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**On hypoelliptic operators with double characteristics**

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# On Hypoelliptic Operators with Double Characteristics (*).

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The object of this paper is to study the hypoellipticity and local solvability of operators with non-negative principal symbols which vanish to exactly second order on their characteristic varieties. Conditions will be imposed on the subprincipal symbol so that parametrices may be constructed. When the subprincipal symbol fails to satisfy these conditions, non-local solvability results will be proved. It may be recalled that the hypoellipticity of an operator implies the local solvability of its adjoint. This means that the results to be given are fairly complete since as it turns out the conditions to be considered on the subprincipal symbol are invariant under taking adjoints. The operators to be investigated are modeled on $P = D_t^2 + a(t, x, D_x)$ where $a$ is a first order pseudo-differential operator, $t \in \boldsymbol{R}^1$, and $x \in \boldsymbol{R}^n$. If $a$ never assumes real negative values then $P$ will be hypoelliptic with loss of one derivative. Section 1 through 3 will be concerned with situations in which $a$ is non-zero but may assume real negative values. In section 1, hypoellipticity is proved if $\operatorname{Im} a$ has constant sign but possibly zeros of finite order in $t$. Local solvability is proved in section 2 for cases in which $\operatorname{Im} a$ vanishes to infinite order. A non-local solvability result is proved in section 3. Section 4 considers cases in which $a$ is allowed to vanish.

Operators which are hypoelliptic with loss of one derivative and whose principal symbols take values in a proper cone of $\boldsymbol{C}^1$ have been characterized by Hörmander [9]. As may be expected, the operators to be considered here will be hypoelliptic with a loss of more than one derivative. Rubenstein [12] studied local solvability in the special case that $P = D_t^2 + a(t)D_x + b(t)D_t$. Weston [14] considered necessary conditions for the local solva-

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bility of operators of the form $P = P_m(x, D)^2 + P_{2m-1}(x, D)$ where $P_m$ is an $m$-th order partial differential operator of principal type and $P_{2m-1}$ is an operator of order $2m-1$. Boutet de Monvel [1] constructed parametrices in the case of hypoellipticity with loss of one derivative case for operators having involutive characteristics and principal symbols which vanish to exactly second order. The results to be given here may also be compared with those of Hörmander [6] and Rothschild-Stein [11], which study operators of the form $P = \sum_1^k X_j^2 + iX_0$ where $X_j$, $j = 0, \ldots, k$ are first order partial differential operators having real symbols. Below the operators to be considered will be restricted to those with their principal part being a single square but whose subprincipal part is not restricted to being only purely imaginary as in [6] and [11]. A similar study of operators which are the sum of two squares will appear in a future paper.

Although the theorems to be proved are stated for operators acting on $\boldsymbol{R}^n$, the hypotheses and conclusions are invariant under smooth changes of coordinates so that these theorems remain valid for manifolds. The notation of [8] will be used for pseudo-differential operators, etc. $C$ will be used to denote any uninteresting constants and may change from line to line.

## 1. – The case of non-vanishing subprincipal symbols.

Let $P(x, D)$ be a classical pseudo-differential operator of order $m$ on $\Omega \subset \boldsymbol{R}^n$. It is of the form $P(x, D) = P_m(x, D) + P_{m-1}(x, D) + \ldots$ where $P_j(x, \xi) \in C^\infty(\Omega \times (\boldsymbol{R}^n \setminus 0))$ is positively homogeneous of degree $j$ in $\xi$. Denote by $P^s_{m-1}$ the subprincipal symbol of $P$ which is defined as

$$(1.1) \qquad P^s_{m-1}(x, \xi) = P_{m-1}(x, \xi) - (2i)^{-1} \sum_{j=1}^n \partial^2 P_m(x, \xi)/\partial x_j \partial \xi_j\,.$$

In this section $P$ will be studied under the assumption that $P_m(x, \xi) \geqslant 0$ and $P_m$ vanishes to exactly second order on $\Sigma$ a smooth submanifold of $\Omega \times (\boldsymbol{R}^n \setminus 0)$ of codimension 1 transverse to the fibers $x = \text{constant}$.

Let $(x_0, \xi_0) \in \Sigma$, then in some conic neighborhood $\Gamma$ of $(x_0, \xi_0)$, $\Sigma$ may be defined by the equation $U(x, \xi) = 0$ where $U$ is homogeneous of degree 1 in $\xi$, and $d_\xi U \neq 0$. Since $P_m$ vanishes to exactly second order $P_m = QU^2$ where $Q(x, \xi) \neq 0$ and is homogeneous of degree $m-2$. Using the ellipticity of $Q$ in $\Gamma$, $P$ may be expressed as $P \equiv Q(x, D) M(x, D) \bmod \Gamma$ where $M(x, D) = U^2 + R_1$ and $R_1$ is a first order operator. ($A \equiv B \bmod \Gamma$ means $A - B \in L^{-\infty}(\Gamma)$). Note that on $\Sigma$, $P^s_{m-1} = QM^s_1$. The problem of constructing a parametrix for $P$ may then be reduced to constructing one for $M$.

Using Fourier integral operators $M$ may be simplified further. A canonical transformation $\varkappa$ from $\Gamma$ to $\boldsymbol{R}^{2n}$ may be found so that $\varkappa \circ U(x, \varepsilon) = \xi_1$ and $(x_0, \xi_0) = (0, \xi^0)$. It may be necessary to reduce the size of $\Gamma$ to find $\varkappa$. An elliptic Fourier integral operator $A \in I^0(\Omega \times \boldsymbol{R}^n, \Lambda)$, where $\Lambda$ is a part of the graph of $\varkappa$ may then be found such that $A^{-1}MA \equiv L \bmod \Gamma'$ where $L = D_{\xi_1}^2 + S_1$, $S_1$ is a first order operator, and $\Gamma'$ is a conic neighborhood of $(0, \xi^0)$. It may be noted that $\varkappa \circ M_1^s = L_1^s$. For details consult sections 5.2 and 6.1 of Duistermaat-Hörmander [4].

The general problem has now been reduced to constructing parametrices for operators of the form

$$K = D_t^2 + S_1(t, x, D_t, D_x) + \dots \tag{1.2}$$

where $(t, x) \in \boldsymbol{R}^{1+n}$, and $S_1$ is a first order operator. In this case, the subprincipal symbol of $K$ on $\Sigma = \{\tau = 0\}$ is $S_1(t, x, 0, \xi) = a(t, x, \xi)$. Expressing $S_1$ as $S_1(t, x, \tau, \xi) = a(t, x, \xi) + \tau b(t, x, \tau, \xi)$, $K$ may be rewritten as

$$K = D_t^2 + a(t, x, D_x) + b(t, x, D_t, D_x)D_t + c(t, x, D_t, D_x) \tag{1.3}$$

where $b$ and $c$ are in $L^0(\boldsymbol{R}^{n+1})$. The existence of a parametrix for $K$ will follow from

PROPOSITION 1.1. *Let $L = D_t^2 + a(t, x, D_x)$ where $a$ is homogeneous of degree 1 in $D_x$, and suppose that in a conic neighborhood $\Gamma$ of $(t_0, x_0, \xi_0)$, $\operatorname{Re} a \neq 0$ and if $\operatorname{Re} a < 0$, then $\operatorname{Im} a(t, x, \xi)$ never changes sign and $\operatorname{Im} a(t, x, \xi)$ has zeros of order at most $k < \infty$ as a function of $t$. Then there are operators $E_i$, $R_i$ and $\varphi(t, x, \xi) \in S^0(\Gamma)$, i.e., with symbols rapidly decreasing outside of $\Gamma$, with $\varphi(t_0, x_0, \xi_0) \neq 0$ such that*

$$E_1 L = \varphi(t, x, D_x) + R_1, \tag{1.4}$$

$$LE_2 = \varphi(t, x, D_x) + R_2, \tag{1.5}$$

*so that*

$$(1 + |D_t|^2 + |D_x|)^{(k+2)/2(k+1)} E_i \qquad i = 1, 2 \tag{1.6}$$

*and*

$$(1 + |D_x|)^{-k/2(k+1)}(1 + |D_t|^2 + |D_x|)^{(k+2)/2(k+1)} R_i \qquad i = 1, 2 \tag{1.7}$$

*are bounded operators on $H_s(\boldsymbol{R}^{n+1})$, and the estimate*

$$\|(1 + |D_t|^2 + |D_x|)^{(k+2)/2(k+1)} \varphi u\|_s \leq C(\|Lu\|_s + \|u\|_s), \qquad u \in C_0^\infty(R^{n+1}) \tag{1.8}$$

*holds.*

It may be remarked that the operators $E_i$ and $R_i$ are pseudo-local in that they diminish wave front sets when applied to a distribution. It follows from (1.4) that $L$ is hypoelliptic in $\Gamma$. The purpose of $\varphi$ in (1.4), (1.5) and (1.8) is to localize the wave front sets to the conic neighborhood $\Gamma$.

In terms of the original operator $P$, Proposition 1.1 may be restated

THEOREM 1.2. *Let $P = P_m + P_{m-1} + \dots$ be an $m$-th order classical pseudo-differential operator and $\Gamma$ a conic neighborhood of $(x_0, \xi_0)$ in which $P_m(x, \xi) = QU^2$ where $Q$ and $U$ are homogeneous of degree $m-2$, and 1 respectively, both $Q$ and $U$ are real, $d_\xi U \neq 0$ and $Q > 0$. Suppose that $\operatorname{Re} P^s_{m-1} \neq 0$ in $\Gamma$ and if $\operatorname{Re} P^s_{m-1} < 0$, then $\operatorname{Im} P^s_{m-1}$ only zeros of even order $\leqslant k$ on the null bicharacteristics of $U$, then*

(i) *$P$ is hypoelliptic and locally solvable,*

(ii) *there exist operators $E_1$ and $E_2$ such that $E_1 P \equiv PE_1 \equiv I \bmod \Gamma'$ where $\Gamma'$ is some conic subneighborhood of $\Gamma$ and $E_i$ are bounded operators from $H_s$ to $H_{s+m-2+(k+1)/2(k+2)}$,*

(iii) *For some $\chi_1, \chi_2$ in $S^0(\Gamma)$ with $\chi_1(x_0, \xi_0) \neq 0$, the estimate*

$$\|\chi_1(x, D)u\|_{s+m-2+(k+1)/2(k+2)} \leqslant C\big(\|\chi_2 Pu\|_s + \|u\|_s\big), \qquad u \in C_0^\infty(\boldsymbol{R}^n) \tag{1.9}$$

*is valid,*

(iv) *if $u \in \mathfrak{D}'$, $Pu \in H_s(\Gamma)$ then $u \in H_{s+m-2+(k+2)/2(k+1)}(\Gamma)$.*

In the opposite direction there is

THEOREM 1.3. *Let $P = P_m + P_{m-1} + \dots$ be such that in a conic neighborhood $\Gamma$ of $(x_0, \xi_0)$ at which $P_m(x_0, \xi_0) = 0$, $P_m = QU^2 \geqslant 0$, $d_\xi U(x, \xi) \neq 0$. If $\operatorname{Re} P^s_{m-1}(x_0, \xi_0) < 0$ and $\operatorname{Im} P^s_{m-1}$ changes sign and has a zero of finite order on the bicharacteristic of $U$ through $(x_0, \xi_0)$ then $P$ is neither locally solvable nor hypoelliptic at $(x_0, \xi_0)$.*

Theorem 1.3 was proved for partial differential operators by Wenston [14]. It will be reproved below in a clearer way.

The parametrices to be constructed will be vector valued pseudo-differential operators. Let $H_i$, $i = 1, 2$ be a pair of Hilbert spaces with a corresponding pair of families of norms $\|\circ\|_{H_1(\xi)}$ and $\|\circ\|_{H_2(\xi)}$ parametrized by $\xi \in \boldsymbol{R}^n$ such that

$$C_1\|u\|_{H_i} \leqslant \|u\|_{H_i(\xi)} \leqslant C_2(1 + |\xi|)^{m_i}\|u\|_{H_i}, \qquad i = 1, 2 \tag{1.10}$$

and denote by $\mathfrak{L}(H_1(\xi), H_2(\xi))$ the space of bounded operators from $H_1(\xi)$ to $H_2(\xi)$ with the uniform operator norm. Given an open set $\Omega \subset \boldsymbol{R}^n$

define the symbol class $S^m_{\varrho\delta}(\Omega\times\boldsymbol{R}^n; H_1(\xi), H_2(\xi))$ to be the space of smooth functions $A(x,\xi)\colon \Omega\times\boldsymbol{R}^n\to\mathcal{L}(H_1(\xi), H_2(\xi))$ such that for any compact subset $K$ of $\Omega$ and any pair of multi-indices $\alpha$, $\beta$ there is a constant $C$ such that

$$\|D^\alpha_x D^\beta_\xi A(x,\xi)\|_{\mathcal{L}(H_1(\xi),H_2(\xi))}\leqslant C(1+|\xi|)^{m+\delta|\alpha|-\varrho|\beta|}. \tag{1.11}$$

holds for $x\in K$, $\xi\in\boldsymbol{R}^n$. The corresponding class of pseudo-differential operators sending $H_1$ valued functions to $H_2$ valued functions of the form

$$A(x,D)u(x) = (2\pi)^{-n}\int\exp(ix,\xi)A(x,\xi)\hat{u}(\xi)\,d_\xi \tag{1.12}$$

with $A$ in $S^m_{\varrho,\delta}(\Omega\times\boldsymbol{R}^n, H_1, H_2)$ will be called $L^m_{\varrho,\delta}(\Omega; H_1(D), H_2(D))$.

