ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

H. BRÉZIS
F. E. BROWDER

Strongly nonlinear elliptic boundary value problems

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 5, nº 3 (1978), p. 587-603
<http://www.numdam.org/item?id=ASNSP_1978_4_5_3_587_0>

Strongly Nonlinear Elliptic Boundary Value Problems.

H. BRÉZIS (*) - F. E. BROWDER (**)

dedicated to Hans Lewy

Let Ω be an open subset of R^n. We consider a nonlinear elliptic partial differential equation of order $2m$, $(m \geqslant 1)$, on Ω of the form

$$A(u) + g(x, u) = f(x)\,, \tag{1}$$

where the principal term of order $2m$ is given in the generalized divergence form

$$A(u) = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, Du, \ldots, D^m u) \tag{2}$$

and the lower-order perturbing term $g(x, u)$ is strongly nonlinear in the sense that we impose relatively weak sign conditions but not an over-all growth condition on the size of $g(x, u)$ as a function of u. In the present discussion, we obtain existence and uniqueness theorems for the solution of the equation (1) under null Dirichlet boundary conditions (as well as other variational boundary conditions). We also obtain related results on existence and uniqueness for general classes of variational inequalities involving the elliptic operator $A(u) + g(x, u)$. These results give a considerable sharpening to earlier results on the existence of solutions for this class of problems obtained in Browder [3], Hess [6], [7], [8], Edmunds-Moscatelli-Webb [5], Webb [11], and Simader [10]. Our treatment includes the case of unbounded domains which previously required a special treatment. Unlike many of the discussions just mentioned, it does not rest upon a generalized

(*) Département de Mathématiques, Université de Paris VI, Paris, Cedex.
(**) Department of Mathematics, University of Chicago, Chicago, Illinois.
Pervenuto alla Redazione il 17 Settembre 1977.

theory of pseudo-monotone operators, nor upon singular perturbation techniques. We employ instead the standard theory of pseudomonotone operators from a reflexive Banach space V to its conjugate space V^* which we apply to the truncated operator $A(u) = g_n(x, u)$ and pass to the limit in n upon the resulting approximate solutions.

We begin by remarking upon our use of the standard notation in this area of discussion. The points of Ω are denoted by $x = (x_1, \ldots, x_n)$ and integration with respect to Lebesgue n-measure on is written $\int dx$. The space L^p for $p \geq 1$ denotes the corresponding Lebesgue space of p-th power summable functions on Ω. We use the conventional notation for differential operators in which α is the n-tuple of non-negative integers $(\alpha_1, \ldots, \alpha_n)$, D^α is the elementary differential operator $D^\alpha = \prod_{j=1}^{n} (\partial/\partial x_j)^{\alpha_j}$, and its order $|\alpha| = \sum_{j=1}^{n} \alpha_j$. Let R^N be the vector space of m-jets on R^n whose elements are denoted by $\xi = \{\xi_\beta\colon |\beta| \leq m.\}$ Each ξ corresponds to a pair (ζ, η) where $\zeta = \{\zeta_\beta\colon |\beta| = m\}$ and $\eta = \{\eta_\beta\colon |\beta| \leq m-1\}$. The ζ form a vector space R^{N_1}, the η a vector space R^{N_2} with $N = N_1 + N_2$.

We denote by $W^{m,p}(\Omega)$ the Sobolev space of functions u in L^p on Ω, all of whose distribution derivatives $D^\beta u$ lie in L^p for $|\beta| \leq m$. $W^{m,p}(\Omega)$ is a reflexive, separable, uniformly convex Banach space with respect to the usual norm

$$\|u\|_{m,p} = \left\{ \sum_{|\beta| \leq m} \|D^\beta u\|_{L^p}^p \right\}^{1/p} .$$

$W_0^{m,p}(\Omega)$ is the space of elements u from $W^{m,p}(\Omega)$ which satisfy the Dirichlet null-boundary conditions of order $m-1$ on the boundary of Ω in the generalized sense, where $W_0^{m,p}(\Omega)$ is defined to be the closure in $W^{m,p}(\Omega)$ of $C_c^\infty(\Omega)$, the testing functions with compact support in Ω.

To define the representation of the operator $A(u)$ in (2) more precisely, we introduce a more precise definition of the functions A_α involved in that representation. Each A_α is a function from $\Omega \times R^N$ to R, the reals, and the family $\{A_\alpha(x, \xi)\}$ satisfies the following assumptions:

Assumptions on $A(u)$:

(I) *Each $A_\alpha(x, \xi)$ is measurable in x for fixed ξ, continuous in ξ for fixed x. For a given real number $p > 1$, there exists a constant c_1 and a function k_1 in $L^{p'}$, with $p' = p(p-1)^{-1}$, such that*

$$|A_\alpha(x, \xi)| \leq c_1 |\xi|^{p-1} + k_1(x)$$

for all α, all x in Ω, and all ξ in R^N.

