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An Inequality for the Rank of a Web and Webs of Maximum Rank (*).

SHIING-SHEN CHERN (**) - PHILLIP A. GRIFFITHS (***)

dedicated to Hans Lewy

1. - Statement of results.

A *web* is given in a neighborhood $U \subset R^N$ by a set of codimension k foliations in general position, a notion we shall make precise in a moment. *Web geometry* is the study of local diffeomorphism invariants of a web ⁽¹⁾; for example, we may ask if it is equivalent to a standard web whose foliations consist of parallel linear spaces of dimension $N - k$. An invariant arises from the consideration of the *abelian q -equations* ($1 \leq q \leq k$) ⁽²⁾ associated to the web. In this paper we will be concerned with the abelian equations when $q = k$. Specifically, we will find a bound on the *rank* or maximum number of linearly independent abelian k -equations, and will show that webs of maximal rank give a very special G -structure in the projectivized tangent spaces PT_x ($x \in U$). In a future paper we hope to use this to show that such webs have a standard local form, generalizing our previous result in the codimension-one case ⁽³⁾.

For simplicity of notation we will carry out our study in detail only in the case $k = 2$. Therefore we now agree, until specified otherwise, that a

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⁽¹⁾ The basic reference is Blaschke-Bol [1].

⁽²⁾ These are defined in general in [4]. The definitions relevant to our present discussion will be given below.

⁽³⁾ Cf. [2], and also [3] for an outline of the main result from [2].

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web will be given in an open set $U \subset R^{2n}$ ⁽⁴⁾ by d foliations by codimension-two submanifolds. The leaves of the i -th foliation will be taken as level sets

$$u_i(x) = \text{const}, \quad v_i(x) = \text{const};$$

these functions are defined up to a local diffeomorphism in (u_i, v_i) -space. The i -th *web normal* is

$$\Omega_i(x) = du_i(x) \wedge dv_i(x).$$

Under a diffeomorphism of (u_i, v_i) , Ω_i is multiplied by a non-vanishing factor so that what is intrinsic is the point

$$\Omega_i(x) \in P(\Lambda^2 T^*),$$

the latter being the projective space associated to the vector space of 2-forms at $x \in U$.

We want to say what it means for the web to be in general position. For this some linear algebra preliminaries are required. It will be convenient not to distinguish between a point $u \in P^N = P(R^{N+1})$ and its homogeneous coordinate vector $u \in R^{N+1} - \{0\}$. A set of points $u_1, \dots, u_d \in P^N$ is in general position in case any $k \leq N + 1$ of them span a P^{k-1} ; i.e. $u_{i_1} \wedge \dots \wedge u_{i_k} \neq 0$ for $1 \leq i_1 < \dots < i_k \leq d$, $k \leq N + 1$.

When we come to the notion of general position of lines some care is necessary. Denote by $G(1, 2n - 1)$ the Grassmannian of lines in $P^{2n-1} = P(R^{2n})$. We will identify $G(1, 2n - 1)$ with its image under the *Plücker embedding*

$$G(1, 2n - 1) \hookrightarrow P^{\binom{2n}{2}-1} = P(\Lambda^2 R^{2n})$$

given by sending the line spanned by points $u, v \in P^{2n-1}$ into $u \wedge v \in P(\Lambda^2 R^{2n})$. A first guess is that a set of lines

$$\Omega_1, \dots, \Omega_d \in G(1, 2n - 1)$$

should be said to be in general position if any $k \leq n$ of them span a P^{2k-1} ; i.e. if all

$$(1.1) \quad \Omega_{i_1} \wedge \dots \wedge \Omega_{i_k} \neq 0 \quad 1 \leq i_1 < \dots < i_k \leq d, \quad k \leq n.$$

This condition is certainly necessary, but for some purposes may not be sufficient.

⁽⁴⁾ The reason for taking the dimension $N = 2n$ will appear in § 4.

For example ⁽⁵⁾, consider a set of four lines $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ in P^3 . The condition (1.1) is equivalent to those lines being pairwise skew. Now it is well-known that there are « in general » two lines meeting each of four skew lines in P^3 . To see better what « in general » means recall that a non-singular quadric surface S in P^3 is doubly ruled by two families of lines, called the A -lines and B -lines. The A -lines (resp. B -lines) are pairwise skew, and any A -line meets any B -line exactly once. All of these facts follow easily by representing S as the image under the Segre embedding

$$P^1 \times P^1 \rightarrow P^3$$

given by

$$(s, t) \rightarrow [1, s, t, st].$$

The A -lines and B -lines are given by $s = \text{const}$ and $t = \text{const}$ respectively. Now there is a unique non-singular quadric surface S containing $\Omega_1, \Omega_2, \Omega_3$ as A -lines.

For the remaining line Ω_4 there are the three possibilities:

Ω_4 meets S in distinct points u_1, u_2 ;

Ω_4 is tangent to S at u ;

Ω_4 is an A -line lying in S .