The calculus of pseudo-differential operators works mutatis mutandis when the $H_i$ are infinite dimensional as when they are finite dimensional. The parametrices to be constructed will be in the space $L^0_{\varrho,\delta}(\boldsymbol{R}^n; H, B(D))$ where $\delta<\varrho$, $H=L^2(\boldsymbol{R}, dt)$ and $B(\xi)$ is the Hilbert space of function $u\colon\boldsymbol{R}\to\boldsymbol{C}$ with the norm

$$\|u\|_{B(\xi)} = \|(1+|\xi|+|D_t|^2)^{(k+2)/2(k+1)}u\|. \tag{1.13}$$

If, for instance, $A\in L^{-m}_{\varrho,\delta}(\boldsymbol{R}^n; H, B(D))$ then the standard result on $H_s$ continuity would become the inequality

$$\|(1+|D_x|)^m(1+|D_t|^2+|D_x|)^{(k+2)/2(k+1)}Au\|\leqslant C\|u\|. \tag{1.14}$$

See also section 4 of [14], where vector valued pseudo-differential operators with norms varying on parameters were first introduced by Sjöstrand.

To construct a parametrix for the operator $L$ of Proposition 1.1 first define a kernel function $e(x,\xi,t,s)$ by

$$e(x,\xi,t,s)=\begin{cases}(\tfrac12)\big(a(t,x,\xi)a(s,x,\xi)\big)^{-\frac14}\exp\Big(-\int_t^s a(t',x,\xi)^{\frac12}\,dt'\Big) & \text{if } t\leqslant s,\\ e(x,\xi,s,t) & \text{if } s\leqslant t.\end{cases} \tag{1.15}$$

The assumptions on a in Proposition 1.1 imply that $a^{\frac12}$ is smooth and its real part has constant sign; choose the square root in (1.15) so that its real part is non-negative. For $g$ in $S^m_{1,0}((\boldsymbol{R}^2\times\Omega)\times\boldsymbol{R}^n)$ define the operator $E_g(x,\xi)$ by

$$E_g(x,\xi)f(t)=\int_{-\infty}^{+\infty}e(x,\xi,t,s)g(x,\xi,t,s)f(s)\,ds. \tag{1.16}$$

It will be shown that $E_g$ is in the symbol class $S^m_{\varrho,\delta}(\Omega\times\boldsymbol{R}^n; H, B(\xi))$ where

$$k/(2(k+1)) = \delta < \tfrac{1}{2} < \varrho = (k+2)/(2(k+1))\,. \tag{1.17}$$

A calculation will verify that

$$(D_t^2 + a(t, x, \xi))\,E_g(x, \xi) = g(x, \xi, t, t)I + R_g(x, \xi) \tag{1.18}$$

where $R_g(x, \xi)$ is the integral operator with kernel

$$r_g(x, \xi, t, s) = \{(g/4)(a''a^{-1} - (\tfrac{5}{4})a'^2a^{-2}) + 2g'(a^{\frac{1}{2}} - a'a^{-\frac{1}{4}}) + g''\}\,e\,, \tag{1.19}$$

the primes in (1.19) denoting differentiation by $t$.

Only the case that $\operatorname{Re} a < 0$ in $\Gamma$ will be treated. The case that $\operatorname{Re} a > 0$ in $\Gamma$ is easier; in fact if $g \in S^m_{1,0}((\boldsymbol{R}^2\times\Omega)\times\boldsymbol{R}^n)$ such that $\operatorname{Re} a > 0$ on its support then $E_g \in S^m_{1,0}(\Gamma; H, B'(\xi))$ where $B'$ is the space with norm

$$\|u\|_{B'(\xi)} = \|(1 + |\xi| + |D_t|^2)\,u\|\,. \tag{1.20}$$

Suppose that $\operatorname{Re} a < 0$, $\operatorname{Im} a \geqslant 0$ and $\operatorname{Im} a$ has a zero of order $k$ in $t$ at $(0, x_0, \xi_0)$. Applying the Malgrange factorization theorem, a conic neighborhood $\Gamma$ of $(0, x_0, \xi_0)$ of the form $I\times V$ where $I = [-T, T]$ and $V$ is a conic neighborhood of $(x_0, \xi_0)$ may be found in which

$$\operatorname{Im} a = J(t, x, \xi)\,p(t, x, \xi) \tag{1.21}$$

where $J \geqslant c|\xi|$ is homogeneous of degree 1 and

$$p(t, x, \xi) = t^k + p_1(x, \xi)\,t^{k-1} + \ldots + p_k(x, \xi)\,. \tag{1.22}$$

Shrink $\Gamma$ so that $p \neq 0$ if $|t| \geqslant T/3$ and $(x, \xi) \in V$. Choose functions $\chi_1$, $\chi_2$ in $C_0^\infty(I)$ so that $\chi_1(t) = 1$ for $|t| \leqslant T/2$, $\chi_2(t) = 1$ for $|t| \leqslant T/3$, $\chi_2(t) = 0$ for $|t| \geqslant T/2$, and $\zeta(x, \xi) \in S^0(V)$ so that $\zeta = 1$ in a smaller neighborhood of $(x_0, \xi_0)$. Set $E = E_g$ where $g = \chi_1(t)\chi_2(s)\zeta(x, \xi)$. Note that for this choice of $g$, $e(x, \xi, t, s) \leqslant \exp(-c|\xi|)$ on the support of $\partial g/\partial t$. Consequently, modulo an $S^{-\infty}$ term $R_g = E_{g'}$ where $g' \in S^0_{1,0}$. That $E$ and $R$ are in $S^0_{\varrho,\delta}(\Gamma; H, B(\xi))$ will follow from

LEMMA 1.4. *Suppose that* $\operatorname{Re} a < 0$, $\operatorname{Im} a > 0$, *and* $\operatorname{Im} a$ *has a factorization of the form* (1.21) *in* $\Gamma$, *then for* $g \in S^m_{1,0}(I^2\times V)$, $E_g \in S^m_{\varrho,\delta}(\Gamma; H, B(\xi))$ *where* $\varrho$ *and* $\delta$ *are given by* (1.17).

PROOF. It may be shown by induction that $D_x^\alpha D_\xi^\beta E_g(x, \xi) = \sum E_{g(\alpha,\beta,\gamma,\delta)}$ where

$$|g(\alpha, \beta, \gamma, \delta)| \leqslant C|t-s|^\gamma |\xi|^\delta |e(x, \xi, s, t)| \tag{1.23}$$

and

$$\delta \leqslant m + \gamma/2 - |\beta| , \qquad \gamma \leqslant |\alpha + \beta| . \tag{1.24}$$

Further estimates on $g(\alpha, \beta, \gamma, \delta)$ will be obtained with the aid of the following lemmas.

LEMMA 1.5. *For any constant $C > 0$ there is a second constant $C'$ such that for any complex number $z$, $|\text{Im}\, z| \leqslant C|\text{Re}\, Z|$, $\text{Re}\, z \leqslant 0$ implies that*

$$|\text{Re}\, z^{\frac{1}{2}}| \geqslant C'|\text{Im}\, z|/|\text{Re}\, z|^{\frac{1}{2}} . \tag{1.25}$$

LEMMA 1.6. *Given any integer $k$ there is a constant $C$ such that for any monic polynomial of degree $k$*

$$C|t-s|^{k+1} \leqslant \int_t^s |p(t')|dt' . \tag{1.26}$$

See Treves [13], Corollary C.1 for a proof of Lemma 1.6.

The last two lemmas may be applied to obtain

$$|e(x, \xi, t, s)| \leqslant C|\xi|^{-\frac{1}{2}} \exp\left(-c|\xi|^{\frac{1}{2}}|t-s|^{k+1}\right) . \tag{1.27}$$

Combining the last inequality with (1.23) gives

$$\begin{aligned} |g(\alpha, \beta, \gamma, \delta)| &\leqslant C|t-s|^\gamma |\xi|^{\delta-\frac{1}{2}} \exp\left(-c|\xi|^{\frac{1}{2}}|t-s|^{k+1}\right) \\ &\leqslant C|\xi|^{\delta-\frac{1}{2}-\gamma/2(k+1)} \exp\left(-c|\xi|^{\frac{1}{2}}|t-s|^{k+1}\right) . \end{aligned} \tag{1.28}$$

To estimate the norm of $E_{g(\alpha,\beta)}$ on $L^2(\boldsymbol{R})$, it will be convenient to use

LEMMA 1.7. *Let $k(t, s)$ be a measurable function on $\boldsymbol{R}^2$ such that*

$$\int |k(t, s)|dt \leqslant B \quad \text{and} \quad \int |k(t, s)|ds \leqslant B , \tag{1.29}$$

*then $Kf(t) = \int k(t, s) f(s)\, ds$ is a bounded operator on $L^2(\boldsymbol{R})$ with norm less than $B$.*

Integrating (1.28) by $t$, it may be seen that

$$\int |g(\alpha, \beta, \delta)(t, s)|dt \leqslant C|\xi|^{\delta-(k+2)/(2(k+1))-\gamma/(2(k+1))}. \tag{1.30}$$

The same inequality holds where integration is with respect to $s$ instead of $t$. Using (1.24), it follows that

$$\begin{aligned}\delta-\gamma/(2(k+1)) &= \delta-\gamma/2+(k/2(k+1))\gamma \\ &\leqslant m-|\beta|+(k/2(k+1))|\alpha+\beta| \\ &\leqslant m+(k/2(k+1))|\alpha|-((k+2)/2(k+1))|\beta|.\end{aligned}$$

Using the last inequality in (1.30), Lemma 1.7 then yields

$$\|D_x^\alpha D_\xi^\beta E(x, \xi)\| \leqslant C|\xi|^{m-(k+2)/2(k+1)+\delta|\alpha|-\varrho|\beta|} \tag{1.31}$$

where $\varrho$ and $\delta$ are given by (1.17).

To complete the proof of Lemma 1.4, it remains to estimate $|D_t|^{(k+2)/(k+1)} D_x^\alpha D_\xi^\beta D_g$. Equation (1.18) may be used to write

$$D_t^2 E_g = -a(t, x, \xi)E_g + g(x, \xi, t, t)I + R_g. \tag{1.32}$$

The last equation may be differentiated by $x$ and $\xi$ to obtain

$$(1+|\xi|)^{-m-\delta|\alpha|+\varrho|\beta|} D_t^2 D_x^\alpha D_\xi^\beta E_g = |\xi|^{1/2(k+2)} S_{g(\alpha,\beta)} \tag{1.33}$$

where $S_{g(\alpha,\beta)}$ is a bounded operator on $L^2(\mathbf{R})$. Set $\Lambda = 1 + D_t^2 + |\xi|$, and multiply (1.32) by $\Lambda^{-k/2(k+1)}$ noticing that $|\xi|\Lambda^{-1}$ is bounded operator to conclude that

$$\|\Lambda^{-k/2(k+1)} D_t^2 D_x^\alpha D_\xi^\beta E_g\| \leqslant C(1+|\xi|)^{m+\delta|\alpha|-\varrho|\beta|}. \tag{1.34}$$

Since

$$\|\Lambda^{-k/2(k+1)}|\xi|\| \leqslant (1+|\xi|)^{(k+2)/2(k+1)}$$

inequality (1.31) implies

$$\|\Lambda^{-k/2(k+1)}|\xi| D_x^\alpha D_\xi^\beta E_g\| \leqslant C(1+|\xi|)^{m+\delta|\alpha|-\varrho|\beta|}. \tag{1.35}$$

Add (1.34) and (1.35) together to yield

$$\|\Lambda^{(k+2)/2(k+1)} D_x^\alpha D_\xi^\beta E_g\| \leqslant C(1+|\xi|)^{m+\delta|\alpha|-\varrho|\beta|}$$

which completes the proof of Lemma 1.4.

Lemma 1.4 may be extended to Sobolev norms, namely,

LEMMA 1.8. *Suppose that the hypotheses of Lemma* 1.4 *hold and* $g \in S^m_{1,0}(I^2 \times V)$, *then for any real number* $s$, $E_g \in S^m_{\varrho,\delta}(\Gamma; H^s(\xi), B^s(\xi))$ *where* $\varrho$ *and* $\delta$ *satisfy* (1.17) *and*

$$H^s(\xi) = (1 + |\xi| + |D_t|)^s L^2 ; \qquad B^s(\xi) = (1 + |\xi| + |D_t|)^s B(\xi) .$$

PROOF. It is sufficient to prove the lemma for $s$ equal to an even integer since it will follow in general by interpolation. For simplicity, only the cases $s = \pm 2$, and $m = 0$ will be considered. Setting $\Delta = 1 + |\xi|^2 + D_t^2$ it is easily seen that the commutator $[\Delta, E_g] = E_{g'}$ where $g' = (a(t) - a(s))h + j$ with $h$, $j$ in $S^0_{1,0}(I^2 \times V)$. Lemma 1.4 then says that $\|\Lambda^{(k+2)/2(k+1)} E_{g'}\| \leqslant \| |\xi|$. From these observations, it follows that

$$\begin{aligned}\|\Lambda^{(k+2)/2(k+1)} \Delta E_g u\| &\leqslant \|\Lambda^{(k+2)/2(k+1)} E_g \Delta u\| + \|\Lambda^{(k+2)/2(k+1)}[E_g, \Delta]u\| \\ &\leqslant C\|\Delta u\| + C|\xi|\|u\| \\ &< 2C\|\Delta u\| ,\end{aligned}$$

The last inequality together with analogous bounds on higher order $x - \xi$ derivatives of $E_g(x, \xi)$ proves the lemma in the case that $s = 2$.