(II) *For each x in Ω, each η in R^{N_2}, and any pair of distinct elements ζ and $\zeta^{(1)}$ of R^{N_1}, we have*

$$\sum_{|\alpha|=m} \left(A_\alpha(x, \eta, \zeta) - A_\alpha(x, \eta, \zeta^{(1)})\right)(\zeta_\alpha - \zeta_\alpha^{(1)}) > 0 .$$

(III) *There exists a constant $c_2 > 0$ and a fixed function k_2 in L^1 such that for all x in Ω and ξ in R^N,*

$$\sum_{|\alpha|\leqslant m} A_\alpha(x, \xi)\xi_\alpha \geqslant c_2|\xi|^p - k_2(x) .$$

In Section 1, we treat the Dirichlet problem for the equation (1) with f in the conjugate space V^* of $V = W_0^{m,p}(\Omega)$ under very general assumptions upon the strongly nonlinear perturbation $g(x, u)$. This set of assumptions is as follows:

Assumptions upon $g(x, u)$:

(1) *The function $g(x, r)$ is measurable in x on Ω for fixed r in R, continuous in r for fixed x. For each x in Ω, $g(x, 0) = 0$, while for all r in R, x in Ω,*

$$g(x, r)\, r \geqslant 0 .$$

(2) *There exists a continuous, nondecreasing function h from R to R with $h(0) = 0$, such that for a given constant C, we have*

$$|g(x, r)| \leqslant |h(r)| ,$$

and

$$|h(r)| \leqslant C\{|g(x, r)| + |r|^{p-1} + 1\}$$

for all x in Ω and all r in R.

Let us note that the second assumption will hold for a function $g(x, r) = g(r)$ independent of x if for any pair of arguments $0 < r < s$

$$g(r) \leqslant Cg(s)$$

with a similar assumption for negative arguments. In particular, it holds if $g(x, r) = h(r)$ is increasing in r and independent of x. The inequalities of (2) when $g(x, r)$ is not independent of x express a comparison of the growth rates of the $g(x, r)$ as x varies over Ω.

Our basic result in Section 1 is given in the following theorem:

THEOREM 1. *Let Ω be a bounded open set, $A(u)$ a differential operator of the form* (2) *which satisfies the Assumptions* (I), (II), *and* (III) *given above. Let $g(x, r)$ satisfy the Assumptions* (1) *and* (2) *given above. If $V = W_0^{m,p}(\Omega)$, then for each f in V^*, the conjugate space of V, there exists u in V with $g(x, u)$ in L^1, $ug(x, u)$ in L^1 such that*

$$A(u) + g(x, u) = f$$

on Ω (in the sense of distributions) while

$$\big(A(u), u\big) + \int g(x, u)u\, dx = (f, u)\,.$$

(Here (w, u) denotes the pairing between an element w of V^* and an element u of V.)

In Section 2, we extend the result of Section 1 to avoid the assumption that the domain Ω is bounded and to cover more general boundary value problems of variational type as well as a rather general class of variational inequalities. This discussion is based upon replacing the assumption (2) on the strongly nonlinear term $g(x, u)$ by another assumption in which $g(x, r)$ is non-decreasing in r, namely:

Alternative Assumption on $g(x, u)$:

(2)′ *The function $g(x, r)$ is non-decreasing in r on R. For each fixed r, $g_r(x) = g(x, r)$ yields a function g_r in $L^1(\Omega)$.*

Note that in this alternative assumption, no comparison is made of the rate of growth of $g(x, r)$ as a function of r for different values of x in Ω. Under this assumption, we may define:

$$G(x, r) = \int_0^r g(x, s)\, ds\,.$$

This function G, the primitive of g with respect to r, is continuous, convex in r, and is non-negative for all arguments with $G(x, 0) = 0$. Its derivative with respect to r is of course $g(x, r)$.

