.3}$$

if $x^3 \neq 0$ holds, in view of (4.1), which is valid when H is the support function of *any* surface with $K > 0$. As was pointed out earlier two other identities like (4.2) and (4.3) hold when the subscripts are changed cyclically. Thus always one at least of the equations of the form (4.2) is elliptic at a given point since $x^1 = x^2 = x^3 = 0$ never occurs.

THEOREM M_1: As in the preceding section, the discussion begins with the uniqueness theorem for the case of two closed convex surfaces S_1, S_2 so that their spherical images cover the entire unit sphere. In that case $H = H^{(1)} + H^{(2)}$ is defined in the entire x^1, x^2, x^3-space, except for the origin, and $W = H^{(1)} - H^{(2)}$ as well. It is to be shown that S_1 and S_2 are identical within a translation. This will be done by considering any one of the Cartesian coordinates $\bar{x}^i = H_i$ of S_1 and S_2, $\bar{x}^3$, say, and showing that $W_3 = H_3^{(1)} - H_3^{(2)}$ is everywhere constant. Once that is done it is clear from symmetry that the same statement would hold for the other coordinates.

LEMMA M: If there are boundaries, symmetry in all three variables does not in general exist, and the latter statement is not necessarily applicable. It does hold, however, as will now be shown. Suppose again, as in Lemma C, that W_3 is found to be identically constant so that W depends only upon x^1 and x^2. If $x^2 \neq 0$ is assumed (the case $x^1 \neq 0$ can be treated in the same fashion) then W would satisfy the appropriate equation of the type (4.2) for this case, i.e.

$$(4.4)\qquad H_{33}W_{11}-2H_{13}W_{13}+H_{11}W_{33}=0\ ,\qquad H_{11}H_{33}-H_{13}^2>0\ .$$

Since $H_{33}\neq 0$ and $W_{31}=0$ and $W_{33}=0$ hold, it follows that $W_{11}=0$, first for $x^2\neq 0$, then for all values of x^2 by continuity. Also, as was noted in the previous section, it follows from Euler's theorem for the first derivatives W_1, W_2 (when $W_3\equiv$ const holds) that

$$W_{11}x^1+W_{12}x^2=0\ ,$$

$$W_{21}x^1+W_{22}x^2=0\ ,$$

for any values x^1, x^2, and as a result $|W_{ik}|=0$. Since $W_{11}=0$, it follows that $W_{12}=0$. Finally, for $x^3\neq 0$, (4.2) holds with $H_{11}H_{22}-H_{12}^2>0$; thus $W_{22}=0$ everywhere, including $x^3=0$, again by continuity. Consequently W_{11}, W_{22}, W_{12} are identically zero, so that W is a linear function of x^1, x^2, x^3, and its first derivatives are constants. In other words, S_1 and S_2 would differ only by a translation. This proves Lemma M, and verifies a statement made in the first section to the effect that two surfaces satisfying the basic condition, i.e. the partial differential equation, imposed in Minkowski's problem would differ at most by a translation if it is assumed that any one of their Cartesian coordinates $\bar{x}^i$ is the same for both at corresponding points.

THEOREM M_1: The uniqueness theorem for closed surfaces is now to be proved with the aid of a maximum principle. To this end consider the « vertical » Cartesian coordinates $\bar{x}^3$ of them at corresponding points; their difference is the function W_3 (cf. (2.2)). If this is not everywhere constant on the unit sphere centered at the origin (which implies that it is constant on every ray from the origin since it is homogeneous of degree zero), it may be assumed that it has a positive maximum at a point p on the sphere, and consequently a maximum at that point in the x^1, x^2, x^3-space. Consider first the case in which p does not lie on the x^3-axis, so that x^1 and x^2 cannot vanish there simultaneously. Assume, say, that $x^2\neq 0$ holds at p. The differential equation (4.4) is valid there, and is elliptic when it is regarded as an equation for W as a function of x^1 and x^3 in the plane $x^2=$ const through the point p since $x^2\neq 0$ at p is assumed. The maximum principle does not immediately apply to this equation since W, being homogeneous of degree one, does not necessarily have a maximum in three-space at p, and thus not necessarily so in the plane $x^2=$ const. However, a device can be introduced so that the maximum principle may be used. That is done by observing first that $H_{33} \neq 0$ holds. The equation (4.3) can therefore be divided by it and then a differentiation of the equation with respect to x^3 yields the following second order differential equation for the derivative W_3:

$$W_{3,11} - \frac{2H_{13}}{H_{33}} W_{3,13} + \frac{H_{11}}{H_{33}} W_{3,33} + AW_{3,1} + BW_{3,3} = 0 . \tag{4.5}$$