In the first case each of the B -lines through u_1, u_2 meets all four Ω_i once, and the second possibility is the limiting case of the first when $u_1 = u_2$. But in the third case there are infinitely many lines meeting the four skew lines Ω_i .

In our study we will give a definition of general position motivated by webs arising from non-degenerate algebraic surfaces in P^{n+1} . For this we assume first the condition (1.1). Given any $n - 1$ of the Ω_i , say $\Omega_1, \dots, \Omega_{n-1}$, spanning a P^{2n-3} we consider any P^{2n-5} contained in this P^{2n-3} and the linear projection

$$\pi: P^{2n-1} - P^{2n-5} \rightarrow P^3.$$

Our second requirement is:

(1.2) the lines $\pi(\Omega_n), \dots, \pi(\Omega_d)$ do not all pass through a common point.

⁽⁵⁾ This observation is due to Ran Donagi.

A set of lines satisfying (1.1) and (1.2) will be said to be in *general position*. It is *not* the case that lines in general position have as Plücker images points in general position in $P^{\binom{2n}{2}-1}$. The linear algebra subtlety here is crucial in our study.

A set of foliations is in general position in case the normals $\Omega_i(x)$ are lines in general position in PT_x^* .

An *abelian equation* is a relation

$$(1.3) \quad \sum_i f_i(u_i(x), v_i(x)) \Omega_i(x) = 0,$$

and the *rank* r of the web is the maximum number of linearly independent abelian equations. Our first result is a bound on this rank. Namely, define the integer t uniquely by the conditions

$$(1.4) \quad t \equiv -d + 1 \pmod{n-1}, \quad 0 < t < n-2$$

and set

$$(1.5) \quad p_s(d, n) = \frac{1}{6(n-1)^2} (d-2n+1+t)(d-n+t)(d-1-2t).$$

THEOREM I. *The rank of a d -web in $U \subset R^{2n}$ satisfies*

$$(1.6) \quad r \leq p_s(d, n)$$

In particular, $r = 0$ when $d < 2n$.

This bound may be seen to be sharp. Webs for which equality holds in (1.6) will be said to be of *maximal rank*. Our remaining results will, in this case, give a particular type of \mathcal{G} -structure in the projectivized cotangent spaces PT_x^* .

Before stating the next theorem we recall a little algebraic geometry. A *ruled surface* S in P^N may be constructed by taking two skew subspaces $P^m, P^{m'}$ ($m + m' = N - 1$) spanning P^N together with rational curves $C \subset P^m, C' \subset P^{m'}$ in projective correspondence and letting S be the surface of ∞^1 lines obtained by joining corresponding points. In case $N = 2n - 1, m = m' = n - 1$, and C together with C' are rational normal curves we obtain what will be termed a *special ruled surface*. In suitable coordinates it is the image of $P^1 \times P^1$ under the map

$$(1.7) \quad (t, s) \rightarrow [t, \dots, t^n; st, \dots, st^n].$$

The lines $\Omega(t)$ on S are obtained by holding t fixed. They all have a common linear parameter s , and are therefore all in projective correspondence with corresponding points spanning a $P^{n-1}(s)$ (where $P^{n-1}(0) = P^{n-1}, P^{n-1} \cdot (\infty) = P^{n-1}$).

THEOREM II. *Assume given a d -web of maximal rank in $U \subset R^{2n}$ where*

$$d \geq 2n + 1, \quad n \geq 3 \text{ (}^6\text{)}$$

Then there is defined a field of special ruled surfaces

$$S(x) \subset PT_x^*$$

such that the web normals are lines belonging to this surface. In particular the web normals are all in projective correspondence, written

$$(1.9) \quad \Omega_i(x) \overline{\wedge} \Omega_j(x) \quad 1 \leq i, j \leq d$$

with corresponding points spanning a P^{n-1} in PT_x^ .*

2. – Proof of the bound on the rank.

Suppose that

$$(2.1) \quad \sum_{i=1}^d f_i^\lambda(u_i(x), v_i(x)) du_i(x) \wedge dv_i(x) = 0 \quad \lambda = 1, \dots, r$$

are r linearly independent abelian equations.

Set $\Omega_i = du_i \wedge dv_i$ and

$$Z_i(x) = [\dots, f_i^\lambda(u_i(x), v_i(x)), \dots] \in P^{r-1}.$$

The abelian equations (2.1) become

$$(2.2) \quad \sum_i Z_i(x) \otimes \Omega_i(x) = 0.$$

As x varies over $U \subset R^{2n}$, $Z_i(x)$ traces out a piece of two-dimensional surface S_i in P^{r-1} . We may take (u_i, v_i) as local coordinates on S_i . There

⁽⁶⁾ The rank is zero when $d < 2n$ and is ≤ 1 when $d = 2n$. These cases are excluded.

is a 1-to- d correspondence

$$x \rightarrow Z_1(x), \dots, Z_d(x),$$

but because of (2.2) corresponding points $Z_i(x)$ are not in general position. At first glance it might appear that there are in fact $\binom{2n}{2}$ independent linear relations among the Z_i , but this is not so. Letting $\{Z_1, \dots, Z_d\}$ denote the linear span in P^{r-1} of $Z_1(x), \dots, Z_d(x)$ we shall prove that

$$(2.3) \quad \dim \{Z_1, \dots, Z_d\} \leq d - 2n,$$

an estimate which will turn out to be sharp.