For the case that $s = -2$, $m = 0$, multiply the commutator $[E_g, \Delta]$ on both sides by $\Delta^{-1}$ to obtain $\Lambda^{(k+2)/2(k+1)}[E_g, \Delta^{-1}] = \Delta^{-1} G \Delta^{-1}$ where $G$ is a bounded operator from $L^2$ to $B$ whose norm is $O(|\xi|)$. Since on $L^2(I)$, $\|\Delta^{-1}\| \leqslant O|\xi|^{-2}$, it may be concluded that

$$\begin{aligned}\|\Lambda^{(k+2)/2(k+1)} \Delta^{-1} E_g u\| &\leqslant \|\Lambda^{(k+2)/2(k+1)} E_g \Delta^{-1} u\| + \|\Lambda^{(k+2)/2(k+1)} \Delta^{-1} G \Delta^{-1} u\| \\ &\leqslant C\|\Delta^{-1} u\| + C|\xi|^{-1}\|\Delta^{-1} u\| \\ &< C\|\Delta^{-1} u\|\end{aligned}$$

Again higher order derivatives of $E_g(x, \xi)$ may be estimated in the same manner to complete the proof in the case of $s = -2$.

The calculus of pseudo-differential operators may now be used to compute $LE(x, D_x)$. The part $D_t^2 E(x, D_x)$ is obtained by composing $D_t^2$ with the symbol of $E(x, D_x)$. To compute $a(t, x, D_x) \circ E(x, D_x)$ consider $a(t, x, \xi)$ as being a symbol in $S^1_{1,0}(\Gamma, B^s(\xi), B^s(\xi))$. It then follows that $a(t, x, D_x) \circ$ $\circ E(x, D_x) = (aE)(x, D)$ modulo an operator in $L^m_{\varrho,\delta}(\Gamma; H^s(D_x), B^s(D_x))$ where $m = 1 - \min(\varrho, 1 - \delta) = k/(2(k+1))$. Applying (1.18) then gives

$$LE = \chi_2(t)\zeta(\chi, D_x) + R \tag{1.36}$$

where

$$(1.37)\qquad (1+|D_t|^2+|D_x|)^{(k+2)/2(k+1)}E$$

and

$$(1.38)\qquad (1+|D_x|)^{-k/2(k+1)}(1+|D_t|^2+|D_x|)^{(k+2)/2(k+1)}R$$

are bounded operators on $H^s(\boldsymbol{R}^{n+1})$. Since $L^*$ satisfies the same hypotheses as $L$, a left parametrix $E'$ may be constructed in a like manner so that

$$(1.39)\qquad E'L = \chi_2(t)\zeta(x, D_x) + R'$$

where $E'$ and $R'$ also satisfy (1.37) and (1.38) respectively.

The hypoellipticity of $L$ in the micro-local sense follows immediately from (1.39). To get local solvability of $L$, suppose that the hypotheses of Proposition 1.1 hold for any $|\xi| = 1$ at $t = 0$, $x = x_0$. Let $\zeta^j(x, \xi)$ be a partition of unity of $W \times \boldsymbol{R}^n$ where $W$ is a neighborhood of $x_0$, by function in $S^0$ such that

$$(1.40)\qquad LE^j = \chi^j(t)\zeta^j(x, D_x) + R^j .$$

Setting $E = \sum E^j$, $R = \sum R^j$, and summing (1.40) over $j$ results in

$$(1.41)\qquad LE = I + R$$

which is valid in $H^s(\bar{\omega})$ where $\omega$ is a sufficiently small neighborhood $(0, x_0)$ in $\boldsymbol{R}^{n+1}$. Since

$$(1.42)\qquad \|Ru\|_{s+1/(k+1)} \leqslant C\|u\|_s$$

it follows that the norm of $R$ on $H_s(\bar{\omega})$ can be made less than 1 by taking $\omega$ to be sufficiently small in diameter. This means that $I + R$ will be invertible on $H^s(\bar{\omega})$ and consequently $L(E(I + R)^{-1}) = I$ on $\omega$, which implies local solvability.

Next $H_s$-estimates will be considered. Multiplying equation (1.39) by $(1 + |D_t|^2 + |D_x|)^{(k+2)/2(k+1)}$ and taking $H_s$-norm yields

$$(1.43)\qquad \|(1 + |D_t|^2 + |D^x|)^{(k+2)/2(k+1)}\chi_2(t)\zeta(x, D_x)u\|_s \leqslant \\ \leqslant (\|Lu\|_s + \|(1 + |D_x|)^{k/2(k+1)}u\|_s) .$$

Note that since $E'$ has its support in $\Gamma$, $\|Lu\|_s$ in (1.43) could just as well have been replaced by $\|\psi(x, t, D_x)Lu\|$ for a $\psi \in S^0(\Gamma)$. The difference between $L$ and $K$, the operator defined by (1.3), is

$$L-K=b(t, x, D_t, D_x)D_t+c(t, x, D_t, D_x) \tag{1.44}$$

where $b$ and $c$ are bounded operators on $H_s$. Putting (1.44) into (1.39), it may be seen that

$$E'K=\chi_2(t)\zeta(x, D_x)+R''$$

where $(1+|D_t|^2+|D^x|)^{1/2(k+1)}R''$ is a bounded operator on $H_s$: This implies the hypoellipticity of $K$ and the estimate

$$\begin{aligned}\|(1+|D_t|^2+|D_x|)^{(k+2)/2(k+1)}\chi_2(t)\zeta(x, D_x)u\| \leqslant \\ C(\|Ku\|_s+\|(1+|D_t|^2+|D_x|)^{\frac{1}{2}}\chi_3(t)\zeta(x, D_x)u\|_s+\|u\|_s)\end{aligned} \tag{1.45}$$

where the support of $\chi_3$ is slightly larger than that of $\chi_2$. Taking the inner product of $\psi K$ and $\psi u$ in $H^s$ where $\psi(t, x, \tau, \xi)\in S^0(\boldsymbol{R}^{2(n+1)})$ gives

$$\begin{aligned}\|\psi D_t u\|_s^2 &\leqslant (\psi Ku, \psi u)_s+C\|\psi u\|_{s+\frac{1}{2}}^2+C\|\psi D_t u\|_s\|u\|_s \\ &\leqslant \varepsilon\|\psi Ku\|_s+\varepsilon\|\psi D_t u\|_s^2+C\|u\|_{s+\frac{1}{2}}\end{aligned}$$

which leads to the estimate

$$\|\psi D_t u\|_s \leqslant \varepsilon\|\psi Ku\|_s+C\|u\|_{s+\frac{1}{2}}\,. \tag{1.46}$$

Taking $\psi=\chi_3(t)\zeta(x, \xi)$, inequality (1.46) may be used to remove the $\|D_t\chi_3\zeta u\|_s$ term on the right side of (1.45) to yield

$$\|(1+|D_t|^2+|D_x|)^{(k+2)/2(k+1)}\chi_2\zeta u\|_s \leqslant C(\|Ku\|_s+\|u\|_{s+\frac{1}{2}})\,. \tag{1.47}$$

In terms of the original operator $P$, (1.47) becomes

$$\|\psi(x, D_x)u\|_{s+m-2+((k+2)/2(k+1))} \leqslant C(\|\psi' Pu\|_s+\|u\|_{s+m-\frac{3}{2}}) \tag{1.48}$$

where $\psi$ and $\psi'$ are in $L^0(\boldsymbol{R}^n)$ having their support in some small conic neighborhood of $(x_0, \xi_0)$. Suppose that the hypotheses of Theorem 1.2 hold at every point $(x_0, \xi_0)\in\Omega\times\boldsymbol{R}^n$ at which $P_m(x_0, \xi_0)=0$, where $\Omega$ is an open subset of $\boldsymbol{R}^n$. Take a partition of unity $\psi_j$ so that (1.48) holds for $\psi_j$ and $\psi_j'$. Let $K$ be a compact subset of $\Omega$. Summing the estimates analogous to (1.48) over the $\psi_j$ whose support intersects $K$ results in the estimate

$$\|u\|_{s+m-2+(k+2)/(2(k+1))} \leqslant C(\|Pu\|_s+\|u\|_s) \tag{1.49}$$

for all $u\in C_0^\infty(K)$.

## 2. – The case that $\operatorname{Im} P^s_{m-1}$ has zeros of infinite order.

In this section, the limiting case as $k \to \infty$ of Theorem 1.2 will be considered, i.e., the imaginary part of the subprincipal symbol of $P$ will be allowed to have zeros of infinite order along the bicharacteristics of $U$. As in the case of operators of principal type, there are local solvability results but there probably are not hypoellipticity results. More exactly what will be proved is

THEOREM 2.1. *Let $P = P_m + P_{m-1} + \dots$ be a classical m-th order pseudo-differential operator defined on $\Omega \subset \mathbf{R}^n$. At $x_0 \in \Omega$, suppose that for every $\xi_0 \in \mathbf{R}^n \setminus 0$ for which $P_m(x_0, \xi_0) = 0$, there is a conic neighborhood $\Gamma$ of $(x_0, \xi_0)$ such that*

(i) *$P_m(x, \xi) = Q(x, \xi)\, U^2(x, \xi)$ in $\Gamma$ where $Q$ and $U$ are homogeneous of degree $m-2$ and $1$ respectively, both real, $Q > 0$ and $d_\xi U \neq 0$ in $\Gamma$;*

(ii) *$\operatorname{Re} P^s_{m-1} \neq 0$ in $\Gamma$, and if $\operatorname{Re} P^s_{m-1} < 0$, then $\operatorname{Im} P^s_{m-1}$ always has the same sign in $\Gamma$, and doesn't vanish identically on any interval of a null bicharacteristic of $U$.*

*Then $P$ is locally solvable at $x_0$, and for any $\varepsilon > 0$ and real number $s$ there is a sufficiently small neighborhood $\omega$ of $x_0$ such that the estimate*

$$\|u\|_{s+m-\frac{3}{2}} \leq \varepsilon \|Pu\|_s + C\|u\|_{s+m-2}, \qquad u \in C_0^\infty(\omega) \tag{2.1}$$

*is valid. If* (i) *and* (ii) *hold at every point $x_0 \in \Omega$, then for any compact sub set $K$ of $\Omega$ there is a constant $C$ such that inequality* (2.1) *holds for all $u \in C_0^\infty(K)$.*

As in the previous section the proof of Theorem 2.1 can be reduced micro-locally to second order operators of the form

$$K = D_t^2 + a(t, x, D_x) + b(t, x, D_x, D_t)D_t + C(t, x, D_x, D_t) \tag{2.2}$$

or rather

$$L = D_t^2 + a(t, x, D_x) \tag{2.3}$$

where $t \in \mathbf{R}$, $x \in \mathbf{R}^n$, $b, c \in L^0(\mathbf{R}^{n+1})$, $a(t, x, \xi) \in S^1_{1,0}(\Gamma)$ $\Gamma$ is a conic neighborhood of $(0, x_0, \xi_0)$ of the form $I \times V$, $I = [-T, T]$, $V$ a conic neighborhood of $(x_0, \xi_0)$. The assumptions of Theorem 1.2 translate to $\operatorname{Re} a \neq 0$ and if $\operatorname{Re} a < 0$ in $\Gamma$ then $\operatorname{Im} a$ has constant sign in $\Gamma$ and $\operatorname{Im} a$ does not vanish identically on any $t$-interval. Only the latter case that

$\operatorname{Re} a < 0$ will be considered since the former is similar but easier. It is then possible to define $a(t, x, \xi)^{\frac{1}{2}}$ to be smooth and have positive real part. Let $E_g$ be defined to be the transpose of the operator defined by (1.15) and (1.16) and set $E = E_g$ where $g = \chi_1(t)\chi_2(s)\zeta(x, \xi)$ with $\chi_i$ and $\zeta$ chosen as in section 1. In this section $H^s(\xi)$ and $B^s(\xi)$ will be the Hilbert spaces of function $u(t)$ on $I$ with norm taken as

$$\|u\|_{H^s(\xi)} = \|(1 + |\xi| + |D_t|)^s u\| \tag{2.4}$$

and

$$\|u\|_{B^s(\xi)} = \|(1 + |\xi|^{\frac{1}{2}} + |D_t|)(1 + |\xi| + |D_t|)^s u\| \tag{2.5}$$

respectively. It will be shown that

$$E(x, \xi)L(x, t, \xi, D_t) = \chi_2(t)\zeta(x, \xi) + R(x, \xi) + R'(x, \xi) \tag{2.6}$$

where $E(x, \xi)$ and $R(x, \xi)$ are in

$$S^0_{\frac{1}{2},\frac{1}{2}}(\Gamma; H^s(\xi), B^s(\xi)) \quad \text{and} \quad R'(x, \xi) \in S^{-\infty}(\Gamma; H^s(\xi), B^s(\xi)) .$$

In fact, in their symbol spaces $E$ and $R$ will have norms $\leqslant O(|I|)$ ($|I| =$ the length of $I$.) More precisely, for any pair of multi-indicies $\alpha$, $\beta$ there is a constant $C$ such that

$$\|D_x^\alpha D_\xi^\beta E(x, \xi)u\|_{B^s(\xi)} \leqslant C|I|\,|\xi|^{(|\alpha| - |\beta|)/2}\|u\|_{H^s(\xi)} . \tag{2.7}$$

Using the vector-valued analogue of the theorem of Calderon and Vaillancourt [2] which says that operators in $L^0_{\frac{1}{2},\frac{1}{2}}(\boldsymbol{R}^n)$ are bounded on $L^2(\boldsymbol{R}^n)$, it will follow that

$$\|(1 + |D_t| + |D_x|^{\frac{1}{2}})E(x, D_x)u\|_s \leqslant C|I|\,\|u\|_s \tag{2.8}$$

and the same estimate for $R(x, D_x)$.

