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A Nonlinear Degenerate Parabolic Equation.

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1. – Introduction.

In this paper we shall prove uniqueness and existence theorems for the Cauchy problem, the first boundary value problem and a mixed Cauchy-Dirichlet problem for the equation

$$u_t = \big(a(u)u_x\big)_x + b(u)u_x \tag{1}$$

in which subscripts denote partial differentiation. The functions a and b are both assumed to be defined, real and continuous on $[0, \infty)$, with

$$a(s) > 0 \quad \text{if } s > 0 \qquad \text{and} \qquad a(0) = 0 .$$

To be precise, we shall study the following three problems for equation (1). Let T be a fixed positive real number.

PROBLEM I (The Cauchy problem). *To find a solution of equation* (1) *in the strip*

$$S = (-\infty, \infty) \times (0, T]$$

satisfying the initial condition

$$u(x, 0) = u_0(x) , \qquad -\infty < x < \infty ,$$

where u_0 is a given function which is defined, real, nonnegative, bounded and continuous on $(-\infty, \infty)$.

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Problem II (The first boundary value problem). *To find a solution of equation* (1) *in the rectangle*

$$R = (-1, 1) \times (0, T]$$

satisfying the conditions

$$\begin{aligned} u(x, 0) &= u_0(x), & -1 \leqq x \leqq 1, \\ u(-1, t) &= \Psi^-(t), & 0 \leqq t \leqq T, \\ u(1, t) &= \Psi^+(t), & 0 \leqq t \leqq T, \end{aligned}$$

where u_0 *is a given function which is defined, real, nonnegative and continuous on* $[-1, 1]$, *and,* Ψ^- *and* Ψ^+ *are given functions which are defined, real, nonnegative and continuous on* $[0, T]$ *and satisfy the compatibility conditions* $\Psi^-(0) = u_0(-1)$ *and* $\Psi^+(0) = u_0(1)$.

Problem III. *To find a solution of equation* (1) *in the half-strip*

$$H = (0, \infty) \times (0, T]$$

satisfying the conditions

$$\begin{aligned} u(x, 0) &= u_0(x), & 0 < x < \infty, \\ u(0, t) &= \Psi(t), & 0 \leqq t \leqq T, \end{aligned}$$

where u_0 *is a given function which is defined, real, nonnegative, bounded and continuous on* $[0, \infty)$, *and* Ψ *is a given function which is defined, real, nonnegative and continuous on* $[0, T]$ *and satisfies the compatibility condition* $\Psi(0) = u_0(0)$.

When $b \equiv 0$, i.e. equation (1) is replaced by the equation

$$u_t = \big(a(u)u_x\big)_x, \tag{1'}$$

we shall call the Problems I, II and III, Problems I′, II′ and III′, respectively.

Equation (1) is nonlinear and of degenerate second order parabolic type. At points where $u > 0$ equation (1) is parabolic, but at points where $u = 0$ it is not. Such equations arise, for example, in the study of the flow of two immiscible fluids in a porous medium [2, p. 511].

It is now well established, no matter how smooth a, b and the boundary data are, that since equation (1) degenerates, the Problems I-III need not have classical solutions [10, 15]. It is therefore necessary to generalize the notion of solutions of these problems. To do this we shall follow Oleinik, Kalashnikov and Yui-Lin [15] who defined classes of weak solutions of the Problems I′-III′ and subsequently proved existence and uniqueness theorems within these classes. Alternative definitions of generalized solutions of the Problems I-III have been given by other authors [3-6, 11, 12, 14, 18-20].

Set

$$A(s)=\int_0^s a(r)\,dr \qquad \text{and} \qquad B(s)=\int_0^s b(r)\,dr\,.$$

DEFINITION I. *A function $u(x,t)$ defined on $\bar{S}$ is said to be a weak solution of Problem I if*: (i) *u is real, nonnegative, bounded and continuous in $\bar{S}$*; (ii) *$A(u)$ has a bounded generalized derivative with respect to x in S*; *and* (iii) *u satisfies the identity*

$$\iint_S \{\varphi_x[(A(u))_x+B(u)]-\varphi_t u\}\,dx\,dt=\int_{-\infty}^{\infty}\varphi(x,0)\,u_0(x)\,dx \tag{2}$$

for all $\varphi\in C^1(\bar{S})$ which vanish for large $|x|$ and for $t=T$.

DEFINITION II. *A function $u(x,t)$ defined on $\bar{R}$ is said to be a weak solution of Problem II if*: (i) *u is real, nonnegative and continuous in $\bar{R}$*; (ii) *$u(-1,t)=\Psi^-(t)$ and $u(1,t)=\Psi^+(t)$ for all $t\in[0,T]$*; (iii) *$A(u)$ has a square-integrable generalized derivative with respect to x in R*; *and* (iv) *u satisfies the identity*

$$\iint_R \{\varphi_x[(A(u))_x+B(u)]-\varphi_t u\}\,dx\,dt=\int_{-1}^{1}\varphi(x,0)\,u_0(x)\,dx \tag{3}$$

for all $\varphi\in C(\bar{R})$ which vanish for $|x|=1$ and for $t=T$, and which have square-integrable generalized first derivatives in R.

DEFINITION III. *A function $u(x,t)$ defined on $\bar{H}$ is said to be a weak solution of Problem III if*: (i) *u is real, nonnegative and continuous in $\bar{H}$*; (ii) *$u(0,t)=\Psi(t)$ for all $t\in[0,T]$*; (iii) *$A(u)$ has a generalized derivative with respect to x in H which is square-integrable in bounded measurable subsets of H and bounded in sets of the form $(\delta,\infty)\times(0,T]$, $\delta\in(0,\infty)$*; *and* (iv) *u satisfies the identity*

$$\iint_H \{\varphi_x[(A(u))_x+B(u)]-\varphi_t u\}\,dx\,dt=\int_0^{\infty}\varphi(x,0)\,u_0(x)\,dx$$

for all $\varphi \in C(\bar{H})$ which vanish for $x = 0$, for large x and for $t = T$, and which have square-integrable generalized first derivatives in H.

Clearly any classical solution of Problem I, II or III is a weak solution of that respective problem.

We shall show that if

$$b^2(s) = O(a(s)) \quad \text{as } s \to 0+ \tag{4}$$

then each of the Problems I-III has at most one weak solution. To prove existence we shall not need condition (4). Instead, we need to assume that a' and b' exist and are locally Hölder continuous on $(0, \infty)$ and that

$$sa'(s) \in L^1(0, 1) \quad \text{and} \quad sb'(s) \in L^1(0, 1)\,. \tag{5}$$

Under this condition, with weak assumptions on the boundary data, we shall show that each of the Problems I-III has at least one weak solution. Thus we shall extend earlier uniqueness and existence theorems which have been proved for the Cauchy problem for the equation

$$u_t = (u^m)_{xx} + (u^n)_x\,, \qquad m,\ n > 1\,, \tag{6}$$

[10] to the Problems I-III.

To prove the existence of a weak solution of Problem I′, Oleinik, Kalashnikov and Yui-Lin [15] do not need condition (5). They do impose regularity and growth conditions on the function a which we do not require, but, these conditions are not essential to their argument. However to prove existence results for the Problems II′ and III′, Oleinik, Kalashnikov and Yui-Lin suppose, in addition to the extra regularity and growth conditions, that

$$a'(s) > 0 \quad \text{if } s > 0 \tag{7}$$

and

$$a^2(s) = O(a'(s)) \quad \text{as } s \to 0+\,.$$

In view of the continuity of a at $s = 0$, (7) implies (5). Thus our conditions for the existence of weak solutions of Problems II′ and III′ are less restrictive.

As an example of a pair of functions which do not satisfy (5), and therefore as an example of an equation to which our existence results do not apply, we may take

$$a(s)=\begin{cases} s\{2+\sin(s^{-2})\} & \text{for } s>0 \\ 0 & \text{for } s=0 \end{cases}$$

and

$$b(s)\equiv 0\,.$$

By a change of variables, it is clear that our results for the Problems II and III apply equally well in the domains $(\eta_1,\eta_2)\times(0,T]$ and $(\eta,\infty)\times(0,T]$ respectively, for any η_1, η_2 and η with $-\infty<\eta_1<\eta_2<\infty$ and $-\infty<<\eta<\infty$.

The paper is structured as follows. First, in section 2 we shall prove two preparatory lemmas. Subsequently, in section 3, we prove our uniqueness theorems. We then turn to the proof of our existence theorems. These require some technical machinery, which we shall set up in section 4. This gives us a clear field in which to prove our existence theorems in section 5. Here, we also prove a result on the local regularity of weak solutions. In section 6, we shall say how we can extract maximum principles for weak solutions of the Problems I-III from our existence proofs. The last section is devoted to a discussion of a necessary and sufficient condition for weak solutions of the Problems I-III to vanish in open subsets of their domains of definition.

2. – Preparatory lemmas.

In this section we prove two preparatory results. The first one is used in the proof of our uniqueness theorems. The second one is used, in section 4, to set up the machinery with which we prove our existence theorems.