To see this we note that since the lines $\Omega_1(x), \dots, \Omega_d(x) \in G_x(1, 2n - 1)$ are in general position we may choose points

$$\omega_1 = du_1 + \lambda_1 dv_1 \in \Omega_1, \dots, \omega_{2n-2} = du_{2n-2} + \lambda_{2n-2} dv_{2n-2} \in \Omega_{2n-2}$$

such that

$$\omega_1 \wedge \dots \wedge \omega_{2n-2} \wedge \Omega_{2n-1} \neq 0.$$

If we multiply (2.2) by $\omega_1 \wedge \dots \wedge \omega_{2n-2}$ the first $2n - 2$ terms drop out and Z_{2n-1} appears with a non-zero coefficient, i.e.

$$(2.4) \quad Z_{2n-1} \in \{Z_{2n}, \dots, Z_d\}.$$

By symmetry it follows that at most $d - 2n + 1$ of $Z_i(x)$ are linearly independent, which proves (2.3).

Now the argument proceeds as in the codimension-one case. By (2.4)

$$(2.5) \quad Z_{2n-1} = \rho_{2n} Z_{2n} + \dots + \rho_d Z_d.$$

Choose $(u_{2n-1}, v_{2n-1}, u_{2n}, v_{2n}, \dots, u_{3n-2}, v_{3n-2})$ as coordinates on U and differentiate (2.5) to obtain

$$(2.6) \quad \frac{\partial Z_{2n-1}}{\partial u_{2n-1}}, \frac{\partial Z_{2n-1}}{\partial v_{2n-1}} \equiv \frac{\partial Z_{3n-1}}{\partial u_{3n-1}}, \frac{\partial Z_{3n-1}}{\partial v_{3n-1}}, \dots, \frac{\partial Z_d}{\partial u_d}, \frac{\partial Z_d}{\partial v_d} \pmod{Z_1, \dots, Z_d}.$$

Let

$$P^{k(0)} = \{Z_1, \dots, Z_d\}, \quad P^{k(1)} = \left\{ Z_1, \frac{\partial Z_1}{\partial u_1}, \frac{\partial Z_1}{\partial v_1}, \dots, Z_d, \frac{\partial Z_d}{\partial u_d}, \frac{\partial Z_d}{\partial v_d} \right\}, \dots,$$

$P^{k(\mu)}$ = span of the μ -th osculating spaces to the surfaces S_i at corresponding points. By (2.4) and (2.6)

$$\begin{aligned} \dim P^{k(0)} &\leq (d - 2n + 1) - 1, \\ \dim P^{k(1)} &\leq (d - 2n + 1) + 2(d - 3n + 2) - 1, \end{aligned}$$

and in general

$$(2.7) \quad \dim P^{k(\mu)} \leq (d - 2n + 1) + 2(d - 3n + 2) + \dots + (\mu + 1)(d - (\mu + 2)n + \mu + 1) - 1.$$

Here we agree that zero is put in whenever one of the first $\mu + 1$ terms on the right becomes ≤ 0 , which obviously happens for large μ .

The $P^{k(\mu)}(x)$ give an increasing sequence of linear subspaces of P^{r-1} , which eventually terminates at say $P^{k(\infty)}(x)$. Since $P^{k(\infty)}(x)$ does not change by differentiation, it must be a constant linear subspace, and hence is all of P^{r-1} since the equations (2.1) were assumed linearly independent. By (2.7)

$$(2.8) \quad r \leq \sum_{\mu \geq 0} \max [(\mu + 1)\{d - n(\mu + 2) + (\mu + 1)\}, 0].$$

It remains to identify this sum with the expression (1.5). Write

$$\tau = \frac{d - 2n + 1}{n - 1} + \frac{t}{n - 1}$$

where t is determined by (1.4); τ is an integer. Put

$$a = d - 2n + 1, \quad s = n - 1.$$

Then the R.H.S. of (2.8) is

$$\begin{aligned} a + 2(a - s) + 3(a - 2s) + \dots + \tau(a - (\tau - 1)s) = \\ = \frac{1}{2} \tau(\tau + 1)a - \frac{1}{3}(\tau - 1)\tau(\tau + 1)s = \frac{1}{6} \tau(\tau + 1)\{3a - 2(\tau - 1)s\}. \end{aligned}$$

Now

$$\begin{aligned} (n - 1)\tau &= d - 2n + 1 + t, \\ (n - 1)(\tau + 1) &= d - n + t, \quad (n - 1)(\tau - 1) = d - 3n + 2 + t, \\ 3a - 2(\tau - 1)s &= 3d - 6n + 3 - 2(d - 3n + 2 + t) = d - 1 - 2t, \end{aligned}$$

so that the R.H.S. of (2.8) is

$$\frac{1}{6(n-1)^2} (d-2n+1+t)(d-n+t)(d-1-2t) = p_\sigma(d, n)$$

according to (1.5). This completes the proof of Theorem I.