To see that $E_g$ is in $S^m_{\frac{1}{2},\frac{1}{2}}(\Gamma; H^s(\xi), B^s(\xi))$ if $g \in S^m(I^2 \times V)$ observe 1) that the support of the kernel of $E_g$ is in $I^2$, 2) the exponential in $e(t, s, x, \xi)$ has negative real part and consequently $e$ is always bounded, and 3) $D_x^\alpha D_\xi^\beta e$ is $O(|\xi|^{(-1 + |\alpha| - |\beta|)/2})$. Putting these facts together and using Lemma 1.7 gives the estimate

$$\|D_x^\alpha D_\xi^\beta E(x, \xi)u\| \leqslant C|I|\,|\xi|^{m + (-1 + |\alpha| - |\beta|)/2}\|u\| . \tag{2.9}$$

The estimate for the rest of the $B(\xi)$ norm and the $B^s(\xi)$ norm follow from (2.9) by the same arguments used in section 1 to prove Lemma 1.4 stating from (1.31). The $R'$ term in (2.6) comes from the analogue of the $g'$ term of (1.19), in the support of which $t \neq s$. The condition that $\operatorname{Im} a$ doesn't vanish on intervals implies that $\operatorname{Re}\int_t^s a^{\frac{1}{2}}(x, \xi, \tau)\, d\tau \geq c|\xi|^{\frac{1}{2}}$ if $t$ and $s$ range over a compact set in which $|t-s| \geq c > 0$. This guarantees that $R'$ is in $S^{-\infty}$.

To compose $E(x, D)$ with $L$ using (2.6), the $D_t^2$ and $a(t, x, D_x)$ parts will be considered separately. The symbol of $ED_t^2$ can be computed exactly. To compose $E$ with $a$, consider $a$ as being in $S^1_{1,0}(\Gamma; H^s(\xi), H^s(\xi))$. Using the standard formula for composing pseudo-differential operators, it follows that

$$(2.10) \qquad E(x, D_x)\circ a(t, x, D_x) = (Ea)(x, D_x) + R'' \bmod L^0_{\frac{1}{2},\frac{1}{2}}(\Gamma; H^s(D), B^s(D))$$

where $R'' \in L^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\Gamma; H^s, B^s)$ and has norm $\leq 0(|I|)$ there. It may be concluded from (2.10) and the symbol identity (2.6) that

$$(2.11) \qquad E(x, D)L(t, x, D_t D_x) = \chi_2(t)\zeta(x, D_x) + S + S'$$

where $S$ is in $L^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\Gamma; H^s(D_x), B^s(D))$ and has small norm, and $S'$ is in $L^0_{\frac{1}{2},\frac{1}{2}}(\Gamma, H^s, B^s)$. Multiplying (2.11) by $1 + |D_t| + |D_x|^{\frac{1}{2}}$ and taking $H_s$ norms results in

$$(2.12) \qquad \|(1 + |D_t| + |D_x|^{\frac{1}{2}})\, \chi_2(t)\zeta(x, D_x)u\|_s \leq \leq \varepsilon\|\psi L u\| + \varepsilon\|\psi(1 + |D_x|)^{\frac{1}{2}} u\|_s + C\|u\|_s$$

where $\psi \in S^0(\Gamma)$ has somewhat larger support than $\chi_2(t)\zeta(x, \xi)$, if $I$ has sufficiently small length. Since from (2.2) and (2.3), it follows that

$$(2.13) \qquad \|(L-K)u\|_s \leq C\|D_t u\|_s + C\|u\|_s$$

(2.12) may be rewritten in terms of $K$ as

$$(2.14) \qquad \|(1 + |D_t| + |D_x|^{\frac{1}{2}})\, \chi_2(t)\zeta(x, D_x)u\|_s \leq \varepsilon\|\psi K u\|_s + \varepsilon\|(1 + |D_t| + |D_x|^{\frac{1}{2}})\, \psi u\|_s + C\|u\|_s .$$

Considering the inner product $(\psi K u, \psi u)_s$ leads to the inequality

$$(2.15) \qquad \begin{aligned} \|\psi D_t u\|_s^2 &\leq (\psi K u, \psi u)_s + C\|\psi u\|^2_{s+\frac{1}{2}} + C\|u\|_s^2 \\ &\leq \|\psi K u\|_s^2 \quad + C\|\psi u\|^2_{s+\frac{1}{2}} + C\|u\|_s^2 . \end{aligned}$$

Making use of (2.15) to simplify (2.14) gives

$$(2.16)\qquad \|\chi_2(t)\zeta(x, D_x)u\|_{s+\frac{1}{2}} \leqslant \varepsilon\|\psi K u\|_s + \varepsilon\|\psi u\|_{s+\frac{1}{2}} + C\|u\|_s .$$

Transforming back to the original operator $P$, (2.16) becomes

$$(2.17)\qquad \|\varphi u\|_{s+m-\frac{3}{2}} \leqslant \varepsilon\|\varphi' P u\|_s + \varepsilon\|\varphi' u\|_{s+m-\frac{3}{2}} + C\|u\|_{s+m-2}$$

where $\varphi(x, \xi)$ and $\varphi'(x, \xi)$ are symbols in $S^0_{1,0}(\Gamma)$, $\Gamma$ being a sufficiently small conic neighborhood of a point at which the hypotheses of Theorem 2.1 hold.

Suppose now that (i) and (ii) hold for every point $(x, \xi)$ in $\Omega\times(\boldsymbol{R}^n\setminus 0)$, and let $K$ be any compact subset of $\Omega$. Find a finite partition of unity of $K\times\boldsymbol{R}^n$ by functions $\varphi_j(x, \xi)\in S^0(\Omega\times\boldsymbol{R}^n)$ for which estimate (2.17) holds. Summing these estimates analogous to (2.17) over $j$ yields

$$(2.18)\qquad \|u\|_{s+m-\frac{3}{2}} \leqslant \varepsilon\|P u\|_s + \varepsilon\|u\|_{s+m-\frac{3}{2}} + C\|u\|_{s+m-2}$$

for all $u\in C_0^\infty(K)$. Taking $\varepsilon = \frac{1}{2}$, (2.18) then implies inequality (2.1) holds for all $u\in C_0^\infty(K)$. If (2.1) holds for $u\in C_0^\infty(\omega)$ where $\omega$ is any neighborhood of $x_0$, it would follow that $P^*$ is locally solvable. Since the hypotheses are invariant under adjoints $P$ is also locally solvable. This completes the proof of Theorem 2.1.

To conclude this section a regularity result weaker than hypoellipticity will be given. It is a consequence of estimate (2.14) rather than (2.1).

PROPOSITION 2.2. *Let $\Omega$ be an open subset of $\boldsymbol{R}^n$ and suppose that $u\in\mathcal{D}'(u)$, $Pu\in C^\infty(\Omega)$, $WF(u)\cap(\Omega\times S^{n-1})$ is compact and that conditions* (i) *and* (ii) *of Theorem* 2.1 *hold in a conic neighborhood $\Gamma$ of $WF(u)$. Then $u\in C^\infty(\Omega)$.*

PROOF. It will be shown that if $u\in H_s(\Gamma)$, then $u\in H_{s+\frac{1}{2}}(\Gamma')$ where $\Gamma'\subset\Gamma$. This will be sufficient to prove the proposition since in general $u\in H^t(\Omega)$ for some sufficiently negative value of $t$.

Let $\zeta(x, \xi)\in S^0(\Omega\times\boldsymbol{R}^n)$ have its support in $\Gamma$ and be equal to 1 on $\Gamma'$ contained in $\Gamma$, and set $R(x, \xi) = \zeta(x, \xi)(1+|\xi|^2/\delta^2)^{-1}$. Note that $R_\delta(x, \xi)\to$ $\to\zeta(x, \xi)$ in $S^0(\Omega\times\boldsymbol{R}^n)$ as $\delta\to 0$ and $R_\delta\in S^{-2}$. If $u\in H^s(\Gamma)$ and $\|R_\delta(s, D)u\|_{s+\frac{1}{2}}$ is bounded independently of $\delta$, it will then follow that $u\in H_{s+\frac{1}{2}}(\Gamma)$.

From the proof of Theorem 2.1, it follows that there is a partition of unity of $\Gamma$ consisting of symbols $\psi_j(x, \xi)\in S_0(\Gamma)$ such that *a*) in the support of $\psi_j$, $P = Q_j(U_j^0 + B_j)$ where $Q_j$ is elliptic, $Q\in L^{m-2}(\Omega)$, $U_j$, $B_j\in L_1(\Omega)$

and $b$) the estimate

$$\|\psi_j(1+U_j^2)^{\frac{1}{2}}v\|_s + \|\psi_j v\|_{s+\frac{1}{2}} \leq \varepsilon\|\psi_j'(U_j^2+B_j)v\|_s + \varepsilon\|\psi_j v\|_{s+\frac{1}{2}} + C\|v\|_s \tag{2.19}$$

is valid for all $v \in C_0^\infty(\Omega)$ where the support of $\psi_j' \subset \Gamma$.

To prove that $\|R_\delta u\|_{s+\frac{1}{2}}$ is bounded, replace $v$ by $R_\delta u$ in (2.19). Rearranging the result gives the inequality

$$\begin{aligned}\|\psi_j R_\delta u\|_{s+\frac{1}{2}} \leq \varepsilon\|\psi_j' R_\delta(U_j^2+B_j)u\|_s + \varepsilon\|\psi_j' R_\delta u\|_{s+\frac{1}{2}} \\ + C\|u\|_s + \varepsilon\|\psi_j'[R_j\, U_j^2]u\|_s + \varepsilon\|\psi'[B_j, R_\delta]u\|_s .\end{aligned} \tag{2.20}$$

Since $Q_j$ is elliptic in the support of $\psi_j'$, it follows that $\psi_j'(U_j^2+B_j)u \in C^\infty$. The fact that the symbol $R_\delta$ is bounded in $S^0(\Gamma)$ implies that $\|\psi_j R_\delta(U_j^2+B_j)u\|_s$ is bounded as $\delta \to 0$. Similarly, since the commutator $[B_j R_\delta]$ is bounded in $L^0(\Gamma)$, it follows that $\|\psi'[B_j, R_\delta]u\|_s$ is bounded independently of $\delta$. From the identities $[R_\delta, U_j^2] = 2[R_\delta, U_j]\, U_j \bmod L^0(\Omega)$ and $[R_\delta, U_j] = CR_s \bmod L^0(\Gamma')$ with $C$ in $L^0(\Omega)$, it follows that

$$\|\psi_j'[R_j\, U_j^2]u\|_s \leq 2\|\psi_j'\, U_j R_\delta u\| + \|u\|_s . \tag{2.21}$$

Analogously to (2.15), it may be shown that

$$\|\psi_j'\, U_j R_\delta u\|_s \leq C\|\psi' R_\delta(U_j^2+B_j)u\|^2 + C\|\psi' R_\delta u\|_{s+\frac{1}{2}} + C\|u\|_s . \tag{2.22}$$

Using (2.21) and (2.22) in (2.20) results in

$$\|\psi_j R_\delta u\|_{s+\frac{1}{2}} \leq \varepsilon\|\psi_j' Pu\|_{s-m+2} + \varepsilon\|\psi_j' R_\delta u\|_{s+\frac{1}{2}} + C\|u\|_s . \tag{2.23}$$

Summing (2.23) over $j$ and recalling that $\Sigma\, \psi_j^2$ is not 0 on the support of $\zeta$ it follows that

$$\begin{aligned}\|R_\delta u\|_{s+\frac{1}{2}} \leq \varepsilon\|Pu\|_{s-m+2} + \varepsilon\|R_\delta u\|_{s+\frac{1}{2}} \\ + C\|u\|_s + C\|\chi u\|_{s+\frac{1}{2}}\end{aligned} \tag{2.24}$$

where $\chi = 0$ in a neighborhood of $WF(u)$. Choosing $\varepsilon = \frac{1}{2}$, makes it possible to absorb the $\varepsilon\|R_\delta u\|_{s+\frac{1}{2}}$ term of (2.24) in the left. This proves that $\|R_\delta u\|_{s+\frac{1}{2}}$ is bounded independently of $\delta$, completing the proof of the proposition.

## 3. – Non-local solvability.

The goal of this section is to prove Theorem 1.3. Suppose that the equation $P(x, D_x)u = f$ is locally solvable at $x_0$. A result of Hörmander ([5], Lemma 6.1.2) states that there is a neighborhood $V$ of $x_0$ and constants $C$ and $N$ such that the inequality

$$\left|\int f\bar{u}\,dx\right| \leqslant C\Big(\sum_{|\alpha|\leqslant N} \sup |D^\alpha f|\Big)\Big(\sum_{|\alpha|\leqslant N} \sup |D^\alpha P^* u|\Big) \tag{3.1}$$

is valid for all $f, u \in C_0^\infty(V)$. The non-solvability result stated in Theorem 1.3 will be proved by contradicting inequality (3.1) when $P$ satisfies the hypotheses of Theorem 1.3. This will be done by constructing approximate solutions of $P^* u = 0$.