LEMMA 1. *Given any $M>0$ there exists a constant C such that*

$$\{B(s_1)-B(s_2)\}^2\leqq C\{s_1-s_2\}\{A(s_1)-A(s_2)\}\qquad \textit{for all } s_1, s_2\in[0, M]$$

if and only if

$$b^2(s)=O\big(a(s)\big)\qquad \textit{as } s\to 0+\,. \tag{4}$$

PROOF. Clearly to prove the lemma, it is sufficient to show that

$$\sup_{0\leqq s_1<s_2\leqq M}\{B(s_1)-B(s_2)\}^2\{s_1-s_2\}^{-1}\{A(s_1)-A(s_2)\}^{-1}<\infty \tag{8}$$

if and only if

$$\sup_{s\in(0,M]} b^2(s)a^{-1}(s) < \infty . \tag{9}$$

Let $0 \leqq s_1 < s_2 \leqq M$. By Cauchy's mean value theorem, there exists a point $r_1 \in (s_1, s_2)$ such that

$$\{B(s_1) - B(s_2)\}^2\{s_1 - s_2\}^{-1}\{A(s_1) - A(s_2)\}^{-1}$$
$$= 2b(r_1)\{B(r_1) - B(s_2)\}\{A(r_1) - A(s_2) + (r_1 - s_2)a(r_1)\}^{-1},$$

but then, by a second application of this theorem, there exists a point $r_2 \in (r_1, s_2)$ such that

$$\{B(s_1) - B(s_2)\}^2\{s_1 - s_2\}^{-1}\{A(s_1) - A(s_2)\}^{-1} = 2b(r_1)b(r_2)\{a(r_1) + a(r_2)\}^{-1}. \tag{10}$$

Note that $0 \leqq s_1 < r_1 < r_2 < s_2 \leqq M$.

Now if (8) holds, it follows, by letting $|s_2 - s_1| \to 0$ in (10), that (9) also holds. On the other hand if (9) holds, since by Young's inequality

$$2\{a(r_1) + a(r_2)\}^{-1} \leqq a^{-\frac{1}{2}}(r_1)a^{-\frac{1}{2}}(r_2)$$

for any $r_1, r_2 \in (0, M)$, using (10) we deduce that (8) also holds. This proves the lemma.

LEMMA 2. *Suppose that $a, b \in C^1(0, \infty)$. Then given any $M > 0$ there exists a function $\theta \in C^2(0, M]$ and a positive constant C such that for all $s \in (0, M]$*

(i) $C \geqq |\theta(s)| \geqq 1/C$,

(ii) $\theta''(s)\theta(s) < 0$,

(iii) $|a'(s)\theta(s) + 2a(s)\theta'(s)| \leqq -C\theta''(s)\theta(s)$,

(iv) $a^2(s) \leqq -C\theta''(s)\theta(s)$,

(v) $|b'(s)| \leqq C|\theta''(s)|$,

(vi) $|a(s)b(s)| \leqq -C\theta''(s)\theta(s)$,

if and only if

$$F(s) = s\{|a'(s)| + |b'(s)|\} \in L^1(0, M). \tag{11}$$

PROOF. We first prove that (11) is a necessary condition for the existence of a function $\theta \in C^2(0, M]$ and a constant C satisfying (i)-(vi).

Observe that if $\theta(s)$ satisfies (i)-(vi) then so does $-\theta(s)$. So without any loss of generality, by (i) and (ii), we shall suppose that $\theta''(s) < 0$ and $C \geqq \theta(s) \geqq 1/C$ for $s \in (0, M]$.

By (iii) there holds

$$|a'(s)| \leqq -C\theta''(s) + 2Ca(s)|\theta'(s)| \qquad \text{for } s \in (0, M]\,,$$

and by (v) there holds

$$|b'(s)| \leqq -C\theta''(s) \qquad \text{for } s \in (0, M]\,.$$

So

$$F(s) \leqq 2C\{-s\theta''(s) + a(s)|s\theta'(s)|\} \qquad \text{for all } s \in (0, M]\,.$$

Integrating this inequality from $\varepsilon \in (0, M)$ to M yields

$$\int_\varepsilon^M F(s)\,ds \leqq 2C\{-M\theta'(M) + \varepsilon\theta'(\varepsilon) + \theta(M) - \theta(\varepsilon) + \\ + [A(M) - A(\varepsilon)][\sup_{\varepsilon \leqq s \leqq M} |s\theta'(s)|]\}\,.$$

Now, because $\theta''(s) < 0$ for $s \in (0, M]$, we deduce that

$$s\theta'(s) = 2\int_{\frac{1}{2}s}^{s} \theta'(s)\,dr \leqq 2\int_{\frac{1}{2}s}^{s} \theta'(r)\,dr = 2\{\theta(s) - \theta(\tfrac{1}{2}s)\} \leqq 2C \qquad \text{for all } s \in (0, M]\,,$$

and moreover that

$$|\theta'(s)| \leqq \theta'(s) - \theta'(M) + |\theta'(M)| \leqq \theta'(s) + 2|\theta'(M)| \qquad \text{for all } s \in (0, M]\,.$$

Thus

$$\int_\varepsilon^M F(s)\,ds \leqq 2C\{|M\theta'(M)| + 3C + 2A(M)[C + |M\theta'(M)|]\}\,.$$

As $\varepsilon \in (0, M)$ is arbitrary it follows that $F \in L^1(0, M)$.

To show that (11) is a sufficient condition for the existence of a function $\theta \in C^2(0, M]$ and a constant C satisfying (i)-(vi) we need only construct an example.

We choose

$$\theta(s) = \Big\{\int_0^s F(r)\,dr + s\int^M \{|a'(r)| + |b'(r)|\}\,dr + sa(M) + s|b(M)| + 1\Big\}^{\frac{1}{2}}\,.$$

We calculate that for $s \in (0, M]$,

$$\theta'(s) = \tfrac{1}{2}\left\{ \int_s^M [|a'(r)| + |b'(r)|]\,dr + a(M) + |b(M)|\right\}\theta^{-1}$$

and

$$\theta''(s) = -\tfrac{1}{2}\{|a'(s)| + |b'(s)| + 2[\theta'(s)]^2\}\theta^{-1}.$$

Hence

$$\theta''(s)\theta(s) = -\tfrac{1}{2}|a'(s)| - \tfrac{1}{2}|b'(s)| - [\theta'(s)]^2 < 0 \quad \text{for } s \in (0, M], \tag{12}$$

and (ii) is satisfied. Next we observe that

$$|a'(s)|, \quad |b'(s)| \leqq -2\theta''(s)\theta(s) \quad \text{for } s \in (0, M]$$

and

$$[\theta'(s)]^2 \leqq -\theta''(s)\theta(s) \qquad \text{for } s \in (0, M],$$

by (12), and also, since

$$|a(s) - a(M)| \leqq \int_s^M |a'(r)|\,dr, \quad |b(s) - b(M)| \leqq \int_s^M |b'(r)|\,dr \quad \text{for } s \in (0, M],$$

that

$$a(s), \quad |b(s)| \leqq 2\theta'(s)\theta(s) \qquad \text{for } s \in (0, M].$$

Thus for all $s \in (0, M]$ there holds

$$|a'(s)\theta(s) + 2a(s)\theta'(s)| \leqq -6\theta''(s)\theta(s)|\theta(s)|,$$

$$a^2(s) \leqq -4\theta''(s)\theta^3(s),$$

$$|b'(s)| \leqq -2\theta''(s)\theta(s)$$

and

$$|a(s)b(s)| \leqq -4\theta''(s)\theta^3(s).$$

It follows that if $|\theta(s)|$ is bounded above and away from zero on $(0, M]$ then it is possible to choose a constant C such that not only (i), but also (iii)-(vi), are satisfied.

Plainly though, θ is bounded below by 1 on $(0, M]$ and since $F \in L^1(0, M)$ it is also bounded above on $(0, M]$.

3. – Uniqueness theorems.

We prove our uniqueness theorems for the Problems I-III in this section. We begin with the Problem I.

THEOREM 1. *If*

$$b^2(s) = O\big(a(s)\big) \qquad as\ s \to 0+ \tag{4}$$

then Problem I has at most one weak solution.

PROOF. Suppose, contrary to the statement of the theorem, that there are two distinct solutions of Problem I, u_1 and u_2. Then we can define $t_0 \in [0, T)$ by

$$t_0 = \sup\{t\colon u_1(x, t) \equiv u_2(x, t)\}\,.$$

By the definition of a weak solution of Problem I there exist positive constants M and K such that

$$0 \leqq u_1\,, \quad u_2 \leqq M \qquad \text{in } \bar{S}$$

and

$$|(A(u_1))_x|\,, \quad |A(u_2))_x| \leqq K \qquad \text{almost everywhere in } S\,.$$

Moreover, u_1 and u_2 must satisfy the identity

$$\iint_S \varphi_t(u_1 - u_2)\,dx\,dt = \iint_S \varphi_x\{(A(u_1) - A(u_2))_x + B(u_1) - B(u_2)\}\,dx\,dt \tag{13}$$

for all $\varphi \in C^1(\bar{S})$ which vanish for large $|x|$ and for $t = T$. Plainly (13) will also hold for all $\varphi \in C(\bar{S})$ which vanish for large $|x|$ and for $t = T$, and which have bounded generalized first derivatives in S.

Let $t_1 \in (t_0, T]$ and define

$$\eta(x, t) = \begin{cases} \displaystyle\int_{t_1}^{t} \{A\big(u_1(x, s)\big) - A\big(u_2(x, s)\big)\}\,ds & \text{for } 0 \leqq t < t_1 \\ 0 & \text{for } t_1 \leqq t \leqq T\,. \end{cases}$$

We also define a sequence of functions $\{\alpha_k\}_{k=1}^{\infty} \subseteq C^{\infty}(-\infty, \infty)$ such that:

$$\begin{aligned} &\alpha_k(x) = 1 && \text{for } |x| \leqq k\,, \\ &0 \leqq \alpha_k(x) \leqq 1 && \text{for } k < |x| < k+1\,, \\ &\alpha_k(x) = 0 && \text{for } k+1 \leqq |x|\,, \end{aligned}$$

and

$$|\alpha_k'(x)| \leqq 2 \qquad \text{for all } k \text{ and all } x \in (-\infty, \infty)\,.$$

It is not difficult to see that the functions $\alpha_k\eta$ are admissible test functions in (13). Thus, choosing $\varphi = \alpha_k\eta$, (13) becomes

$$(14) \quad \iint_{S_{t_1}} \alpha_k\{u_1 - u_2\}\{A(u_1) - A(u_2)\}\,dx\,dt + \tfrac{1}{2}\int_{-\infty}^{\infty} \alpha_k(x)\eta_x^2(x, 0)\,dx$$
$$= \iint_{S_{t_1}} \alpha_k\eta_x\{B(u_1) - B(u_2)\}\,dx\,dt + \iint_{S_{t_1}} \alpha_k'\eta\{(A(u_1) - A(u_2))_x + B(u_1) - B(u_2)\}\,dx\,dt\,,$$

where $S_t = (-\infty, \infty) \times (0, t]$. We shall denote the first integral on the right hand side of (14) by I_1 and the second one by I_2.

