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ANDREW JOHN SOMMESE

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Real Algebraic Spaces.

ANDREW JOHN SOMMESE (*)

It is well-known [cf. § 1 for definitions] that if two holomorphic vector bundles on a complex manifold X are topologically equivalent on a submanifold $\mathfrak{R}$ without complex tangents then there is a Stein neighborhood of $\mathfrak{R}$ in X on which they are holomorphically equivalent. This article treats the algebraic analogue of this fact.

In § 1 notation and background material are collected.

In § 2 it is shown that two algebraic vector bundles on a Zariski neighborhood U of the real points $X_{\boldsymbol{R}}$ of a projective analytic space X defined over $\boldsymbol{R}$, are algebraically equivalent on a possibly smaller Zariski open set $V \supseteq X_{\boldsymbol{R}}$ if they are topologically equivalent on $X_{\boldsymbol{R}}$. Various extensions of this result are given.

It is shown that the set of algebraic sections of an algebraic bundle E over a compact real algebraic space X is dense, for $1 \leqslant k \leqslant \infty$, in the space of C^k topology.

In § 3 it is shown that given a compact Kaehler manifold X with an anti-holomorphic involution and with a trivial canonical bundle, one has:

$$0 \to H^0(X, \Omega_X^q) \to H({}^q\mathcal{C}, \boldsymbol{C})$$

where $0 \leqslant q \leqslant \dim_{\boldsymbol{C}} X$, $\mathcal{C}$ is any connected component of the real points of X, and Ω_X^q is the q-th exterior power of the holomorphic cotangent sheaf.

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(*) The Institute for Advanced Study, Princeton, New Jersey 08540 and Cornell University, Ithaca, New York.

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§ 1. – In this section background material is collected and notation fixed. *All analytic spaces are assumed reduced.*

If $\mathcal{S}$ is an analytic coherent sheaf on an analytic space X, then $\Gamma_h(X, \mathcal{S})$ the set of sections of $\mathcal{S}$ over X possesses a functorial Fréchét space structure that coincides with the compact-open topology when $\mathcal{S}$ is the locally free sheaf associated to a holomorphic vector bundle on X [cf. 4, Chapter 8]. If $f\colon Y \to X$ is a holomorphic map where Y is an analytic space, then the natural map $f^*\colon \Gamma_h(X, \mathcal{S}) \to \Gamma_h(Y, f^*\mathcal{S})$ is continuous. If $\phi\colon \mathcal{S} \to \mathcal{T}$ is an $\mathcal{O}_X$ linear sheaf map where $\mathcal{T}$ is an analytic coherent sheaf over X, then the natural map $\phi_*\colon \Gamma_h(X, \mathcal{S}) \to \Gamma_h(X, \mathcal{T})$ is continuous.

If X is a quasi-projective analytic space (i.e. a Zariski open set of a projective analytic space) and $\mathcal{S}$ is an algebraic coherent sheaf on X, then $\Gamma_a(X, \mathcal{S})$ denotes the space of sections of $\mathcal{S}$ on X. Regarding X as an analytic space and letting $\mathcal{S}_h$ be the analytic coherent sheaf associated to $\mathcal{S}$ one has the injective map:

$$i\colon \Gamma_a(X, \mathcal{S}) \to \Gamma_h(X, \mathcal{S}_h)\,.$$

By an affine algebraic space X one will simply mean an algebraic subspace of $\mathbf{C}^N$ with the induced reduced algebraic structure sheaf $\mathcal{O}_X$. The associated analytic space, also denoted X, with the analytic structure sheaf ${}_a\mathcal{O}_X$ is a Stein space.

One has the following useful lemma of Cornalba and Griffiths:

LEMMA I-*A*. *Let X be an affine algebraic space and $\mathcal{F}$ an algebraic coherent sheaf on X. Then $\Gamma_a(X, \mathcal{T})$ is dense in $\Gamma_h(X, \mathcal{T}_h)$.*

PROOF. – One has X algebraically embedded in $\mathbf{C}^N$. Let $\mathcal{E}$ be a locally free algebraic coherent sheaf in $\mathbf{C}^N$ and $\lambda\colon \mathcal{E} \to \mathcal{F}$ a $\mathcal{O}_{\mathbf{C}^N}$ linear surjective sheaf map where $\mathcal{F}$ is regarded as a sheaf on $\mathbf{C}^N$. Consider the commutative diagram:

$$\begin{array}{ccc}
\Gamma_a(\mathbf{C}^N, \mathcal{E}) & \xrightarrow{\lambda} & \Gamma_h(\mathbf{C}^N, \mathcal{E}_h) \\
\downarrow & & \downarrow{\scriptstyle r} \\
\Gamma_a(\mathbf{C}^N, \mathcal{F}) & \to & \Gamma_h(\mathbf{C}^N, \mathcal{F}_h) \\
\Vert & & \Vert \\
\Gamma_a(X, \mathcal{F}) & \to & \Gamma_h(X, \mathcal{F}_h)\,.
\end{array}$$

Now $\lambda(\Gamma_a(\boldsymbol{C}^N, \mathcal{E}))$ is dense in $\Gamma_h(\boldsymbol{C}^N, \mathcal{E}_h)$ by [2, Prop. 17.1]. Now r is a surjection since $\boldsymbol{C}^N$ is a Stein space, and thus the image of $r\circ\lambda$ is dense in $\Gamma_h(X, \mathcal{F}_h)$. Q.E.D.