With the aid of Fourier integral operators, the proof may be reduced to considering the special case of operators of the form

$$L^* = D_t^2 + b(t, x, D_t, D_x) \tag{3.2}$$

where $b$ is a first order operator. Setting $a(t, x, \xi) = b(t, x, 0, \xi)$, the assumptions of Theorem 1.3 become that there is a point $(t_0, x_0, \xi_0)$ at which $\operatorname{Re} a < 0$ and $\operatorname{Im} a$ changes sign. Suppose that $t_0 = 0$, and $x_0 = 0$. Using the reasoning of Lemma 5.1 in Cardosa-Treves [3], it may be further assumed that

$$\operatorname{Im} a(t, x, \xi) = t^k c(t, x, \xi) \tag{3.3}$$

where $C(t, x, \xi) > 0$ and $k$ is odd.

Approximate solutions of $L^* u = 0$ will be sought of the form

$$U(x, t, \lambda) \sim \exp\left(i\lambda x\xi_0 - \lambda^{\frac{1}{2}}\beta(x, t)\right) \sum_{0\leqslant j} g_j(x, t)\,\lambda^{-j/2}\,. \tag{3.4}$$

For this purpose it is necessary to have the asymptotic expansion

$$L\{\exp\left(i\lambda x\xi - \lambda^{\frac{1}{2}}\beta(t, x)\right) g(x, t)\} = \sum_{j=0}^{\infty} C_j(x, t)\,\lambda_{1-j/2} \exp\left(i\lambda x\xi - \lambda^{\frac{1}{2}}\beta(x, t)\right). \tag{3.5}$$

The $C_j(x, t)$ depend on derivatives of $\beta$ and $g$ of order $\leqslant j$. The first two are

$$C_0 = \left(a(t, x, \xi) - \beta_t^2\right) g \tag{3.6}$$

and

$$C_1 = 2\beta_t g_t + (\beta_{tt} + ib_\tau(x, t, \xi, 0)\beta_t + i\sum b_{\xi_j}\beta_{x_j})g\,. \tag{3.7}$$

To obtain (3.5), recall the expansion for an $m$-th order operator $a(x, D)$ applied to a rapidly oscillation function given by:

$$a(x, D)\{\exp(i\lambda x\xi)g(x)\} \sim \sum \lambda^{m-|\alpha|} a^{(\alpha)}(x, \xi) D^\alpha g/\alpha \tag{3.8}$$

where the error made by breaking the sum after $|\alpha| < N$ falls off like

$$O(\lambda^{-N} \sup_{|\alpha| \leqslant N} |D^\alpha g|)\,. \tag{3.9}$$

(See Hörmander [7], Theorem 2.6.) If in (3.8), $g$ is replaced by $\exp(-\lambda^{\frac{1}{2}}\beta)g$, the error after $N$ terms given by (3.9) is $O(\lambda^{-N/2})$, i.e., the expansion is still asymptotic. Using (3.8) and adding in the $t$-derivatives gives (3.5).

For the expansion (3.4), $\beta(x, t)$ will be chosen as

$$\beta(x, t) = \int_0^t a(t', x, \xi)^{\frac{1}{2}}\, dt' + x^2 \tag{3.10}$$

which makes $C_0 = 0$ in (3.5). From (3.3) and Lemma 1.5, it follows that

$$\operatorname{Re}\beta(x, t) \geqslant ct^{k+1} + x^2 \tag{3.11}$$

if the square root with positive real part when $t > 0$ chosen in (3.10). With this choice of $\beta$, if $L^*$ is applied to (3.4) and (3.5) is used, then $L^*(u(x, t, \lambda)) \sim 0$ is equivalent to

$$2\beta_t g_{j,t} + (\beta_{tt} + i\beta_\tau\beta_t + ib_{\xi_j}\beta_{x_j})g_j = F_j(x, t) \qquad j = 0, 1, \ldots \tag{3.12}$$

where $F_j$ depends on $g_0, \ldots, g_{j-1}$. Functions of compact support $g_j$ may be found so that $g_0(0, 0) = 1$, $g_j(0, 0) = 0$ for $j > 0$ and (3.12) is satisfied in some fixed neighborhood of the origin. The approximate solutions constructed in this way satisfy

$$\sup_{l+|\alpha| \leqslant N} |D_x^\alpha D_t^l L^* u(\lambda)| \leqslant C\lambda^{-M} \tag{3.13}$$

for any choice of $M$ and $N$. Inequality (3.13) holds near $(0, 0)$ because of (3.12), and away from $(0, 0)$ because of (3.11). Choosing $f_\lambda(x, t) = f(\lambda x)g(\lambda t)$ where $f, g \in C_0^\infty$ with $\hat{f}(\xi_0) \neq 0$ and $g(0) = 1$, and putting $f_\lambda$ and $u(\lambda)$ in (3.1) then leads to a contradiction in the usual way. This proves Theorem 1.3.

## 4. – Operators with vanishing subprincipal symbols.

Operators will now be treated which have the same form as those above except that the subprincipal symbols will be allowed to vanish. Let $P(x, D)$ be an $m$-th order classical pseudo-differential operator of $\Omega \subset \boldsymbol{R}^n$. At a point $(x_0, \xi_0)$ at which the principal symbol $P_m(x, \xi)$ vanishes, it will be assumed that there is a conic neighborhood $\Gamma$ of $(x_0, \xi_0)$ such that

(4.1) *$P_m(x, \xi) = Q(x, \xi)\big(U(x, \xi)\big)^2$ in $\Gamma$ where $Q$ and $U$ are homogeneous of degree $m-2$ and 1 respectively, both are real, $Q$ is elliptic in $\Gamma$, $d_\xi U \neq 0$ in $\Gamma$ and $Q > 0$; and*

(4.2) *in $\Gamma$, the equations $P_m(x, \xi) = 0$ and $P^s_{m-1}(x, \xi) = 0$ define a smooth manifold $\Sigma$ of codimension 2 such that $H^j P^s_{m-1} = 0$ on $\Sigma \cap \Gamma$ if $j < k$ an $H^k_u P^s_{m-1} \neq 0$ on $\Sigma \cap \Gamma$.* ($H_a b$ denotes the Hamiltonian vector field of $a$ applied to $b$ and equals $\{a, b\}$ the Poisson bracket of $a$ and $b$.)

Letting $c = H^{k-1}_u P^s_{m-1}$, note that on $\Sigma$, $u = c = 0$ and $\{u, c\} \neq 0$. It follows that the differentials of $c$ and $u$ are independent and that $\{c = u = 0\} = \Sigma$ in $\Gamma$ since both varieties have the same co-dimension. That $\{u, c\} \neq 0$ means that $H_u$ is transverse to the manifold $c = 0$ (shrinking $\Gamma$ if necessary). Let $v$ be the solution of $H_u v = 1$ and $v = 0$ on $c = 0$. Find a canonical transformation so that $\tau = u(x, \xi)$ and $t = v(x, \xi)$ are canonical coordinates in the new system. Changing notation, call $(x, \xi) \in \boldsymbol{R}^{2n}$ the remaining variables, also change $n$ to $n+1$ as the dimension of $\Omega$. Factoring $Q(x, D)$ out of $P$ and applying an elliptic Fourier integral operator corresponding to the above canonical transformation, the result is an operator of the form

$$K = D_t^2 + b(t, x, D_t, D_x) \tag{4.3}$$

where

$$b(t, x, \tau, \xi) = t^k a(t, x, \xi) + \tau c(t, x, \tau, \xi)\,, \tag{4.4}$$

$a \neq 0$ is homogeneous of degree 1, and $c$ homogeneous of degree 0. Constructing a parametrix for $P$ may then be reduced to constructing one for

$$L = D_t^2 + t^k a(t, x, D_x) \tag{4.5}$$

in a conic neighborhood of $(0, x_0, 0, \xi_0)$.

As was done in the first section, consider $L$ to be a pseudo-differential operator in $x-\xi$ acting on functions of $t$. Constructing a parametrix for $L$ may be attempted by finding an approximation of the Green's function of the operator $\mathfrak{L}=D_t^2+t^k a(t,x,\xi)$, considering $x$ and $\xi$ as parameters. For this purpose approximate solutions of $\mathfrak{L}u=0$ are needed. Trying a solution of the form

$$u(t)=g(t,x,\xi)\,V(\varphi(t,x,\xi)) \tag{4.6}$$

where $V(s)$ is a solution of

$$V''-s^k V=0 \tag{4.7}$$

it is seen that if

$$\varphi(t,x,\xi)=\left((k+2)/2\int_0^t t'^{k/2}a(t',x,\xi)^{\frac12}\,dt'\right)^{2/k+2} \tag{4.8}$$

and

$$g(t,x,\xi)=\varphi_t^{-\frac12} \tag{4.9}$$

then

$$\mathfrak{L}u=-(d^2g/dt^2)\,V(\varphi)\,. \tag{4.10}$$

Notice that if $a(t,x,\xi)$ is expanded as

$$a(t,x,\xi)=a_0(x,\xi)+a_1t^l+O(t^{l+1}) \tag{4.11}$$

then a simple calculation will show that

$$\varphi(t,x,\xi)=a_0^{1/(k+2)}t\{1+(2/(k+2l+2)(a_1/a_0)\,t^l+O(t^{l+1})\} \tag{4.12}$$

and that $g(t)=a_0^{1/(k+2)}(1+O(t))$ and is a smooth function. Letting $V_1$ and $V_2$ be two independent solutions of (4.7), and $u_1$ and $u_2$ be defined as in (4.6), another simple calculation will show that the Wronskian

$$W(u_1,u_2)=W(V_1,V_2)=c\neq 0\,. \tag{4.13}$$

Suppose that there are two independent solutions of (4.7) such that $u_1(t,x,\xi)$ is exponential decreasing as $t\to-\infty$, exponentially increasing as $t\to+\infty$ and vice-versa for $u_2$. Set

$$\begin{cases} e(x,\xi,t,s)=u_1(t,x,\xi)\,u_2(s,x,\xi)/c & \text{if } t\leqslant s\,,\\ e(x,\xi,s,t) & \text{if } t\geqslant s\,,\end{cases} \tag{4.14}$$

and

$$E(x, \xi) f(t) = \int_{-\infty}^{+\infty} e(x, \xi, t, s) f(s) \, ds . \tag{4.15}$$

It may be verified that

$$\mathfrak{L} E = I + R \tag{4.16}$$

where $R$ is an integral operator with the kernel

$$r(x, \xi, t, s) = -\big(g(t, x, \xi)\big)^{-1} (d^2 g/dt^2) e(x, \xi, t, s) . \tag{4.17}$$

$E(x, D)$ will then be a candidate for a parametrix of $L$. The exponential growth and decay condition is needed to be able to localize $E$ as in section 1.

Before proceeding further, the behavior of solutions $V(z)$ of (4.7) as $z$ moves along the curve $z = a_0^{1/k+2} t + O\big(|\xi|^{1/k+2} t^2\big)$ in the complex plane must be analyzed. First the behavior of $V(z)$ on lines will be studied.

LEMMA 4.1. *For any non-trivial solution $V$ of* (4.7), $V\big(\exp(i\theta) t\big)$ *cannot be exponentially decreasing as* $|t| \to \infty$ *if* $(k+2)\theta \not\equiv 0 \bmod \pi$.

PROOF. Making the substitution $z = \exp(-i\theta) s$, the lemma resolves into whether

$$V'' - a z^k V = 0 \tag{4.18}$$

can have an exponentially decreasing solution on the real axis if $\operatorname{Im} a \neq 0$. Multiply (4.18) by $\bar{V}$, forming the integral over the reals, integrating by parts, and separating real and imaginary parts gives

$$\int_{-\infty}^{\infty} |V'|^2 + \operatorname{Re} a z^k |V|^2 \, dz = 0 \quad \text{and} \quad \operatorname{Im} a \int_{-\infty}^{\infty} t^k |V|^2 \, dz = 0 .$$

If $\operatorname{Im} a \neq 0$, this shows that if $V$ is exponential decreasing solution of (4.18) then $V \equiv 0$, which proves the lemma.

