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A. V. FERREIRA
G. TOMASSINI

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Finiteness Properties of Topological Algebras, I (1).

A. V. FERREIRA (*) - G. TOMASSINI (**)

Doubtless the most challenging problem in commutative topological algebra theory is the following: how are related the algebraic properties of such an algebra A, the nature of its structure space and the behaviour of the Gel'fand transform of the elements of A? An answer to this question for Banach algebras is given by a theorem due to Gleason which became classical and states that the subset of the structure space of a complex Banach algebra A formed by the finitely generated (over A) maximal ideals is open and can be given a (finite-dimensional) complex analytic space structure which makes the Gel'fand transforms of the elements of A holomorphic.

Results of this kind for general topological algebras are lacking, even for Fréchet algebras of holomorphic functions. A theorem similar to the Gleason's one just quoted, would provide a powerful tool for the study of the envelope of holomorphy of general complex analytic spaces, an essentially open domain of research (cf. [1], [11], [14]).

This paper concerns the above mentioned problem. It contains preliminary results which are important for the later work and are of an independent interest.

The contents of the paragraphs included in this article can be summarized as follows. § 0 contains only some notations, conventions and known results used in the paper. In § 1 we prove two structure theorems (Th. 1.3 and Th. 1.5) for topological algebras which satisfy a « maximum modulus principle », and as a consequence we obtain a finiteness principle of Banach algebra theory under much weaker hypothesis than those of Douady (cf. [6] and [8]). In § 2 we establish that barrelled noetherian algebras have compact structure space (Th. 2.1) and we give some simple properties of noethe

(1) Lavoro eseguito nell'ambito del gruppo G.N.S.A.G.A. del C.N.R.

(*) Istituto Matematico « L. Tonelli », Università di Pisa, Pisa.

(**) Istituto Matematico « U. Dini », Università di Firenze, Firenze.

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rian topological algebras. This allows us to formulate the following conjecture: every Fréchet noetherian algebra with a topology defined by algebra semi-norms is semi-local. However for the time being we could only prove our conjecture by adding some natural hypothesis on the Fréchet topology.

The importance of the conjecture consists in that it is a crucial test for the possibility of the formulation for Fréchet algebras of some « Gleason type » theorem: indeed in view of § 2 the structure space $\Sigma(A)$ of a noetherian Fréchet algebra A is compact; hence, if a Gleason type theorem holds, $\Sigma(A)$ must consists of a finite number of points.

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0. – Preliminaries.

Let A be a unitary $\boldsymbol{C}$-algebra and p a semi-norm on A; we say that p is an algebra semi-norm iff $p(1)=1$ and $p(xy) \leqslant p(x)p(y)$ for every x, y in A. In the sequel, if not otherwise stated, « topological algebra » means a $\boldsymbol{C}$-algebra with a multiplicative identity 1 and endowed with a locally convex Hausdorff topology defined by a system of algebra semi-norms that is filtering for the relation $\leqslant$. We shall identify the set $\boldsymbol{C}1$ of scalar elements of such an algebra with the complex field and the expression « morphism of topological algebras » will stand for a not necessarily continuous $\boldsymbol{C}$-algebra homomorphism.

It is easy to see that our topological algebras are just the Hausdorff topological unitary complex algebras called usually « locally m-convex » in the literature. For a detailed account of the basic properties of these algebras we refer the reader to [10] and in this § we merely introduce some notations and recall a few results systematically used in the text.

Let A be a topological algebra and $(p_\gamma)_{\gamma\in\Gamma}$ be a family of algebra semi-norms which is filtering for the relation $\leqslant$, and defines the topology of A. For any $\gamma\in\Gamma$, $\operatorname{Ker} p_\gamma$ is a closed ideal in A and p_γ induces on $A/\operatorname{Ker} p_\gamma$ an algebra norm so that the completion A_γ of $(A/\operatorname{Ker} p_\gamma,\ p_\gamma/\operatorname{Ker} p_\gamma)$ is a Banach algebra. If $\gamma, \delta\in\Gamma$ are such that $p_\gamma \leqslant p_\delta$, then $\operatorname{Ker} p_\delta \subset \operatorname{Ker} p_\gamma$ and we have a continuous morphism $h_{\gamma\delta}$ of the Banach algebra A_δ into the Banach algebra A_γ. The mappings $h_{\gamma\delta}$ form a projective system of morphisms and the natural monomorphism $h\colon A\to \varprojlim (A_\gamma, h_{\gamma\delta})_{\gamma,\delta\in\Gamma}$ is an topological isomorphism of A onto $h(A)$ so that $\varprojlim (A, h_{\gamma\delta})_{\gamma.\delta\in\Gamma}$ is a completion for A and we have $A\simeq \varprojlim (A_\gamma, h_{\gamma\delta})_{\gamma.\delta\in\Gamma}$ whenever A is complete.

Now, let X be the set-valued functor defined on the category of commutative unitary complex algebras which associates to each algebra the set of its characters, i.e. the set $X(A)$ of all (unitary) homomorphisms of the given algebra onto $\boldsymbol{C}$. For every topological algebra A, we denote by $X_0(A)$ the subset of the dual space A' of A formed by the elements of $X(A)$ that are continuous. If we write $h'_{\delta\gamma}$ for the transposed linear mapping $A'_\gamma \to A'_\delta$, $\gamma, \delta \in \Gamma$, we get an inductive system $(A'_\gamma, h'_{\delta\gamma})$ and an isomorphism h' of $\varinjlim (A'_\gamma, h'_{\delta\gamma})_{\gamma,\delta\in\Gamma}$ onto A'. Every $h'_{\delta\gamma}$ maps $X_0(A_\gamma)$ injectively into $X_0(A_\delta)$ and h' induces a bijection of $\varinjlim (X(A_\gamma), h'_{\delta\gamma})_{\gamma,\delta\in\Gamma}$ onto $X_0(A)$. This fact gives us the possibility to put on $X_0(A)$ several (in general not equivalent) topologies which are induced by the natural topologies currently considered on the space A'. In the next §§ 1, 2, the topology considered on X_0 (A) is that induced by the weak topology of A' and $X_0(A)$ will be a compact space for this topology.

Let A be a topological algebra. $X_0(A)$ is clearly a weakly closed subset of A', and by an easy argument we can see that, if the set A^* of the units of A is open, then $X_0(A)$ is an equicontinuous set and so a weakly compact subset of A'. For barrelled algebras we have a converse to this statement:

LEMMA 0.1. *Let A be a complete barrelled topological algebra. Then the following conditions are equivalent*:

(i) *A^* is open,*

(ii) *$X_0(A)$ is a compact subset of A' for the weak topology,*

(iii) *the topology of A can be defined by a family $(p_\gamma)_{\gamma\in\Gamma}$ of algebra semi-norms such that for each $\gamma \in \Gamma$, $X_0(A)$ is homeomorphic to $X(A_\gamma)$ when these spaces are endowed with the weak topologies.*

PROOF. The proof that the chain of implications (i) $\to$ (ii) $\to$ (iii) $\to$ (i) holds is simple and we just outline it.