The following is quite useful in conjunction with the above:

RUNGE'S THEOREM. *Let X be a Stein space and let $\varphi: X \to \boldsymbol{R}$ be an at least C^2 strictly plurisubharmonic exhaustion function* [4, pg. 273 ff]. *The restriction map $r: \Gamma_h(X, \mathcal{S}) \to \Gamma_h(X_c, \mathcal{S}|_{X_c})$ has a dense image where $\mathcal{S}$ is an analytic coherent sheaf on X and $X_c = \{x \in X | \varphi(x) < c\}$ for a real number c.*

PROOF. Let d be a real number $c < d < \infty$. X_d is relatively compact in X and by basic Stein space theory one can choose global sections $\{s_1, \ldots, s_N\}$ of $\mathcal{S}$ such that they span $\mathcal{S}|_{X_d}$ over the structure sheaf $\mathcal{O}_{X_d}$ of X_d. Now let $\mathcal{F}$ be the rank N free sheaf on X; one the above has an $\mathcal{O}_X$ linear sheaf map $\mathcal{F} \to \mathcal{S}$ which is surjective on X_d. One has the commutative diagram

$$\begin{array}{ccc}
\Gamma_h(X, \mathcal{S}) & \longleftarrow & \Gamma_h(X, \mathcal{F}) \\
\downarrow & & \downarrow{\scriptstyle r_2} \\
\Gamma_h(X_d, \mathcal{S}|_{X_d}) & \leftarrow & \Gamma_h(X_d, \mathcal{F}) \\
\downarrow & & \downarrow{\scriptstyle r_1} \\
\Gamma_h(X_c, \mathcal{S}|_{X_c}) & \leftarrow & \Gamma_h(X_c, \mathcal{F})
\end{array}$$

Now the lower two horizontal arrows are surjective since X_d and X_c are Stein spaces [4, pgs. 275-276] and higher cohomology groups of coherent sheaves vanish. Further r_1 has dense image by [4, pg. 275]. Thus if one shows that r_2 has dense image then since r_1 and r_2 are continuous $r_1 \circ r_2$ will have dense image and the theorem will follow.

Since $\mathcal{F}$ is a direct sum of trivial line bundles it suffices to prove this for holomorphic functions. Let $\{d_i\}$ for $i = 1, 2, 3, \ldots$ be a sequence of real numbers such that $c < d_1 < d_2 < \ldots$ and $d_i \to \infty$. Now given an $\varepsilon > 0$, a holomorphic function f on X_c and a compact set $K \Subset X_c$ one can find a holomorphic function f_1 on X_{d_1} such that $\sup\limits_K |f - f_1| < \varepsilon/2$ [4, pg. 275]. Similarly one can find a sequence $\{f_j\}$ with f_j holomorphic on X_{d_j} and $\sup\limits_{X_{d_{j-1}}} |f_j - f_{j+1}| < \varepsilon/2^{j+1}$. Thus f_j converges to a global holomorphic function g on X. Note $\sup\limits_K |f - g| < \varepsilon$. Q.E.D.

The following is one of Grauert's theorems on holomorphic vector bundles on Stein spaces [cf. 2, : 19-20 for a nice summary with proofs].

GRAUERT'S THEOREM. *Let X be a Stein space and let E be a holomorphic vector bundle on X. If E is topologically trivial then E is holomorphically trivial. Given any differentiable complex vector bundle F on X, then there is a holomorphic bundle on X with the underlying differentiable vector bundle structure of F.*

I need the concept of a submanifold without complex tangents of a complex manifold and a theorem about it due to Range and Siu [7]. The implications of the result for algebraic geometry will become clear in the next section.

Recall [6, II-9.2], that given a complex manifold X one has an associated almost complex structure J on T_X, the real tangent bundle of X. J is a fiberwise linear map of T_X to itself such that $J^2 = -I$ where I is the identity on T_X.

DEFINITION. *Let X be a complex manifold and M a C^k submanifold where $1 \leqslant k \leqslant \infty$. M is said to be without complex tangents if given $m \in M$, then $J(T_{M|m}) \cap (T_{M|m}) = \{m\}$ where $\{m\}$ is identified with the origin of $T_{M|m}$, the real tangent space of M at m.*

PROPOSITION [Range-Siu]: *Let $1 \leqslant k \leqslant \infty$ and let M be a C^k submanifold without complex tangents in a complex manifold X. Then there exists a Stein open neighborhood U of M in X such that the restrictions to M of all holomorphic sections of any holomorphic vector bundle E on U are dense in the Frechet space of all C^k sections of $E|_M$.*

PROOF. Range and Siu prove this for the trivial bundle but the above extension is easy.

First note that one can find a holomorphic vector bundle F on U such that $E \otimes F'$ is topologically trivial and then use Grauert's theorem above.

Next note given any section of E on M one can lift it to $E \oplus F$, use [7] to approximate and then project down to E. Q.E.D.

Associated to any analytic space X is an analytic space X'', the conjugate analytic space. The topological space of X'' is the same as X and the holomorphic structure sheaf of X'' consists of the conjugates of the elements of the holomorphic structure sheaf of X. If X is a complex manifold, then the holomorphic transition functions of X'' are simply the conjugates of the holomorphic transition functions of X. *A conjugation $\sigma: X \to X$ is an anti-holomorphic involution, i.e. a map σ, whose square is the identity and which is holomorphic when considered as a map from X to X''.*

LEMMA I-*B*. *Let $\sigma: X \to X$ be a conjugation of a connected complex manifold X, with fixed point set $X_{\mathbf{R}}$. If $X_{\mathbf{R}}$ is non-empty, then each connected* *component is a submanifold of X without complex tangents with real dimension equal to* $\dim_C X$.

PROOF. X_R is a manifold since it is the fixed point set of the finite group $\{1, \sigma\}$ where $1: X \to X$ is the identity transformation.

Associated to the complex structure on X one has an almost complex structure J. J is a fiberwise linear C^∞ transformation $J: T_X \to T_X$ with $J^2 = -I$ where T_X is the real tangent bundle of X. To say σ is antiholomorphic is equivalent to saying $d\sigma \circ J = -J \circ d\sigma$. At a point $x \in C$, a connected component of X_R, this implies that J interchanges the plus and minus one eigenspace of $d\sigma|_x$. Thus $\dim_R \mathcal{C} = \dim_C X$, and further $(J|_x)(T_{\mathcal{C}}|_x) \cap (T_{\mathcal{C}}|_x) = \{x\}$ for each $x \in \mathcal{C}$. This is equivalent to $\mathcal{C}$ and hence X_R being totally real submanifolds of X, i.e. having no complex tangents. Q.E.D.