It now remains to analyze what happens when $a_0$ is real. Note that $V(s) = s^{\frac{1}{2}} W_{1/(k+2)}\big(2 s^{(k+2)/2}/(k+2)\big)$ is a solution of

$$V'' + s^k V = 0 \tag{4.19}$$

if $W_\nu$ is any solution of Bessel's equation

$$z^2 W'' + zW' + (z^2 - \nu^2) W = 0 . \tag{4.20}$$

Suppose that $k > 0$ is even. One solution of (4.20) is Hankel's function $H_\nu^{(1)}(z)$ whose asymptotic expansion for $|\arg z| < \pi$ as $|z| \to \infty$ starts with $cz^{-\frac{1}{2}} \exp(iz)$. (The notation for Bessel functions as well as the necessary formulas to be used can be found in [10], especially sections 3.1.2 and 3.14.1) As $s$ changes from positive to negative real values $z = 2s^{(k+2)/2}/(k+2)$ increases its argument by $(k+2)\pi/2$. Using the formula

$$H_\nu^{(1)}(z \exp(im\pi)) = -\frac{\sin[(m-1)\pi\nu]}{\sin(\pi\nu)} \cdot H_\nu^{(1)}(z) - \exp(-i\pi\nu) \frac{\sin(\pi\nu m)}{\sin \pi\nu} H_\nu^{(2)}(z) \tag{4.21}$$

for integral $m$, it may be seen that

$$H_{1/(k+2)}^{(1)}\big(z \exp(i(k+2)\pi/2)\big) = aH_{1/k+2}^{(1)}(z) + bH_{1/k+2}^{(2)}(z)$$

where both $a$ and $b$ are not 0. The first term of the expansion of $H_\nu^{(2)}(z)$ is $cz^{-\frac{1}{2}} \exp(-iz)$. This yields the following conclusion: there is a solution $V$ of (4.19) which is exponentially decreasing along $t \exp(i\theta)$ as $t \to \infty$ if $\theta > 0$ and small and $V$ is exponentially increasing along $t \exp(i\theta)$ as $t \to \infty$ if $\theta \neq 0$ is near $\pi$. Taking the conjugate of $V$ gives a solution with similar growth but which decays along $\arg z < 0$ and small. If $k$ is even and $a > 0$, it is elementary to see that an exponentially decaying solution of (4.18) on one side of the real axis must have the opposite behavior on the other side. From the asymptotic solutions of (4.20), it follows that this behavior continues to hold in a conic neighborhood of the real axis.

Next suppose that $k$ is odd. One solution of (4.7) is

$$V(s) = K_{1/(k+2)}\big(2s^{(k+2)/2}/(k+2)\big)$$

where $K_\nu$ is a modified Bessel function which for $|\arg z| < 3\pi/2$ satisfies $K_\nu(z) \sim cz^{-\frac{1}{2}} \exp(-z)$ as $|z| \to \infty$. To study what happens when $s$ changes from $+$ to $-$, i.e., $\arg z$ increases by $(k+2)\pi/2$ use the formula

$$K_\nu(t \exp[im\pi]) = \exp(-im\nu\pi) K_\nu(z) - i\pi \frac{\sin(m\pi\nu)}{\sin(\pi\nu)} I_\nu(z) \tag{4.22}$$

with $m = (k+1)/2$ and the asymptotic expansion of $K_\nu$ and $I_\nu$ to obtain that

$$K_{1/k+2}(z \exp[i(k+2)\pi/2]) = z^{-\frac{1}{2}}(a \exp(iz) + b \exp(-iz))$$

where $a \neq 0$ and $b \neq 0$. From the last equation it follows that when $k$ is odd, (4.7) has a solution which is exponentially decreasing in a conic neighborhood of the positive reals and exponential increasing along $0 < |\arg z - \pi| <$ a small number. The above considerations may be summarized as

LEMMA 4.2. *If $k$ is even and $a$ is real, then*

(i) *$V'' - a^2 t^k V = 0$ has two independent solutions $V_1$ and $V_2$ such that*

$$V_1(t) \sim t^{-k/4} \exp(-2at^{(k+2)/2}/k+2) \qquad \text{as } t \to +\infty \tag{4.23}$$

$$V_2(t) \sim t^{-k/4} \exp(-2a(-t)^{(k+2)/2}/k+2) \qquad \text{as } t \to -\infty \tag{4.24}$$

(ii) *$V'' + a^2 t^k V = 0$ has two independent solutions $V_1^{\pm}$ and $V_2^{\pm}$ such that*

$$V_1^{\pm}(t) \sim t^{-k/4} \exp(\pm 2iat^{(k+2)/2}/k+2) \qquad \text{as } t \to +\infty \tag{4.25}$$

$$V_2^{\pm}(t) \sim t^{-k/4}(\exp(2ia(-t)^{(k+2)/2}/k+2) + c \exp(-2ia(-t)^{(k+2)/2}/k+2) \tag{4.26}$$

*as $t \to -\infty$ where $c \neq 0$ ;*

*If $k$ is odd and $a$ is real, then $V'' - a^2 t^k V = 0$ has two independent solutions $V_1$ and $V_2$ such that*

$$V_1(t) \sim t^{-k/4} \exp(-2at^{(k+2)/2}/k+2) \qquad \text{as } t \to +\infty, \tag{4.27}$$

$$V_2(t) \sim t^{-\frac{1}{4}}\left[\exp(2ai(-t)^{(k+1)/2}/k+2) + c \exp(-2ai(-t)^{(k+2)/2}/k+2)\right] \tag{4.28}$$

*as $t \to -\infty$ with $c \neq 0$.*

Equations (4.23)-(4.28) are asymptotic in the sense that the quotient of both sides of an equation is equal to $1 + O(|t|^{-(k+2)/2})$. The asymptotic formulas are derived from those for Bessel functions and consequently are valid in a conic neighborhood of the real axis.

The study of $E(x, \xi)$ may now be continued. In this section, let $I = [-T, T]$ be an interval on the $t$-axis, with $T$ to be specified later, and define $H^s(\xi)$ to be the completion of $C_0^\infty(I)$ with respect to the norm

$$\|u\|_s = \|(1 + |\xi| + |D_t|)^s u\|$$

and $B^s(\xi)$ to be the completion of $C_0^\infty(I)$ with the norm

$$\|u\|_{B^s(\xi)} = \|(1 + |\xi|^{2/(k+2)} + |D_t|^2 + |t^k\xi|)(1 + |\xi| + |D_t|)^s u\|$$

The symbol of $E(x, \xi)$ must be modified slightly to make it microlocal. Change the kernel defined by (4.14) by multiplying it by $\zeta(x, \xi)\chi_1(t)\chi_2(s)$ where $\zeta \in S^0_{1,0}$ is 1 in a conic neighborhood of $(x_0, \xi_0)$ and has its support in another conic neighborhood $\Gamma$, and $\chi_1, \chi_2 \in C_0^\infty(I)$, $\chi_2 = 1$ in a neighborhood of the support of $\chi_1$. The symbol identity (4.16) then becomes

$$\mathfrak{L}E = \zeta(x, \xi)\chi_1(t) + R \bmod S^{-\infty} \tag{4.29}$$

where $R$ has kernel (4.17) multiplied by $\zeta\chi_1\chi_2$. It is easy to see that $\mathfrak{L} \in S^0_{1,0}(\Gamma; B^s(\xi), H^s(\xi))$. The symbol class to which $E$ belongs is identified in

LEMMA 4.3. *If $a(0, x_0, \xi_0)$ is not real, or if $k$ is even and $a(0, x_0, \xi_0) > 0$, then for some sufficiently narrow conic neighborhood $\Gamma$ of $(x_0, \xi_0)$, $E(x, \xi)$ $\varepsilon$ $S^0_{1,0}(\Gamma; H^s(\xi), B^s(\xi))$.*

PROOF. Using Lemmas 4.1 and 4.2, there are solutions $V_1$ and $V_2$ of (4.7) so that the corresponding $u_1$ and $u_2$ given by (4.6) satisfy

$$u_1(t, x, \xi) \sim c_+(|\xi|^{1/k+2}t)^{-k/4} \exp\left(\int_0^t q(t, x, \xi)^{\frac{1}{2}}\, dt'\right) \tag{4.30}$$

for $|\xi|^{1/k+2}t > O(1)$, and

$$u_2(t, x, \xi) \sim c_-(|\xi|^{1/k+2}t)^{-k/4} \exp\left(\int_0^t q(t', x, \xi)^{\frac{1}{2}}\, dt'\right) \tag{4.31}$$

for $|\xi|^{1/k+2}t < -O(1)$ where $q(t, x, \xi) = t^k a(t, x, \xi)$, and the square root is taken to have positive real part. The expressions for $u_2$ are the same except there is a minus sign in the exponential and there are different functions $c_\pm(x, t, \xi)$ which in both cases are bounded functions of all their arguments.

First the $L^2$ norm of $E$ will be estimated using Lemma 1.7. To do this it is necessary to have the following variant of Lemma 1.6 which is proved in the same way as that lemma.

LEMMA 4.4. *Suppose that $a(t, x, \xi)$ has a zero of order $l$ in $t$ at $(0, x_0, \xi_0)$, then there is a conic neighborhood $\Gamma$ of $(x_0, \xi_0)$, an interval $I$ containing 0, and* *a constant $C$ such that*

$$c|\xi|^{\frac{1}{2}}|t-s|^{(k+l+2)/2} \leqslant \int_t^s |t'^k a(t', x, \xi)|^{\frac{1}{2}}\, dt' \tag{4.32}$$

*holds for $t, s \in I$ and $(x, \xi) \in \Gamma$.*

To apply Lemma 1.7 in this situation, an estimate is required for the expression

$$\int_t^\infty |u_1(t, x, \xi)\, u_2(s, x, \xi)|\, ds \tag{4.33}$$

and three others which look similar. Three cases will need to be considered.

In the first case, suppose that $|\xi|^{1/k+1} t \geqslant O(1)$. Using (4.30) and the corresponding bound for $u_2$ gives that the integrand in (4.33) is bounded by

$$O(1)|\xi|^{-1/k+1}\big(|\xi|^{2/k+1} ts\big)^{-k/4} \exp -\Big(\int_t^s q(t', x, \xi)^{\frac{1}{2}}\, dt'\Big)\,. \tag{4.34}$$

Applying Lemma 4.4 with $l = 0$ gives that

$$c|\xi|^{\frac{1}{2}}|t-s|^{\frac{1}{2}(k+2)} \leqslant \operatorname{Re}\int_t^s (q(t', x, \xi))^{\frac{1}{2}}\, dt'\,. \tag{4.35}$$

Using the last inequality in (4.34), it follows that expression (4.33) is less than

$$C|\xi|^{-1/k+1}\int_t^\infty \exp\,(-c|\xi|^{\frac{1}{2}}|t-s|^{(k+2)/2}\, ds = C'|\xi|^{-2/(k+2)}\,. \tag{4.36}$$

In the second case, suppose that $|\xi^{1/k+2} t| \leqslant O(1)$. When $|\xi|^{1/k+2}|t| \leqslant O(1)$, then $|u_j(t, x, \xi)| \leqslant O\big(|\xi|^{1/(k+2)}\big)$. Consequently,

$$\int_t^{O(|\xi|^{-1/k+2})} |u_1(t, x, \xi)\, u_2(s, x, \xi)|\, ds \leqslant O\big(|\xi|^{-2/(k+2)}\big)\,. \tag{4.37}$$

The part of (4.33) when $s \geqslant O\big(|\xi|^{-1/k+2}\big)$ contributes

$$C|\xi|^{-1/k+2}\int_{O(|\xi|^{-1/k+2})}^\infty \exp\Big(-\int_0^s q(t')^{\frac{1}{2}}\, dt'\Big)\, ds \leqslant C|\xi|^{-1/k+2}\int_{O(|\xi|^{-1/k+2})}^\infty \exp\,\big(-\,(|\xi|^{\frac{1}{2}} s^{(k+2)/2}\big)\, ds \leqslant C|\xi|^{-2/k+2}\,.$$

In the third case, suppose that $|\xi|^{1/k+2}t\leqslant -O(1)$. The integral in (4.33) may be broken up into three parts: the first integrating over $t\leqslant s\leqslant \leqslant -O(|\xi|^{-1/k+2})$, the second over $|s|\leqslant O(|\xi|^{-1/k+2})$, and the third over $O(|\xi|^{-1/k+2})\leqslant s$. Reasoning as above it may be shown that each part contributes a term $\leqslant O(|\xi|^{-2/k+2})$.

Combining the above three cases gives the estimate

$$\|E(x,\xi)u\|\leqslant C|\xi|^{-2/k+2}\|u\| . \tag{4.38}$$

Next it will be shown that

$$\|t^k|\xi|E(x,\xi)u\|\leqslant C\|u\| . \tag{4.39}$$

For this, it is necessary to obtain estimates on the operator whose kernel is $t^k|\xi|e(x,\xi,t,s)$. The argument will be divided into the same three cases which were used to show (4.38).