Here the coefficient B, for example, is given by $\partial/\partial x^3(H_{11}/H_{33})$. Since W_3, being homogeneous of degree one, has a maximum at p in three-space, it has also a maximum at p in the plane $x^2 = \text{const}$. But that, as is well known (and not difficult to prove), is not possible because the discriminant of the relevant quadratic form for (4.5), i.e.

$$\frac{H_{11}}{H_{33}} - \left(\frac{H_{13}}{H_{33}}\right)^2 = \frac{1}{H_{33}^2}(H_{11}H_{33} - H_{13}^2) ,$$

is clearly positive. Thus $W_3 = \text{const}$ would hold everywhere, and Lemma M would yield the proof of Theorem M_1.

The uniqueness proof will therefore be completed once the special case in which p is at the point $(0, 0, 1)$ on the unit sphere has been dealt with. This is done by showing that a contradiction results also in this case, as follows. It may be assumed through an appropriate translation of S_2 that $W_3 = 0$ at $(0, 0, 1)$, i.e. at p (it is legitimate to assume $x^3 = 1$ at p since W_3 is in any case constant along the x^3-axis).

Consider the function $W_3(x^1, x^2, 1)$; it satisfies, from $\sum_{i=1}^{3} W_i x^i = W$, the relation

$$W_3 = W - W_1 x^1 - W_2 x^2 , \tag{4.6}$$

in which W_1 and W_2 depend only upon x^1 and x^2. In addition, the equation (4.2) holds for W. The basic fact is that W_3 has $W_3 = 0$ as a maximum at p, hence also in the plane $x^3 = 1$; it follows from (4.6) that $W = 0$ at $x^1 = 0$, $x^2 = 0$, i.e. at the origin in the plane $x^3 = 1$. It is a matter of simple direct verification to see that the right-hand side of (4.6) is unaltered if W is replaced by $W + ax^1 + bx^2$, i.e. if an arbitrary linear function is added to it. By a proper choice of the constants a and b the first derivatives W_1 and W_2 of the new function would have the value zero at the origin. It follows that W may be assumed to vanish to second order at the origin. The existence of the maximum of $W_3 = 0$ at p means that the following inequality holds:

$$W_3 = W - W_1 x^1 - W_2 x^2 \leqslant 0 . \tag{4.7}$$

This can be put into a useful form by noting that

$$W_1 x^1 + W_2 x^2 = (x^1, x^2) \cdot \operatorname{grad} W = r W_r , \qquad r^2 = (x^1)^2 + (x^2)^2 ,$$

so that

$$W - W_1 x^1 - W_2 x^2 = W - r W_r = -r^2 \left(\frac{W}{r}\right)_r \tag{4.8}$$

holds. From (4.7) it then follows that

$$\left(\frac{W}{r}\right)_r \geqslant 0 \tag{4.9}$$

is valid. Since W vanishes to second order for $r = 0$, it follows that $W/r \to 0$ as $r \to 0$, and $W/r \geqslant 0$ results since W/r is a non-decreasing function of r in view of (4.9) (*). Consequently W also has that property; in other words W would take on the minimum value zero at $r = 0$. But that is impossible unless $W \equiv 0$ holds since W satisfies (4.2) in the plane $x^3 = 1$ and $H_{11}H_{22} - H_{12}^2 > 0$ holds.

For the closed surfaces S_1 and S_2 therefore, $W_3 \equiv 0$ holds, and W_1 and W_2 would be constant, as was proved earlier. This completes the uniqueness proof of theorem M_1.