It follows trivially from the last paragraph that $J|_{X_R}$ gives an isomorphism between T_{X_R}, the real tangent bundle of X_R, and N_{X_R}, the normal bundle of X_R in X.

It should be noted that if X is singular then X_R might have components of different dimensions. For example C with the conjugation $\sigma(z) = \bar{z}$ has the real line as fixed pointset. Now consider the analytic space C^ν gotten from C by identifying $\sqrt{-1}$ and $-\sqrt{-1}$ defining a germ of a function at the new point $\{\sqrt{-1}, -\sqrt{-1}\}$ as germs of functions f and g at $\sqrt{-1}$ and $-\sqrt{-1}$ respectively such that $f(\sqrt{-1}) = g(-\sqrt{-1})$. The involution σ descends to the analytic space C^ν but C^ν_R the fixed point set is the real line and the point $\{\sqrt{-1}, -\sqrt{-1}\}$.

X_R can be empty as the conjugation on CP^1 given by $\sigma(z) = -1/\bar{z}$ shows.

If X is a compact connected complex manifold with conjugation σ, then [1, pg. 64] $X_R \times X_R$ is unoriented cobordant to X. This implies that X_R is non-empty if some Pontryagin number of X is odd. For example if $\dim_C X = 2n$, then $\dim_C H^n(X, \Lambda^n \Omega^1_X)$ being odd implies by means of the Hodge decomposition that the Euler characteristic of X is odd and hence by the above that X_R is non-empty for any conjugation of X.

If X is quasi-projective it is not hard to see (cf. Lemma I-C' below) that X'' has a quasi-projective structure also. A holomorphic conjugation σ on a quasi-projective analytic space is said to be algebraic if $\sigma: X \to X''$ is algebraic. If X is a projective analytic space one can see from Chow's lemma that every holomorphic conjugation is algebraic.

LEMMA I-C. *Let σ be an algebraic conjugation of a quasi-projective analytic space X with fixed point set X_R. There exists an algebraic embedding Φ from X*

into CP^N *for some* N *such that*:

a) $\Phi \circ \sigma = \tau \circ \Phi$ *where* $\tau\colon CP^N \to CP^N$ *is the conjugation*

$$(z_0, \ldots, z_N) \to (\bar{z}_0, \ldots, \bar{z}_N),$$

b) *a hyperplane* H *invariant under* τ *can be chosen so that*

$$X_{\boldsymbol{R}} \subseteq X - \Phi^{-1}(H);$$

c) *if* X *is a projective manifold, then the non-primitive cohomology of* $H^*(X, \boldsymbol{C})$ *with respect to the Kaehler class that is Poincaré dual to* H, *is in the kernel of the restriction map* $r\colon H^*(X, \boldsymbol{C}) \to H^*(X_{\boldsymbol{R}}, \boldsymbol{C})$.

PROOF. $\sigma\colon X \to X''$ is algebraic and thus one has an algebraic embedding $\varphi = (1, \sigma)\colon X \to X \times X''$ where 1 is the identity on X. On noting that $\varphi \circ \sigma = \sigma' \circ \varphi$ where $\sigma'(x, y) = (y, x)$, and that $\varphi^{-1}(\Delta \cap \varphi(X)) = X_{\boldsymbol{R}}$ where Δ is the diagonal of $X \times X''$ and $X_{\boldsymbol{R}}$ are the fixed points of σ, one sees that parts *a*) and *b*) of the lemma follow from:

LEMMA I-C'. *If* X *be a quasi-projective analytic space, then so is* X''. *Further there exists an algebraic embedding* $\Phi\colon X \times X'' \to CP^N$ *for some* N *such that* $\Phi(x, y) = \tau \circ \Phi(y, x)$ *and where there is some hyperplane* H *of* CP^N *such that* $\Phi^{-1}(H)$ *is disjoint from the diagonal of* $X \times X''$.

PROOF. Let L be a very ample line bundle on X, e.g. the pullback to X of the hyperplane section bundle of the CP^n in which it is assumed that X is embedded; i.e. $\varphi\colon X \to CP^n$. Now let $\{s_0, \ldots, s_n\}$ be the sections of L that give the homogeneous coordinates of CP^n restricted to X. Now $\bar{L}$ is the holomorphic line bundle on X'' with transition functions that are conjugate to those of L. $\{\bar{s}_0, \ldots, \bar{s}_n\}$ are sections of $\bar{L}$ that give rise to a holomorphic map $\bar{\varphi}\colon X'' \to CP^n$. The image of $\bar{\varphi}$ is the conjugate of the image of X under φ. The closure $\overline{\bar{\varphi}(X'')}$ of $\bar{\varphi}(X'')$ is the conjugate of the closure of $\varphi(X)$. Thus $\overline{\bar{\varphi}(X'')}$ is a projective analytic space and so is $\overline{\bar{\varphi}(X'')} - \bar{\varphi}(X'')$ since $\overline{\varphi(X)} - \varphi(X)$ is. This implies X'' is quasi-projective.,

Consider the map $\Phi\colon X \times X'' \to CP^{n^2+2n}$ given by

$$\Phi(x, y) =$$
$$= \left(\ldots, s_i(x) \otimes \bar{s}_j(y) + s_j(x) \otimes \bar{s}_i(y), \ldots, \sqrt{-1}\left(s_i(x) \otimes \bar{s}_j(y) - s_j(x) \otimes \bar{s}_i(y)\right), \ldots\right).$$

It is clearly an embedding since it is a composition of the embedding

$$B = (\varphi, \bar{\varphi})\colon X \times X'' \to \boldsymbol{C}P^n \times \boldsymbol{C}P^n ,$$

the Segre embedding of $\boldsymbol{C}P^n \times \boldsymbol{C}P^n \to \boldsymbol{C}P^{n^2+2n}$, and an automorphism of $\boldsymbol{C}P^{n^2+2n}$. Note that $\overline{\Phi(x, y)} = \Phi(y, x)$. The set

$$\mathcal{H} = \left\{ (x, y) \in X \times X'' \,\middle|\, \sum_i s_i(x) \otimes \bar{s}_i(y) = 0 \right\}$$

is the pullback of a hyperplane H of $\boldsymbol{C}P^{n^2+2n}$ to $X \times X''$. $\mathcal{H} \cap \Delta = \phi$ where Δ is the diagonal of $X \times X''$. Q.E.D.