We know already that (i) $\to$ (ii). Suppose $X_0(A)$ is weakly compact; since A is barrelled, $X_0(A)$ is equicontinuous. Let $(p_\gamma)_{\gamma\in\Gamma}$ be a filtering family of algebra semi-norms which defines the topology of A; since $X_0(A)$ is equicontinuous, there exists γ_0 in Γ and a positive number α such that $X_0(A)$ is contained in the polar set of $\{a \in A : p_{\gamma_0}(a) \leqslant \alpha\}$. Therefore $\operatorname{Ker} p_{\gamma_0} \subset \operatorname{Ker} \chi$ for every χ in $X_0(A)$, and every χ induces a character on $A/\operatorname{Ker} p_{\gamma_0}$ which is continuous for $p_{\gamma_0}/\operatorname{Ker} p_{\gamma_0}$. Hence, if we denote by Γ_0 the subset formed by the γ for which $p_\gamma \geqslant p_{\gamma_0}$, we see that every operator $h'_{\delta\gamma}$ with $\gamma, \delta \in \Gamma$ maps $X(A_\gamma)$ bijectively onto $X(A_\delta)$, $X(A_\gamma) \cong X(A_\delta)$ and $X_0(A)$ is homeomorphic to each $X(A_\gamma)$, $\gamma \in \Gamma_0$, with respect to their weak topologies.

To see that (iii) $\to$ (i) it is enough to remark that, when (iii) holds an element a in A is invertible iff $\chi(a) \neq 0$ for every $\chi \in X_0(A)$ and since there is also a neighborhood V of 0 such that $X_0(A)$ is contained in the polar set of V, we must have $1+\frac{1}{2}V\subset A^*$.

By the general Gel'fand-Mazur theorem, every topological algebra which is a field, is necessarily isomorphic to $\boldsymbol{C}$. It follows that every closed maximal ideal is an hyperplane and the kernel of a continuous character. Thus the correspondence $\chi\mapsto\operatorname{Ker}\chi$ is a bijection β of $X_0(A)$ onto the set $\Sigma(A)$ formed by the closed maximal ideals of A. $\Sigma(A)$ will be called the *structure space* of A. Let $(p_\gamma)_{\gamma\in\Gamma}$ be a filtering family of algebra semi-norms which defines the topology of A; consider on each $X(A_\gamma)$ the topology induced by the weak topology of A'_γ for each $\gamma\in\Gamma$, and put on $\varinjlim\left(X(A_\gamma),\, h'_{\delta\gamma}\right)$ the inductive limit topology. The image by h' of this topology will be called the *inductive limit topology* of $X_0(A)$ and does not depend upon the filtering family of algebra semi-norms chosen to define the topology of A; the image of the inductive limit topology of $X_0(A)$ by β is called the *inductive limit topology* of $\Sigma(A)$.

These considerations enable us to define the Gel'fand transform on A and to interpret A as an algebra of continuous functions on a nice space. For each element a in A we shall denote by $\hat{a}$ the complex-valued function on $\Sigma(A)$ which sends every maximal ideal M to $\beta^{-1}(M)(a)$; $\hat{a}$ is said to be the *Gel'fand transform* of a and it is continuous for the inductive limit topology of $\Sigma(A)$. From now on $\Sigma(A)$ is assumed to carry its inductive limit topology.

Now, let us denote by $C(\Sigma(A))$ the space of the complex-valued functions on $\Sigma(A)$ which are continuous for the inductive limit topology of $\Sigma(A)$. Take a filtering family of algebra semi-norms $(p_\gamma)_{\gamma\in\Gamma}$ which defines the topology of A and for each $\gamma\in\Gamma$ let $\varkappa_\gamma$ be the canonical morphism $A\to A/\operatorname{Ker}p_\gamma$. The topology on $C(\Sigma(A))$ of the uniform convergence on the sets $\beta\left({}^t\varkappa_\gamma(X(A_\gamma))\right)$ depends only on the topology of A and not on the particular system $(p_\gamma)_{\gamma\in\Gamma}$. Endowed with the topology just described, $C(\Sigma(A))$ becomes a topological algebra and it is straightforward to verify that $\Lambda\colon a\mapsto\hat{a}$ is a continuous homomorphism of A into $C(\Sigma(A))$. The kernel of Λ is the *topological radical* $R(A)$. When A is complete, $R(A)$ is the *Jacobson radical* of A and $\hat{A}=\operatorname{Im}\Lambda$ is a full subalgebra of $C(\Sigma(A))$ $\left(\text{i.e. for every } a\in A,\ \hat{a}\in\hat{A}^* \text{ iff } \hat{a}\in C(\Sigma(A))^*\right)$. The *joint spectrum* $\operatorname{sp}_A(a_1,\dots,a_n)$ for any family $a_1,\dots,a_n$ of elements in A is also equal to $\left\{(\hat{a}_1(M),\dots,\hat{a}_n(M))\colon M\in\Sigma(A)\right\}$ and, in particular, for each $a\in A$, $\operatorname{sp}_A(a)=\hat{a}(\Sigma(A))$.

Let us assume moreover that A^* is open in A. Then, the closure of a proper ideal is a proper ideal, each maximal ideal is closed, every character of A is continuous, and the proof of 0.1 we have given previously, shows that the conditions (ii), (iii) hold. It follows that $\Sigma(A)$ is just the maximal spectrum of A, the inductive limit topology is the same as the image by β

of the weak topology induced on $X_0(A)=X(A)$ by A', and the topological algebra $C(\Sigma(A))$ is the Banach algebra of the complex-valued continuous functions on the compact space $\Sigma(A)$ with the uniform norm.

We round off this § 0 by recalling a symbolic calculus Lemma. This is a consequence of the general theory of Waelbroeck [17], but in the special situation that concerns us in this paper, it can be deduced at once from the Banach algebra version.

In the statement below, A denotes a complete barrelled topological algebra with weakly compact character space $X_0(A)$, and K will be a compact subset of $\boldsymbol{C}^n$ which is the joint spectrum of a family of elements of A. We consider the algebra $\mathcal{O}(K)$ of germs of holomorphic functions on K endowed with its natural topology of Silva inductive limit and put z_i for the i-th–coordinate function on $\boldsymbol{C}^n$. We have the following

Lemma 0.2 (*Šilov-Arens-Calderón Theorem and Šilov idempotent Theorem*). *There is a unique coherent way to associate to each finite family $a_1, \ldots, a_n$, of elements of A, a continuous unitary $\boldsymbol{C}$-algebra homomorphism $\vartheta\colon \mathcal{O}(K)\to A$, $K=\operatorname{sp}_A(a_1, \ldots, a_n)$, such that $\vartheta(z_i)=a_i$, $i=1, \ldots, n$. If e is an idempotent of A, $\hat{e}$ is the characteristic function of a closed and open subset of $\Sigma(A)$. Conversely if Ω is a closed and open subset of $\Sigma(A)$ there is a unique idempotent element e in A such that $\hat{e}$ is the characteristic function of Ω.*

Proof. Let $(p_\gamma)_{\gamma\in\Gamma}$ be a filtering family of algebra semi-norms which defines the topology of A and satisfies the conditions (iii) of 0.1.