THEOREM 3. *Let Ω be an arbitrary open set in R^n, $A(u)$ a differential operator of the form* (2) *which satisfies the Assumptions* (I), (II), *and* (III). *Let $g(x, r)$ satisfy the Assumptions* (1) *and* (2)′, *$G(x, r)$ its primitive with respect to r as defined above. Let V be any closed subspace of $W^{m,p}(\Omega)$, K a closed convex subset of V, $(0 \in K)$, f a given element of V^*.*

Then there exists u in K such that $g(x, u)$ lies in $L^1(\Omega)$, $g(x, u)u$ lies in in $L^1(\Omega)$ and $\int G(x, u) < +\infty$, while u satisfies both of the following variational inequalities:

(i) For each v in $K \cap L^\infty(\Omega)$,

$$\bigl(A(u) + g(x, u) - f, v - u\bigr) \geqslant 0 .$$

(ii) For each v in K,

$$\int G(x, v) - \int G(x, u) + \bigl(A(u) - f, v - u\bigr) \geqslant 0 .$$

(By $\bigl(A(u), v - u\bigr)$ for general V, we mean $a(u, v - u)$ as defined in the discussion of Section 1.)

THEOREM 4. *Suppose that the hypotheses of Theorem* 3 *hold while in addition A is monotone*, i.e.,

$$\bigl(A(u) - A(v), u - v\bigr) \geqslant 0$$

for all u and v in V. Then for two solutions u_1 and u_2 of the problem considered in Theorem 3 *for a given f, we have*

$$\bigl(A(u_1) - A(u_2), u_1 - u_2\bigr) = 0 ,$$

and

$$G\left(x, \frac{u_1 + u_2}{2}\right) = \frac{1}{2}\{G(x, u_1) + G(x, u_2)\} .$$

If these two conditions imply that $u_1 = u_2$, then the solution u of Theorem 3 *is unique.*

The relation between the two classes of problems considered in Sections 1 and 3 is clarified in Section 4 by the following result:

THEOREM 5. *Suppose $g(x, r)$ satisfies the assumptions* (1) *and* (2) *of Theorem* 1 *on a bounded open set Ω of R^n and that $g(x, r)$ is also non-decreasing in r for each fixed x. Suppose that u is a solution of*

$$A(u) + g(x, u) = f$$

for a given f in V^* with u in $V = W_0^{m,p}(\Omega)$, $g(x, u)$ and $g(x, u)u$ in L^1, and that we have the equality

$$\bigl(A(u), u\bigr) + \int g(x, u)u = (f, u) .$$

Then u is a solution of the variational inequalities (i) *and* (ii) *of Theorem* 4 *for* $K = V$.

In particular, u is unique under the hypotheses of Theorem 4.

As we show in a paper to follow the present one, the methods which we have applied to elliptic problems can be adapted in a suitably modified form to the treatment of a corresponding broad class of strongly nonlinear parabolic initial-boundary value problems of variational type.

§ 1. – We now proceed to the proof of Theorem 1. We begin by noting that for u in $W_0^{m,p}(\Omega)$ and v in $C_c^\infty(\Omega)$, if we denote by (w, w_1) the pairing between elements of L^p spaces given by

$$(w, w_1) = \int w w_1 \, dx ,$$

and similarly for the pairing between a distribution and a testing function, then

$$\big(A(u), v\big) = a(u, v) = \sum_{|\alpha| \leqslant m} \big(A_\alpha(\xi_m(u)), D^\alpha v\big)$$

where $\xi_m(u)$ is the function from Ω to R^N given by

$$\xi_m(u)(x) = \{D^\alpha u(x) \colon |\alpha| \leqslant m\} .$$

By part (I) of the Assumptions on $A(u)$, it follows that for each u in $V = W_0^{m,p}(\Omega)$, (and indeed for each u in $W^{m,p}(\Omega)$), $A_\alpha(\xi_m(u))$ lies in the space $L^{p'}$, the conjugate space to the space L^p for the p described in that Assumption. It follows by the Hölder inequality that

$$|a(u, v)| \leqslant c\big(\|u\|_{m,p}\big) \|v\|_{m,p}$$

and that for each fixed u in V, $a(u, v)$ is a well-defined bounded linear functional of v in V. This functional we denote once more by $A(u)$, so that $A(u)$ is an element of V^* and also is a distribution on Ω. It follows by the standard arguments that as a mapping from V to V^*, A is a continuous mapping which maps bounded sets of V into bounded sets of V^*.

If V is a general subspace of $W_0^{m,p}(\Omega)$, we define $A(u)$ as the element of V^* such that

$$\big(A(u), v\big) = a(u, v) , \qquad (v \in V) .$$

Here, $A(u)$ is no longer a distribution.