When $n = 2$, we have $t = 0$ and

$$(2.9) \quad p_\sigma(d, n) = \frac{1}{6} (d-1)(d-2)(d-3) = \\ = \text{geometric genus of a smooth surface of degree } d \text{ in } P^3.$$

Exactly the same considerations can be carried out for a d -web of codimension k in a neighborhood $U \subset R^{kn}$, $k < n$. Let r_k be its rank, i.e., the maximum number of linearly independent abelian k -equations. Then we have

$$(2.10) \quad r_k \leq \pi(d, n, k)$$

where

$$(2.11) \quad \pi(d, n, k) = \sum_{\mu \geq 0} \max \left(\binom{k+\mu-1}{\mu} \{d - (k+\mu)n + k - 1 + \mu\}, 0 \right).$$

The first term in $\pi(d, n, k)$ is

$$d - kn + k - 1.$$

Hence we have

$$(2.12) \quad \pi(d, n, k) = 0 \text{ when and only when } d < kn - k + 1.$$

The first two terms in $\pi(d, n, k)$ are

$$d - kn + k - 1 + k\{d - (k+1)n + k\} = \\ = (k+1)d - k(k+2)n + k^2 + k - 1.$$

Hence we have

$$(2.13a) \quad \pi(d, n, k) = n + k,$$

when and only when

$$(2.13b) \quad d = (k+1)n - (k-1).$$

3. – Proof of Theorem II.

By the assumption of maximal rank

$$(3.1) \quad \{Z_1, \dots, Z_d\} = \{Z_{2n}, \dots, Z_d\} = P^{d-2n}(x),$$

i.e. there are exactly $(2n - 1)$ independent relations among the $Z_i(x)$. On the other hand, (2.2) gives what appears to be $\binom{2n}{2}$ relations, and consequently some of these must be dependent. We will see that the geometrical consequence of this is the presence in $P(T_x^*)$, $x \in U \subset R^{2n}$, of a field of special ruled surfaces. An intermediate step is the normal form (3.16), which we will derive first.

To carry this out we choose $u_1, v_1, \dots, u_n, v_n$ as coordinate system and write, at a fixed point $x_0 \in U$,

$$(3.2) \quad \begin{cases} du_s = du_1 + \sum_{\lambda=2}^n A_{s\lambda} du_\lambda + \sum_{\lambda=2}^n B_{s\lambda} dv_\lambda, \\ dv_s = dv_1 + \sum_{\lambda=2}^n C_{s\lambda} du_\lambda + \sum_{\lambda=2}^n D_{s\lambda} dv_\lambda, \end{cases} \quad s = n + 1, \dots, d.$$

This is possible since by general position all

$$\frac{\partial(u_s, v_s)}{\partial(u_1, v_1)} \neq 0.$$

We set

$$A_\lambda = (A_{n+1,\lambda}, \dots, A_{d,\lambda}) \in R^{d-n}$$

and similarly for $B_\lambda, C_\lambda, D_\lambda$.

(3.3) LEMMA. *The vectors A_λ, B_λ are multiples of a vector E_λ , and the E_λ are linearly independent (here $\lambda = 2, \dots, n$).*

PROOF. The abelian equations (2.2) are

$$(3.4) \quad \sum_{\alpha=1}^n Z_\alpha du_\alpha \wedge dv_\alpha + \sum_{s=n+1}^d Z_s du_s \wedge dv_s = 0.$$

The coefficient of $du_\alpha \wedge dv_\alpha$ gives Z_α , $\alpha = 1, \dots, n$, as a linear combination of Z_s , $s = n + 1, \dots, d$, so that by (3.1) there are at most $n - 1$ inde-

pendent relations among Z_s . In particular, the coefficient of $du_1 \wedge dv_1$ gives

$$(3.5) \quad Z_1 + \sum_{s=n+1}^d Z_s = 0,$$

and the coefficients of $dv_1 \wedge du_\lambda$ and $dv_1 \wedge dv_\lambda$ give

$$(3.6)_\lambda \quad \begin{cases} \sum_{s=n+1}^d A_{s\lambda} Z_s = 0, \\ \sum_{s=n+1}^d B_{s\lambda} Z_s = 0, \quad \lambda = 2, \dots, n. \end{cases}$$