By Young's inequality and the definition of t_0,

$$I_1 \leqq \varepsilon \int_{t_0}^{t_1}\int_{-\infty}^{\infty} \alpha_k\{B(u_1) - B(u_2)\}^2\,dx\,dt + \frac{1}{4\varepsilon}\int_{t_0}^{t_1}\int_{-\infty}^{\infty} \alpha_k\,\eta_x^2\,dx\,dt$$

for any $\varepsilon > 0$. However, because (4) holds, by Lemma 1 there exists a constant C which only depends on M such that

$$\{B(u_1) - B(u_2)\}^2 \leqq C\{u_1 - u_2\}\{A(u_1) - A(u_2)\} \quad \text{in } \bar{S}\,.$$

Hence, setting $\varepsilon = 1/(2C)$, we obtain

$$(15) \qquad I_1 \leqq \tfrac{1}{2}\iint_{S_{t_1}} \alpha_k\{u_1 - u_2\}\{A(u_1) - A(u_2)\}\,dx\,dt + \tfrac{1}{2}C\int_{t_0}^{t_1}\int_{-\infty}^{\infty} \alpha_k\eta_x^2\,dx\,dt\,.$$

In view of the bound on $|(A(u_1))_x|$ and $|(A(u_2))_x|$, if we choose a positive constant C_1 such that

$$\sup_{s\in[0,M]} a(s)\,, \qquad \sup_{s\in[0,M]} |B(s)| \leqq C_1\,,$$

we can estimate I_2 by

$$
\begin{aligned}
(16)\qquad I_2 &\leqq \iint_{S_{t_1,k}} |\alpha_k'| \{ |(A(u_1) - A(u_2))_x| + |B(u_1) - B(u_2)| \} |\eta| \, dx\, dt \\
&\leqq 4(K + C_1) \iint_{S_{t_1,k}} \left\{ \int_t^{t_1} |A(u_1(x, s)) - A(u_2(x, s))| \, ds \right\} dx\, dt \\
&\leqq 4(K + C_1) t_1 \iint_{S_{t_1,k}} |A(u_1(x, s)) - A(u_2(x, s))| \, dx\, ds \\
&\leqq 4\sqrt{2}(K + C_1) t_1^{\frac{3}{2}} \left\{ \iint_{S_{t_1,k}} |A(u_1(x, s)) - A(u_2(x, s))|^2 \, dx\, ds \right\}^{\frac{1}{2}} \\
&\leqq 4\sqrt{2C_1}(K + C_1) t_1^{\frac{3}{2}} \left\{ \iint_{S_{t_1,k}} \{u_1 - u_2\}\{A(u_1) - A(u_2)\} \, dx\, dt \right\}^{\frac{1}{2}},
\end{aligned}
$$

where $S_{t,k} = \{(-k-1, -k) \cup (k, k+1)\} \times (0, t]$.

From (14), (15) and (16) we derive the estimate

$$
\begin{aligned}
(17)\qquad &\iint_{S_{t_1}} \alpha_k \{u_1 - u_2\}\{A(u_1) - A(u_2)\} \, dx\, dt + \int_{-\infty}^{\infty} \alpha_k(x) \eta_x^2(x, 0) \, dx \\
&\leqq C \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \alpha_k \eta_x^2 \, dx\, dt + C_2 t_1^{\frac{3}{2}} \left\{ \iint_{S_{t_1,k}} \{u_1 - u_2\}\{A(u_1) - A(u_2)\} \, dx\, dt \right\}^{\frac{1}{2}},
\end{aligned}
$$

in which C_2 is a constant which depends only on M and K.

Next we define the function

$$
\zeta(x, t) = \int_0^t \{ A(u_1(x, s)) - A(u_2(x, s)) \} \, ds .
$$

Then $\eta(x, t) = \zeta(x, t) - \zeta(x, t_1)$ for $0 \leqq t \leqq t_1$. Substitution into (17) yields, when $t_1 \leqq T_0 = \min\{t_0 + 1/(4C), T\}$,

$$
\begin{aligned}
(18)\qquad &\iint_{S_{t_1}} \alpha_k \{u_1 - u_2\}\{A(u_1) - A(u_2)\} \, dx\, dt + \tfrac{1}{2} \int_{-\infty}^{\infty} \alpha_k(x) \zeta_x^2(x, t_1) \, dx \\
&\leqq 2C \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \alpha_k(x) \zeta_x^2(x, t) \, dx\, dt + C_2 t_1^{\frac{3}{2}} \left\{ \iint_{S_{t_1,k}} \{u_1 - u_2\}\{A(u_1) - A(u_2)\} \, dx\, dt \right\}^{\frac{1}{2}}.
\end{aligned}
$$

Hence,

$$\int_{-\infty}^{\infty} \alpha_k(x)\zeta_x^2(x, t_1)\,dx \leqq 4C\int_{t_0}^{t_1}\int_{-\infty}^{\infty} \alpha_k(x)\zeta_x^2(x, t)\,dx\,dt + 4MC_1^{\frac{1}{2}}C_2 t_1^2$$

for all $t_1 \in (t_0, T_0]$ and all k. This implies by Gronwall's lemma [13, p. 94] that

$$\int_{t_0}^{t_1}\int_{-\infty}^{\infty} \alpha_k(x)\zeta_x^2(x, t)\,dx\,dt \leqq 2MC_1^{\frac{1}{2}}C_2 t_1^3 \exp(4Ct_1)$$

for all $t_1 \in (t_0, T_0]$ and all k. It follows that $\zeta_x \in L^2(S_{T_0})$ and hence, by (18), that $\{u_1 - u_2\}\{A(u_1) - A(u_2)\} \in L^1(S_{T_0})$. Consequently we derive from (18), by use of the dominated convergence theorem, that

$$\iint_{S_{t_1}} \{u_1 - u_2\}\{A(u_1) - A(u_2)\}\,dx\,dt + \tfrac{1}{2}\int_{-\infty}^{\infty} \zeta_x^2(x, t_1)\,dx \leqq 2C\int_{t_0}^{t_1}\int_{-\infty}^{\infty} \zeta_x^2(x, t)\,dx\,dt$$

for all $t_1 \in (t_0, T_0]$. A second application of Gronwall's lemma now yields

$$\int_{t_0}^{T_0}\int_{-\infty}^{\infty} \zeta_x^2(x, t)\,dx\,dt = 0\,,$$

and hence

$$\iint_{S_{T_0}} \{u_1 - u_2\}\{A(u_1) - A(u_2)\}\,dx\,dt = 0\,.$$

Thus, in view of the continuity of u_1 and u_2, we have shown that $u_1 \equiv u_2$ in $\bar{S}_{T_0}$. This contradicts the definition of t_0. We can therefore only assume that Problem I has at most one weak solution.

To prove that if (4) holds then Problem II can have at most one weak solution we use many ideas from the proof of the last theorem. We shall therefore only sketch the proof.

THEOREM 2. *If* (4) *holds then Problem II has at most one weak solution.*

PROOF. Suppose that u_1 and u_2 are two distinct solutions of Problem II. Then we can define $t_0 \in [0, T)$ by

$$t_0 = \sup\{t\colon u_1(x, t) \equiv u_2(x, t)\}\,.$$

Now, it follows from (3) that u_1 and u_2 must satisfy the identity

$$\iint_R \varphi_t(u_1 - u_2)\,dx\,dt = \iint_R \varphi_x\{(A(u_1) - A(u_2))_x + B(u_1) - B(u_2)\}\,dx\,dt \tag{19}$$

for all $\varphi \in C(\bar{R})$ which vanish for $|x| = 1$ and for $t = T$, and have square-integrable generalized first derivatives in R.

Let $t_1 \in (t_0, T]$ and define

$$\eta(x, t) = \begin{cases} \int_{t_1}^{t} \{(A(u_1(x, s)) - A(u_2(x, s))\}\,ds & \text{for } 0 \leqq t < t_1 \\ 0 & \text{for } t_1 \leqq t \leqq T\,. \end{cases}$$

It is not difficult to see that η is an admissible test function in (19). Setting $\varphi = \eta$ in (19) gives

$$\iint_{R_{t_1}} \{u_1 - u_2\}\{A(u_1) - A(u_2)\}\,dx\,dt + \tfrac{1}{2}\int_{-1}^{1} \eta_x^2(x, 0)\,dx = \iint_{R_{t_1}} \{B(u_1) - Bu_2)\}\eta_x\,dx\,dt\,, \tag{20}$$

where $R_{t_1} = (-1, 1) \times (0, t_1]$. However, because (4) holds and because u_1 and u_2 are bounded in R, arguing as we did in the proof of Theorem 1 we can show that there is a positive constant C such that

$$\iint_{R_{t_1}} \{B(u_1) - B(u_2)\}\eta_x\,dx\,dt \leqq \tfrac{1}{2}\iint_{R_{t_1}} \{u_1 - u_2\}\{A(u_1) - A(u_2)\}\,dx\,dt + \tfrac{1}{2}C\int_{t_0}^{t_1}\int_{-1}^{1} \eta_x^2\,dx\,dt\,. \tag{21}$$

Thus, from (20) and (21), we have

$$\iint_{R_{t_1}} \{u_1 - u_2\}\{A(u_1) - A(u_2)\}\,dx\,dt + \int_{-1}^{1} \eta_x^2(x, 0)\,dx \leqq C\int_{t_0}^{t_1}\int_{-1}^{1} \eta_x^2\,dx\,dt\,. \tag{22}$$

Next we define the function

$$\zeta(x, t) = \int_0^{t_1} \{A(u_1(x, s)) - A(u_2(x, s)\}\,ds\,.$$

By manipulation in (22), we deduce that

$$\iint_{R_{t_1}} \{u_1 - u_2\}\{A(u_1) - A(u_2)\}\, dx\, dt + \tfrac{1}{2}\int_{-\infty}^{\infty} \zeta_x^2(x, t_1)\, dx \leqq 2C \int_{t_0}^{t_1}\int_{-1}^{1} \zeta_x^2(x, t)\, dx\, dt$$

whenever $t_1 \leqq T_0 = \min\{t_0 + 1/(4C), T\}$.