Part *c*) of Lemma I-*C* follows from part *b*) and the definition of the primitive decomposition [9, pg. 75]. Q.E.D.

Let me now justify the title of this paper. Usually by a quasi-projective analytic space X defined over $\boldsymbol{R}$ one means a quasi-projective analytic space and an algebraic embedding of X into $\boldsymbol{C}P^N$ such that the set X is invariant under the natural conjugation of $\boldsymbol{C}P^N$. Another way of saying this is there exists an algebraic embedding $\Phi\colon X \to \boldsymbol{C}P^N$ such that $\Phi(X)$ and $\overline{\Phi(X)} - \Phi(X)$ are defined by the vanishing of homogeneous polynomials with real coefficients. This former is precisely what was shown.

If E is a holomorphic vector bundle over an analytic space X with conjugation $\sigma\colon X \to X$, then E is said to have a conjugation $\mathcal{L}$ defined over (or simply over) σ if there exists a conjugation $\mathcal{L}\colon E \to E$ of E as an analytic space such that the diagram:

$$\begin{array}{ccc} E & \xrightarrow{\mathcal{L}} & E \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\sigma} & X \end{array}$$

commutes where the vertical arrows are the bundle projections and $\mathcal{L}$ restricted to any fiber is conjugate linear.

Note the trivial bundle has a conjugation over σ.

If X is quasi-projective and σ is an algebraic conjugation then one says a pair $(E, \mathcal{L})$ is an algebraic vector bundle on X defined over σ if E is an algebraic vector bundle and $\mathcal{L}$ is an algebraic conjugation of E defined over σ.

It is easily seen by the reader acquainted with quasi-projective spaces X defined over $\boldsymbol{R}$ that every algebraic vector bundle on X defined over $\boldsymbol{R}$ gives rise to such a pair. A slight extension of the reasoning of Lemma I-*C* shows the converse also holds. I will not use either of these facts and thus will not prove them.

Finally, let me describe the C^k topology $(1 \leqslant k \leqslant \infty)$ on C^k sections of a C^k differentiable vector bundle E over a compact real algebraic space $X_{\boldsymbol{R}}$. The construction is entirely analogous to that of the topology of $\Gamma_h(X, \mathcal{S})$ mentioned in the second paragraph of this section. One can C^k embed $X_{\boldsymbol{R}}$ in a Grassmannian $\mathcal{G}$ such that E is the restriction of the universal bundle $\mathcal{E}$ on $\mathcal{G}$. The space of C^k sections of $\mathcal{E}$ restrict to give the space of C^k sections of E on $X_{\boldsymbol{R}}$: Put the quotient topology of the space of sections of $\mathcal{E}$. It is the usual argument to show this topology doesn't depend on the embedding and that it coincides with the usual C^k topology if $X_{\boldsymbol{R}}$ is a manifold.

§ 2. – In this section X will always denote a projective analytic space defined over $\boldsymbol{R}$ with conjugation σ and real points $X_{\boldsymbol{R}}$: *All spaces are as always assumed reduced.*

Let $\mathfrak{Z}(X_{\boldsymbol{R}})$ and $\mathfrak{Z}(X_{\boldsymbol{R}}, \sigma)$ be the sets of Zariski open sets and σ invariant Zariski open sets of X respectively that contain $X_{\boldsymbol{R}}$.

Let $\mathrm{Vect}(U)$ for $U \in \mathfrak{Z}(X_{\boldsymbol{R}})$ be the set of algebraic vector bundles on U. Let $\mathrm{Vect}(U, \sigma)$ for $U \in \mathfrak{Z}(X_{\boldsymbol{R}}, \sigma)$ be the set of algebraic vector bundles defined over σ; thus an element of $\mathrm{Vect}(U, \sigma)$ is a pair $(E, \mathfrak{L})$ as at the end of § 1. Given such an $(E, \mathfrak{L})$ let $E_{\boldsymbol{R}}$ be the real vector bundle over $X_{\boldsymbol{R}}$ left fixed by $\mathfrak{L}$ (note that $E_{\boldsymbol{R}} \otimes_{\boldsymbol{R}} \boldsymbol{C} \approx E|_{X_{\boldsymbol{R}}}$).

Now an algebraic section of $F|_{X_{\boldsymbol{R}}}$ for $F \in \mathrm{Vect}(U)$ is any section of $F|_{X_{\boldsymbol{R}}}$ that is the restriction of an algebraic section of $F|_V$ where $V \subseteq U$ and $V \in \mathfrak{Z}(X_{\boldsymbol{R}})$. Denote these sections by $\Gamma_a(X_{\boldsymbol{R}}, F|_{X_{\boldsymbol{R}}})$. Similarly an algebraic section of $E_{\boldsymbol{R}}$ for $(E, \mathfrak{L}) \in \mathrm{Vect}(U, \sigma)$ where $U \in \mathfrak{Z}(X_{\boldsymbol{R}}, \sigma)$ is a section of $E_{\boldsymbol{R}}$ that is the restriction of any section s of E over V where $V \subseteq U$, $V \in \mathfrak{Z}(X_{\boldsymbol{R}}, \sigma)$, and $\overline{\mathfrak{L}(s)} = s$; denote these sections by $\Gamma_a(X_{\boldsymbol{R}}, E_{\boldsymbol{R}})$.