For any $\gamma\in\Gamma$ we have $K=\operatorname{sp}_{A_\gamma}(a_1, \ldots, a_n)$ for every finite family of elements of A, and by the Šilov-Arens-Calderón theorem for Banach algebras, there exists a unique coherent way to associate to each such a family a continuous unitary $\boldsymbol{C}$-algebra homomorphism $\vartheta_\gamma\colon \mathcal{O}(K)\to A_\gamma$ that maps z_i onto a_i, $i=1,\ldots,n$. If γ, δ are such that $p_\gamma\leqslant p_\delta$, we must have $\vartheta_\gamma=h_{\gamma\delta}\circ\vartheta_\delta$; hence, the family $(\vartheta_\gamma)_{\gamma\in\Gamma}$ defines a continuous $\boldsymbol{C}$-algebra unitary homomorphism $\vartheta\colon \mathcal{O}(K)\to A$ which maps z_i onto a_i. The uniqueness part is also clear.

The first assertion of the second part is obvious. The second follows immediately from the corresponding theorem for the Banach algebras and from the fact that $A\simeq \varprojlim\,(A_\gamma, h_{\gamma\delta})_{\gamma,\delta\in\Gamma}$.

1. – Some structure theorems.

Let X be a topological space, $C(X)$ the algebra of complex-valued continuous functions on X and consider a subalgebra A of $C(X)$.

We say that A satisfies the *maximum modulus principle* (m.m.p.) at a point $x_0\in X$ iff each $f\in A$ such that $|f(x_0)|=\sup_{x\in X}|f(x)|$ is constant on a neighborhood of x_0.

We say that A satisfies the *maximum modulus principle locally* at a point $x_0 \in X$ iff there is a neighborhood V of x_0 such that A satisfies the m.m.p. on V, *i.e.* A satisfies the m.m.p. at each point of V.

It is easily seen that the following property holds: If $Y \subset X$ is connected and A satisfies the m.m.p. on Y, any $f \in A$ such that $|f|$ reaches its maximum value at a point $x_0 \in Y$ is constant on a neighborhood of Y.

Let X be a topological space, Y a subspace of X and Y_0 a subset of Y. For an algebra A of complex-valued continuous functions neither of the statements (a), (b) below implies the other:

(a) A satisfies the m.m.p. on Y_0,

(b) $A|_Y$ satisfies the m.m.p. on Y_0.

However, for any algebra A of complex-valued continuous functions which contains the constant functions and separates the points of a compact (Hausdorff) space X we can prove the following:

LEMMA 1.1. *If A satisfies the* m.m.p. *on a subset S of its Šilov boundary $\check{S}(A)$, then $A|_{\check{S}(A)}$ also satisfies the* m.m.p. *on S.*

PROOF. Let $x_0 \in S$ and $f \in A$ be such that $|f(x_0)| = \sup_{x \in \check{S}(A)} |f(x)|$. Also we obtain $|f(x_0)| = \sup_{x \in X} |f(x)|$, so that f is constant on some neighborhood $V \cap \check{S}(A)$ of x_0 in $\check{S}(A)$.

Before stating the main theorem of this section it is useful to prove the following

LEMMA 1.2. *Let Δ be the closed unit disc in $\mathbf{C}$ and let $z_1, \ldots, z_k \in \partial\Delta$. There exists a rational function h such that*:

(i) *h is holomorphic on a neighborhood of Δ,*

(ii) $h(\Delta) \subset \Delta$, $h(\partial\Delta) \subset \partial\Delta$,

(iii) $h(z_1) = \ldots = h(z_k) = h(1) = 1$.

PROOF. First of all let us observe that any two-factor Blaschke product

$$\Phi_{\lambda,\mu}(w) = \frac{w-\lambda}{1-\bar{\lambda}w}\,\frac{w-\mu}{1-\bar{\mu}w}$$

with $(1-\lambda)(1-\mu) = 1$, $|\lambda|, |\mu| < 1$ has the properties

$$\Phi_{\lambda,\mu}(\Delta) \subset \Delta\,, \qquad \Phi_{\lambda,\mu}(\partial\Delta) \subset \partial\Delta\,, \qquad \Phi_{\lambda,\mu}(1) = 1$$

and, moreover $\Phi_{\lambda,\mu}(z^0_{\lambda,\mu}) = 1$ where

$$z^0_{\lambda,\mu} = -\frac{1-\lambda\mu}{1-\bar{\lambda}\bar{\mu}}\,.$$

The value of μ given by the equation $(1-\lambda)(1-\mu) = 1$ is $\mu = -\lambda(1-\lambda)^{-1}$ and $|\mu|$ is certainly less than 1 if λ belongs to the domain $\Omega = \{\lambda \in \boldsymbol{C}\colon |\lambda| < 1, |1-\lambda| > 1\}$. For real $\lambda \in \Omega$ we get $z^0_{\lambda,\mu} = -1$ and it can be easily verified that the image of Ω by $\lambda \mapsto z^0_{\lambda,\mu}$ (with $\mu = -\lambda(1-\lambda)^{-1}$) contains a neighborhood U^0 of the point -1. Now let us suppose that $z' \in \partial\Delta$. If z' can be expressed in the form $z' = \exp[2\alpha\pi i]$, $\alpha = p/q \in \boldsymbol{Q}$, we have $z'^q = 1$ and the function $h_{z'}(z) = z^q$ satisfies the conditions (i), (ii), (iii) with $k = 1$, $z' = z_1$.

Let us consider the case when $z' = \exp[2\alpha\pi i]$, $\alpha \in \boldsymbol{R}\setminus\boldsymbol{Q}$. From the approximation theorem of Kronecker it follows that for some integer $q > 0$, $z'^q \in U^0$. Now take λ, μ with $\lambda \in \Omega$, $\mu = -\lambda(1-\lambda)^{-1}$ such that $z^0_{\lambda,\mu} = z'^q$ and put $w = z^q$. It is clear that $h_{z'}(z) = \Phi_{\lambda,\mu}(z^q)$ is a rational function, regular on Δ, which satisfies the relations $h_{z'}(\Delta) \subset \Delta$, $h_{z'}(\partial\Delta) \subset \partial\Delta$, $h_{z'}(1) = 1$, $h_{z'}(z') = 1$.