THEOREM M_2: For surfaces S_1 and S_2 with boundaries, such that any one of their Cartesian coordinates is assumed to be the same for both at corresponding points, the uniqueness theorem is proved in the same way as above by using the maximum principle: a positive maximum of W_3, say, could be assumed to exist in the spherical image Ω, and the discussion above would once more lead to a uniqueness proof. The observations made concerning Theorem C_3 (but restricted to the case $K > 0$) are also relevant here.

(*) This device is adapted to the present problem from a much more general unpublished lemma by D. Koutroufiotis and L. Nirenberg. The lemma states: Let $u(x, y)$ be such that $u_{xx}u_{yy} - u_{xy}^2 \leqslant 0$ holds in a bounded star-shaped domain D relative to the origin. Then the function $u - xu_x - yu_y$ takes on any maximum value in D also at some point on the boundary of D. The lemma has an intricate proof except when the maximum occurs at the origin (which is the case in the present discussion).

In case it is the value of the support functions $H^{(1)}$ and $H^{(2)}$ of S_1 and S_2 that is prescribed at boundaries, as is done by Oliker and Volkov, rather than a Cartesian coordinate, it is also possible and advantageous in some cases to give uniqueness proofs by working in three-space rather than on the surface of the unit sphere, since the proofs then require only elementary procedures, rather than such things as the use of Hilbert's theorem, proved with the aid of his theory employing integral equations, to make a statement about relevant linear eigenvalue problems on the sphere. A few examples will be given.

THEOREM M_3: Suppose that the spherical image Ω lies in a hemisphere without touching the great circle that bounds the hemisphere, and this in turn is assumed to lie in the plane $x^3 = 0$. In that case the rays from the origin to Ω will cut out a bounded domain D in the plane $x^3 = 1$, say, with any number of distinct curves forming the boundary ∂D of D, on which $W = H^{(1)} - H^{(2)}$, when solutions $H^{(1)}$ and $H^{(2)}$ are support functions of two surfaces satisfying the boundary conditions, would have the value zero. On that plane (4.2) holds, with $H_{11}H_{22} - H_{12}^2 > 0$, when H is defined as $H = H^{(1)} + H^{(2)}$ since $x^3 = 0$ does not occur in Ω. Multiplication of (4.2) by W and integration over D yields, evidently

$$\iint_D W(H_{22}W_{11} - 2H_{12}W_{12} + H_{11}W_{22})\,dx^1\,dx^2 = 0\,.$$

The parenthesis can be written as a divergence expression, i.e. $(H_{22}W_1 - H_{12}W_{21})_1 + (H_{11}W_2 - H_{12}W_1)_2 = F_1 + G_2$ since third derivatives cancel out. Integration by parts then results in the identity:

$$\iint_D (H_{22}W_1^2 - 2H_{12}W_1W_2 + H_{11}W_2^2)\,dx^1\,dx^2 = \int_{\partial D} W(F\,dx^2 - G\,dx^1)\,,$$

with F and G defined as indicated above. Since $W = 0$ on ∂D it follows that

$$\iint_D (H_{22}W_1^2 - 2H_{12}W_1W_2 + H_{11}W_2^2)\,dx^1\,dx^2 = 0\,,$$

and since the integrand is a positive definite quadratic form in W_1 and W_2, it follows that W_1 and W_2 vanish everywhere, and consequently W_3 also. Thus the uniqueness proof of Theorem M_3 is carried out in elementary classical style.

THEOREM M_4: Another interesting case in which H is prescribed at a boundary is that of bounded surfaces with a spherical image Ω that is a hemisphere, and the uniqueness within translations of such surfaces is to be proved. This special case has received a good deal of attention in the literature, as was pointed out earlier. If the great circle arc that is the boundary of Ω lies in the plane $x^3 = 0$, it follows that $W = H^{(1)} - H^{(2)}$ vanishes on every ray from the origin lying in that plane, as was pointed out in the same situation in the preceding section. Thus $W \equiv 0$ in the plane $x^3 = 0$ holds. (This is the main reason why this case is so special.) But if $W \equiv 0$ holds for all x^1, x^2 in the plane $x^3 = 0$, its first derivatives W_1 and W_2 also vanish everywhere there. Consequently both W_1 and W_2, if not zero everywhere, could be assumed to have a positive maximum for $x^3 = c \neq 0$ on the unit sphere, hence also in the plane $x^3 = c$. As a result the same discussion as was carried out on the basis of equation (4.5), but with subscripts 1 and 2 rather than 1 and 3, would lead to a proof of uniqueness.