We recall that a mapping T of V into V^* is said to be pseudo-monotone if it is continuous from finite dimensional subspaces of V to the weak topology of V^* and satisfies the following condition:

(p-m) For any sequence $\{u_j\}$ in V which converges weakly to u in V and for which $\limsup\, (T(u_j),\ u_j - u) \leq 0$, $T(u_j)$ converges weakly to $T(u)$ in V^* and $(T(u_j), u_j)$ converges to $(T(u), u)$.

We recall that T is said to be coercive if $\|u\|^{-1}(T(u), u) \to +\infty$ as $\|u\| \to +\infty$, i.e., if there exists a function c from R^+ to R with $c(r) \to +\infty$ as $r \to +\infty$ such that for all u

$$(T(u), u) \geq c(\|u\|)\, \|u\| .$$

PROPOSITION 1. *Let Ω be any open subset of R^n, V a closed subspace of $W^{m,p}(\Omega)$, A an operator which satisfies the Assumptions* (I), (II), (III) *given above. Then A is a continuous coercive mapping of V into V^* which maps bounded sets of V into bounded sets of V^*. Moreover, A is pseudo-monotone from V to V^*.*

PROOF TO PROPOSITION 1. The coercivity of A follows from the hypothesized inequality (III) by integration. The continuity and boundedness of A follow by standard arguments as already noted.

The pseudo-monotonicity of A from V to V^* is proved in Browder [4]. Q.E.D.

We now introduce the truncated functions $g_n(x, r)$ in the usual way by setting

$$g_n(x, r) = \begin{cases} g(x, r) & \text{if } |g(x, r)| < n , \\ n & \text{if } g(x, r) \geq n , \\ -n & \text{if } g(x, r) \leq -n . \end{cases}$$

PROPOSITION 2. *Let Ω be a bounded open set in R^n, a function from $\Omega \times R$ to R which satisfies the Assumptions* (1) *and* (2) *above. Then for each n the mapping which assigns to each u, the element $A(u) + g_n(x, u)$ of V^* is a continuous coercive, pseudomonotone mapping of V into V^*.*

PROOF OF PROPOSITION 2. If Ω is bounded and g satisfies the conditions (1) and (2), then the operator $A(u) + g_n(x, u)$ will satisfy the Assumptions (I), (II), and (III) if $A(u)$ does.

Hence, the conclusion of Proposition 2 follows from that of Proposition 1. Q.E.D.

PROPOSITION 3. *Let V be a reflexive Banach space, T a coercive, bounded pseudo-monotone mapping of V into V^*. Then T is surjective.*

PROOF. OF PROPOSITION 3. This is a standard result of the theory of pseudo-monotone mappings [1].

Using Propositions 1 and 2, we see that the result of Theorem 1 follows from the semi-abstract statement which we formulate as Theorem 2:

THEOREM 2. *Let* $V = W_0^{m,p}(\Omega)$, *where* Ω *is a bounded open subset of* R^n, g *a function from* $\Omega \times R$ *into* R *which satisfies the Assumptions* (1) *and* (2) *stated above. Suppose that* A *is a coercive pseudo-monotone mapping of* V *into* V^* *which maps bounded sets of* V *into bounded sets in* V^*. *Then for each* f *in* V^*, *there exists* u *in* V *such that*

$$A(u) + g(x, u) = f$$

with $g(x, u)$ *in* L^1, $g(x, u)u$ *in* L^1, *and*

$$(A(u), u) + \int_\Omega g(x, u)u = (f, u)\,.$$

PROOF OF THEOREM 2. By Proposition 3, for each positive integer n and for the given element f of V^*, there exists an element u_n of V such that

$$A(u_n) + g_n(x, u_n) = f(x)\,.$$

Since $g_n(x, u_n) = w_n$ is automatically an element of V^*, we know moreover that

$$(A(u_n), u_n) + \int_\Omega g_n(x, u_n)u_n = (f, u_n)\,.$$

From the definition of the truncation and the assumption that $g(x, r)r \geqslant 0$, it follows immediately that $g_n(x, u_n)u_n \geqslant 0$. Hence

$$c(\|u_n\|)\,\|u_n\| \leqslant (A(u_n), u_n) \leqslant (f, u_n) \leqslant \|f\|_{V^*}\|u_n\|\,.$$

Hence $c(\|u_n\|) \leqslant \|f\|$. Since $c(r) \to +\infty$ as $r \to \infty$, it follows that there exists a constant M such that $\|u_n\|_V \leqslant M$ for all n. Since A maps bounded sets into bounded sets, it follows that $\|A(u_n)\|_{V^*} \leqslant M_1$ for all n for a suitable constant M_1. Using the reflexivity of V, it follows that for an infinite subsequence of the integers n (which we denote without loss of generality as the original sequence) u_n converges weakly in V to an element u, while $A(u_n)$ converges weakly in V^* to an element w.