By the above remark at most $(n-1)$ of the $2(n-1)$ equations $(3.6)_\lambda$ can be independent. In other words, if $R_\lambda \subset R^{d-n}$ is the span of A_λ, B_λ then

$$\dim \left(\sum_{\lambda=2}^n R_\lambda \right) \leq n-1.$$

The lemma amounts to

$$\sum_{\lambda=2}^n R_\lambda = \bigoplus_{\lambda=2}^n R_\lambda \cong R^{n-1},$$

which is implied by

$$R_\lambda \not\subset \sum_{\lambda \neq \gamma} R_\gamma$$

for fixed λ . If, on the contrary, the equations $(3.6)_\lambda$ are linear combinations of $(3.6)_\gamma$ for $\gamma \neq \lambda$, then taking $\lambda = n$ we will have

$$\begin{aligned} A_n &= \sum_{\gamma=2}^{n-1} a_\gamma A_\gamma + b_\gamma B_\gamma, \\ B_n &= \sum_{\gamma=2}^{n-1} c_\gamma A_\gamma + d_\gamma B_\gamma. \end{aligned}$$

By (3.2)

$$du_s = du_1 + \sum_{\gamma=2}^{1-n} A_{s\gamma} (du_\gamma + a_\gamma du_n + c_\gamma dv_n) + \sum_{\gamma=2}^n B_{s\gamma} (dv_\gamma + b_\gamma du_n + d_\gamma dv_n).$$

In the R^4 defined by

$$(3.7) \quad \begin{cases} du_\gamma + a_\gamma du_n + c_\gamma dv_n = 0, \\ dv_\gamma + b_\gamma du_n + d_\gamma dv_n = 0, \quad \gamma = 2, \dots, n-1, \end{cases}$$

we have

$$(3.8) \quad du_s = du_1, \quad s = n + 1, \dots, d,$$

contradicting general position. More precisely, the $2(n - 2)$ one-forms (3.7) span a P^{2n-5} in $P^{2n-1} = P(T_x^*)$. This P^{2n-5} , does not meet any of the web normals $\Omega_1, \Omega_{n+1}, \dots, \Omega_d$. Under the linear projection

$$P^{2n-1} - P^{2n-5} \xrightarrow{\pi} P^3$$

(3.8) says exactly the lines $\pi(\Omega_1), \pi(\Omega_{n+1}), \dots, \pi(\Omega_d)$ all pass through a common point, and this contradicts general position. Thus Lemma (3.3) is proved.

By the lemma we have $A_\lambda = \alpha_\lambda E_\lambda, B_\lambda = \beta_\lambda E_\lambda, \alpha_\lambda, \beta_\lambda$ not both zero. Replacing du_λ by $\alpha_\lambda du_\lambda + \beta_\lambda dv_\lambda$ we obtain

$$A_\lambda = E_\lambda, \quad B_\lambda = 0$$

in (3.2). After a similar argument applied to C_λ, D_λ we may assume

$$C_\lambda = 0, \quad D_\lambda = F_\lambda,$$

so that (3.2) is now

$$(3.9) \quad \begin{cases} du_s = du_1 + \sum_{\lambda=2}^n E_{s\lambda} du_\lambda, \\ dv_s = dv_1 + \sum_{\lambda=2}^n F_{s\lambda} dv_\lambda. \end{cases}$$

(3.10) LEMMA. E_λ is a non-zero multiple of F_λ .

PROOF. By the proof of Lemma (3.3) the $2(n - 1)$ vectors E_γ, F_γ span an R^{n-1} in R^{d-n} . Thus, if R_λ is the span of E_λ, F_λ

$$\dim \sum_{\lambda=2}^n R_\lambda = n - 1.$$

If some E_λ and F_λ are linearly independent, i.e.

$$\dim R_\lambda = 2,$$

then for some other γ we must have

$$R_\gamma \subset \sum_{\lambda \neq \gamma} R_\lambda .$$

Taking $\gamma = n$ we obtain a relation

$$(3.11) \quad \begin{cases} E_n = \sum_{\gamma=2}^{n-1} a_\gamma E_\gamma + b_\gamma F_\gamma , \\ F_n = \sum_{\gamma=2}^{n-1} c_\gamma E_\gamma + d_\gamma F_\gamma , \end{cases}$$

which we will show leads to a contradiction.

Using (3.9) the coefficients of $du_\gamma \wedge dv_n$ and $du_n \wedge dv_\gamma$ in (3.4) give

$$(3.12) \quad \begin{cases} \sum_{s=n+1}^d (E_{s\gamma} F_{s n}) Z_s = 0 , \\ \sum_{s=n+1}^d (E_{s n} F_{s\gamma}) Z_s = 0 , \quad 2 \leq \gamma \leq n-1 . \end{cases}$$

The coefficient of $du_\lambda \wedge dv_\gamma$ gives

$$(3.13) \quad \delta_\gamma^\lambda Z_\gamma + \sum_{s=n+1}^d (E_{s\gamma} F_{s\lambda}) Z_s = 0 , \quad 2 \leq \gamma, \lambda \leq n-1 .$$

Finally the coefficient of $du_n \wedge dv_n$ gives, after we plug in (3.11),

$$(3.14) \quad Z_n + \sum_{\lambda,\gamma=2}^{n-1} \sum_{s=n+1}^d (a_\lambda E_{s\lambda} + b_\lambda F_{s\lambda})(c_\gamma E_{s\gamma} + d_\gamma F_{s\gamma}) Z_s = 0 .$$