THEOREM M_5: If the spherical image Ω lies on a hemisphere bounded by $x^3 = 0$, but a part of $\partial\Omega$ lies on $x^3 = 0$, a more complicated discussion is required to prove the uniqueness theorem for bounded surfaces when the support function is prescribed at boundaries. The earlier discussion of a similar case in which Green's theorem sufficed to prove uniqueness would not yield the desired result in this case since the projection on a plane $x^3 =$ const by rays from the origin to points of Ω would result in an unbounded domain D in the plane $x^3 =$ const, and the boundary integrals in Green's theorem would not necessarily tend to zero at ∞. However, this difficulty can be overcome because the differential equation in the present case has the special form

$$(4.10) \qquad H_{11} W_{22} - 2H_{12} W_{12} + H_{22} W_{11} = 0 \,, \qquad H_{11} H_{22} - H_{12}^2 > 0$$

for $W = W(x^1, x^2)$ in any plane $x^3 = \text{const} \neq 0$, and because it is natural to make the assumption that $W \to 0$ upon approaching boundary points of Ω in the plane $x^3 = 0$.

The basic idea is to avoid boundary integrals by introducing an auxilliary C^∞-function $\zeta_0(x, y) = \zeta_0(r)$, $r^2 = (x^1)^2 + (x^2)^2$, that has the value 1 for $r < 1$ and the value 0 for $r > 2$ and integrating by parts twice in order to get rid of boundary integrals altogether. A function $\zeta(x, y)$ (*) is defined as

$$\zeta = \zeta_0\left(\frac{r}{R}\right) = \zeta_0\left(\frac{x^1}{R}, \frac{x^2}{R}\right)$$

(*) The basic idea of getting rid of boundary integrals by successive integration by parts is due to S. BERNSTEIN. The generalization of it through the use of the auxiliary multiplier ξ in what follows is due to L. NIRENBERG.

for a positive value of R. It follows that second derivatives of ζ are bounded for R large by const/R^2. Multiplication of (4.10) by $W\zeta$ (instead of W, as in the preceding) and use of Green's theorem leads to

$$0 = \iint [(H_{11}W_2 - H_{12}W_1)_2 W\zeta - (H_{12}W_2 - H_{22}W_1)_1 W\zeta]\, dx^1\, dx^2 =$$
$$= \iint [-(H_{11}W_2 - H_{12}W_1)(W\zeta)_2 + (H_{12}W_2 - H_{22}W_1)(W\zeta)_1]\, dx^1\, dx^2 .$$

The integrals are taken over a part of the domain D that lies inside of a sufficiently large circle so that on its circumference $\zeta = 0$ holds and boundary integrals vanish either because $W = 0$ on any boundaries in that circle, or because $\zeta = 0$ on the intersection of D with it. The above identity can be written as follows:

$$0 = \iint \zeta(H_{11}W_2^2 - 2H_{12}W_1W_2 + H_{22}W_1^2)\, dx^1\, dx^2 +$$
$$+ \iint [-W(H_{11}W_2 - H_{12}W_1)\zeta_2 + W(H_{12}W_2 - H_{22}W_1)]\, dx^1\, dx^2 =$$
$$= I + \tfrac{1}{2}\iint [-(H_{11}(W^2)_2 - H_{12}(W_1)_1^2)\zeta_2 + (H_{12}(W^2)_2 - H_{22}(W^2)_1)\zeta_1]\, dx^1\, dx^2 =$$
$$= I + J .$$

The symbols I and J have an obvious significance. Green's theorem is used once again with respect to J, with the result

$$J = \tfrac{1}{2} \int\limits_{\substack{R<r<2R \\ (x^1,x^2)\in\Omega}} W^2(H_{11}\zeta_{22} - 2H_{12}\zeta_{12} + H_{22}\zeta_{11})\, dx^1\, dx^2 .$$