In the first case,

$$\int_t^\infty |\xi||t^k u_1(t,x,\xi)u_2(s,x,\xi)|ds \tag{4.40}$$

has an integrand less than

$$|\xi|^{\frac12}t^{k/2}(t/s)^{k/4}\exp\left(-C|\xi|^{\frac12}\int_t^s|t'|^{k/2}dt'\right) . \tag{4.41}$$

Using the inequality

$$|t/s|^{k/4}\leqslant C\left(\frac{1+|\xi|^{1/k+2}|t|}{1+|\xi|^{1/k+2}|s|}\right)^{k/4}\leqslant C(1+|\xi|^{1/k+2}|t-s|)^{k/4} \tag{4.42}$$

which is valid for $s,t\geqslant O|\xi|^{-1/k+2}$, it follows that the fraction $(t/s)^{k/4}$ may be absorbed into the exponential part of (4.41). Another variant of Lemma 1.6 says that for any polynomial $p(t)$ of degree less than a fixed bound, there is a constant $C$ such that

$$C|p(t)|^{\frac12}|t-s|\leqslant\int_t^s|p(t')|^{\frac12}dt' . \tag{4.43}$$

Applying (4.43) to the exponential gives that (4.40) is bounded by

$$|\xi|^{\frac12}t^{k/2}\int_t^\infty \exp(-c|\xi|^{\frac12}|t-s|)\,ds=1/c . \tag{4.44}$$

Case two is trivial since when $|t| \leqslant O(|\xi|^{1/k+2})$, the added factor $|\xi t^k| \leqslant O|\xi|^{2/k+2}$, and the estimate in case two above shows that (4.40) is less than a constant. Case three is similar to the two previous ones. Applying Lemma 1.7 once more yields (4.39). From (4.29), it follows using (4.28) and (4.39) that $D_t^2 E$ is bounded on $L^2(I)$. Together this gives

$$\|(1 + |D_t|^2 + |\xi t^k| + |\xi|^{1/k+2}) E(x, \xi) u\| \leqslant C\|u\| . \tag{4.45}$$

Estimates of derivatives of the symbol will follow inductively from (4.29). For example, apply $D_x^\alpha$ to (4.29). It is seen that $\mathfrak{L} D_x^\alpha E =$ a bounded operator on $L^2 + D_x^\alpha R$. Differentiating $R$ by $x$ brings down at worst in case one above a factor bounded by $|\xi|^{\frac{1}{2}} \int_t^s |t'|^{k/2}\, dt'$ to the kernel of $R$ which can be absorbed into the exponential part of the kernel of $R$. Note that the kernels of $R$ and $E$ are similar. Consequently, $D_x^\alpha R$ is bounded on $L^2$. Take an operator $E'$ which has a kernel like $E$ except with a cutoff function $\varphi(t, x, \xi)$ which is on the support of $\zeta(x, \xi)\chi_1(t)$ and such that $E'\mathfrak{L} = \varphi(t, x, \xi) + R'$. Then applying $E'$ on the left to the result of (4.29) differentiated gives $E'\mathfrak{L} D_x^\alpha E = D_x^\alpha E + R' D_x^\alpha E + E' \circ$a bounded operator which equals a bounded operator from $H(\xi)$ to $B(\xi)$. Consequently, $D_x^\alpha E$ is a bounded operator from $H(\xi)$ to $B(\xi)$. It is also not difficult to get similar bounds on $E$ and its derivatives as operators from $H^s$ to $B^s$, which will complete the proof of Lemma 4.3. Likewise it may be shown that

$$R \in S^0_{1,0}(\Gamma;\, H^s(\xi),\, B^s(\xi)).$$

Lemma 4.3 may now be used to compute the composition of $L$ and $E(x, D_x)$. The symbol of $D_t^2$ composed with $E(x, D_x)$ may be obtained exactly by composing $D_t^2$ with the symbol of $E$. Considering $t^k a(t, x, D_x)$ to be in $L^1_{1,0}(\Gamma; B^s, B^s)$, it follows that its composition with $E$ is the operator with symbol $t^k a(t, x, \xi) E(x, \xi)$ plus an operator in $L^0_{1,0}(\Gamma; H^s, B^s)$. Combining the last two observations with (4.29) gives that

$$L \circ E(x, D_x) = \chi_2(t)\zeta(x, D_x) + R' \tag{4.46}$$

where $R' \in L^0_{1,0}(\Gamma; H^s, B^s)$, i.e., $(1 + |D_t|^2 + |t^k D_x| + |D_x|^{2/k+2}) R'$ is a bounded operator on $H_s$. Since $K$ differs from $L$ by a bounded operator time $D_t$, it may be concluded that

$$KE = \chi_2(t)\zeta(x, D_x) + R'' \tag{4.47}$$

where $(1+|D_t|+|t^kD_x|^{\frac{1}{2}}+|D_x|^{1/k+2})R$ is a bounded operator on $H_s$. The above observations may be summed up in

PROPOSITION 4.5. *Suppose that* $K$ *has the form of* (4.3), *and that at* $(0, x_0, \xi_0)$, *either* $a(0, x_0, \xi_0)$ *is not real or* $k$ *is even and* $a(0, x_0, \xi_0) > 0$, *then there is a conic neighborhood* $\Gamma$ *of* $(x_0, \xi_0)$ *and an interval* $I = [-T, T]$ *such that*

(1) *there are operators* $E_i$ *and* $R_i$, $i = 1, 2$ *and* $\psi \in L_0(I\times\Gamma)$ *satisfying*

$$E_1K = \psi(t, x, D_x) + R_1 \tag{4.48}$$

$$KE_2 = \psi(t, x, D_x) + R_2 \tag{4.49}$$

*such that* $\psi(0, x_0, \xi_0) \neq 0$, $(1+|D_t|^2+|t^kD_x|+|D_x|^{2/k+2})E_i$ *and* $(1+|D_t|+ +|t^kD_x|^{\frac{1}{2}}+|D_x|^{1/k+2})R_i$ *are bounded operators on* $H_s$ *for* $i = 1, 2$;

(2) *there are* $\psi_1$ *and* $\psi_2 \in L^0(I\times\Gamma)$ *with* $\psi_1(0, x_0, \xi_0) \neq 0$, $\psi_2 \neq 0$ *on the support of* $\psi_1$ *such that for any* $s$ *there is a constant* $C$ *such that*

$$\|\psi_1(1+|D_t|^2+|t^kD_x|+|D_x|^{2/k+2})u\|_s \leqslant C\|\psi_2Ku\|_s + \|\psi_2|D_t|u\|_s + \|u\|_s \tag{4.50}$$

*is valid for any* $u \in C_0^\infty(I\times\mathbf{R}^n)$;

(3) $K$ *is hypoelliptic near* $(0, x_0, \xi_0)$.

The construction of a left parametrix for $K$ is analogous to the construction of a right parametrix. Hypoellipticity and estimate (4.50) are immediate consequences of equation (4.48).

The conclusion of Proposition 4.3 may be translated back to the operator $P$ resulting in

THEOREM 4.6. *Suppose that* $P$ *is an* $m$*-th order pseudo-differential operator on* $\Omega \subset \mathbf{R}^n$ *such that at every point* $(x_0, \xi_0) \in \Omega\times(\mathbf{R}^n\setminus 0)$ *at which the principal symbol* $P_m$ *of* $P$ *vanishes,* (4.1) *and* (4.2) *are satisfied and in addition, at* $(x_0, \xi_0)$, *either* $H_u^kP_{m-1}^s$ *is not real or it is positive and* $k$ *is even. Then,*

(1) $P$ *is hypoelliptic*;

(2) *for any compact subset* $K$ *of* $\Omega$, *and real number* $s$, *there is a constant* $C$ *such that the estimate*

$$\|u\|_{s+m-2+2/(k+2)} \leqslant C(\|Pu\|_s + \|u\|_{s+m-2}), \quad u \in C_0^\infty(K) \tag{4.51}$$

*holds.*

(3) *$P$ has right and left paramatrices which are bounded operators from $H_s$ to $H_{s+m-2+2/(k+2)}$.*

Full use of Lemma 4.2 has not yet been many. Returning to $L$ of the form (4.5), suppose that it satisfies the assumption

(4.52) *$a(0, x_0, \xi_0) \neq 0$ and real, $\operatorname{Im} a(t, x_0, \xi_0)$ has a first order zero in $t$ at 0, and there is a neighborhood of $(0, x_0, \xi_0)$ of the form $I \times \Gamma$ where $I = [-T, T]$ and $\Gamma$ is a conic neighborhood of $(x_0, \xi_0)$ such that (a) if $k$ is odd then $\operatorname{Im} a(t, x, \xi)$ does not change sign in $\{t \in I: a(0, x_0, \xi_0)t < 0\} \times \Gamma$, or (b) if $k$ is even, $a(0, x_0, \xi_0) < 0$ and $\operatorname{Im} a(t, x, \xi)$ does not change sign in both the sets $[0, T] \times \Gamma$ and $[-T, 0] \times \Gamma$.*

If $L$ satisfies (4.52), it then follows from Lemma 4.2 that it is possible to find solutions $V_1$ and $V_2$ of (4.7) so that the corresponding $u_1$, $u_2$ given by (4.6) satisfy (4.30) and (4.31) for $|\xi|^{1/k+2}|t| \geqslant O(1)$, $i = 1$ and its analogue for $i = 2$, the square root of $q$ having non-negative real part in (4.30), etc. Note that as a consequence of (4.52), $a(t, x, \xi)$ has either the form $\operatorname{tg}(t, x, \xi)$ with $g \neq 0$ if $k$ is even or $h(x, \xi) + \operatorname{tg}(t, x, \xi)$ with $gh > 0$ if $k$ is odd. Consider the symbol $E(x, \xi)$ with kernel defined by (4.14) multiplied by $\zeta(x, \xi)\chi_1(t)\chi_2(\xi)$ as above. It will be shown that $E$ is a parametrix for $L$ and that $L$ is hypoelliptic. The first step is to show

LEMMA 4.7. *Suppose that $a$ satisfies assumption (4.52) in $I \times \Gamma$, then for any pair of multi-indices $\alpha$ and $\beta$ there is a constant $C$ such that*

$$\|D_x^\alpha D_\xi^\beta E(x, \xi) u\|_s \leqslant C|\xi|^{-1/(k+2)-1/(k+4)+\delta|\alpha|-\varrho|\beta|}\|u\|_s \tag{4.53}$$

*for $u \in C_0^\infty(I)$ where $\|u\|_s = \|(1 + |\xi| + |D_t|)^s u\|$, and*

$$\varrho = (1 + 1/(k+4))/2 > \delta = (1 - 1/(k+4))/2 . \tag{4.54}$$

PROOF. The lemma will be proved only for the case $s = 0$. The norm of $E$ will be estimated with the aid of Lemma 1.7. For $\alpha = \beta = 0$, it is required to estimate the integral

$$I = \int_t^\infty \chi_1(t)\chi_2(s)\zeta(x, \xi)|u_1(t, x, \xi)u_2(s, x, \xi)|ds$$

and three other analogous expressions. The estimation will be split into three cases: that when $t \geqslant 0(|\xi|^{1/k+2})$, when $|t| \leqslant O(|\xi|^{1/k+2})$, and when $t \leqslant -O(|\xi|^{1/k+2})$.

In case 1, the integrand of $I$ may be bounded by

$$C|\xi|^{-\frac{1}{2}}|ts|^{-k/4}\exp\left(-\int_t^s q^{\frac{1}{2}}(t', x, \xi)\,dt'\right)$$
$$\leqslant |\xi|^{-1/k+2}\exp\left(-|\xi|^{\frac{1}{2}}\int_t^s |t'|^{k/2+1}\,dt'\right).$$

Lemma 1.5 has been used together with (4.52) to estimate the real part of $q^{\frac{1}{2}}$. Making use of Lemma 4.4 gives that

$$I \leqslant C|\xi|^{-1/k+2}\int_t^\infty \exp\left(-|\xi|^{\frac{1}{2}}|t-s|^{(k+4)/2}\right) ds$$
$$\leqslant C|\xi|^{-1/(k+2)-1/(k+4)}.$$

In case 2, the integral between $t$ and $O(|\xi|^{-1/k+2})$ contributes $O(|\xi|^{-2/k+2})$ to $I$. When $s \geqslant O(|\xi|^{-1/k+2})$, the integrand is less than

$$C|\xi|^{-1/k+2}\exp\left(-|\xi|^{\frac{1}{2}}s^{(k+4)/2}\right),$$

which when integrated over $s \geqslant O(|\xi|^{-1/(k+2)})$ adds $|\xi|^{-1(k+2)-1/(k+4)}$ to $I$.

The third case is similar to the first two. Combining the estimates of all three cases show (4.53) when $\alpha = \beta = 0$.

Applying $D_x^\alpha D_\xi^\beta$ to $E(x, \xi)$ adds the factor

$$|\xi|^{(|\alpha|-|\beta|)/2}|t-s|^{(\alpha+\beta)}$$

to the integrand of $I$ in case 1. Each factor of $|t-s|$ may be absorbed into the exponential adding a factor of $|\xi|^{-1/2(k+4)}$ to $I$. Since $u_i(t) = V_i(\varphi(t, x, \xi))$ applying $D_x^\alpha D_\xi^\beta$ adds the factor $|\xi|^{|\alpha|/k+2-k|\beta|/k+2}$ to $I$ when $|t|, |s| \leqslant O(|\xi|^{-1/lk+2})$. Combining these observations with the first part of the proof establishes (4.53) in general.

It may be noted that the proof of Lemma 4.7 doesn't require that $\operatorname{Im} a(t, x, \xi)$ has a first order zero, but only that $\operatorname{Im} a(t, x, \xi)$ has constant sign in the same regions as considered in condition (4.52). When $\operatorname{Im} a(t, x, \xi)$ has a zero of order $l$ in $t$ at $(0, x_0, \xi_0)$ it may be shown that

$$\|D_x^\alpha D_\xi^\beta E(x, \xi)\| \leqslant C|\xi|^{-1/(k+2)-1/(k+2+2l)+\delta'|\alpha|-\varrho'|\beta|} \tag{4.55}$$

where

$$\varrho' = (1 + 1/(k+2l+2))/2 > \delta' = (1 - 1/(k+2l+2))/2$$

LEMMA 4.8. *If $a(t, x, \xi)$ satisfies the assumptions of Lemma 4.7, then for any pair of multi-indices $\alpha$, $\beta$ there is a constant $C$ such that*

$$\| |\xi t^k|^{\frac{3}{4}} D_x^\alpha D_\xi^\beta E(x, \xi) u \|_s \leqslant C |\xi|^{k/(2(k+4)(k+2)) + \delta|\alpha| - \varrho|\beta|} \|u\|_s \tag{4.56}$$

*for $u \in C_0^\infty(I)$ where $\varrho$ and $\delta$ are given by* (4.54).