Substituting (3.11) into (3.12) and using (3.13) gives

$$(3.15) \quad \begin{cases} \sum_{\lambda=2}^{n-1} \sum_{s=n+1}^d (c_\lambda E_{s\gamma} E_{s\lambda}) Z_s = -d_\gamma Z_\gamma , \\ \sum_{\lambda=2}^{n-1} \sum_{s=n+1}^d (b_\lambda F_{s\lambda} F_{s\gamma}) Z_s = -a_\gamma Z_\gamma , \quad 2 \leq \gamma \leq n-1 . \end{cases}$$

Expanding out (3.14) and plugging in (3.15) and (3.13) we obtain

$$Z_n - \sum_{\gamma=2}^{n-1} (3a_\gamma d_\gamma + b_\gamma c_\gamma) Z_\gamma = 0,$$

which contradicts the maximal rank assumption (3.1). This proves Lemma (3.10).

We now arrive at our *normal form* for the du_i, dv_i . Namely, we may multiply by a scale factor and assume $E_\lambda = F_\lambda$. If we relabel and define $A_{i\alpha}$ by

$$\begin{aligned} A_{s\lambda} &= E_{s\lambda}, & n+1 \leq s \leq d, \quad 2 \leq \lambda \leq n, \\ A_{s1} &= 1, & n+1 \leq s \leq d, \\ A_{\alpha\beta} &= \delta_{\alpha\beta}, & 1 \leq \alpha, \beta \leq n, \end{aligned}$$

then (3.9) becomes

$$(3.16) \quad \begin{cases} du_i = \sum_{\alpha=1}^n A_{i\alpha} du_\alpha, \\ dv_i = \sum_{\alpha=1}^n A_{i\alpha} dv_\alpha, & 1 \leq i \leq d. \end{cases}$$

(3.17) LEMMA. *The vectors $A_i = [\dots, A_{i\alpha}, \dots] \in P^{n-1}$ lie on $\infty^{(n-1)(n-2)/2}$ linearly independent quadrics.*

PROOF. The basic abelian equation (3.4) gives upon substituting in (3.16)

$$(3.18) \quad \sum_{i=1}^d A_{i\alpha} A_{i\beta} Z_i = 0, \quad 1 \leq \alpha, \beta \leq n.$$

These are $n(n+1)/2$ relations among the Z_i , and by (3.1) only $2n-1$ of the equations (3.18) can be independent. In other words we have

$$\frac{n(n+1)}{2} - (2n-1) = \frac{(n-1)(n-2)}{2}$$

linearly independent relations

$$\sum_{\alpha, \beta=1}^n k_{\alpha\beta} A_{i\alpha} A_{i\beta} = 0, \quad k_{\alpha\beta} = k_{\beta\alpha},$$

among the coefficients in (3.18), and this gives the lemma.

Now we can complete the proof of Theorem II. Namely, by (3.17) the A_i lie on a rational normal curve C in P^{n-1} . After a linear change of coordinates we may assume that C is given parametrically by

$$(3.19) \quad t \rightarrow [t, t^2, \dots, t^n].$$

According to (3.16) we have now written $P^{2n-1} = P(T_x^*)$ as the span of the P^{n-1} determined by the du_α and P'^{n-1} determined by the dv_α , and in each of P^{n-1} , P'^{n-1} we have the rational normal curve (3.19) such that setting $t = t_i$ gives $du_i \in P^{n-1}$ and $dv_i \in P'^{n-1}$ respectively. The i -th web normal Ω_i is the line $du_i + s dv_i$, which is just the line $t = t_i$ on the standard ruled surface given parametrically by

$$(s, t) \rightarrow [t, t^2, \dots, t^n; st, st^2, \dots, st^n].$$

4. - Webs defined by algebraic varieties.

A projective algebraic variety of dimension k , $V_k \subset P^m$ is non-degenerate in case it does not lie in a P^{m-1} . The degree d is the number of intersections with a generic P^{m-k} , written

$$(4.1) \quad V \cdot P^{m-k} = p_1 + \dots + p_d.$$

(Here and in what follows, we frequently omit the index of V_k .) For non-degenerate V , which we will always assume, the $p_i \in P^{m-k}$ are in general position (c.f. Lemma 1.8 in [2]).

We continue to denote by $G(m-k, m)$ the Grassmannian of P^{m-k} 's in P^m , and for fixed $p \in P^m$ we let $\Sigma(p)$ designate the Schubert variety of all P^{m-k} 's which pass through the point p . Note that $\Sigma(p) \cong G(m-k-1, m-1)$ and has codimension k in $G(m-k, m)$. The algebraic variety V defines a web in open sets $U \subset G(m-k, m)$ by specifying the i -th web leaf through P^{m-k} to be $\Sigma(p_i)$ where the p_i are given by (4.1). The basic geometric object here is the incidence correspondence

$$(4.2) \quad I_V \subset V \times G(m-k, m)$$

defined by V , where $I_V = \{(p, A) : p \in V, A \in G(m-k, m), p \in A \cap V\}$. By taking V and the p_i to be defined over the real numbers, we have associated to a projective variety $V_k \subset P^m$ a d ($=$ degree V) web of codimension k ($=$ $\dim V$) submanifolds in $U \subset R^{k(m-k+1)}$.