Now we complete the proof by induction on k. The case $k = 1$ is obvious in view of the above remarks. Suppose that $k > 1$ and let us assume that the lemma is true for any system of $k-1$ points of $\partial\Delta$. Let $z_1, \ldots, z_{k-1}$, z_k be k points on $\partial\Delta$. By the induction hypothesis there exists a rational function $'h$, regular on Δ, such that: $'h(\Delta) \subset \Delta$, $'h(\partial\Delta) \subset \partial\Delta$, $'h(z_1) = \ldots$ $\ldots = {}'h(z_{k-1}) = {}'h(1) = 1$.

Put $z' = {}'h(z_k)$ and take a rational function $h_{z'}$ as above. The rational function $h = h_{z'} \circ {}'h$ satisfies the conditions (i), (ii), (iii).

We can state the following « first structure theorem »:

THEOREM 1.3. *Let A be a barrelled topological algebra with compact connected structure space $\Sigma(A)$. Suppose that the restriction algebra $\hat{A}|_{\check{S}(A)}$ satisfies the maximum modulus principle locally at a point $M_0 \in \check{S}(A)$ and A is complete (or more generally $\hat{A}$ is full in $C(\Sigma(A))$). Then $\Sigma(A)$ reduces to a single point M_0 and A is local.*

PROOF. The proof reduces essentially to showing that $\check{S}(A) = \{M_0\}$.

Indeed let us suppose this has been proved and let $a \in M_0$. Since $\hat{a}$ vanishes on $\check{S}(A)$ we must have $\hat{a} = 0$; it follows that $M_0 \subset R(A) \subset M_0$ and $\Sigma(A) = \{M_0\}$ as well.

If $\check{S}(A) \neq \{M_0\}$ we can choose an open neighborhood Y of M_0 in $\check{S}(A)$, $Y \neq \check{S}(A)$, such that if $|\hat{a}|$, $a \in A$, reaches its maximum value at a point of $\overline{Y}$, the closure of Y in $\check{S}(A)$, $\hat{a}$ is constant on some neighborhood of that point in $\check{S}(A)$. We will prove that this will lead to a contradiction.

For the sake of clarity we shall split our argument into two steps (A) and (B).

(A) For each $a \in A$ and $X \subset \check{S}(A)$ we put:

$$\gamma_a = \sup_{X(A)} |\hat{a}|\,, \qquad T_{X,a} = \{\hat{a}(M) \in \boldsymbol{C}\colon M \in X,\ |\hat{a}(M)| = \gamma_a\}\,,$$
$$W_{X,a} = \hat{a}^{-1}(T_{X,a})\,, \qquad T_a^* = \{M \in \Sigma(A)\colon |\hat{a}(M)| = \gamma_a\}\,.$$

The cardinal number of $T_{X,a}$ will be denoted by $k_X(a)$. We have the obvious relation $W_{X,a} \cap X = T_a^* \cap X$.

We shall check first that $T_{\overline{Y},a}$ is actually finite for every $a \in A$. Assume the contrary, that $T_{\overline{Y},a}$ is an infinite set for some $a \in A$. Then we can pick up a sequence (M_n) from $W_{\overline{Y},a} \cap \overline{Y}$ such that $M_n \mapsto \hat{a}(M_n)$ is a $1-1$ correspondence between (M_n) and $\{\hat{a}(M_n)\}$.

Let M' be a cluster point of (M_n) in the compact space $\overline{Y} \cap T_a^* = W_{\overline{Y},a} \cap \overline{Y}$. From the maximum modulus principle it follows that $\hat{a}$ is constant on some neighborhood U of M' in $\check{S}(A)$ because $|\hat{a}(M')| = \gamma_a$. Since the sequence $(\hat{a}(M_n))$ is injective this contradicts the fact that M_n must be in U for infinitely-many integers n. Thus $k_{\overline{Y}}(a)$ is finite for any $a \in A$.

(B) Let us consider an element $a \in A$ such that $|\hat{a}|$ takes its maximum value at some point in Y i.e. $T_a^* \cap Y \neq \emptyset$. Our aim, at this stage of the proof is to show that $|\hat{a}|$ also takes its maximum value at some point in $\check{S}(A) \setminus Y$.

This is obvious if $\hat{a}$ is constant. Thus we can assume that $\hat{a} \neq 0$ and also $\gamma_a = 1$. Suppose there is an $a \in A$ such that $|\hat{a}|$ does not reach its maximum value at any point of $\check{S}(A) \setminus Y$. We first show that for such an element a, $W_{Y,a} \cap Y = W_{\overline{Y},a} \cap \overline{Y}$ is closed and open in $\check{S}(A)$. $W_{Y,a} \cap Y = T_a^* \cap \check{S}(A)$ is clearly closed in $\check{S}(A)$. On other hand, if $M \in W_{Y,a} \cap Y$, $\hat{a}$ is constant on some neighborhood U of M in $\check{S}(A)$. Hence $U \subset W_{Y,a} \cap Y$ and this shows that $W_{Y,a} \cap Y$ is open in $\check{S}(A)$ (we must have $U \subset Y$ because $|\hat{a}| \neq 1$ off Y).

By 1.2 we can take a rational function h, holomorphic on Δ which satisfies the conditions: $h(\Delta) \subset \Delta$, $h(\check{\Delta}) \subset \mathring{\Delta}$ and $h(T_{Y,a}) = 1$. We can evaluate h at a; let us put $b = h(a)$; we have $\hat{b} = h \circ \hat{a}$ and this relation implies $W_{Y,b} \cap Y = W_{Y,a} \cap Y$, $T_{Y,b} = \{1\}$ and $k_Y(b) = 1$, so that the function $\hat{b}$ peaks on $W_{Y,a} \cap Y$ within $\check{S}(A)$, i.e. $\hat{b}(W_{Y,a} \cap Y) = \{1\}$, and $|\hat{b}(M)| < 1$ for any M in $\check{S}(A) \setminus (W_{Y,a} \cap Y) = V_{Y,a}$. Thus from the fact that $f = \hat{b} - 1$ vanishes on $W_{Y,a} \cap Y$, a closed and open subset of $\check{S}(A)$, we can apply Lemma 2.1 of I. Glicksberg (cf. [5]) which states that f must also vanish on the set $Z = \{M \in \Sigma(A): |\hat{b}(M)| > \alpha\}$ where $\alpha < 1$ is the constant $\sup_{V_{Y,a}} |\hat{b}|$. Z is closed and open in $\Sigma(A)$ because we also have $Z = \{M \in \Sigma(A): \hat{b}(M) = 1\}$. Since $\Sigma(A)$ is connected and $Z \neq \emptyset$, Z must be all of $\Sigma(A)$ which implies that $\hat{b} = 1$. This is impossible because by our hypothesis $|\hat{a}|$ does not take its maximum value at any point of $\check{S}(A) \setminus Y$.

Thus $|\hat{a}|$ for every $a \in A$, takes its maximum value at some point of the closed subset $\check{S}(A) \setminus Y \neq \emptyset$. This is in contradiction to the definition of the Šilov boundary and the proof is complete.