The proof of the theorem is completed similarly to the proof of Theorem 1.

By a combination of the techniques used in the proofs of Theorems 1 and 2, we can also prove a uniqueness result for weak solutions of the Problem III. We omit the proof.

THEOREM 3. *If* (4) *holds then Problem III has at most one weak solution.*

4. – Machinery for existence.

We shall prove our existence theorems using a method due to Oleinik, Kalashnikov and Yui-Lin [15]. Thus, we construct weak solutions of the Problems I-III as the pointwise limit of sequences of positive classical solutions of equation (1). The strategy is as follows. In this section we first show that the requisite positive classical solutions of equation (1) exist. Next, we find an interior smoothness estimate of them which only depends on a number of given parameters. After this we find an L^2 estimate of their first derivative with respect to x. In the next section we shall show how, for each of the Problems I-III, these estimates enable us to construct a sequence of positive classical solutions of equation (1) which converges pointwise, and guarantee that its limit function is indeed a weak solution of the problem.

Throughout this section, we shall denote by Q the rectangle $(\eta_1, \eta_2) \times (0, T]$, where $-\infty < \eta_1 < \eta_2 < \infty$, and by Q_δ the rectangle $(\eta_1 + \delta, \eta_2 - \delta) \times (0, T]$ where $\delta \in (0, \frac{1}{2}(\eta_2 - \eta_1))$.

LEMMA 3. *Let* $\varepsilon, \alpha \in (0, 1]$ *and* $M \in (0, \infty)$ *be fixed arbitrary real constants. Suppose that* $u_0(x)$ *is a* $C^{2+\alpha}[\eta_1, \eta_2]$ *function, and, that* $\Psi_1(t)$ *and* $\Psi_2(t)$ *are* $C^{1+\alpha}[0, T]$ *functions such that*

$$\varepsilon \leqq u_0 \leqq M \qquad \text{on } [\eta_1, \eta_2],$$

$$\varepsilon \leqq \Psi_1, \quad \Psi_2 \leqq M \qquad \text{on } [0, T],$$

and

$$\Psi_i(0) = u_0(\eta_i),$$

$$\Psi_i'(0) = \big(A(u_0)\big)''(\eta_i) + \big(B(u_0)\big)'(\eta_i),$$

for $i = 1, 2$. *Then if* a' *and* b' *exist and are locally Hölder continuous on* $(0, \infty)$ *there exists a unique function* $u(x, t)$ *such that*:

(i) $u \in C^{2,1}(\bar{Q})$;

(ii) $(A(u))_x \in C^{2,1}(Q)$;

(iii) $\varepsilon \leqq u \leqq M$ *in* $\bar{Q}$;

(iv) $u_t = (a(u)u_x)_x + b(u)u_x$ *in* Q;

$$\text{(v)} \quad u(x, 0) = u_0(x), \quad \eta_1 \leqq x \leqq \eta_2, \tag{23}$$

$$u(\eta_i, t) = \Psi_i(t), \quad 0 \leqq t \leqq T, \text{ for } i = 1, 2. \tag{24}$$

PROOF. Because a' and b' exist and are locally Hölder continuous on $(0, \infty)$ there exists a $\beta \in (0, 1]$ and functions $f, g \in C^{1+\beta}(-\infty, \infty)$ such that

$$f(s) = a(s), \quad g(s) = b(s) \qquad \text{for } \varepsilon \leqq s \leqq M;$$

$$f'(s) = 0, \quad g'(s) = 0 \qquad \text{for } s \leqq \tfrac{1}{2}\varepsilon \text{ and } s \geqq 2M;$$

and

$$\inf_{s \in (-\infty, \infty)} f(s) > 0.$$

Then by [13, p. 452] there exists a $\gamma \in (0, 1]$ and a function $u(x, t) \in C^{2+\gamma}(\bar{Q})$ which satisfies the equation

$$u_t = (f(u)u_x)_x + g(u)u_x \quad \text{in } Q \tag{25}$$

and the boundary condition (23), (24). Moreover by a straightforward application of the maximum principle $\varepsilon \leqq u \leqq M$ in $\bar{Q}$. Hence u satisfies (1) as well. Thus, we have shown that there is a function $u(x, t)$ satisfying (i), (iii), (iv) and (v). If there were two such functions, then by retracing the above argument, we could find two $C^{2,1}(\bar{Q})$ solutions of problem (23)-(25), which is not possible [13, p. 455].

To complete the proof of the lemma it therefore remains to show that (ii) is satisfied. Set $v = A(u)$. Then v satisfies the equation

$$v_t = f(u)v_{xx} + g(u)v_x \quad \text{in } Q.$$

We observe that $u \in C^{2+\gamma}(\bar{Q})$ and that $f, g \in C^{1+\beta}(-\infty, \infty)$ for some $\beta, \gamma \in (0, 1]$. Thus by a series of elementary computations we can show that $f(u)$, $(\partial/\partial x)(f(u))$, $g(u)$ and $(\partial/\partial x)(g(u)) \in C^{0+\beta\gamma}(\bar{Q})$. By a result from the standard theory of uniformly parabolic equations it follows that $v_x \in C^{2,1}(Q)$ [7, p. 72].

In Lemma 3 we have established the existence of a class of positive classical solutions of equation (1) which we shall use later to construct weak solutions of the Problems I-III. First, however, we must make some regularity estimates of these positive classical solutions.

LEMMA 4. *Let the assumptions of Lemma 3 hold and let $u(x, t)$ be the function exhibited in Lemma 3. Suppose that*

$$|(A(u_0))'| \leqq K_0 \qquad \text{in} \qquad \left[\eta_1 + \frac{1}{2}\delta, \eta_2 - \frac{1}{2}\delta\right].$$

Then if

$$sa'(s) \in L^1(0, 1) \qquad \text{and} \qquad sb'(s) \in L^1(0, 1) \tag{5}$$

there exists a constant K which depends only on K_0, M and δ, such that

$$|A(u(x_1, t_1)) - A(u(x_2, t_2))| \leqq K\{|x_1 - x_2|^2 + |t_1 - t_2|\}^{\frac{1}{2}}$$

for all (x_1, t_1), $(x_2, t_2) \in \bar{Q}_\delta$.

PROOF. We use a Bernstein-type technique.

Since (5) holds and a' and b' are continuous on $(0, \infty)$ it follows that

$$F(s) = s\{|a'(s)| + |b'(s)|\} \in L^1(0, M).$$

Thus by Lemma 2 there exists a function $\theta \in C^2(0, M]$ and a positive constant C such that for all $s \in (0, M]$:

(i) $C \geqq |\theta(s)| \geqq 1/C$,

(ii) $\theta''(s)\theta(s) < 0$,

(iii) $|a'(s)\theta(s) + 2a(s)\theta'(s)| \leqq -C\theta''(s)\theta(s)$,

(iv) $a^2(s) \leqq -C\theta''(s)\theta(s)$,

(v) $|b'(s)| \leqq C|\theta''(s)|$,

(vi) $|a(s)b(s)| \leqq -C\theta''(s)\theta(s)$.

Set

$$w(x, t) = \int_0^{u(x,t)} a(s)\theta^{-1}(s)\, ds .$$

Noting (i) we see that w is as smooth as $A(u)$. Furthermore, from (1) we have

$$w_t = a(u) w_{xx} + \theta'(u)(w_x)^2 + b(u) w_x .$$

We differentiate this equation with respect to x and multiply through by w_x. Writing $p = w_x$ we obtain

$$(26) \qquad \tfrac{1}{2}(p^2)_t - a(u) p p_{xx} = a^{-1}(u)\theta(u)\theta''(u) p^4 + \\ + \{a'(u) a^{-1}(u)\theta(u) + 2\theta'(u)\} p^2 p_x + b'(u) a^{-1}(u)\theta(u) p^3 + b(u) p p_x .$$

Consider the function

$$z(x, t) = \zeta^2(x) p^2(x, t)$$

in which $\zeta \in C^2[\eta_1, \eta_2]$ is a cut-off function such that:

$$\zeta(x) = 1 \quad \text{if } x \in [\eta_1 + \tfrac{3}{4}\delta, \eta_2 - \tfrac{3}{4}\delta],$$

$$\zeta(x) = 0 \quad \text{if } x \in [\eta_1, \eta_1 + \tfrac{1}{2}\delta] \cup [\eta_2 - \tfrac{1}{2}\delta, \eta_2],$$

$$0 \leqq \zeta(x) \leqq 1 \text{ for } x \in [\eta_1, \eta_2] .$$

If z attains a positive maximum at a point in Q, then at that point we have

$$z_x = 0 \qquad \text{and} \qquad a(u) z_{xx} - z_t \leqq 0 ,$$

or in other words

$$\zeta p_x = -\zeta' p$$

and

$$\zeta^2\{\tfrac{1}{2}(p^2)_t - a(u) p p_{xx}\} \geqq p^2 a(u)\{\zeta\zeta'' - 2(\zeta')^2\} .$$