On the other hand, we also know that

$$\int_\Omega g_n(x, u_n)u_n = (f, u_n) - (A(u_n), u_n) \leqslant M\|f\| + MM_1 = M_2$$

for all n. For each positive integer R and all n, we have

$$R|g_n(x, u_n)| \leqslant u_n g_n(x, u_n) + R\{h(R) + |h(-R)|\}$$

since

$$|g_n(x, u_n)| \leqslant |g(x, y_n)| \leqslant |h(u_n)| \leqslant \{h(R) + |h(-R)|\}$$

if $|u_n| \leqslant R$, while $u_n g_n(x, u_n) = |u_n|\,|g_n(x, u_n)|$ for all points of Ω. Hence for any subset B of Ω, we have

$$\int_B |g_n(x, u_n)|\, dx \leqslant R^{-1}\int u_n g_n(x, u_n) + \operatorname{meas}(B)\, M_3(R) \leqslant R^{-1}M_2 + M_3(R)\operatorname{meas}(B).$$

Hence by choosing R sufficiently large, and then making meas (B) sufficiently small, we find that the sequence $\{g_n(x, u_n)\}$ of L^1 is equi-uniformly integrable. (This type of argument is some-times referred to as the principle of De La Vallee Poussin [9], p. 159.)

We may choose an infinite subsequence of the original sequence (which we denote once more for simplicity of notation as $\{u_n\}$) such that u_n converges to u almost everywhere to u in Ω.

It follows immediately for this new sequence by the continuity of $g(x, r)$ in r and the definition of truncation that $g(x, u_n(x))$ converges almost everywhere to $g(x, u(x))$ and furthermore that $g_n(x, u_n(x))$ converges almost everwhere to $g(x, u(x))$ in Ω. By Fatou's Lemma, it follows that

$$\int g(x, u)u \leqslant M_2\,,$$

i.e., $g(x, u)u$ lies in L^1. Moreover, by the equi-uniform integrability of $\{g_n(x, u_n)\}$ and their convergence to $g(x, u)$ a.e., it follows from Vitali's Theorem that $g_n(x, u_n)$ converges to $g(x, u)$ in L^1, where $g(x, u)$ is itself an element of L^1.

To continue our argument, we shall need to apply the following result:

PROPOSITION 4. *Let H be a continuous convex functions on the reals with $H(0) = 0$. Let u be an element of V with $H(u)$ in L^1. Then there exists a sequence $\{v_j\}$ in $C_c^\infty(\Omega)$ such that v_j converges to u in V, v_j converges to u almost everywhere in Ω, and $H(v_j)$ is bounded for all j by a fixed function in L^1.*

PROOF OF PROPOSITION 4. This is Lemma 3, p. 11, of Brézis [2].

PROOF OF THEOREM 2 CONTINUED. We consider the infinite subsequence $\{u_n\}$ at which we had arrived during the course of the argument, and in order to apply the pseudo-monotonicity of the mapping A, we seek to show that $\limsup\,(A(u_n), u_n - u) \leqslant 0$. For any v in $V \cap L^\infty$, we have

$$(A(u_n), u_n - u) = (A(u_n), u_n - v) + (A(u_n), v - u) =$$

$$= (f, u_n - v) - \int g_n(x, u_n)(u_n - v) + (A(u_n), v - u)\,.$$

By Fatou's Lemma,

$$\int g(x, u)\,u \leqslant \liminf \int g_n(x, u_n)\,u_n\,.$$

By the L^1 convergence of $g_n(x, u_n)$ to $g(x, u)$,

$$\int g_n(x, u_n)\,v \to \int g(x, u)\,v\,.$$

Hence

$$\limsup\,(A(u_n), u_n - u) \leqslant (f - w, u - v) - \int g(x, u)(u - v)\,.$$

In particular, we may choose $v = v_j$ for any element of the sequence described in Proposition 4 where we choose for H the convex function

$$H(r) = \int_0^r h(s)\,ds\,.$$

H is continuous and convex, while by construction $H(0) = 0$. Moreover

$$|H(u)| \leqslant |h(u)\,u| \leqslant C|u|\{g(x, u) + |u|^{p-1} + 1\}$$

where the function on the right-hand side of the inequality lies in L^1. Hence $H(u)$ lies in L^1, and the sequence $\{v_j\}$ converging to u in the sense of Proposition 4 may be constructed.