As a consequence, we obtain a second « structure theorem »:

THEOREM 1.4. *Let A be a Banach algebra with connected structure space and let us suppose that there exist some $M_0 \in \check{S}(A)$ such that M_0 is finitely generated over A. Then A is a finite-dimensional vector space over $\mathbf{C}$.*

PROOF. According to Gleason's theorem (cf. [16]), it follows from our hypothesis that there is an open neighborhood U of M_0 in $\Sigma(A)$ which can be given a structure of complex space in such a way that $\hat{A}|_U$ becomes an algebra of holomorphic functions. Hence $\hat{A}$ verifies the m.m.p. locally at M_0 and, by Lemma 1.2, $\hat{A}|_{\check{S}(A)}$ also verifies the m.m.p. locally at M_0. Thus, Theorem 1.3 applies and we have $\Sigma(A) = \{M_0\}$ so that A is a local $\mathbf{C}$-algebra. We prove that A is actually a finite-dimensional $\mathbf{C}$-vector space.

Let $m_1, \ldots, m_r$ be a system of generators for M_0 over A. If we consider the algebra $C\{z_1, \ldots, z_r\}$ of convergent power series endowed with the usual Silva topology we will have a natural continuous homomorphism $\sigma: C\{z_1, \ldots$ $\ldots, z_r\} \to A$ which sends each z_i to m_i, $i = 1, \ldots, r$, since the joint spectrum of $m_1, \ldots, m_r$ reduces to $(0, \ldots, 0)$. Let us prove that σ is onto.

For this, let us consider the A-linear mapping $\vartheta: A^r \to M_0$ given by $(a_1, \ldots, a_r) \mapsto \sum_{i=1}^{r} a_i m_i$. The Banach-Schauder theorem tells us that ϑ (which is onto) is an open mapping and so a positive constant δ can be chosen in such a way that for any $m \in M_0$ there is some $a \in \vartheta^{-1}(m)$, $a = (a_1, \ldots, a_r)$ which satisfies the relation: (α) $\max_{1 \leqslant i \leqslant r} \|a_i\| \leqslant \delta \|m\|$. Now take $a \in A$. We must find a convergent power series which is mapped by σ onto a. We establish first, by induction, that the following holds: for any $h \in \mathbf{N}$, there is a polynomial $P_h = \sum_{i_1 + \ldots + i_r \leqslant h} \lambda_{i_1 \ldots i_r} m_1^{i_1} \ldots m_r^{i_r}$ in $\mathbf{C}[m_1, \ldots, m_r]$ of degree $\leqslant h$ and a homogeneous polynomial $R_{h+1} = \sum_{j_1 + \ldots + j_r = h+1} a_{j_1 \ldots j_r} m_1^{j_1} \ldots m_r^{j_r}$ in $A[m_1, \ldots$ $\ldots, m_r]$ of degree $h+1$ which satisfy the relations:

(1) $\quad a = P_h + R_{h+1}\,,$

(2) $\quad |\lambda_{i_1 \ldots i_r}| \leqslant (2r\delta)^h \|a\|$, $\|a_{j_1 \ldots j_r}\| \leqslant (2r\delta)^{h+1} \|a\|$ for each $(i_1, \ldots, i_r)$ and $(j_1, \ldots$ $\ldots, j_r)$ such that $i_1 + \ldots + i_r \leqslant h$, $j_1 + \ldots + j_r = h + 1$ respectively.

Let $\chi_0 \in X(A)$ be such that $M_0 = \operatorname{Ker} \chi_0$. The case $h = 0$ is obvious in view of the previous inequality (α) and the inequality $|\chi_0(a)| \leqslant \|a\|$. Suppose that our assertion has been proved for h and consider $h+1$. We have $a = P_h + R_{h+1}$

$$P_h = \sum_{i_1+\ldots+i_r \leqslant h} \lambda_{i_1\ldots i_r} m_1^{i_1} \ldots m_r^{i_r}, \qquad R_{h+1} = \sum_{j_1+\ldots+j_r = h+1} a_{j_1\ldots j_r} m_1^{j_1} \ldots m_r^{j_r}.$$

Put $a_{j_1\ldots j_r} - \chi_0(a_{j_1\ldots j_r}) = \sum_{k=1}^{r} b_{j_1\ldots j_r,k} m_k$, with

$$\max_{1\leqslant k\leqslant r} \|b_{j_1\ldots j_r,k}\| \leqslant \delta \|a_{j_1\ldots j_r} - \chi_0(a_{j_1\ldots j_r})\|.$$

By substituting we have

$$R_{h+1} = \sum_{\substack{j_1+\ldots+j_r=h+1\\ 1\leqslant k\leqslant r}} b_{j_1\ldots j_r,k} m_1^{j_1} \ldots m_r^{j_r} m_k + \sum_{j_1+\ldots+j_r=h+1} \chi_0(a_{j_1\ldots j_r}) m_1^{j_1} \ldots m_r^{j_r}.$$

We put also

$$R_{h+2} = \sum_{\substack{j_1+\ldots+j_r=h+1\\ 1\leqslant k\leqslant r}} b_{j_1\ldots j_r,k} m_1^{j_1} \ldots m_r^{j_r} m_k = \sum_{k_1+\ldots+k_r=h+2} a_{k_1\ldots k_r} m_1^{k_1} \ldots m_r^{k_r},$$

$$P_{h+2} = P_h + \sum_{j_1+\ldots+j_r=h+1} \chi_0(a_{j_1\ldots j_r}) m_1^{j_1} \ldots m_r^{j_r} = \sum_{i_1+\ldots+i_r\leqslant h+1} \lambda_{i_1\ldots i_r} m_1^{i_1} \ldots m_r^{j_r}.$$

For any $(k_1, \ldots, k_r)$ with $k_1 + \ldots + k_r = h+2$, each $a_{k_1\ldots k_r}$ is the sum of at most r coefficients $b_{j_1\ldots j_r,k}$ with $j_1 + \ldots + j_r = h+1$, $1 \leqslant k \leqslant r$, therefore

$$\|a_{k_1\ldots k_r}\| \leqslant r \max \|b_{j_1\ldots j_r,k}\| \leqslant r\delta \max \|a_{j_1\ldots j_r} - \chi_0(a_{j_1\ldots j_r})\| \leqslant$$
$$\leqslant 2r\delta \max \|a_{j_1\ldots j_r}\| \leqslant (2r\delta)^{h+2} \|a\|$$

by the induction hypothesis. Proceeding analogously for P_{h+1}, (2) is easily proved. As a consequence we obtain that $\varphi = \sum_{i_1+\ldots+i_r\geqslant 0} \lambda_{i_1\ldots i_r} z_1^{i_1} \ldots z_r^{i_r}$ is a convergent power series and that $\sigma(\varphi) = a$, since the generators can be so chosen in order that $\delta < 1/2r$.