Using this in (26), we deduce that at a positive maximum of z in Q,

$$-a^{-1}(u)\theta''(u)\theta(u) p^2 \zeta^2 \leqq -\{a'(u) a^{-1}(u)\theta(u) + 2\theta'(u)\}\zeta\zeta' p + \\ + a(u)\{2(\zeta')^2 - \zeta\zeta''\} + b'(u) a^{-1}(u)\theta(u)\zeta^2 p - b(u)\zeta\zeta' .$$

So at this point, by (ii)

$$p^2\zeta^2 \leqq \{a'(u)\theta(u) + 2a(u)\theta'(u)\}\{\theta(u)\theta''(u)\}^{-1}\zeta\zeta' p + a^2(u)\cdot \cdot\{\zeta\zeta'' - 2(\zeta')^2\}\{\theta(u)\theta''(u)\}^{-1} - b'(u)\{\theta''(u)\}^{-1}\zeta^2 p + + a(u)b(u)\{\theta(u)\theta''(u)\}^{-1}\zeta\zeta' ,$$

which by (iii)-(vi) means

$$p^2\zeta^2 \leqq C\{|\zeta\zeta' p| + |\zeta p| + |\zeta\zeta'' - 2(\zeta')^2| + |\zeta'|\}.$$

Thus, by Young's inequality

$$\tfrac{1}{2}z = \tfrac{1}{2}p^2\zeta^2 \leqq C\{C(\zeta')^2 + C + |\zeta\zeta'' - 2(\zeta')^2| + |\zeta'|\}. \tag{27}$$

On the other hand, if z does not attain a positive maximum in Q, then z must take its maximum value in $\bar{Q}$ on the lower boundary of Q. However, by definition

$$z(x, 0) = \zeta^2(x)\theta^{-2}\big(u_0(x)\big)\{\big(A(u_0)\big)'(x)\}^2 \leqq K_0^2 C^2 . \tag{28}$$

It follows from (27) and (28) that there exists a constant C_1 which depends on K_0, M and δ, but not on ε, such that

$$\sup_{\bar{Q}_{\frac{3}{4}\delta}} |w_x| \leqq C_1 .$$

However, by (i) this implies that

$$\sup_{\bar{Q}_{\frac{3}{4}\delta}} |\big(A(u)\big)_x| \leqq C_1 C .$$

Now set $v(x, t) = A\big(u(x, t)\big)$. Then v satisfies the equation

$$v_t = a(u)v_{xx} + b(u)v_x \quad \text{in } Q ,$$

and v has a bound which only depends on M in Q. Moreover, from the above

$$|v(x_1, t) - v(x_2, t)| \leqq C_1 C|x_1 - x_2| \quad \text{for all } (x_1, t),\ (x_2, t) \in \bar{Q}_{\frac{3}{4}\delta} . \tag{29}$$

It follows from [8] that there exists a constant C_2 which depends on $C_1 C$,

M and δ, but not on ε, such that

$$|v(x, t_1) - v(x, t_2)| \leqq C_2|t_1 - t_2|^{\frac{1}{4}} \quad \text{for all } (x, t_1), (x, t_2) \in \bar{Q}_\delta . \tag{30}$$

Combining (29) and (30) produces the required result.

LEMMA 5. *Let the assumptions of Lemma 3 hold and let $u(x, t)$ be the function exhibited in Lemma 3. Suppose that there are positive constants K_0 and K_0' such that*

$$|(A(u_0))'| \leqq K_0 \qquad \text{in} \qquad \left[\eta_1 + \frac{1}{2}\delta, \eta_2 - \frac{1}{2}\delta\right]$$

and

$$\int_0^T |(A(\Psi_i))'| dt \leqq K_0' \qquad \text{for } i = 1, 2 .$$

Then if (5) *holds there exists a constant L which depends only on K_0, K_0', M, T and δ such that*

$$\iint_{Q \setminus Q_\delta} \{(A(u))_x\}^2 \, dx \, dt \leqq L .$$

PROOF. We shall only prove that

$$\int_0^T \int_{\eta_1}^{\eta_1 + \delta} \{(A(u))_x\}^2 \, dx \, dt \leqq \tfrac{1}{2} L .$$

In an identical way one can show that

$$\int_0^T \int_{\eta_2 - \delta}^{\eta_2} \{(A(u))_x\}^2 \, dx \, dt \leqq \tfrac{1}{2} L .$$

Let

$$\chi(x, t) = A(u(x, t)) - A(\Psi_1(t)) .$$

Then, using (1) we have

$$\int_0^T \int_{\eta_1}^{\eta_1 + \delta} \{u_t - (A(u))_{xx} - (B(u))_x\} \chi \, dx \, dt = 0 .$$

Integrating by parts with respect to x, this is equivalent to:

$$\int_0^T\int_{\eta_1}^{\eta_1+\delta}\{(A(u))_x\}^2\,dx\,dt=\int_0^T\{(A(u))_x(\eta_1+\delta,t)+B(u(\eta_1+\delta,t))\}\chi(\eta_1+\delta,t)\,dt \tag{31}$$

$$-\int_0^T\int_{\eta_1}^{\eta_1+\delta}B(u)(A(u))_x\,dx\,dt-\int_0^T\int_{\eta_1}^{\eta_1+\delta}u_tA(u)\,dx\,dt+\int_0^T\int_{\eta_1}^{\eta_1+\delta}u_t(x,t)A(\Psi_1(t))\,dx\,dt\,.$$

Denote the four integrals on the right hand side of (31) by I_1, I_2, I_3 and I_4 respectively. We shall estimate these integrals in turn.

First, we introduce some notation. Let

$$P(s)=\int_0^s A(r)\,dr\,,\qquad Q(s)=\int_0^s B(r)a(r)\,dr\,,$$

and choose a positive real constant C_1 such that

$$C_1\geqq\sup_{s\in(0,M]}A(s)\,,\qquad \sup_{s\in(0,M]}|B(s)|\,,\qquad \sup_{s\in(0,M]}P(s)\,,\qquad \sup_{s\in(0,M]}|Q(s)|\,.$$

Then

$$|I_1|\leqq 2C_1\int_0^T\{|(A(u))_x(\eta_1+\delta,t)|+C_1\}\,dt\,.$$

But by Lemma 4, there exists a positive constant K which depends on K_0, M and δ, but not on ε, such that

$$|(A(u))_x(\eta_1+\delta,t)|\leqq K\qquad\text{for } t\in[0,T]\,.$$

Thus

$$|I_1|\leqq 2C_1T(K+C_1)\,.$$

Next, we observe that

$$I_2=-\int_0^T\int_{\eta_1}^{\eta_1+\delta}(Q(u))_x\,dx\,dt\,,$$

and so

$$|I_2|\leqq 2C_1T\,.$$

Now,

$$I_3 = -\int_0^T \int_{\eta_1}^{\eta_1+\delta} (P(u))_t \, dx\, dt$$

and therefore

$$|I_3| \leqq 2C_1\delta .$$

Whereas

$$I_4 = \int_{\eta_1}^{\eta_1+\delta} \{u(x, T) A((\Psi_1(T)) - u_0(x) A(\Psi_1(0))\} \, \mathrm{d}x - \int_0^T \int_{\eta_1}^{\eta_1+\delta} u(x, t)\{(A(\Psi_1))'(t)\} dx dt;$$

and so

$$|I_4| \leqq M\delta(2C_1 + K_0') .$$

From the estimates for I_1, I_2, I_3 and I_4, we deduce that

$$\int_0^T \int_{\eta_1}^{\eta_1+\delta} \{(A(u))_x\}^2 \, dx\, dt \leqq 2C_1(KT + C_1T + T + \delta + M\delta) + K_0' M\delta = \tfrac{1}{2}L .$$

5. – Existence theorems.

We are now in a position to prove our first existence theorem.

THEOREM 4. *Suppose that a' and b' exist and are locally Hölder continuous on $(0, \infty)$ and that*

$$sa'(s) \in L^1(0, 1) \qquad \text{and} \qquad sb'(s) \in L^1(0, 1). \tag{5}$$

Then if $A(u_0)$ satisfies a Lipschitz condition on $(-\infty, \infty)$ Problem I has at least one weak solution.

PROOF. Because $A(u_0)$ satisfies a Lipschitz condition on $(-\infty, \infty)$ we can choose positive constants M and K_0, sequences of positive constants $\{\varepsilon_k\}_{k=1}^\infty$, $\{\alpha_k\}_{k=1}^\infty$ and a sequence of functions $\{u_{0,k}\}_{k=1}^\infty$ such that:

(i) $\varepsilon_k, \ \alpha_k \in (0, 1]$ for all k;

(ii) $u_{0,k} \in C^{2+\alpha_k}(-\infty, \infty)$ for all k;

(iii) $\varepsilon_k \leqq u_{0,k}(x) \leqq M$ if $|x| < k$, for all k,

$u_{0,k}(x) = M$ if $|x| \geqq k$, for all k;

(iv) $u_{0,k+1}(x) \leqq u_{0,k}(x)$ for all $x \in (-\infty, \infty)$ and all k;

(v) $|(A(u_{0,k}))'(x)| \leqq K_0$ for all $x \in (-\infty, \infty)$ and all k;

(vi) $u_{0,k} \to u_0$ as $k \to \infty$ uniformly on compact subsets of $(-\infty, \infty)$;

(vii) $\varepsilon_k \to 0$ as $k \to \infty$.