We remark in addition that since $h(r)$ is the derivative of H at r,

$$h(u)(v - u) \leqslant H(v) - H(u)\,.$$

Therefore,

$$h(u)\,v \leqslant H(v) - H(u) + h(u)\,u \leqslant H(v) + h(u)\,u\,.$$

Moreover, by the inequality $|g(x, r)| \leqslant |h(r)|$ and the sign conditions on the two quantities $g(x, r)$ and $h(r)$, we see that

$$g(x, u)v \leqslant H(v) + h(u)u .$$

(Indeed, if u and v have the same sign, this is a consequence of the fact that $|g(x, u)v| \leqslant |h(u)v| = h(u)v$. In the other case, $g(x, u)v$ is negative, and the right side of the inequality is positive.)

For each j,

$$g(x, u)v_j \leqslant (g(x, u)v_j)^+ ,$$

where $(g(x, u)v_j)^+$ denotes the non-negative part of the function. By the inequality we have just derived

$$(g(x, u)v_j)^+ \leqslant H(v_j) + h(u)u .$$

The term on the right by Proposition 4 is dominated by an L^1 function. The sequence of function $(g(x, u)v_j)^+$ converges almost everywhere to $(g(x, u)u)^+ = g(x, u)u$. Hence by the Lebesgue dominated convergence theorem,

$$\int (g(x, u)v_j)^+ \to \int g(x, u)u .$$

On the other hand

$$\limsup\, (A(u_n), u_n - u) \leqslant (f - w, u - v_j) + \int g(x, u)v_j - \int g(x, u)u$$

where $(f - w, u - v_j) \to 0$ since $v_j - u \to 0$ in V, while

$$\int g(x, u)(v_j - u) \leqslant \int (g(x, u)v_j)^+ - \int g(x, u)u$$

where the difference of the integrals on the right approaches 0 as $j \to +\infty$. Hence,

$$\limsup\, (A(u_n), u_n - u) \leqslant 0 .$$

Since A is pseudo-monotone, it follows that $w = A(u)$ and that $(A(u_n), u_n - u) \to 0$. Since $A(u_n)$ converges to $A(u)$ in V^* and hence in the sense of distributions while

$$A(u_n) = f - g_n(x, u_n) \to f - g(x, u)$$

in the sense of distributions, it follows that

$$A(u) + g(x, u) = f .$$

From the equality,

$$\big(A(u_n), u_n\big) + \int g_n(x, u_n) u_n = (f, u_n) ,$$

it follows by Fatou's Lemma that

$$\big(A(u), u\big) + \int g(x, u) u \leqslant (f, u) .$$

To complete the proof of Theorem 2, we wish to show the reverse of this last inequality. For each element of the sequence $\{v_j\}$ constructed as above using Proposition 4, we have

$$\big(A(u), v_j\big) + \int g(x, u) v_j = (f, v_j) .$$

Thus,

$$\big(A(u), v_j\big) + \int \big(g(x, u) v_j\big)^+ \geqslant (f, v_j) .$$

Since

$$\big(g(x, u) v_j\big)^+ \to g(x, u) u$$

in L^1 as above, it follows that

$$\big(A(u), u\big) + \int g(x, u) u \geqslant (f, u) .$$

Combining this fact with the previously established inequality, we see that

$$\big(A(u), u\big) + \int g(x, u) u = (f, u) . \qquad \text{Q.E.D.}$$

§ 2. – Proof of Theorem 3. We shall employ the general procedure used in the proof of Theorems 1 and 2. By the theory of variational inequalities for pseudo-monotone mappings on reflexive Banach spaces [1], for each positive integer n, there exists a solution u_n in K of the variational inequality

$$\big(A(u_n) + g_n(x, u_n) - f, v - u_n\big) \geqslant 0 , \qquad (v \in K) .$$

Since A is assumed to be coercive and 0 lies in K, we know that

$$c\big(\|u_n\|\big)\, \|u_n\| \leqslant \big(A(u_n), u_n\big) \leqslant \big(A(u_n) + g_n(x, u_n), u_n\big) \leqslant (f, u_n) .$$

Hence, it follows as before that the sequence $\{u_n\}$ is bounded in V, and that the sequence $\{A(u_n)\}$ is bounded in V^*. Therefore, we may assume by passing to an infinite subsequence that u_n converges weakly in V to an element u of K, and $A(u_n)$ converges weakly in V^* to an element w of V^*. We shall show that u is a solution of the problem posed in Theorem 3, and that $w = A(u)$. By the same argument as in the proof of Theorem 2, $\int g(x, u_n)u_n$ is uniformly bounded for all n.