We now wish to verify that the web defined by a non-degenerate algebraic variety is non-degenerate according to our definition, which we shall do for a surface $S \subset P^{n+1}$. For this consider the linear projection

$$(4.3) \quad \pi: P^{n+1} - P^{n-3} \rightarrow P^3$$

with center P^{n-3} defined by

$$\pi(p) = (p \wedge P^{n-3}) \cdot P^3$$

where $p \wedge P^{n-3}$ is the P^{n-2} spanned by $p \in P^{n+1} - P^{n-3}$ and the center. Under such a projection, $\pi(S) = S'$ is a non-degenerate surface in P^3 of degree

$$d' = d - \# \text{ of points in } S \cap P^{n-3}.$$

The projection induces an inclusion

$$(4.4) \quad \pi^{-1}: G(1, 3) \rightarrow G(n-1, n+1)$$

whose image is the Schubert cycle of all P^{n-1} 's containing the center P^{n-3} . Our first observation is that the web in $G(1, 3)$ defined by S' is the intersection of $\pi^{-1}G(1, 3)$ with the web in $G(n-1, n+1)$ defined by S , even in case there are finitely many points in $S \cap P^{n-3}$.

Now consider the web in $G(1, 3)$ defined by a non-degenerate surface $S' \subset P^3$. For a generic line P^1 in P^3 the intersection

$$P^1 \cdot S' = p_1 + \dots + p_d$$

where the p_i are distinct. The Schubert cycle $\Sigma(p)$ consists of all lines passing through $p \in P^3$, and under the Plücker embedding

$$G(1, 3) \rightarrow P^5$$

$\Sigma(p)$ is a plane. If $\Sigma(p)$ and $\Sigma(q)$ fail to intersect transversely, then they must have in common a line in P^5 . Any line on $G(1, 3)$ is the $\Sigma(p, P^2)$ ($p \in P^2 \subset P^3$) of lines in P^3 passing through p and contained in P^2 . Consequently $\Sigma(p)$ and $\Sigma(q)$ meet transversely unless $p = q$. From this we deduce that the normals to the web defined by $S' \subset P^3$ are skew lines in the projectivized cotangent spaces to $G(1, 3)$, these being P^3 's.

Finally, for the web defined by a non-degenerate surface S in P^{n+1} , the projection

$$P^{2n-1} - P^{2n-5} \rightarrow P^3 \quad (P^{2n-1} = P(T_x^*))$$

in the definition of web non-degeneracy corresponds to the transposed differential of the inclusion (4.4) induced by the linear projection (4.3) whose center P^{n-3} contains p_1, \dots, p_{n-1} . But since $S' = \pi(S)$ is still non-degenerate we deduce that the web defined by S in the neighborhood of a generic $P^{n-1} \in G(n-1, n+1)$ is non-degenerate according to the definition used in Theorems I and II.

Given $V_k \subset P^m$ we consider a meromorphic k -form ω on V and define the *trace* ω , a meromorphic k -form on the Grassmannian $G(m-k, m)$, by

$$\omega(A) = \sum_{i=1}^d \omega(p_i(A)), \quad A \in G(m-k, m),$$

where the intersection

$$A \cdot V = p_1(A) + \dots + p_d(A)$$

for a variable $(m-k)$ -plane A . In terms of the diagram (4.2)

$$\omega = (\pi_2)_* \pi_1^* \omega$$

where π_1, π_2 are respectively the projections $I_V \rightarrow V, I_V \rightarrow G(m-k, m)$. The form ω is a *differential of the first kind* (d.f.k.) if ω is holomorphic (cf. § II of [4]). The space of d.f.k. will be denoted by $H^{k,0}(V)$ and its dimension by $h^{k,0}(V)$. In case V is non-singular $H^{k,0}(V)$ are just the holomorphic k -forms and $h^{k,0}(V)$ is the usual Hodge number.

Since there are no holomorphic forms on $G(m-k, m)$, for ω a d.f.k. we have $\omega = 0$, which is *Abel's theorem*

$$(4.5) \quad \sum_{i=1}^d \omega(p_i(A)) = 0.$$

Clearly (4.5) gives an abelian k -equation on the web defined by V . Conversely, it is not difficult to see that every abelian k -equation is of this form, and consequently the rank of the web is equal to $h^{k,0}(V)$. From Theorem I we deduce the bound

$$(4.6) \quad h^{k,0}(V) \leq \pi(d, m-k+1, k)$$

on the number of linearly independent d.f.k. of a non-degenerate $V_k \subset P^m$.