Let $Q_k = (-k-1, k+1) \times (0, T]$. Then by Lemma 3 there exists a unique function $u_k \in C^{2,1}(\bar{Q}_k)$ such that:

$$(A(u_k))_x \in C^{2,1}(Q_k)$$

$$\varepsilon_k \leqq u_k(x, t) \leqq M \quad \text{for all } (x, t) \in \bar{Q}_k,$$

$$u_k \text{ satisfies (1) in } Q_k,$$

$$u_k(x, 0) = u_{0,k}(x) \quad \text{for } |x| \leqq k+1,$$

$$u_k(\pm(k+1), t) = M \quad \text{for } 0 \leqq t \leqq T.$$

We note that, in view of condition (iv), by a standard application of the maximum principle, $u_{k+1}(x, t) \leqq u_k(x, t)$ for all $(x, t) \in \bar{Q}_k$. Hence we can define a real, nonnegative, bounded function $u(x, t)$ on $\bar{S}$ by

$$u(x, t) = \lim_{k \to \infty} u_k(x, t).$$

We assert that u is a weak solution of Problem I.

Since (5) holds, by Lemma 4 there exists a constant K which only depends on M and K_0 such that for any $k \geqq 1$ there holds

$$|A(u_{k+1}(x_1, t_1)) - A(u_{k+1}(x_2, t_2))| \leqq K\{|x_1 - x_2|^2 + |t_1 - t_2|\}^{\frac{1}{4}} \tag{32}$$

for all (x_1, t_1), $(x_2, t_2) \in \bar{Q}_k$. Hence

$$|A(u(x_1, t_1)) - A(u(x_2, t_2))| \leqq K\{|x_1 - x_2|^2 + |t_1 - t_2|\}^{\frac{1}{4}}$$

for all (x_1, t_1), $(x_2, t_2) \in \bar{S}$. It follows that u is continuous in $\bar{S}$ and moreover that $A(u)$ has a bounded generalized derivative with respect to x in S. Thus to show that u is a weak solution of Problem I it remains to show that u satisfies the integral identity (2).

Let φ be an admissible test function in (2). Then there exists an $m \geqq 1$ such that the support of φ is entirely contained in $\bar{Q}_m$. From (32) we know that $\{(A(u_k))_x\}_{k=1}^{\infty}$ has a weakly convergent subsequence in $L^2(Q_m)$. We denote this subsequence by $\{(A(u_{k'}))_x\}_{k'=1}^{\infty}$ and its weak limit by w. Then since $u_{k'} \in C^{2,1}(\bar{Q}_m)$ for any $k' \geqq m$ there holds

$$\iint_{Q_m} \{A(u_{k'})\xi_x + (A(u_{k'}))_x \xi\}\, dx\, dt = 0$$

for all $\xi \in C_0^1(Q_m)$. Hence, taking the limit, we obtain

$$\iint_{Q_m} \{A(u)\xi_x + w\xi\}\, dx\, dt = 0$$

for all $\xi \in C_0^1(Q_m)$. It follows that $w = (A(u))_x$ and hence that the entire sequence $\{A(u_k)_x\}$ converges weakly to $(A(u))_x$ in $L^2(Q_m)$.

Now, as the functions u_k are solutions of (1), for all $k \geqq m$ there holds

$$\iint_{Q_m} \{[(A(u_k))_x + B(u_k)]\varphi_x - u_k\varphi_t\}\, dx\, dt = \int_{-\infty}^{\infty} u_{0,k}(x)\varphi(x, 0)\, dx\,. \tag{33}$$

However, as $k \to \infty$, u_k converges pointwise to u in Q_m, $u_{0,k}$ converges uniformly to u_0 in $[-m-1, m+1]$, and from the above, $(A(u_k))_x$ converges weakly in $L^2(Q_m)$ to $(A(u))_x$. Thus letting $k \to \infty$ in (33) we obtain

$$\iint_{S} \{[(A(u))_x + B(u)]\varphi_x - u\varphi_t\}\, dx\, dt = \int_{-\infty}^{\infty} u_0(x)\varphi(x, 0)\, dx\,.$$

Since φ was an arbitrary test function the proof of the theorem is complete.

We note that clearly the continuity condition on u_0 in Theorem 4 is minimal.

We now turn to the question of existence for the Problem II.

THEOREM 5. *Suppose that a' and b' exist and are locally Hölder continuous on $(0, \infty)$ and that (5) is satisfied. Then if $A(u_0)$ is locally Lipschitz continuous on $(-1, 1)$, and, $A(\Psi^-)$ and $A(\Psi^+)$ are absolutely continuous on $[0, T]$ Problem II has at least one weak solution.*

PROOF. Because of the conditions on u_0, Ψ^- and Ψ^+, we can choose positive constants M and K_0', sequences of positive constants $\{\varepsilon_k\}_{k=1}^{\infty}$, $\{\alpha_k\}_{k=1}^{\infty}$ and sequences of functions $\{u_{0,k}\}_{k=1}^{\infty}$, $\{\Psi_k^-\}_{k=1}^{\infty}$, $\{\Psi_k^+\}_{k=1}^{\infty}$ such that:

(i) $\varepsilon_k, \alpha_k \in (0, 1]$ for all k;

(ii) $u_{0,k} \in C^{2+\alpha_k}[-1, 1]$ for all k;

(iii) $\Psi_k^-, \Psi_k^+ \in C^{1+\alpha_k}[0, T]$ for all k;

(iv) $\varepsilon_k \leqq u_{0,k}(x) \leqq M$ for all $x \in [-1, 1]$ and all k,
$\varepsilon_k \leqq \Psi_k^-(t),\ \Psi_k^+(t) \leqq M$ for all $t \in [0, T]$ and all k;

(v) $u_{0,k+1}(x) \leqq u_{0,k}(x)$ for all $x \in [-1, 1]$ and all k,
$\Psi_{k+1}^-(t) \leqq \Psi_k^-(t),\ \Psi_{k+1}^+(t) \leqq \Psi_k^+(t)$ for all $t \in [0, T]$ and all k;

(vi) $\Psi_k^-(0) = u_{0,k}(-1),\ \Psi_k^+(0) = u_{0,k}(1)$ for all k,
$(\Psi_k^-)'(0) = \big(A(u_{0,k})\big)''(-1) + \big(B(u_{0,k})\big)'(-1)$ for all k,
$(\Psi_k^+)'(0) = \big(A(u_{0,k})\big)''(1) + \big(B(u_{0,k})\big)'(1)$ for all k;

(vii) given any $\delta \in (0, 1)$ there exists a constant $K_0(\delta)$ such that
$|\big(A(u_{0,k})\big)'(x)| \leqq K_0(\delta)$ for all $x \in (-1 + \delta, 1 - \delta)$ and all k;

(viii) $\int_0^T |\big(A(\Psi_k^-)\big)'|\,dt,\ \int_0^T |\big(A(\Psi_k^+)\big)'|\,dt \leqq K_0'$ for all k;

(ix) $u_{0,k} \to u_0$ as $k \to \infty$ uniformly on $[-1, 1]$;

(x) $\Psi_k^- \to \Psi^-$ and $\Psi_k^+ \to \Psi^+$ as $k \to \infty$ uniformly on $[0, T]$;

(xi) $\varepsilon_k \to 0$ as $k \to \infty$.

By Lemma 3, there exists a unique function $u_k(x, t) \in C^{2,1}(\bar{R})$ such that

$$\big(A(u_k)\big)_x \in C^{2,1}(R),$$
$$\varepsilon_k \leqq u_k(x, t) \leqq M \quad \text{for all } (x, t) \in \bar{R},$$
$$u_k \text{ satisfies (1) in } R,$$
$$u_k(x, 0) = u_{0,k}(x) \quad \text{for } -1 \leqq x \leqq 1,$$
$$u_k(-1, t) = \Psi_k^-(t) \quad \text{for } 0 \leqq t \leqq T,$$
$$u_k(1, t) = \Psi_k^+(t) \quad \text{for } 0 \leqq t \leqq T.$$

In view of the monotonicity conditions on $\{u_{0,k}\}$, $\{\Psi_k^-\}$ and $\{\Psi_k^+\}$, by the standard maximum principle for uniformly parabolic equations $u_{k+1}(x, t) \leqq u_k(x, t)$ for all $(x, t) \in \bar{R}$ and all k. Hence we can define a real nonnegative bounded function on $\bar{R}$ by

$$u(x, t) = \lim_{k\to\infty} u_k(x, t) .$$

Lemma 4 implies that given any $\delta \in (0, \frac{1}{2})$, there exists a constant K which depends on M, $K_0(\delta)$ and δ such that

$$|A(u_k(x_1, t_1)) - A(u_k(x_2, t_2))| \leqq K\{|x_1 - x_2|^2 + |t_1 - t_2|\}^{\frac{1}{4}}$$

for all (x_1, t_1), $(x_2, t_2) \in [-1 + 2\delta, 1 - 2\delta] \times [0, T]$ and all k. Taking the limit, we see that for any $\delta \in (0, \frac{1}{2})$, there exists a constant $K = K(M, \delta, K_0(\delta))$ such that

$$|A(u(x_1, t_1)) - A(u(x_2, t_2))| \leqq K\{|x_1 - x_2|^2 + |t_1 - t_2|\}^{\frac{1}{4}} \tag{34}$$

for all (x_1, t_1), $(x_2, t_2) \in [-1 + 2\delta, 1 - 2\delta] \times [0, T]$. It follows that u is continuous in $(-1, 1) \times [0, T]$ and that $A(u)$ has a generalized derivative with respect to x in R.