We now deduce the equi-uniform integrability of the sequence $\{g_n(x, u_n)\}$ on the (possibly) unbounded open set Ω by a variant of the De La Valle Poussin principle applied in the preceding case. For each positive integer R,

$$R|g_n(x, u_n)| \leqslant u_n g_n(x, u_n) + R\{g(x, R) + |g(x, -R)|\} .$$

Hence, for each set B with meas (B) sufficiently small, $\int_B |g_n(x, u_n)|$ may be made small uniformly in n. In addition, for each given $\varepsilon > 0$, there exists a subset B_ε of finite measure in Ω such that $\int_{\Omega - B} |g_n(x, u_n)| < \varepsilon$. Thus the hypotheses of the Vitali convergence theorem hold since we can show using the local form of the Sobolev imbedding theorem that for a suitable infinite subsequence $g_n(x, u_n)$ converges almost everywhere in Ω to $g(x, u)$. It follows as in the proof of Theorem 2 that $g(x, u)$ lies in L^1, that $g(x, u)u$ lies in L^1 by the Fatou Lemma, and that $g_n(x, u_n)$ converges to $g(x, u)$ strongly in $L^1(\Omega)$.

Let v be any element of K and set

$$G_n(x, r) = \int_0^r g_n(x, s)\, ds .$$

Each G_n is a convex, non-negative, differentiable function on R for fixed x. Hence for any pair of arguments r and s

$$G_n(x, r) - G_n(x, s) \geqslant g_n(x, s)(r - s) .$$

If we substitute for r and s, $v(x)$ and $u_n(x)$ respectively, we obtain

$$G_n(x, v) - G_n(x, u_n) \geqslant g_n(x, u_n)(v - u_n) .$$

We now integrate over Ω and obtain the inequality

$$\int G_n(x, v) - \int G_n(x, u_n) \geqslant \int g_n(x, u_n)(v - u_n) \geqslant (f - A(u_n), v - u_n) .$$

Suppose that

$$\int G(x, v) < +\infty .$$

Then

$$|G_n(x, v)| \leqslant |G(x, v)|$$

implies that

$$\int G_n(x, v) \to \int G(x, v) .$$

Since $G_n(x, u_n)$ converges almost everywhere to $G(x, u)$ it follows that

$$\int G(x, v) - \int G(x, u) \geqslant \limsup \left(A(u_n) - f, u_n - v\right)$$

for every v in K such that $\int G(x, v) < +\infty$. Setting $v = u$, however, we obtain

$$0 \geqslant \limsup \left(A(u_n) - f, u_n - u\right) = \limsup \left(A(u_n), u_n - u\right) .$$

By the pseudo-monotonicity of the mapping A from V to V^*, it follows that $w = A(u)$, i.e., $A(u_n)$ converges weakly to $A(u)$ in V^* while

$$\left(A(u_n), u_n\right) \to \left(A(u), u\right) .$$

Hence, for any v in K with $\int G(x, v) < +\infty$, we have by a preceding inequality

$$G(x, v) - G(x, u) \geqslant \left(A(u) - f, u - v\right) .$$

Thus the variational inequality (ii) of the conclusion of Theorem 3 has been established.

To complete the proof of Theorem 3, it suffices to establish the variational inequality (i) for the case in which v lies in $K \cap L^\infty$. To obtain this conclusion, however, it suffices to take the inequality

$$\int g_n(x, u_n)(v - u_n) \geqslant \left(A(u_n) - f, u_n - v\right) .$$

Using Fatou's Lemma and the strong convergence of $g_n(x, u_n)$ to $g(x, u)$, we obtain

$$\int g(x, u)v - \int g(x, u)u \geqslant \left(A(u) - f, u - v\right) ,$$

as desired. Q.E.D.

PROOF OF THEOREM 4. Suppose that u_1 and u_2 satisfy the conclusions fo Theorem 3 for f a given element of V^* and K a given convex subset of K. Suppose that A is monotone. For any element v of K with $\int G(x, v) < +\infty$,

$$\int G(x, v) - \int G(x, u_1) \geq (A(u_1) - f, u_1 - v)$$

and

$$\int G(x, v) - \int G(x, u_2) \geq (A(u_2) - f, u_2 - v) \, .$$

Since $G(x, r)$ is convex in r, if we set

$$v = (\tfrac{1}{2})(u_1 + u_2) \, ,$$

then v is a permissible element, and we have

$$u_1 - v = (\tfrac{1}{2})(u_1 - u_2) = -(u_2 - v) \, .$$

Therefore

$$G(x, v) - G(x, u_1) \geq (\tfrac{1}{2})(A(u_1) - f, u_1 - u_2) \, ,$$
$$G(x, v) - G(x, u_2) \geq (\tfrac{1}{2})(A(u_2) - f, u_2 - u_1) \, .$$

Adding, we obtain the inequality

$$(A(u_1) - A(u_2), u_1 - u_2) + 4\int \left\{ \frac{G(x, u_1) + G(x, u_2)}{2} - G\left(x, \frac{u_1 + u_2}{2}\right) \right\} \leq 0 \, .$$

The conclusion then follows from the monotonicity of A and the convexity of $G(x, r)$ in r. Q.E.D.