In case $k=1$ and $m=n$ we obtain *Castelnuovo's bound* (cf. [2])

$$(4.7) \quad \pi(C) \leq \pi(d, n) = \pi(d, n, 1)$$

on the genus of a curve of degree d in P^n . The curves for which equality holds in (4.1) were extensively discussed in our previous paper [2], where in fact we proved that their properties could be deduced by web-theoretic methods.

When $k = 2$ we set $m = n + 1$ so that our variety is a surface $S \subset P^{n+1}$ corresponding to a codimension-2 web in $U \subset R^{2n}$. We denote by $p_g(S)$ the number $h^{2,0}(S)$ of d.f.k.; for smooth S this is the *geometric genus*. Theorem I gives the bound

$$(4.8) \quad p_g(S) \leq p_g(d, n) = \pi(d, n, 2).$$

This inequality has been proved algebro-geometrically by Joe Harris in his Harvard thesis, which contains general methods of estimating the superabundance (= « number of relations among conditions imposed by ») of linear systems with base conditions imposed.

A special case of (4.7) is (7)

$$p_g(S) = 0 \quad \text{for degree } S < 2n.$$

The general statement for a non-degenerate $V_k \subset P^m$ is, by (2.12) and with $n = m - k + 1$,

$$(4.8) \quad h^{k,0}(V) = 0 \quad \text{for } d < k(m - k) + 2.$$

These bounds are sharp. For example, for each $n \geq 2$ there are $K3$ surfaces $S \subset P^{n+1}$ of degree $2n$, characterized by having as hyperplane sections canonical curves of genus $n + 1$. In general

$$h^{k,0}(V) \leq 1 \quad \text{for } d = k(m - k) + 2,$$

and if V is smooth and if $h^{k,0}(V) = 1$, then V is simply-connected (for $k \geq 2$) with trivial canonical bundle.

To give another application we first observe that, by (2.13a) and (2.13b), there is, for each k a unique function $m \rightarrow \bar{d}(m)$ satisfying

$$\pi(\bar{d}(m), m - k + 1, k) = m + 1.$$

(7) After we mentioned this result to R. Hartshorne, he showed us an algebraic-geometric proof, together with the result that S must be a $K3$ -surface, if $\text{deg } S = 2n$, $p_g(S) = 1$.

(Notice that $n = m - k + 1$). For example we have

$$(4.9) \quad \begin{cases} d(m) = 2m & \text{when } k = 1, \\ d(m) = 3m - 4 & \text{when } k = 2, \end{cases}$$

and in general

$$(4.10) \quad d(m) = (k + 1)m - (k - 1)(k + 2).$$

Next we remark that (4.6) can be inverted to

$$(4.11) \quad d > d(h^{k,0}, m)$$

bounding from below the degree of a non-degenerate $V_k \subset P^m$ with fixed $h^{k,0}$.

In particular we consider *canonical algebraic varieties*, defined by the property that their canonical linear system $|K|$ gives a birational and biregular mapping of the abstract variety onto its image in P^m ($m + 1 = h^{k,0}$). For such varieties the degree of the canonical image is

$$(-1)^k c_1^k$$

where c_1 is the 1-st Chern class, and by combining (4.10) and (4.11) we deduce the bound

$$(4.12) \quad h^{k,0}(V) \leq \frac{1}{k+1} [(-1)^k c_1^k + k^2 + 2k - 1]$$

on the Hodge number of a canonical variety. For $k = 1, 2$ we may use (4.9) to obtain

$$\pi(C) \leq \frac{-c_1}{2} + 1, \quad p_d(S) \leq \frac{c_1^2}{3} + \frac{7}{3}.$$

The first is an equality due to $c_1 = 2 - 2\pi$, but the second is in general an inequality. It may be compared with Max Noether's estimate

$$(4.13) \quad p_\sigma(S) \leq \frac{c_1^2}{2} + 2$$

valid for any surface. We remark that here the factor $\frac{1}{2}$ ultimately comes from the 2 in

CLIFFORD'S THEOREM:

$$\dim |L| \leq \frac{\deg L}{2}$$

for a special linear series $|L|$ on a curve, and consequently the generalization of (4.13) to higher dimension is

$$h^{k,0} \leq \frac{(-1)^k \sigma_1^k}{2} + \text{const},$$

which is sharp for suitable double coverings of P^k .

The estimates (4.6)-(4.8), (4.11)-(4.13) were consequences of Theorem I. It is of course, interesting to ask whether or not these bounds are sharp, and if so to determine the structure of the *extremal varieties* defined as those for which equality holds. Now Theorem II gives at least the infinitesimal structure of extremal surfaces $S \subset P^{n+1}$ where the degree $d > 2n$. By continuing the reasoning in the proof of that result we may show that an extremal surface lies in a very special way as a divisor on a threefold $V \subset P^{n+1}$ of minimal degree $n-1$, and this leads to an effective determination of all extremal surfaces. These matters will be taken up in a future paper.

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