PROPOSITION I *If $F \in \mathrm{Vect}(U)$ where $U \in \mathfrak{Z}(X_{\boldsymbol{R}})$ then $\Gamma_a(X_{\boldsymbol{R}}, F|_{X_{\boldsymbol{R}}})$ is dense in the C^k sections of $F|_{X_{\boldsymbol{R}}}$. If $(E, \mathfrak{L}) \in \mathrm{Vect}(U, \sigma)$ where $U \in \mathfrak{Z}(X_{\boldsymbol{R}}, \sigma)$ then $\Gamma_a(X_{\boldsymbol{R}}, E_{\boldsymbol{R}})$ is dense in the C^k sections of $E_{\boldsymbol{R}}$.*

PROOF. – I will prove the latter statement since the proof of the former is exactly the same but with a few steps less. By Lemma I-B there exists an embedding Φ of a σ invariant affine $V \supseteq X_{\boldsymbol{R}}$ into $\boldsymbol{C}^N$ such that $\Phi \circ \sigma = \overline{\Phi}$ and thus $\Phi(X_{\boldsymbol{R}}) \subseteq \boldsymbol{R}^N$. Let $z_i = x_i + (\sqrt{-1})\, y_i$ with $i = 1, \ldots, N$ be the usual coordinates on $\boldsymbol{C}^N$; with $\boldsymbol{R}^N = \{p \in \boldsymbol{C}^N | y_i(p) = 0, \forall i\}$. Note $\boldsymbol{C}^N \subseteq \boldsymbol{CP}^N$ where $\{w_0, \ldots, w_n\}$ are homogeneous coordinates on $\boldsymbol{CP}^N$, $\boldsymbol{C}^N = \{p \in \boldsymbol{CP}^N | w_0 \neq 0\}$, and $z_j = w_j / w_0$: Now $\varphi = \sum_{i=1}^{N} y_i^2$ defines a strongly pseudoconvex function on $\boldsymbol{C}^N$ and $\tilde{\varphi} = \varphi \circ \Phi$ on V. $\tilde{\varphi}$ is actually a non-negative exhaustion function on V that vanishes precisely on $X_{\boldsymbol{R}}$. To see it is an exhaustion function it suffices to show there is no closed set $\{p_n \in V | n = 1, 2, 3, \ldots\}$ with $\tilde{\varphi}(p_n) \leqslant C$ for some C independent of n but where $|x_i(p_n)| \to \infty$ for some index i. If this happened it is easily seen that there is some index j and infinitely many p_n' such that $|x_j(p_n')| \geqslant |x_\lambda(p_n')| \geqslant 0$ for all λ. Thus by rearranging the coordinates and renumbering the $\{p_n'\}$ one has that there exists a sequence $\{p_n\} \subseteq V$ such that $\tilde{\varphi}(p_n) \leqslant C$ independently of n and $|x_1(p_n)| \to \infty$ where $|x_1(p_n)| \geqslant |x_\lambda(p_n)| > 0$ for all λ and n. Now in homogeneous coordinates in $\boldsymbol{C}P^N$, p_n is

$$\left(1/x_1(p_n), 1 + (\sqrt{-1})\,(y_1(p_n)/x_1(p_n)), \ldots, (x_N(p_n) + \sqrt{-1}\,y_N(p_n))/x_1(p_n)\right).$$

The imaginary parts of these coordinates go to 0 since $\tilde{\varphi}(p_n) \leqslant C$ and for some subsequence q_n of the $\{p_n\}$ the real parts converge since they are bounded. This gives the absurd conclusion that $\overline{V} \cap \boldsymbol{R}P^N = X \cap \boldsymbol{R}P^N = X_{\boldsymbol{R}}$ contains a point in $\boldsymbol{R}P^N - \boldsymbol{R}^N$ and hence not in $X_{\boldsymbol{R}}$.

Now let B be a Stein neighborhood of $\boldsymbol{R}^N$ in $\boldsymbol{C}^N$ such that the Range-Siu theorem holds. Assume B is chosen so that $V \cap B \subseteq U$ and $\tilde{\varphi}(B \cap V)$ is bounded. Now let G be a holomorphic bundle on $B \cap X$ such that $F \oplus G$ is holomorphically trivial. This is clear differentiably-now use Grauert's theorem. Now $E \oplus G$ is the restriction of the trivial bundle $\mathcal{T}$ on B. Let s be a C^k section of $E|_{X_{\boldsymbol{R}}}$, then s has a C^k extension $\tilde{s}$ as a section of $\mathcal{T}$ on $\boldsymbol{R}^N$. Apply Range-Siu to get a holomorphic section of $\mathcal{T}$ on B that is C^k close to $\tilde{s}$. Thus by restriction to $B \cap X$ and projection to $E|_{B\cap X}$ one has a section of E that is C^k close to s on $X_{\boldsymbol{R}}$, i.e. the image of $\Gamma_h(B \cap V, \mathcal{E}_{h|B\cap V})$ is dense in $C^k(X_{\boldsymbol{R}}, E|_{X_{\boldsymbol{R}}})$ where $\mathcal{E}_h$ is the analytic coherent sheaf on V associated to the algebraic coherent sheaf $\mathcal{E}_a$ on V induced by algebraic sections of E and $C^k(X_{\boldsymbol{R}}, E|_{X_{\boldsymbol{R}}})$ is the C^k sections of $E|_{X_{\boldsymbol{R}}}$.

Now choose a $c \in \boldsymbol{R}$ such that $V_c = \{x \in V | \tilde{\varphi}(x) < c\} \subseteq B \wedge V$. Then Runge's theorem says $\Gamma_h(V, \mathcal{E}_h)$ is dense in $\Gamma_h(V_c, \mathcal{E}_{h|V_c})$ and hence by the last paragraph in $C^k(X_{\boldsymbol{R}}, E|_{X_{\boldsymbol{R}}})$. Now by Lemma I-$A$, $\Gamma_a(V, \mathcal{E})$ is dense in $\Gamma_h(V, \mathcal{E}_h)$ and hence in $C^k(X_{\boldsymbol{R}}, E|_{X_{\boldsymbol{R}}})$.

Now there is a natural continuous map from $\Gamma_a(V, \mathcal{E})$ into $C^k(X_{\boldsymbol{R}}, E_{\boldsymbol{R}})$. Namely $f \to [f + \overline{\mathcal{L}(f)}]/2$. This factors through the surjective $C^k(X_{\boldsymbol{R}}, E|_{X_{\boldsymbol{R}}}) \to$ $\to C^k(X_{\boldsymbol{R}}, E_{\boldsymbol{R}})$ given by the same formula. But $\Gamma_a(V, \mathcal{E})$ was just shown to be dense in $C^k(X_{\boldsymbol{R}}, E|_{X_{\boldsymbol{R}}})$ and hence in $C^k(X_{\boldsymbol{R}}, E_{\boldsymbol{R}})$. Q.E.D.