PROOF. The proof will be restricted to only the case that $s = 0$ and $\alpha = \beta = 0$. As in the proof of the last lemma, it will be sufficient to estimate the integrals

$$I = \int_t^\infty |t^k \xi|^{\frac{3}{4}} \chi_1(t) \chi_2(s) \zeta(x, \xi) |u_1(t, x, \xi) u_2(s, x, \xi)| ds$$

and

$$J = \int_s^\infty |t^k \xi|^{\frac{3}{4}} \chi_1(t) \chi_2(s) \zeta(x, \xi) |u_1(s, x, \xi) u_2(t, x, \xi)| dt$$

and two other similar expressions.

Considering $I$ first, the estimation will be divided into the same three cases used for Lemma 4.7. When $t \geqslant O(|\xi|^{1/k+2})$, the integrand of $I$ may be bounded by

$$|\xi t^k|^{\frac{1}{4}} |t/s|^{k/4} \exp\left(- |\xi|^{\frac{1}{2}} \int_t^s \tau^{k/2+1} d\tau\right).$$

From inequality (4.42) and the inequality

$$|t - s|^{(k+2l+2)/2} \leqslant \int_t^s |\tau|^{k/2+1} d\tau \tag{4.57}$$

it follows that the factor $|t/s|^{k/4}$ may be absorbed in the exponential at the expanse of adding the factor $|\xi|^{k/2(k+2)(k+4)}$ to the integrand. The remaining part of $I$ is bounded by

$$\int_t^\infty |\xi t^k|^{\frac{1}{4}} \exp\left(- C |\xi|^{\frac{1}{2}} t^k |t - s|^2\right) ds \leqslant O(1)$$

having used the inequality

$$c |t|^{k/2} |t - s|^2 \leqslant \int_t^s |\tau|^{k/2+1} d\tau. \tag{4.58}$$

Consequently, in the first case

$$|I| \leqslant C|\xi|^{k/2(k+2)(k+4)} . \tag{4.59}$$

In the second case, when $|t| \leqslant O(|\xi|^{1/k+2})$ then $|\xi t^k|^{\frac{3}{4}} \leqslant O(|\xi|^{3/2(k+2)})$. The remaining integral was estimated in case 2 of Lemma 4.7 to be less than $|\xi|^{-1/(k+2)-1/(k+4)}$ and so $|I| \leqslant O(1)$ in this case. The third case is similar to the first two.

The considerations for $J$ will be divided into three cases similar to those above: $s \geqslant O(|\xi|^{-1/k+2})$, etc. In the first case when $s \geqslant O(|\xi|^{-1/k+2})$ the integrand of $J$ is bounded by

$$|\xi t^k|^{\frac{1}{4}} |t/s|^{k/4} \exp - c|\xi|^{\frac{1}{2}} \int_s^t \tau^{k/2+1}\, d\tau .$$

Absorbing the factor $|t/s|^{k/4}$ into the exponential as above gives

$$J \leqslant |\xi|^{k/2(k+2)(k+4)} \int_s^\infty |\xi t^k|^{\frac{1}{4}} \exp\left(- c|\xi|^{\frac{1}{2}} \int_t^s \tau^{k/2+1}\, d\tau\right) dt .$$

Replacing $s$ and $t$ by $|\xi|^{-1/(k+4)}s$ and $|\xi|^{-1/(k+4)}t$, the above integral becomes

$$\int_s^\infty t^{k/4} \exp\left(- c\int_s^t \tau^{k/2+1}\right) d\tau .$$

That the last integral is bounded independently of $s$, may be seen by applying Hölder's inequality to obtain

$$\int_s^\infty t^{k/2} \exp\left(- \int_s^t \tau^{(k/2)+1}\, d\tau\right) dt \leqslant \left( \int_s^\infty t^{(k/2)+1} \exp\left(- \int_s^t \tau^{(k/2)+1}\, d\tau\right) dt\right)^{k/(k+2)} \cdot$$

$$\cdot \left( \int_s^\infty \exp\left(- \int_s^t \tau^{(k/2)+1}\, d\tau\right) dt\right)^{2/k+2} \leqslant \int_0^\infty \exp\left(- c|t|^{(k+4)/2}\right) dt \leqslant O(1) .$$

The other two cases may be treated similarly to those for $I$. Altogether these estimates give

$$|J| \leqslant C|\xi|^{k/2(k+2)(k+4)} ,$$

which combined with (4.59) proves the lemma.

Lemma 4.8 may be improved to yield the estimate

$$\left\| (1 + |\xi t^k|^{\frac{1}{2}} + |D_t|)^{\frac{1}{2}} D_x^\alpha D_\xi^\beta E(x, \xi) u \right\|_s \leqslant C|\xi|^{k/2(k+2)(k+4) + \delta|\alpha| - \varrho|\beta|} \|u\|_s . \tag{4.60}$$

This may be seen by multiplying the identity

$$D_t^2 E = t^k a(t, x, \xi) E + \chi\zeta + R$$

by $(1 + |\xi t^k| + |D_t|^2)^{-\frac{1}{4}}$ to get the estimate

$$\begin{aligned} (4.61) \qquad & \|(1 + |\xi t^k| + |D_t|^2)^{-\frac{1}{4}} D_t^2 E u\|_s \\ & \leqslant \|(1 + |\xi t^k| + D_t^2)^{-\frac{1}{4}} |\xi t^k| E u\|_s + C\|u\| \\ & < \|\{(1 + |\xi t^k| + D_t^2)^{-\frac{1}{4}} |\xi t^k|^{\frac{1}{4}}\} |\xi t^k|^{\frac{3}{4}} E u\|_s + C\|u\|_s \\ & < C|\xi|^{k/2(k+2)(k+4)} \|u\|_s \end{aligned}$$

(Operators with symbols based on $1 + |\xi t^k| + |D_t|^2$ were essentially studied in Lemma 4.3 and Proposition 4.5.) Inequality (4.60) is obtained by adding the inequality

$$\|(1 + |\xi t|^k + D_t^2)^{\frac{1}{4}} |\xi t^k| E u\|_s \leqslant \| |\xi t^k|^{\frac{3}{4}} E u\|_s \leqslant C|\xi|^{k/2(k+2)(k+4)} \|u\|_s$$

to (4.61).

The estimate (4.60) may be combined with Lemma 4.7 to give

$$(4.62) \qquad \|\Lambda^{\frac{3}{2}} E u\|_s < C|\xi|^{k/2(k+2)(k+4)} \|u\|_s$$

where

$$\Lambda = 1 + |\xi|^{(5k+12)/3(k+2)(k+4)} + |\xi t^k|^{\frac{1}{2}} + |D_t| \, .$$

But since

$$\|\Lambda^{3k/2(5k+12)} w\|_s \geqslant |\xi|^{k/2(k+2)(k+4)} \|w\|_s$$

(4.62) implies

$$\|\Lambda^{(6k+18)/(5k+12)} E u\|_s \leqslant C\|u\|_s \, .$$

The last inequality together with similar ones for $D_x^\alpha D_\xi^\beta E$ may be restated as saying that $E$ is in the symbol class $S^0_{\varrho,\delta}(\Gamma; H^s(\xi), V^s(\xi))$ where $V^s(\xi)$ is the Hilbert space on functions on $I$ with norm

$$\|u\|_{V^s(\xi)} = \|\Lambda^{(6k+18)/(5k+12)} u\|_{H^s(\xi)}$$

and $\varrho$ and $\delta$ are given by (4.54).

The composition of $L$ and $E$ may now be computed. Composing $t^k a(t, x, D_x)$ with $E(x, D_x)$ results in

$$(4.63) \qquad (t^k a E)(x, D_x) + t^k (\partial_\xi a)(D_x E)(x, D_x) \bmod L^0_{\varrho,\delta}(\Gamma; H^s, V^s) \, .$$

But since $D_x E(x,\xi) \in S^{\frac{1}{2}}_{\varrho,\delta}(\Gamma; H^s, V^s)$, it follows that $t^k(D_x E)(x, D_x)$ is a bounded operator from $H_s$ to $\Lambda^{(k+6)/(5k+12)} H_s$. Combining these last observations with the symbol identity (4.29) leads to

$$L \cdot E = \chi_2(t)\xi(x, D_x) + R' \tag{4.64}$$

where

$$\left(1 + |D_x|^{(5k+12)/3(k+2)(k+4)} + |t^k D_x|^{\frac{1}{2}} + |D_t|\right)^{(k+6)/(5k+12)} R' . \tag{4.65}$$

is a bounded operator of $H_s$. Since $|D_t| E$ is also an operator satisfying (4.65), this shows

$$K \circ E = \chi_2(t)\zeta(x, D_x) + R'' \tag{4.66}$$

where $R''$ satisfies (4.65). This shows that $E$ is a right parametrix for $K$. A left parametrix may be constructed in a similar fashion. The above arguments may be summed up in

THEOREM 4.9. *If $a(t, x, \xi)$ satisfies condition* (4.52) *at* $(0, x_0, \xi_0)$ *and* $K$ *is given by* (4.3), *then*

(i) *$K$ is hypoelliptic near* $(0, x_0, \xi_0)$

(ii) *$K$ has right and left parametrices near* $(0, x_0, \xi_0)$

(iii) *$K$ satisfies the estimate,*

$$\|\varphi u\|_{s+(k+6)/3(k+2)(k+4)} \leqslant C'\left(\|Ku\|_s + \|u\|_s\right), \qquad u \in C_0^\infty(\Omega) \tag{4.67}$$

*where $\varphi \in S^0_{1,0}$ having its support in a sufficiently small conic neighborhood of* $(0, x_0, \xi_0)$.

Theorem 4.9 may easily be translated into a hypoellipticity result for the original operator $P$. When $\operatorname{Im} a$ has zero of order greater than one there is the weaker result of

PROPOSITION 4.10. *Suppose that $M = D_t^2 + a(t, x, D_x)t^k + ib(t, x, D_x)t^{k+l}$ in $I \times \Omega \subset \mathbf{R}^{n+1}$, where $a$ and $b$ are first order elliptic operators with real symbols, and $l \leqslant k + 2$, then $M$ is locally solvable.*

To prove the proposition, let $E(x, \xi)$ symbol be constructed as above. As remarked, $E$ satisfies (4.55). (It is also easy to see that $|D_t|^{\frac{1}{2}} E$ is bounded, but this is not strong enough to conclude the hypoellipticity or even the local solvability of $M$.) If it can be shown that $E \circ t^k a(t, x, D_x) = (Et^k a)(x, D_x) + a$ small operator, it would follow from the analogue of (4.63) that

$$(4.68)\qquad \|(1+|D_x|)^{(1/k+2)+1/(k+2+2l)}uMu\|_s, \qquad u\in C_0^\infty(\omega)$$

if the $x$-diameter of $\omega$ is small enough. From (4.67) the local solvability of $M^*$ would follow, and also that of $M$ since $M^*$ satisfies the hypotheses of the proposition.

Since $a(x,t,\xi)t^k\in S^1$, it follows from (4.55) that

$$(4.69)\qquad E\circ a(x,t,D_x)t^k=(Eat^k)(x,D_x)+t^k(\partial_\xi E)(D_x a)(x,D_x)$$

modulo an operator in $L_{\varrho',\delta'}^{-1/k+2l+2}(\Omega;H^s,H^s)$. For $s\geqslant t\geqslant O(|\xi|^{-1/k+2})$, the kernel of the integral operator which is the symbol of $t^k\,\partial_\xi E\cdot D_x a$ may be bounded by

$$(4.70)\qquad |t/s|^{k/4}\int_t^s\tau^{k/2}\,d\tau\exp\Big(-|\xi|^{\frac12}\int_t^s\tau^{k/2+l}\,d\tau\Big)\,.$$

To estimate one contribution to the $L^2$-norm of $t^k\,\partial_\xi E\cdot D_x a$, the integral from $t$ to $\infty$ of (4.70) with respect to $s$ needs to be estimated. In the range of $s$ and $t$ being discussed, inequality (4.42) holds. Absorbing the factor $(t/s)^{k/4}$ into the exponential of (4.69) adds a factor of $|\xi|^{kl/2(k+2)(k+2l+2)}$. Using Hölder's inequality gives

$$\int_t^s\tau^{k/2}\,d\tau\leqslant\Big(\int_t^s\tau^{k/2+l}\,d\tau\Big)^{k/k+2l}|t-s|^{2l/k+2l}\,.$$

If the above expression is absorbed into the exponential of (4.70), the result is decreased by more than the factor $|\xi|^{-k/2(k+2l)}$. The integral of (4.70) is consequently bounded by

$$C|\xi|^{-1/(k+2l+2)+kl/2(k+2)(k+2l+2)-k/l(k+2l)}\,.$$

If $l\leqslant k+2$, then $kl/2(k+2)(k+2l+2)-k/2(k+2l)\leqslant 0$ which gives that the integral of (4.70) is bounded by $O(|\xi|^{-1/(k+2l+2)})$. Similar bounds may be obtained for the other contributions. These will show that

$$t^k\,\partial_\xi E\cdot D_x a\in S_{\varrho',\delta'}^{-1/k+2l+2}(\Omega\times\boldsymbol{R}^n,H^s,H^s)$$

which is a small operator. This completes a sketch of the proof of Proposition 4.10.

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