By Lemma 5 there exists a constant L which depends only on $K_0(\frac{1}{4})$, K_0', M and T such that for all k there holds

$$\int_{-1}^{-\frac{1}{2}} \int_0^T \{(A(u_k))_x\}^2 \, dx\, dt + \int_{\frac{1}{2}}^{1} \int_0^T \{(A(u_k))_x\}^2 \, dx\, dt \leqq L .$$

Moreover, by (34) there exists a constant C_1 which depends only on $K_0(\frac{1}{4})$ and M such that for all k:

$$|(A(u_k))_x(x, t)| \leqq C_1 \quad \text{for all } (x, t) \in [-\tfrac{1}{2}, \tfrac{1}{2}] \times (0, T] .$$

This means that $\{(A(u_k))_x\}_{k=1}^{\infty}$ has a weakly convergent subsequence in $L^2(R)$. By an argument similar to that used in the proof of Theorem 4 its weak limit can only be $(A(u))_x$. So $(A(u))_x \in L^2(R)$.

Proceeding as in the proof of Theorem 5, it is not difficult to show that u satisfies the integral identity (3). Furthermore by definition $u(-1, t) = \Psi^-(t)$, $u(1, t) = \Psi^+(t)$ for all $t \in [0, T]$. Thus in order to show that u is a weak solution of Problem II it remains to show that u is continuous in $\bar{R}$.

We make the observation that if $u(x, t)$ were a weak solution of Problem II, then $v(x, t) = u(-x, t)$ would be a weak solution of the equation

$$v_t = (a(v)v_x)_x - b(v)v_x \quad \text{in } R$$

with data

$$\begin{aligned} v(x, 0) &= u_0(-x), & -1 \leqq x \leqq 1, \\ v(-1, t) &= \Psi^+(t), & 0 \leqq t \leqq T, \\ v(1, t) &= \Psi^-(t), & 0 \leqq t \leqq T. \end{aligned}$$

Thus, because of this symmetry, to show that u is a weak solution of Problem II it becomes sufficient to show that u is continuous in $[-1, 1) \times \times [0, T]$. However we already know that u is continuous in $(-1, 1) \times [0, T]$ and that $u(x, 0) = u_0(x)$ is continuous on $[-1, 1]$. Thus to show that u is a weak solution of Problem II it is enough to prove that for any fixed $t_0 \in [0, T]$:

$$\limsup_{\substack{(x,t)\to(-1,t_0) \\ (x,t)\in R}} u(x, t) \leqq \Psi^-(t_0) \tag{35}$$

and

$$\liminf_{\substack{(x,t)\to(-1,t_0) \\ (x,t)\in R}} u(x, t) \geqq \Psi^-(t_0). \tag{36}$$

We will demonstrate that this is true below and thereby complete the proof of the theorem.

For any k:

$$u(x, t) \leqq u_k(x, t) \quad \text{for all } (x, t) \in \bar{R}.$$

Hence

$$\limsup_{\substack{(x,t)\to(-1,t_0) \\ (x,t)\in R}} u(x, t) \leqq \limsup_{\substack{(x,t)\to(-1,t_0) \\ (x,t)\in R}} u_k(x, t) = \Psi_k^-(t_0).$$

Letting $k \to \infty$ in the latter inequality yields (35).

If $\Psi^-(t_0) = 0$ then trivially (36) is true. When, however, $\Psi^-(t_0) > 0$ we must use a comparison function argument to show that (36) holds. We shall show that given any $\varepsilon \in (0, \Psi^-(t_0))$ we can define a function $w(x, t)$ on $\bar{R}$ such that

$$\liminf_{\substack{(x,t)\to(-1,t_0) \\ (x,t)\in R}} w(x, t) = \Psi^-(t_0) - \varepsilon \tag{37}$$

and such that, for all sufficiently large k, there holds

$$u_k(x, t) \geqq w(x, t) \quad \text{for all } (x, t) \in \bar{R}\,. \tag{38}$$

Clearly this suffices to show that (36) also holds when $\Psi^-(t_0) > 0$.

Suppose therefore that $\Psi^-(t_0) > 0$ and let $\varepsilon \in (0, \Psi^-(t_0))$ be a fixed arbitrary constant. Set

$$\beta = 1 + \sup_{0 \leqq r \leqq M} |B(r)|\,.$$

Then the following functions are well defined on $(0, \infty)$:

$$\varrho(c) = \int_0^M a(r)\{cr + B(r) + \beta\}^{-1} dr$$

and

$$\lambda(c) = t_0 - c^{-1} \int_0^{\Psi^-(t_0) - \varepsilon} a(r)\{cr + B(r) + \beta\}^{-1} dr\,.$$

We observe that

$$c\{t_0 - \lambda(c)\} \to 0+ \quad \text{as } c \to \infty\,.$$

If $t_0 > 0$, we choose and fix c so large that

$$\begin{cases} \lambda(c) > 0\,, \\ c\{t_0 - \lambda(c)\} < 2\,, \\ \Psi^-(t) \geqq \Psi^-(t_0) - \frac{1}{2}\varepsilon \quad \text{for all } t \in [\lambda(c), t_0]\,; \end{cases} \tag{39a}$$

and set

$$t_1 = \lambda(c)\,.$$

On the other hand if $t_0 = 0$, we choose and fix c so large that

$$\begin{cases} -c\lambda(c) = c\{t_0 - \lambda(c)\} < 2\,, \\ u_0(x) \geqq u_0(0) - \frac{1}{2}\varepsilon = \Psi^-(t_0) - \frac{1}{2}\varepsilon \quad \text{for } x \in [-1, -1 - c\lambda(c)]\,; \end{cases} \tag{39b}$$

and set

$$t_1 = 0 = t_0\,.$$

Next we define an increasing function $y:[0, \varrho(c)] \to [0, M]$ by

$$\eta = \int_0^{y(\eta)} a(r)\{cr + B(r) + \beta\}^{-1} dr .$$

By the definition of $\varrho(c)$, y is well-defined and moreover it is a bijection from $[0, \varrho(c)]$ to $[0, M]$. Also, it is not difficult to check that $y \in C^2(0, \varrho(c)]$, with

$$(A(y))' = a(y)y' = cy + B(y) + \beta > 0 \quad \text{on } [0, \varrho(c)]$$

and

$$(40) \qquad (A(y))'' = y'\{c + b(y)\} \quad \text{on } [0, \varrho(c)] .$$

Note that by the definition of $\lambda(c)$:

$$(41) \qquad y(c\{t_0 - \lambda(c)\}) = \Psi^-(t_0) - \varepsilon .$$

Now if $t_0 < T$, by (39a, b) and (41), we can pick t_2 such that

$$t_0 < t_2 \leqq T ,$$

$$c\{t_2 - \lambda(c)\} < 2 ,$$

$$y(c\{t_2 - \lambda(c)\}) < \Psi^-(t_0) - \tfrac{1}{2}\varepsilon ,$$

$$\Psi^-(t) \geqq \Psi^-(t_0) - \tfrac{1}{2}\varepsilon \quad \text{for all } t \in [t_0, t_2] .$$

If $t_0 = T$ we set

$$t_2 = T = t_0 .$$

Let m be chosen so large that $\varepsilon_k < \Psi^-(t_0) - \varepsilon$ for all $k \geqq m$ and for each $k \geqq m$ define the point η_k by

$$y(\eta_k) = \varepsilon_k .$$

We set

$$\Omega_k = \{(x, t) : t_1 < t \leqq t_2 \text{ and } -1 < x < c\{t - \lambda(c)\} - \eta_k - 1\}$$

and

$$\Gamma_k = \{(x, t) : t_1 < t \leqq t_2 \text{ and } x = c\{t - \lambda(c)\} - \eta_k - 1\} .$$

Let $\Omega = \bigcup_{k=m}^{\infty} \Omega_k$. Noting that $\eta_k \to 0$ as $\varepsilon_k \to 0$ we see that

$$\Omega = \{(x, t) : t_1 < t \leqq t_2 \text{ and } -1 < x < c\{t - \lambda(c)\} - 1\} .$$

Plainly, since by definition $c\{t_2 - \lambda(c)\} < 2$, $\Omega \subseteq R$.

We are now in a position to define the function w. We set

$$w(x, t) = \begin{cases} y(-x - 1 + c\{t - \lambda(c)\}) & \text{if } (x, t) \in \bar{\Omega} \\ 0 & \text{if } (x, t) \in \bar{R} \setminus \bar{\Omega} . \end{cases}$$

To check that w satisfies (37) we utilize the choice of t_1 and t_2, and (41). We find that

$$\liminf_{\substack{(x,t)\to(-1,t_0)\\(x,t)\in R}} w(x, t) = \liminf_{\substack{(x,t)\to(-1,t_0)\\(x,t)\in \Omega}} w(x, t) = w(-1, t_0) = \Psi^-(t_0) - \varepsilon .$$

We therefore now have to check that (38) holds.

Fix $k \geqq m$. Then if $t \in [t_1, t_2]$:

$$\text{(42)} \quad u_k(-1, t) = \Psi_k^-(t) \geqq \Psi^-(t) \geqq \Psi^-(t_0) - \tfrac{1}{2}\varepsilon > \\ > y(c\{t_2 - \lambda(c)\}) > y(c\{t - \lambda(c)\}) = w(-1, t) .$$

Whereas if $(x, t) \in \Gamma_k$:

$$\text{(43)} \quad u_k(x, t) \geqq \varepsilon_k = y(\eta_k) = w(x, t) .$$

We also note that if $t_1 = 0$, i.e. if $t_0 = 0$, for $x \in [-1, -1 - c\lambda(c)]$:

$$\text{(44)} \quad u_k(x, t) = u_{0,k}(x) \geqq u_0(x) \geqq u_0(0) - \tfrac{1}{2}\varepsilon = \Psi^-(t_0) - \tfrac{1}{2}\varepsilon > \\ > y(c\{t_2 - \lambda(c)\}) > y(-c\lambda(c)) > y(-x - 1 - c\lambda(c)) = w(x, 0) .$$

Thus, by (42)-(44), there holds

$$u_k(x, t) \geqq w(x, t) \quad \text{for all } (x, t) \in \bar{\Omega}_k \setminus \Omega_k .$$

We now apply the maximum principle. Using (40) we observe that w is a classical solution of (1) in Ω_k. Furthermore w is bounded away from zero in $\bar{\Omega}_k$ by ε_k. Thus,

$$u_k(x, t) \geqq w(x, t) \quad \text{for all } (x, t) \in \bar{\Omega}_k .$$

However,

$$u_k(x, t) \geqq \varepsilon_k = y(\eta_k) \geqq w(x, t) \quad \text{for all } (x, t) \in \bar{R} \setminus \bar{\Omega}_k .$$

Plainly then

$$u_k(x, t) \geqq w(x, t) \quad \text{for all } (x, t) \in \bar{R} .$$

We have therefore shown that the function w satisfies (37) and (38). Since this was the last step in the verification of (35) and (36), the proof of the theorem is complete.