Having proved the surjectivity of σ we can now complete at once our proof. Recall that $\boldsymbol{C}\{z_1, \ldots, z_r\}$ with its Silva topology is a direct limit of a sequence of separable Banach spaces. Hence $\boldsymbol{C}\{z_1, \ldots, z_r\}$ is a Suslin locally convex space. Since A is ultra-bornological we must have $A \simeq \boldsymbol{C}\{z_1, \ldots, z_r\}/$ $/\operatorname{Ker} \sigma$ as topological vector spaces, by the Schwartz-Martineau or Grothendieck open mapping theorem (cf. [7], [9]). As a consequence A is a metrizable Silva space and Corollary 2 (p. 401) of [12] implies that A is a finite dimensional $\boldsymbol{C}$-vector space. This completes the proof.

Let us denote by $\boldsymbol{C}^{(\boldsymbol{N})}$ the $\boldsymbol{C}$-vector space which is the direct sum of a family of complex lines indexed by $\boldsymbol{N}$. In the sequel we shall suppose that $\boldsymbol{C}^{(\boldsymbol{N})}$ is equipped with the $\boldsymbol{C}$-algebra structure for which the product of any couple $a = (a_n)$, $b = (b_n)$ of elements of $\boldsymbol{C}^{(\boldsymbol{N})}$ is (c_n) where $c_0 = a_0 b_0$ and $c_n = a_0 b_n + a_n b_n + a_n b_0$ for $n \neq 0$.

Our third « structure theorem » is as follows:

THEOREM 1.5. *Let A be a barrelled complete topological algebra with compact structure space $\Sigma(A)$ and suppose that the restriction algebra $\hat{A}|_{\check{S}(A)}$ satisfies the maximum modulus principle. Then, if A does not contain a subalgebra isomorphic to $\boldsymbol{C}^{(\boldsymbol{N})}$, $\Sigma(A)$ is a finite set and A is a semi-local algebra which splits into a finite direct sum of complete barrelled local algebras. If moreover A is integral, then A is local.*

PROOF. We divide the proof into two parts (A) and (B). In (A) we show that any « decreasing chain » of closed and open distinct subsets of $\Sigma(A)$ is finite. With part (B) we can easily complete our proof.

(A) Suppose there exists a sequence $X_1 \supsetneqq X_2 \supsetneqq \ldots$ where the X_n are closed and open subsets of $\Sigma(A)$. If we put $Z_n = X_n \setminus X_{n+1}$, $n = 1, 2, \ldots$, we get a sequence of pairwise disjoint closed and open non-void subsets of $\Sigma(A)$. By Šilov theorem (cf. 0.2) we can associate to the sequence (Z_n) a sequence of idempotents in A, (j_n) such that we have for every n, $Z_n = \hat{j}^{-1}(1)$ and, as a consequence, $\hat{j}_n(\bigcup_{m \neq n} Z_m) = \{0\}$. Let us remark that such a sequence (j_n) of idempotents is necessarily orthogonal, i.e. $j_m j_n = 0$ for $m \neq n$. Indeed, if $m \neq n$, $j_m j_n$ is an idempotent which belongs to $R(A)$ and clearly this implies $j_m j_n = 0$. From the orthogonality of (j_n) we can deduce at once that the family $\mathfrak{J}$ consisting of 1 and the terms of the sequence (j_n) is linearly independent over $\boldsymbol{C}$. It is easy to check that the linear space J spanned by $\mathfrak{J}$ is actually a subalgebra of A. J is obviously isomorphic to $\boldsymbol{C}^{(\boldsymbol{N})}$ and this is in contradiction with our assumption. Thus every decreasing chain of closed and open distinct subsets of $\Sigma(A)$ must be finite.

(B) From (A) it follows that the set $\mathcal{C}$ formed by the closed and open non-void subsets partially ordered by inclusion is inductive. Hence by Zorn lemma every closed and open non-void subset contains a minimal element. For any $X \in \mathcal{C}$ we shall denote by $\mu(X)$ an arbitrary minimal non-void closed and open subset of $\Sigma(A)$ contained in X. If X itself is minimal in $\mathcal{C}$ we will have necessarily $\mu(X) = X$ and X must be connected. Let us consider the following inductive construction. We put $C_0 = \mu(\Sigma(A))$ and for any integer n we define C_{n+1} from $C_0, \ldots, C_n$ by the relations $C_{n+1} = \mu\left(\Sigma(A) \setminus \bigcup_{k=0}^{n} C_k\right)$ if $\Sigma(A) \neq \bigcup_{k=0}^{n} C_k$, and $C_{n+1} = \emptyset$ otherwise. For every n put

$X_n = \Sigma(A) \setminus \bigcup_{k=0}^{n} C_k$. (X_n) is a sequence of closed and open subsets of $\Sigma(A)$ which is decreasing. Hence by (A), we must have $X_n = \emptyset$ for all but a finite number of integers n. Thus we can fix an integer n_0 such that $\Sigma(A) \neq \neq \bigcup_{k=0}^{n} C_k$ for $n < n_0$ and $\Sigma(A) = \bigcup_{k=0}^{n_0} C_k$. $(C_k)_{0 \leqslant k \leqslant n_0}$ is a partition of $\Sigma(A)$ into pairwise disjoint minimal subsets of $\Sigma(A)$ and, as the C_k, $k = 0, \dots, n_0$, are connected they are just the connected components of $\Sigma(A)$. Using again the theorem of Šilov we may take idempotents $j_0, \dots, j_{n_0}$ in A such that $\hat{j}_k^{-1}(1) = C_k$ for $k = 0, \dots, n_0$. Thus $A = Aj_0 + \dots + Aj_{n_0}$ is isomorphic to $A_0 \times \dots \times A_{n_0}$ where for each k, $0 \leqslant k \leqslant n_0$, A_k is a complete barrelled algebra isomorphic to Aj_k and $\Sigma(A_k)$ is homeomorphic to C_k. $\Sigma(A)$ is a topological sum of the $\Sigma(A_k)$ and it is easy to see that every algebra A_k satisfies the m.m.p. on $\check{S}(A_k)$. In view of the Theorem 1.3 each algebra A_k is local. If A is integral, it can have no other idempotents than 0 and 1. In this case A must be local and we are done.

COROLLARY 1.6. *Let A be a barrelled complete topological algebra such that its topology is coarser than a locally convex metrizable topology. Suppose that the structure space of A is compact and that $\hat{A}|\check{S}(A)$ satisfies the maximum modulus principle on $\check{S}(A)$. Then A is the direct sum of a finite number of local complete and barrelled algebras. If A is integral then A is local.*

PROOF. By Theorem 1.5 it is enough to show that A cannot contain a subalgebra isomorphic to $\boldsymbol{C}^{(\boldsymbol{N})}$.