The limit of J when $R \to \infty$ is now considered. The term $H_{11}\zeta_2$ behaves like const/R^3 since $H_{11} \sim 1/r$ (since H_{11} is homogeneous of degree -1) and $\zeta_{22} \sim 1/r^2$, as was noted above. Thus if W were uniformly bounded in the x_1, x_2-plane as $R \to \infty$ the term $W^2 H_{11}\zeta_{22}$ would be such that $R^2 W^2 H_{11}\zeta_{22} \to 0$ uniformly as $R \to \infty$, and the same observation would apply to the other terms in the integrand of J. Thus for $R < r < 2R$, $J \to 0$ as $R \to \infty$. Consequently $I \to 0$ also. As a result, since $\zeta = 1$ in this range of r, it follows that

$$\iint\limits_D (H_{11}W_2^2 - 2H_{12}W_1W_2 + H_{22}W_1^2)\, dx^1\, dx^2 = 0 ,$$

hence $W \equiv \text{const} = 0$, in case boundaries of D occur. If Ω is a hemisphere, hence $K_T = 2\pi$, so that D is the entire plane $x^3 = \text{const}$, only $W = \text{const}$

is proved, but the desired uniqueness theorem for the surfaces results—provided that the boundedness of W at ∞ is established.

In the present case this can be done. At first sight it seems not reasonable since W is homogeneous of degree one in three-space and hence becomes linearly unbounded at ∞ along rays from the origin. However in the plane $x^3 = \text{const.}$ W is not homogeneous. The circumstance that its value on the unit sphere is assumed to be zero where $x^3 = 0$, together with the assumption (which is now made) that W has continuous first derivatives upon approaching the boundary of Ω leads to the boundedness of W at ∞ in a plane $x^3 = \text{const.}$ In the accompanying figure, the r-axis passes through a boundary point of Ω in $x^3 = 0$ at the point in the intersection of the r, x^3-plane and the plane $x^3 = a$. It is known that $\lim_{\theta \to 0} W(1, \theta) = 0$, and $W(p) = W(\varrho, \theta) = \varrho W(1, \theta)$, with $\varrho = a/\sin\theta$. The behaviour of $W(p)$ is to be investigated when $\theta \to 0$, hence $p \to \infty$. Thus for $W(\varrho, \theta)$ the equation

$$W(\varrho, \theta) = \frac{aW(1, \theta)}{\sin\theta}$$

holds and

$$\lim_{\theta \to 0} W(\varrho, \theta) = \lim_{a \to 0} \frac{aW_\theta(1, \theta)}{\cos\theta} = aW_\theta(1, 0)\,,$$

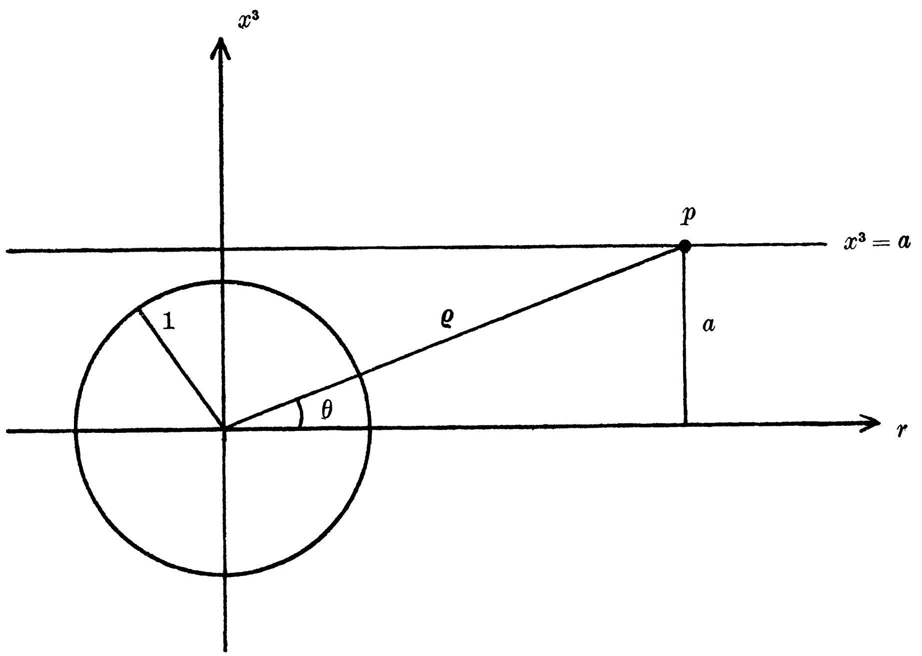

Fig. 1. – Behavior of $W(\varrho, \theta)$ in $x^3 = a$.

since $W_\theta(1, \theta)$ is assumed to be continuous at $\theta = 0$. Hence $W(\varrho, \theta)$ is uniformly bounded in the plane $x^3 = a$ when $p \to \infty$. This concludes the proof of Theorem M_5.