§ 3. – We now give the proof of Theorem 5 on the relation of the procedures of Sections 1 and 2.

PROOF OF THEOREM 5. Suppose that u is a solution in $V = W_0^{m,p}(\Omega)$ of the differential equation

$$A(u) + g(x, u) = f$$

with $g(x, u)$ and $g(x, u)u$ in L^1, and with

$$(A(u), u) + \int g(x, u) u = (f, u) \, .$$

For any testing function v in Ω, we have

$$(A(u), v) + \int g(x, u)v = (f, v) .$$

Hence

$$G(x, v) - G(x, u) \geqslant g(x, u)(v - u)$$

implies that

$$\int G(x, v) - \int G(x, u) \geqslant \int g(x, u)(v - u) = (f - A(u), v - u) .$$

Suppose that v is an element of V with $\int G(x, v) < +\infty$. We know that $H(v) \in L^1$ since

$$H(r) \leqslant C\left\{G(x,r) + \frac{1}{p}|r|^p + |r|\right\}$$

and therefore by Proposition 4 we may construct a sequence of testing functions v_j converging to v in V such that $H(v_j)$ is dominated by a fixed L^1 function; thus $G(x, v_j)$ converges to $G(x, v)$ in L^1. Taking the limit of the inequality for $v = v_j$ given above, we see that

$$\int G(x, v) - \int G(x, u) \geqslant (f - A(u), v - u) ,$$

so that the inequality (ii) holds for u.

To obtain the inequality (i), we consider v in $V \cap L^\infty$ and choose a sequence of testing functions $\{v_j\}$ in V converging a.e. and boundedly to v. If we consider the inequality

$$\int g(x, u)(v_j - u) = (f - A(u), v_j - u)$$

and take the limit, we obtain the equality

$$\int g(x, u)(v - u) = (f - A(u), v - u) . \qquad \text{Q.E.D.}$$

BIBLIOGRAPHY

[1] H. Brézis, *Equations et inéquations non linéaires dans les espaces vectoriels en dualité*, Ann. Institut Fourier (Grenoble), **18** (1968), pp. 115-175.

[2] H. Brézis, *Integrales convexes dans les espaces de Sobolev*, Israel Jour. Math., **13** (1972), pp. 9-23.

[3] F. E. BROWDER, *Existence theory for boundary value problems for quasilinear elliptic systems with strongly lower order terms*, Proc. Symposia in Pure Mathematics, vol. 23, American Mathematical Society, Providence, R. I. (1971), pp. 269-286.

[4] F. E. BROWDER, *Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains*, Proc. Nat. Acad. Sci., **74** (1977), pp. 2659-2661.

[5] D. E. EDMUNDS - V. B. MOSCATELLI - J. L. R. WEBB, *Strongly nonlinear elliptic operators in unbounded domains*, Publications Math. de l'Université Bordeaux I, **4** (1974), pp. 5-32.

[6] P. HESS, *A strongly nonlinear elliptic boundary value problem*, Jour. Math. Analysis and Appl., **43** (1973), pp. 241-249.

[7] P. HESS, *On nonlinear mappings of monotone type with respect to two Banach spaces*, Jour. math. pure et appl., **52** (1973), pp. 13-26.

[8] P. HESS, *Variational inequalities for strongly nonlinear elliptic operators*, Jour. Math. pure et appl., **52** (1973), pp. 285-298.

[9] I. NATANSON, *Theory of functions of a real variable*, Ungar, New York, vol. 1 (1955).

[10] C. G. SIMADER, *Über schwache Losungen des Dirichletproblems für streng nichtlineäre Differentialgleichungen*, Math. Zeitschrift, **150** (1976), pp. 1-26.

[11] J. R. L. WEBB, *On the Dirichlet problem for strongly nonlinear elliptic operators in unbounded domains*, Jour. London Math. Soc., II Ser., **10** (1975), pp. 163-170.