Assume the contrary, that there exists a $\boldsymbol{C}$-algebra isomorphism Φ of $\boldsymbol{C}^{(\boldsymbol{N})}$ onto a subalgebra B of A. Let us denote by (δ_{mn}) the Kronecker family $\delta_{mn} = 0$, $m \neq n$, $\delta_{nn} = 1$, and let us consider the family (e_n) of elements of $\boldsymbol{C}^{(\boldsymbol{N})}$ where for each n, $e_n = (\delta_{nm})$; we have $\Phi(e_0) = 1$ and for each n, $j_n = \Phi(e_n)$ is an idempotent of A which is $\neq 0, 1$. The family (j_n) is clearly orthogonal. Take an increasing sequence (p_n) of seminorms for a finer metrizable topology τ. The partial sums of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{1 + p_n(j_n)} j_n$$

form a Cauchy sequence for τ as a consequence of the inequalities

$$\sum_{n=l}^{l+l'} \frac{1}{2^n} \frac{p_k(j_n)}{1 + p_n(j_n)} < \sum_{n=l}^{l+l'} \frac{1}{2^n},$$

$l \geqslant k$, $l' \in \boldsymbol{N}$, which hold for any semi-norm p_k.

Therefore the sequence of the partial sums is a Cauchy sequence in A and the series converges in A to an element a since A is complete.

Now from the definition of $\check{S}(A)$ it follows that each set $\hat{j}_n^{-1}(1) \cap \check{S}(A)$ $n \in \boldsymbol{N}$, is non-void. For any $n \in \boldsymbol{N}$ take M_n in $\hat{j}_n^{-1}(1) \cap \check{S}(A)$ and consider a cluster point M' of the sequence (M_n) in the compact space $\check{S}(A)$. We have the relations:

$$(1-\hat{a})(M_n) = 1 - \frac{1}{2^n}\frac{1}{1+p_n(j_n)}$$

for every $n \in \boldsymbol{N}$, $(1-\hat{a})(M') = 1$ and moreover $(1-\hat{a})(M) = 1$ for any $M \in \Sigma(A) \setminus \bigcup_{n=1}^{+\infty} \hat{j}_n^{-1}(1)$. Thus $|1-\hat{a}|$ takes its maximum value at $M' \in \check{S}(A)$ so that $1-\hat{a}$ must be constant on some neighborhood U in $\check{S}(A)$ of M'. This is not possible because U contains M_n for infinitely-many integers n. It follows that A cannot contain a subalgebra isomorphic to $\boldsymbol{C}^{(\boldsymbol{N})}$, and the corollary is proved.

COROLLARY 1.7. *Let A be a Banach algebra such that every maximal ideal in $\check{S}(A)$ is finitely generated. Then A is a finite product of finite-dimensional local algebras.*

PROOF. In view of Gleason's theorem this follows from the Corollary 1.6 and Theorem 1.4.

We end this section with following remarks.

REMARKS: 1. The statement contained in 1.6 and the Theorem 1.4 hold for a real Banach algebra A. They can be easily deduced by applying to $A \otimes_{\boldsymbol{R}} \boldsymbol{C}$ the results of the complex case.

2. $\boldsymbol{C}^{(\boldsymbol{N})}$ endowed with the topology locally convex direct sum of the topology of $\boldsymbol{C}$, turns out to be a topological algebra which is barrelled, complete and has a compact structure space; moreover, the Šilov boundary of $\boldsymbol{C}^{(\boldsymbol{N})}$ is all of the structure space and the Gelfand transform of the elements of $\boldsymbol{C}^{(\boldsymbol{N})}$ are locally constant. This shows that the hypothesis in the statement of the Theorem 1.5 that A does not contain any subalgebra isomorphic to $\boldsymbol{C}^{(\boldsymbol{N})}$ cannot be dropped without further assumptions on the topology of A.

2. – Noetherian topological algebras.

We begin by stating the following general structure theorem:

THEOREM 2.1. *Let A be a barrelled complete noetherian topological algebra. Then*

(i) *the structure space of A is compact, A^* is open and the topology of A can be defined by a filtering family (p_γ) of algebra semi-norms such that the natural mappings $\Sigma(A_\gamma) \to \Sigma(A)$ are homeomorphisms,*

(ii) *there exists a finite family $A_1, \ldots, A_n$ of barrelled complete topological noetherian algebras with connected structure space such that $A \simeq A_1 \times \ldots \times A_n$.*

PROOF. To prove (i), according to Lemma 0.1, it is enough to check that $X_0(A)$ is a compact subset of A' for the weak topology and, since A is barrelled, we can restrict ourselves to check that $X_0(A)$ is weakly bounded i.e. for any $a \in A$ the set $\hat{a}(\Sigma(A))$ is bounded in $\boldsymbol{C}$. Assume in the contrary that there is a in A with $\hat{a}(\Sigma(A))$ unbounded. Then it is easy to construct by an inductive procedure a sequence (χ_n) of continuous characters of A, such that: $|\chi_0(a)| \geqslant 1$ and $|\chi_n(a)| \geqslant 2^n \max\big(|\chi_0(a)|, \ldots, |\chi_{n-1}(a)|\big)$ for $n = 1, \ldots$.

For any continuous semi-norm p on A we have for every $n \in \boldsymbol{N}$,

$$\sum_{m=n}^{+\infty} p\Big(\big(\chi_m(a)\big)^{-1} a\Big) \leqslant p(a) \sum_{m=n}^{+\infty} 2^{-m}.$$

Hence for each $n \in \boldsymbol{N}$ the product $\prod_{m=n}^{+\infty} \Big(1 - \big(\chi_m(a)^{-1} a\big)\Big)$ is absolutely convergent in A to an element a_n. Now (Aa_n) is an increasing sequence of ideals so that $I = \bigcup_{n \in \boldsymbol{N}} Aa_n$ is an ideal of A and, since A is noetherian, there is a natural member $\bar{n}$ such that $I = Aa_{\bar{n}}$. Let us fix an integer $n > \bar{n}$ and let us choose b in A satisfying the relation $a_n = ba_{\bar{n}}$. We have clearly $\chi_{\bar{n}}(a_n) = 0$ and it follows that

$$0 = \chi_{\bar{n}}\Big(\prod_{m=n}^{+\infty}\Big(1 - \big(\chi_m(a)\big)^{-1} a\Big)\Big) = \prod_{m=n}^{+\infty} |1 - \chi_{\bar{n}}(a)\chi_m(a)^{-1}|.$$

This is impossible because from the hypothesis on the family (χ_n)

$$\prod_{m=n}^{+\infty} |1 - \chi_{\bar{n}}(a)\,\chi_m(a)^{-1}| \geqslant \prod_{m=n}^{+\infty} (1 - 1/2^m) > 0.$$

Hence the subset $X_0(A)$ of A' must be weakly bounded.

The proof of (ii) is similar to the proof of the Theorem 1.5. We prove first that A cannot contain an infinite sequence of pairwise orthogonal idempotents. Once we have verified this, we can conclude that the class of the closed and open subset of $\Sigma(A)$ has the property that every decreasing sequence of elements of this class is stationary, then the argument used in part (B) of the proof of 1.5 will establish (ii). So let us assume that (j_n) is a sequence of non-zero orthogonal idempotents in A. Let J be the ideal generated by $j_1, \ldots, j_n, \ldots$; since A is noetherian, we will have $J = Aj_{n_1} + \ldots + Aj_{n_k}$ for some increasing finite sequence integers $n_1, \ldots, n_k$.