5. – The problems of Weyl and Liebmann

The uniqueness theorems to be discussed here for these problems have, in a sense, already been proved. It remains only to recall the partial formulation of these problems in Section 2. The isometric surfaces S_1, S_2 defined by vectors $\boldsymbol{X}(u, v)$, $\boldsymbol{Y}(u, v)$ were supposed given. The mean surface $\boldsymbol{Z} = \frac{1}{2}(\boldsymbol{X} + \boldsymbol{Y})$ was introduced and assumed to be regular in the parameters (u, v), and in that case its curvature K is positive since S_1 and S_2 have that property (cf. [9]). The difference $\boldsymbol{V} = \boldsymbol{X} - \boldsymbol{Y}$ was also introduced. From the isometry the existence of a rotation vector $\boldsymbol{\delta}$ can be proved such that

$$d\boldsymbol{V} = \boldsymbol{\delta} \times d\boldsymbol{Z} \tag{5.1}$$

holds when the differentials are taken with respect to u and v, and the symbol $\times$ denotes the vector product. A scalar function $W(u, v)$ is introduced as follows:

$$W = \boldsymbol{\delta} \cdot \boldsymbol{n}\,, \tag{5.2}$$

in which W is the scalar product of $\boldsymbol{\delta}$ with the unit normal $\boldsymbol{n}$ of the mean surface given by $\boldsymbol{Z}(u, v)$. Thus W can be considered as a function of the directions of the normals of the mean surface, hence as a point function on its spherical image. It can therefore be extended into three-space as a homogeneous function of degree one. It then is found that the components δ^i in x^i-space of $\boldsymbol{\delta}$ are given by

$$\delta^i = \frac{\partial W}{\partial x^i} = W_i\,. \tag{5.3}$$

Since $d\boldsymbol{V}$ in (5.1) is an exact differential, a compatibility condition is satisfied. The mean surface can be represented by its support function H, and the compatibility condition then takes the form

$$H_{22} W_{11} - 2H_{12} W_{12} + H_{11} W_{22} = 0\,, \tag{5.4}$$

with two other like conditions obtained by permutation of subscripts, all of which are equivalent. This theory was derived by H. Weyl [15] for

Liebmann's problem, but it is valid also for Weyl's problem (cf. [9]). Thus a uniqueness theorem proved for one is also valid for the other, although for Liebmann's theorem no restriction involving the mean surface is needed since the original unperturbed surface takes its place.

In other words the differential equation for W is the same as it was for Minkowski's problem, although W now determines through $W_i = \delta^i$ the components of the rotation vector instead of the difference of the Cartesian coordinates of the surfaces S_1 and S_2 to be compared. As was already noted in Section 2 the surfaces S_1 and S_2 will differ at most by a rigid rotation if $\boldsymbol{\delta}(u, v)$ is independent of u and v.

THEOREM W_1: If the surfaces considered are closed and convex, the function W defined in (5.2) is defined in the entire x^1, x^2, x^3-space except at the origin, and the vector $\boldsymbol{\delta} = (\delta^1, \delta^2, \delta^3)$ also. It is to be shown that they are congruent. The derivatives W_i are continuous on the whole unit sphere, hence bounded. The uniqueness theorems are therefore proved for Weyl's problem for closed surfaces by resorting once more to the device that leads to equation (4.5), and the discussion that follows it, and the proof by the maximum principle for that equation leads to the result that the components δ^i of the rotation vector are of necessity everywhere constant.