Now for $U \in \mathfrak{Z}(X_{\boldsymbol{R}}, \sigma)$ let $(L, \tau) \in \mathrm{Vect}(U, \sigma)$. Define $\mathrm{Vect}(U, \sigma, L, \tau)$ as the set of triples $(E, \mathcal{L}, q)$ where $(E, \mathcal{L}) \in \mathrm{Vect}(U, \sigma)$ and q is a non-degenerate symmetric bilinear pairing $q\colon E \otimes E \to L$ such that $q \circ (\mathcal{L} \otimes \mathcal{L}) = \tau \circ q$.

Note that elements of $\mathrm{Vect}(U, \sigma)$ for $U \in \mathfrak{Z}(X_{\boldsymbol{R}}, \sigma)$ give rise to well defined elements of $\mathrm{Vect}_{\boldsymbol{R}}(X_{\boldsymbol{R}})$ the differentiable real vector bundles on $X_{\boldsymbol{R}}$.

To see this simply note that given $(E, \mathfrak{L}) \in \mathrm{Vect}(U, \sigma)$ and any $x \in X_{\boldsymbol{R}}$, then $\mathfrak{L}\colon E|_x \to E|_x$ is a conjugation and thus the fixed points $E_{\boldsymbol{R}}$ over $X_{\boldsymbol{R}}$ are well defined. Note $E_{\boldsymbol{R}} \otimes_{\boldsymbol{R}} \boldsymbol{C} = E|x_{\boldsymbol{R}}$. Thus letting $\mathrm{Vect}_{\boldsymbol{C}}(X_{\boldsymbol{R}})$ denote the differential complex vector bundles in X– one has the commutative diagram

$$\begin{array}{ccc} \mathrm{Vect}(U, \sigma) & \longrightarrow & \mathrm{Vect}(U) \\ \downarrow & & \downarrow \\ \mathrm{Vect}_{\boldsymbol{R}}(X_{\boldsymbol{R}}) & \xrightarrow{\otimes_{\boldsymbol{R}}\boldsymbol{C}} & \mathrm{Vect}_{\boldsymbol{C}}(X_{\boldsymbol{R}}) \end{array}$$

where the vertical arrow on the right is restriction. Now if $\mathrm{Vect}_{\boldsymbol{R}}(X_{\boldsymbol{R}}, L_{\boldsymbol{R}})$ denotes the set of differentiable real vector bundles $E_{\boldsymbol{R}}$ on $X_{\boldsymbol{R}}$ with symmetric bilinear pairings $q\colon E_{\boldsymbol{R}} \otimes E_{\boldsymbol{R}} \to L_{\boldsymbol{R}}$ nondegenerate then one has a natural map from

$$\mathrm{Vect}'(U, \sigma, L, \tau) \to \mathrm{Vect}_{\boldsymbol{R}}(X_{\boldsymbol{R}}, L_{\boldsymbol{R}})\,.$$

These give rise to various maps in the limit over $U \in \mathfrak{Z}(X_{\boldsymbol{R}})$ and $U \in \mathfrak{Z}(X_{\boldsymbol{R}}, \sigma)$. For simplicity let $V(X_{\boldsymbol{R}}) = \varinjlim_{U \in \mathfrak{Z}(X_{\boldsymbol{R}})} \mathrm{Vect}(U)$, $V(X_{\boldsymbol{R}}, \sigma) = \varinjlim_{U \in \mathfrak{Z}(X_{\boldsymbol{R}}, \sigma)} \mathrm{Vect}(U, {}_m)$ and $V(X_{\boldsymbol{R}}, \sigma, L_{\boldsymbol{R}}) = \varinjlim_{U \in \mathfrak{Z}(X_{\boldsymbol{R}}, \sigma)} \mathrm{Vect}(U, \sigma, L|_U, \tau)$ where one has but an element $(L, \tau) \in \mathrm{Vect}(U', \mathrm{g})$ and one takes the limit only over $U \in \mathfrak{Z}(X_{\boldsymbol{R}}, \sigma)$ such that $U' \subset U$.

Evans [3] did proposition I with $k = 0$.

Corollary I. *With $X, \sigma X_{\boldsymbol{R}}$-as above and for (L, τ) a line bundle in* $\mathrm{Vect}(U, \sigma)$ *one has the following diagrams*:

$$0 \to V(X_{\boldsymbol{R}}) \longrightarrow \mathrm{Vect}_{\boldsymbol{C}}(X_{\boldsymbol{R}})$$

$$\begin{array}{ccc} 0 \to V(X_{\boldsymbol{R}}, \sigma, L_{\boldsymbol{R}}) & \to & \mathrm{Vect}_{\boldsymbol{R}}(X_{\boldsymbol{R}}, L_{\boldsymbol{R}}) \\ \downarrow & & \downarrow \\ 0 \to V(X_{\boldsymbol{R}}, \sigma) & \longrightarrow & \mathrm{Vect}_{\boldsymbol{R}}(X_{\boldsymbol{R}}) \end{array}$$

with exact rows. The above square expresses $V(X_{\boldsymbol{R}}, \sigma, L_{\boldsymbol{R}})$ as a fiber product of the two diagonal groups over $\mathrm{Vect}_{\boldsymbol{R}}(X_{\boldsymbol{R}})$, *i.e. given $y \in V(X_{\boldsymbol{R}}, \sigma)$ and $z \in \mathrm{Vect}_{\boldsymbol{R}}(X_{\boldsymbol{R}}, L_{\boldsymbol{R}})$ that have the same image in* $\mathrm{Vect}_{\boldsymbol{R}}(X_{\boldsymbol{R}})$, *then there exists a unique element $\mu \in V(X_{\boldsymbol{R}}, \sigma, L_{\boldsymbol{R}})$ that goes into y and z.*

Proof. Let E and F be elements of $V(X_{\boldsymbol{R}})$ that are the same over $X_{\boldsymbol{R}}$. Then one has a section s of $\mathrm{Hom}(E, F)|x_{\boldsymbol{R}}$ that gives an isomorphism of

$E|_{X_R}$ and $F|_{X_R}$. There exists a $U \in \mathfrak{Z}(X_R)$ on which both E and F are defined and by proposition I, there exists an algebraic section s of $\mathrm{Hom}(E, F)$ on U that is C^k close to s in X_R for any k one wants. Thus $\tilde{s}$ is an isomorphism in a Zariski neighborhood of X_R.