Take $n > n_k$. There are $a_1, \ldots, a_k \in A$ which satisfy the relation $j_n = a_1 j_{n_1} + \ldots + a_k j_{n_k}$; this is absurd because

$$j_n = j_n^2 = a_1 j_{n_1} j_n + \ldots + a_k j_n j_n = 0 .$$

The proof of the theorem is now complete.

A simple and interesting application of this theorem is contained in the following corollaries

COROLLARY 2.2. *For any connected reduced complex space* $(X, \mathcal{O})$ *the algebra of the constant functions over* X *is the unique noetherian closed subalgebra of* $\mathcal{O}(X)$ *for the natural topology of* $\mathcal{O}(X)$.

PROOF. It is enough to assume that X is not compact. Let $A \subset \mathcal{O}(X)$ be a closed noetherian subalgebra of $\mathcal{O}(X)$. In view of Theorem 2.1 there is a compact subset K of X such that $\Sigma(A) \simeq \Sigma(A_K)$ where A_K is the completion of the algebra A with respect to the norm $|\cdot|_K = \sup_K |\cdot|$. Let $x \in X \setminus K$ and $f \mapsto f(x)$ be the continuous character of A determined by x and so a continuous character of A_K. It follows that there exists a constant c such that $|f(x)| \leq c|f|_K$ for each $f \in A$.

For any $n \in \boldsymbol{N}$ we have $|f(x)|^n \leq c|f^n|_K$ so that $|f(x)| \leq \lim_{n\to\infty} |f^n|_K^{1/n} = |f|_K$.

Thus $|f|$ takes its maximum value on K and must be constant.

Let n be a natural number. The algebra $\boldsymbol{C}[\![z_1, \ldots, z_n]\!]$ of formal power series endowed with the product topology of the topology of $\boldsymbol{C}$ is a Fréchet topological noetherian algebra and its topology can be defined by the following sequence of algebra semi-norms, which we will call the canonical system of semi-norms on $\boldsymbol{C}[\![z_1, \ldots, z_n]\!]$: for any $k \in \boldsymbol{N}$ we denote by p_k the semi-norm that associates to each power series $\sum_{i_1, \ldots, i_n \in \boldsymbol{N}} \alpha_{i_1 \ldots i_n} z_1^{i_1} \ldots z_n^{i_n}$ the number $\sum_{i_1 + \ldots + i_n \leq k} |\alpha_{i_1 \ldots i_n}|$.

COROLLARY 2.3. *Every closed noetherian topological subalgebra of* $\boldsymbol{C}[\![z_1, \ldots, z_n]\!]$ *is local.*

PROOF. Let M be the maximal ideal of $\boldsymbol{C}[\![z_1, \ldots, z_n]\!]$. First note that for every $k \in \boldsymbol{N}$ we have $\operatorname{Ker} p_k = M^{k+1}$. We apply the results of Theorem 2.1.

As A is integral, $\Sigma(A)$ must be connected. On other hand, we can take k_0 in $\boldsymbol{N}$ such that $\Sigma(A)$ is homeomorphic to $\Sigma(A_{k_0})$. We have $A_{k_0} = A/A \cap \operatorname{Ker} p_{k_0}$ because $\dim_{\boldsymbol{C}} A/A \cap \operatorname{Ker} p_{k_0} < +\infty$, so that A_{k_0} is a finite dimensional Banach algebra with connected structure space. It follows that $\Sigma(A_{k_0})$ has a single point by 1.7 and this implies that A is local.

This § 2 contains essentially the first general statements about noetherian topological algebras. Finer results require a more sophisticated machinery which is too lenght to describe here and will be the subject of our next paper. Thus we close up the present exposition with some consequences of the closed-graph theorem and Krull intersection theorem.

Let us recall that all the known categories of topological vector spaces in which holds the closed-graph theorem (such as Fréchet spaces, Suslin ultra-bornological spaces, (LF)-ultrabornological spaces, etc. ...) have finite products. So, in the sequel, we shall say that a topological algebra A satisfies the open mapping property iff for every pair $s, t \in \boldsymbol{N}$, any surjective continuous $\boldsymbol{C}$-linear mapping $f: A^t \oplus \boldsymbol{C}^s \to A$ is open. Similarly we say that a couple A, B of topological algebras has the closed-graph property iff every $\boldsymbol{C}$-linear mapping $A \to B$ with closed graph in the product space $A \times B$ is continuous.

LEMMA 2.4. *Let A a complete barrelled topological algebra which is noetherian and satisfies the open mapping property. Then, for each maximal ideal M of A, M^n is closed for every $n \in \boldsymbol{N}$.*

PROOF. Let M be a maximal ideal of A. M is closed and is the kernel of a character of A. Denote by $m_1, \ldots, m_k$ a finite system of generators for M. It is easy to check by induction on n that given $n \in \boldsymbol{N}$ every a in A can be written in the form $a = P_n + R_n$ with P_n a polynomial in $\boldsymbol{C}[m_1, \ldots, m_k]$ of degree $n-1$ and R_n a homogeneous polynomial of degree n in $A[m_1, \ldots, m_k]$.

Hence A can be obtained as the image of a linear continuous mapping $f: A^s \times \boldsymbol{C}^t \to A$ in such a way that we have also $M^n = f(A^s)$, $\operatorname{Ker} f \subset A^s$ and $A = M^n \oplus f(\boldsymbol{C}^t)$ as $\boldsymbol{C}$-vector spaces. By the topological hypothesis on A the last equality holds also in the category of the topological $\boldsymbol{C}$-vector spaces and so M^n is closed because A is complete.

A beautiful property of the algebras we are considering is the following

THEOREM 2.5. *Suppose A, B are noetherian barrelled topological algebras and that the closed-graph property holds for the couple A, B. Then, if both A and B satisfy the open mapping property, every morphism $A \to B$ is continuous.*

THEOREM 2.6. *Let A be a complete barrelled topological algebra which is* *noetherian and satisfies the open mapping property. Then every ideal of A is closed.*

PROOF. By the primary decomposition theorem (cf. [6]) it is enough to prove that each primary ideal of A is closed. Let Q be a primary ideal of A and denote by P the radical of Q. By the intersection theorem of Krull, if M is a maximal ideal of A that contains P, we have $Q = \bigcap_{n \in N} (Q + M^n)$. Thus the fact that Q is closed is established if we can prove that for every $n \in N$, $Q + M^n$ is closed, but this is an immediate consequence of the Lemma 2.4.

Added in proofs.

In next paper of this series now in preparation, an example of a non-semi-local noetherian Fréchet algebra will be presented, and also the metrical conditions will be given which assure the semi-locality of a Fréchet noetherian algebra. The subsequent paper will establish the structure of general noetherian topological algebras.

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