THEOREM L_1: For Liebmann's problem (5.4) is also valid, as Weyl showed (although he was apparently not aware that the theorem is valid for the problem named for him), with H now the support function of the original surface S and $\boldsymbol{\delta}$ an infinitesimal rotation vector when $S \to S + \delta S$, for δS small of first order in a deformation parameter. Thus for closed surfaces the uniqueness theorem for this problem is proved in the same way as for Weyl's problem, although they are basically quite different since an exact isometry is not required in Liebmann's problem.

THEOREMS W_2, L_2: For surfaces with boundaries uniqueness theorems can also be proved in various cases for Weyl's problem, and of course also for Liebmann's problem. If, for example, it is assumed that the surfaces S_1 and S_2 have the same fixed boundaries in space, which means that their three Cartesian coordinates $\bar{x}^i$ are the same at corresponding points, the uniqueness theorems of the present section can be proved in the same way as they were proved in section 4 for various cases in which the support function was prescribed at boundaries. That can be seen as follows. At boundary curves the vector function $\boldsymbol{V} = \boldsymbol{X} - \boldsymbol{Y}$ would be the zero vector, since $\boldsymbol{X}$ and $\boldsymbol{Y}$ are by assumption identical on them. It then follows from (5.1) that $\boldsymbol{\delta}$ and $d\boldsymbol{Z}$ are linearly dependent along such a curve, or put differently, $\mathbf{\delta}$ falls along a tangent vector of the mean surface and thus its component in the direction of the normal to this surface vanishes. Consequently $W = 0$ along the boundary, in view of (5.2). Thus theorems W_2, L_2 hold (cf. Theorems M_3, M_4, M_5).

Theorems of this kind were proved by Hsiung [2] by the method using an integral of the exterior derivative of a properly chosen invariant differential form of degree one. However, such theorems were proved first for rather general cases by Rellich [8] who, however assumed only that *one* of the Cartesian coordinates was prescribed at a boundary. (As was noted earlier, and is anyway well known, if one such coordinate $\bar{x}^k$ is assumed known for a surface S then any other surface isometric to it which has the same $\bar{x}^k$ coordinate will differ from S only by an orthogonal transformation.) Thus to prescribe all three coordinates at boundaries, as was done above, and as is also done by Hsiung would seem to be unnecessarily restrictive.

The formulation of Weyl's problem in terms of the rotation vector is not well adapted for dealing in general with uniqueness questions when boundaries occur and a particular coordinate in three-space of the boundary is prescribed, since it is not (for the author at least) easy to see how that results in a suitable boundary condition on W in terms of a mean surface and its unknown normals at the boundary.

THEOREMS W_3, L_3: However, in the special, but interesting, case of what is called in the literature a convex cap (konvexe Mütze) with total curvature 2π, i.e. a convex surface S with spherical image a hemisphere, and which has a boundary in the plane $x^3 = 0$, say, a uniqueness theorem can be proved. Surfaces isometric to S are to be considered which also have their boundaries in $x^3 = 0$ and with spherical image a hemisphere. Thus the Cartesian coordinate $\bar{x}^3$ would be zero at such boundaries. Once more this proves to be an exceptional case; it is dealt with as follows. It is first of all readily seen that the mean surface, if regular, is also a convex cap, with its boundary in the plane $x^3 = 0$. Boundary curves of S_1 and S_2 are in isometric correspondence on S_1 and S_2, hence their tangent vectors also correspond in the respective tangent planes of S_1 and S_2. Thus in these planes the tangent vectors of S_1 and S_2 orthogonal to the plane $x^3 = 0$ would also correspond in the isometry since they are both at right angles to directions in the plane $x^3 = 0$ that are known to correspond. It is then clear that the rotation vector $\mathbf{\delta}$ (known to be uniquely determined) must be orthogonal to the plane $x^3 = 0$ for all boundary points of S_1 and S_2. This, however, means that $\mathbf{\delta}$ is in the tangent plane of the mean surface at its boundary. Consequently, from (5.2), $W = 0$ at the boundary, and the uniqueness theorem is proved in the same way as for this case in Minkowski's problem, since the differential equation and boundary condition are the same for both problems (cf. Theorem M_5).

E. Kann [3], [4] established new proofs for uniqueness theorems for Liebmann's problem and Weyl's problem for convex surfaces with boundaries; his method, which employs projective transformations and is rather complicated, is different from all others known to the author.

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