Now assume $(E, \mathfrak{L}) \in V(X_R, \sigma)$ and that one has a nondegenerate symmetric bilinear pairing $q: E_R \otimes E_R \to L_R$. Regard q as a linear map $q': E_R \to E_R^* \otimes L_R$. Now regard $E^* \otimes L$ with the involution $\mathfrak{L}^* \otimes \tau$ as an element of $V(X_R, \sigma)$. By the above one has a $U \in \mathfrak{Z}(X_R, \sigma)$ on which E and $E^* \otimes L$ are defined and on which there is an algebraic map λ' defined over σ between E and $E^* \otimes L$. $\lambda'|_{X_R}$ is C^k close to q' on X_R, when considered as a map from E_R to $E_R^* \otimes L_R$. Associated to λ' one has a not necessarily symmetric pairing $\lambda: E \otimes F \to L$ which when restricted to X_R gives a not necessarily symmetric pairing $E_R \otimes E_R \to L_R$ that is C^k close to q. Upon symmetrizing λ the theorem is proved. Q.E.D.

COROLLARY II. *Let X be a connected irreducible projective analytic space with conjugation σ and fixed points X_R. Let $(E, \mathfrak{L}) \in V(X_R, \sigma)$. Let Y be a C^k submanifold ($k \geq 2$) of X_R that is disjoint from the singular set of X_R. Assume $\dim_R Y_i = \dim_R C_i$-rank E_R where $\{C_i\}$ are the connected components of X_R and $Y_i = C_i \cap Y$. Further assume Y is defined by the vanishing of a C^k section f of E_R that vanishes to the first order on Y. Then there exists an embedding ϕ of Y into X_R that is as close to the original embedding as one wants in the C^k topology and where $\phi(Y)$ is real algebraic.*

PROOF. One notes that the algebraic sections of E_R are dense in $C^k(X_R, E_R)$ so it suffices to show for all sections $\tilde{f}$ of E_R that are sufficiently close to f it follows that the zero set of $\tilde{f}$ is C^k diffeomorphic to Y. This is easily seen to be purely local around any components of Y. Note f has no other critical points that Y is a small neighborhood of Y since it vanishes to the first order there. Thus one is reduced to the following lemma.

LEMMA. *Let T be a connected n real-dimensional C^∞ manifold and let E be a rank r real C^k vector bundle on T. Let s be a C^k section with $k \geq 2$ of E on T that vanishes to the first order on a compact connected submanifold Y with $\dim_R Y = n$-rank where s has no other critical points on T. Then any C^k section of E that is near enough in the C^2 topology vanishes to the first order on a submanifold diffeomorphic to Y.*

PROOF. Choose any section f near enough to s such that $s_\lambda^{-1}(0)$ is compact for $\lambda \in [-\varepsilon, 1+\varepsilon]$ for some $\varepsilon > 0$ and where $s_\lambda = (1-\lambda)s + \lambda f$ and where s_λ has no critical points other than $s_\lambda^{-1}(0)$ where s_λ vanishes to the first order. $\bigcup_{\lambda\in[-\varepsilon,1+\varepsilon]} s_\lambda^{-1}(0) = X$ is clearly a C^k submanifold of $[-\varepsilon, 1+\varepsilon]\times T$. By construction the projection $p\colon X \to [-\varepsilon, 1+\varepsilon]$ is of maximal rank. Q.E.D.

This has as a consequence the classical result [5; 16.5.4] of Hilbert that any finite disjoint union of C^2 circles in $\boldsymbol{R}^2$ can be approximated by an algebraic curve—it was this result that suggested the above; [cf. 8].

§ 3. – PROPOSITION II. *Let X be a compact Kaehler manifold with a conjugation σ with non-empty fixed point set $X_{\boldsymbol{R}}$. If K_X, the canonical bundle is holomorphically trivial, then given any connected component $\mathcal{C}$ of $X_{\boldsymbol{R}}$ one has an injection:*

$$0 \to H^0(X, \Omega_X^q) \xrightarrow{r} H^q(X, \boldsymbol{C})$$

where Ω_X^q is the q-th exterior power of the holomorphic cotangent sheaf of X and $0 \leqslant q \leqslant \dim_{\boldsymbol{C}} X$. In particular $X_{\boldsymbol{R}}$ is orientable.

PROOF. The restriction map r makes sense since all holomorphic forms on X are closed. Note that if $p+q=\dim_{\boldsymbol{C}} X$, then exterior multiplication gives a perfect pairing:

$$(*)\qquad H^0(X, \Omega_X^q)\otimes H^0(X, \Omega_X^p) \to H^0(X, K_X)\,.$$

To see this note that by basic Hodge theory one has a perfect pairing:

$$(**)\qquad H^0(X, \Omega_X^p)\otimes H^n(X, \Omega_X^q) \to H^n(X, K_X)$$

where $n=\dim_{\boldsymbol{C}} X$. Now to see that $(*)$ and $(**)$ are equivalent it suffices to construct an isomorphism from $H^0(X, \Omega_X^q)$ to $H^n(X, \Omega_X^q)$ and an isomorphism from $H^0(X, K_X)$ to $H^n(X, K_X)$ that are both compatible with the pairing. Let η be a non-vanishing section of K_X and note that exterior multiplying with $\bar\eta$ is compatible with the pairing. Since $\eta\wedge\bar\eta$ is some constant multiple of a volume form it is clear that this mapping gives an isomorphism of $H_0(X, K_X)$ with $H^n(X, K_X)$. To see the latter isomorphism note that since X is Kaehler one can by Hodge theory use conjugation of harmonic forms to reduce to the question whether exterior multiplying with η gives an isomorphism of $H^q(X, \mathcal{O}_X)$ and $H^q(X, K_X)$ where $\mathcal{O}_X$ is the holomorphic structure sheaf of X. This is a trivial consequence of the long exact cohomology sequence associated to the short exact sequence:

$$0 \to 0 \to \mathcal{O}_X \xrightarrow{\eta} K_X \to 0\,.$$

Now note that if w is a holomorphic section of K_X over U then $\overline{\sigma^* w}$ is a holomorphic section of K_X over $\sigma^{-1}(U)$. Thus σ gives rise to a conjugation of $H_0(X, K_X)$. Let η be a holomorphic section invariant under this conjugation. Now one can choose a neighborhood U of a point $x \in C$ with coordinates $\{z_1, \ldots, z_n\}$ such that $z_i(x) = 0$ and $\sigma(z_1, \ldots, z_n) = (\bar{z}_1, \ldots, z_n)$ for example use lemma I-C. Now $\eta|_U$ is of the form $a\, dz_1 \ldots dz_n$ where $a(\bar{z}_1, \ldots, \bar{z}_n) = \overline{a(z_1, \ldots, z_n)}$ and thus $\eta|_{\mathcal{C}\cap U} = a\, dx_1 \wedge \ldots \wedge dx_n$ with a real. Since η is nowhere zero on X, it follows that a is nowhere zero on U and thus that $\eta|_{\mathcal{C}\cap U}$ is nowhere zero. Thus since a is real one has a nowhere vanishing section $\eta|_{\mathcal{C}}$ of $\Lambda^n T^*_{\mathcal{C}}$ where $T^*_{\mathcal{C}}$ is the real cotangent bundle of $\mathcal{C}$. Thus $\eta|_{\mathcal{C}}$ or its opposite is a volume form and represents a nontrivial element of $H^n(\mathcal{C}, \boldsymbol{R})$ and thus of $H^n(\mathcal{C}, \boldsymbol{C})$.

Now let $\alpha \in H^0(X, \Omega^p_X)$ and assume $\alpha|_{\mathcal{C}}$ gave a trivial element of $H^p(\mathcal{C}, \boldsymbol{C})$. One gets an immediate contradiction by noting that due to the perfect pairing there exists $\beta \in H^0(X, \Omega^{n-p}_X)$ such that $\alpha \wedge \beta = \eta$. Thus $\beta|_{\mathcal{C}} \in H^{n-p}(\mathcal{C}, \boldsymbol{C})$ and thus $\eta|_{\mathcal{C}} = (\alpha|_{\mathcal{C}}) \wedge (\beta|_{\mathcal{C}})$ would be trivial but by the last paragraph it isn't. Q.E.D.

It is an interesting question whether the canonical bundle in some direct way controls the orientability of $X_{\boldsymbol{R}}$. One easy result is:

PROPOSITION III. *Let X be a compact complex manifold with $H^1(X, \mathcal{O}_X) = 0$ (e.g. X compact Kaehler and $H^1(X, \boldsymbol{C}) = 0$). If the first Chern class of K_X is $2a$ where $a \in H^2(X, \boldsymbol{Z})$ and if $H^2(X, \boldsymbol{Z})$ has no 2 torsion then $X_{\boldsymbol{R}}$ is orientable.*

PROOF. Looking at the Kummer sequence

$$0 \to \boldsymbol{Z}_2 \to \mathcal{O}^*_X \xrightarrow{z^2} \mathcal{O}^*_X \to 0$$

one notes that the above conditions let us find a holomorphic line bundle L on X such that $L^2 = K_X$. Further $\overline{\sigma L}$, the holomorphic line bundle on X with transition functions translated by σ and then conjugated, is isomorphic to L. This follows since $H^1(X, \mathcal{O}_X) = 0$ implies holomorphic line bundles are totally determined by their first integral Chern classes. Now $\overline{\sigma K_X} = K_X$ as we observed in the last proof. Thus if α is the class of $\overline{\sigma L}$ and a in the class of L one has $2a = 2\alpha$.

In particular this implies $K_X \approx L \otimes (\overline{\sigma L})$. Thus one can choose positive transition functions for K_X on $X_{\boldsymbol{R}}$. Thus the real form of $K_X|_{X_{\boldsymbol{R}}}$ associated to the natural conjugation is the trivial bundle. But the same reasoning as in the last proof lets us identify this with $\Lambda^n T^*_{X_{\boldsymbol{R}}}$. Thus $X_{\boldsymbol{R}}$ has a nowhere vanishing volume form and is orientable. Q.E.D.

As a consequence of the theorem of Conner and Floyd (cf. § 1) that X is unoriented cobordant to $X_{\mathbf{R}} \times X_{\mathbf{R}}$ one sees that the restriction mod 2 of any Chern number of X gives the corresponding Stiefel-Whitney number of $X_{\mathbf{R}}$. Thus Dwyer pointed out to me that in particular $c_1^n[X]$, the first Chern class of X raised to the $n = \dim_{\mathbf{C}} X$ power and evaluated on X being odd implies that the first Stiefel-Whitney number of $X_{\mathbf{R}}$ is non-zero and thus $X_{\mathbf{R}}$ is non-orientable. Now if H_d is a non-singular hypersurface of degree d in $\mathbf{C}P^{n+1}$, then $c_1^n[H_d] = d(d-n-2)^n$. Thus by proposition III and the above remarks one sees for $n > 1$, if H_d has a conjugation then its real form is orientable if d-$\dim_{\mathbf{C}} H_d$ is 0 mod 2 and not orientable if d is odd and $\dim_{\mathbf{C}} H_d$ is even.

Note added in proof. It has been pointed out to me by S. Akbulut that Proposition I can be proved by reducing to the case of a Grassmannian and proving it there. The perfect pairing used in the proof of Proposition II is studied in *Kähler manifolds with trivial canonical class*, by F. Bogomolov, Math. USSR Izv., **8** (1974), pp. 9-20 (= Izv. Akad. Nauk SSSR Ser. Math., **38** (1974), pp. 11-21).

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