To prove the existence of weak solutions of Problem III we combine the techniques used in the proofs of Theorems 4 and 5. We omit the details.

THEOREM 6. *Suppose that a' and b' exist and are locally Hölder continuous on $(0, \infty)$ and that (5) is satisfied. Then if $A(u_0)$ satisfies a Lipschitz condition on any set of the form (δ, ∞), $\delta > 0$, and $A(\Psi)$ is absolutely continuous on $[0, T]$ Problem III has at least one weak solution.*

To conclude this section we shall prove a result concerning the local differentiability of the weak solution of the Problem I which we have constructed.

PROPOSITION 1. *Let the assumptions of Theorem 4 hold and let $u(x, t)$ be the weak solution of Problem I exhibited in Theorem 4. Then u is a classical solution of equation (1) in a neighbourhood of any point $(x_0, t_0) \in S$ where $u(x_0, t_0) > 0$.*

PROOF. Let (x_0, t_0) be a fixed point in S where $u(x_0, t_0) > 0$. We shall denote by $\mathcal{B}(\gamma)$ the open ball in $\boldsymbol{R}^2$ with centre at (x_0, t_0) and radius γ. Otherwise throughout the proof of this theorem we shall adopt the notation used in the proof of Theorem 4.

Since $u(x_0, t_0) > 0$ there exists a $\gamma_0 > 0$ and a $\mu > 0$ such that

$$u(x, t) \geqq \mu \quad \text{for all } (x, t) \in \mathcal{B}(\gamma_0) \cap S .$$

Hence we can choose m so large that $\mathcal{B}(\gamma_0) \cap S \subseteq Q_m$ and

$$u_k(x, t) \geqq \mu \quad \text{for all } (x, t) \in \mathcal{B}(\gamma_0) \cap S \text{ and all } k \geqq m .$$

Fix $\gamma_1 \in (0, \gamma_0)$. Then by a generalization of Nash's theorem [13, p. 204] we can find a constant $\beta \in (0, 1]$ and a positive constant χ such that

$$|u_k(x_1, t_1) - u_k(x_2, t_2)| \leqq \chi\{|x_1 - x_2|^2 + |t_1 - t_2|\}^{\frac{1}{2}\beta}$$

for all (x_1, t_1), $(x_2, t_2) \in \mathcal{B}(\gamma_1) \cap S$ and all $k \geqq m$. Here the constants β and χ may depend only on μ, $\gamma_0 - \gamma_1$ and M. It follows that for all $k \geqq m$, $a(u_k)$ and $b(u_k)$ are Hölder continuous in $\mathcal{B}(\gamma_1) \cap S$ with exponent β.

We set $v_k = A(u_k)$ and note that

$$(v_k)_t = a(u_k)(v_k)_{xx} + b(u_k)(v_k)_x \quad \text{in } \mathcal{B}(\lambda_1) \cap S .$$

Thus by the standard theory of linear parabolic equations [7, p. 72] $v_k \in C^{2+\beta}(\mathcal{B}(\gamma_1) \cap S)$ for all $k \geqq m$. This means, by a second application of this theory [7, p. 64] that there exists a $\gamma_2 \in (0, \gamma_1)$ for which $\|v_k\|^{2+\beta}_{\mathcal{B}(\gamma_2)\cap S}$ may be estimated independently of k. So $A(u) \in C^{2+\beta}(\mathcal{B}(\gamma_2) \cap S)$, and therefore $u \in C^{2,1}(\mathcal{B}(\gamma_2) \cap S)$.

To show that u is a classical solution of (1) in a neighbourhood of (x_0, t_0) we now only have to show that u satisfies (1) in $\mathcal{B}(\gamma_2) \cap S$, but this follows immediately from the identity (2).

Similarly, we can prove the following results.

PROPOSITION 2. *Let the assumptions of Theorem* 5 *hold and let* $u(x, t)$ *be the weak solution of Problem II exhibited in Theorem* 5. *Then* u *is a classical solution of equation* (1) *in a neighbourhood of any point* $(x_0, t_0) \in R$ *where* $u(x_0, t_0) > 0$.

PROPOSITION 3. *Let the assumptions of Theorem* 6 *hold and let* $u(x, t)$ *be the weak solution of Problem III exhibited in Theorem* 6. *Then* u *is a classical solution of equation* (1) *in a neighbourhood of any point* $(x_0, t_0) \in H$ *where* $u(x_0, t_0) > 0$.

6. – Maximum principles for weak solutions.

We have proved our existence theorems following the method which Oleinik, Kalashnikov and Yui-Lin [15] used to prove the existence of weak solutions of the Problems I′-III′. Thus, subject to the constraints of our existence and uniqueness theorems we may prove weak maximum principles for the Problems I-III in exactly the same way as Oleinik, Kalashnikov and Yui-Lin [15] have for the Problems I′-III′.

PROPOSITION 4. *Suppose that* a' *and* b' *exist and are locally Hölder continuous on* $(0, \infty)$ *and that* (4) *and* (5) *are satisfied. Let* u_1 *and* u_2 *denote two weak solutions of Problem I with respective initial data* u_{01}, u_{02}. *Suppose that* $A(u_{01})$ *and* $A(u_{02})$ *satisfy Lipschitz conditions on* $(-\infty, \infty)$ *and that*

$$u_{01}(x) \leqq u_{02}(x) \qquad \textit{for all } x \in (-\infty, \infty) .$$

Then

$$u_1(x,t) \leqq u_2(x,t) \qquad \text{for all } (x,t) \in \bar{S}.$$

PROPOSITION 5. *Suppose that a' and b' exist and are locally Hölder continuous on $(0, \infty)$ and that (4) and (5) are satisfied. Let u_1 and u_2 denote two weak solutions of Problem II with respective data $u_{01}, \Psi_1^-, \Psi_1^+$ and $u_{02}, \Psi_2^-, \Psi_2^+$. Suppose that $A(u_{01})$ and $A(u_{02})$ are locally Lipschitz continuous on $(-1, 1)$; that $A(\Psi_1^-)$, $A(\Psi_1^+)$, $A(\Psi_2^-)$ and $A(\Psi_2^+)$ are absolutely continuous on $[0, T]$; and that*

$$u_{01}(x) \leqq u_{02}(x) \qquad \text{for all } x \in [-1, 1],$$

and

$$\Psi_1^-(t) \leqq \Psi_2^-(t), \quad \Psi_1^+(t) \leqq \Psi_2^+(t) \qquad \text{for all } t \in [0, T].$$

Then

$$u_1(x,t) \leqq u_2(x,t) \qquad \text{for all } (x,t) \in \bar{R}.$$

PROPOSITION 6. *Suppose that a' and b' exist and are locally Hölder continuous on $(0, \infty)$ and that (4) and (5) are satisfied. Let u_1 and u_2 denote two weak solutions of Problem III with respective data u_{01}, Ψ_1 and u_{02}, Ψ_2. Suppose that $A(u_{01})$ and $A(u_{02})$ satisfy Lipschitz conditions on any set of the form (δ, ∞), $\delta > 0$, that $A(\Psi_1)$ and $A(\Psi_2)$ are absolutely continuous on $[0,T]$; and that*

$$u_{01}(x) \leqq u_{02}(x) \qquad \text{for all } x \in [0, \infty),$$

and

$$\Psi_1(t) \leqq \Psi_2(t) \qquad \text{for all } t \in [0, T].$$

Then

$$u_1(x,t) \leqq u_2(x,t) \qquad \text{for all } (x,t) \in \bar{H}.$$

A detailed proof of Proposition 4 for equation (6) is given in [9].

7. – Compact support of weak solutions.

Consider Problem I′ and suppose that u_0 is nontrivial and has compact support. Then for the weak solution $u(x,t)$ of Problem I′ to have compact support it is necessary and sufficient that

$$a(s)/s \in L^1(0,1) \tag{45}$$

[15, 17]. Similarly if we let $u(x,t)$ be a weak solution of Problem III′ for which Ψ is positive and u_0 has compact support, then for $u(x,t)$ to have compact support (45) is also necessary and sufficient [15, 16]. In this section we shall extend these results to the Problems I and III and give a comparable result for the Problem II. However we shall be limited by our existence and uniqueness theorems. Thus we must suppose that a' and b' exist and are locally Hölder continuous on $(0,\infty)$ and that (4) and (5) are satisfied.

We make the preliminary observation that if $u(x,t)$ is a weak solution of Problem I with initial data u_0, then

$$v(x,t) = u(-x,t)$$

is the weak solution of Problem I for the equation

$$v_t = \big(a(v)v_x\big)_x - b(v)v_x$$

with the initial data

$$v(x,0) = u_0(-x)\,, \qquad -\infty < x < \infty\,.$$

Thus, if we suppose that u_0 has compact support, to prove that the weak solution $u(x,t)$ of Problem I has compact support if (45) holds, it is enough to prove the following result.

LEMMA 6. *Let $u(x,t)$ be a weak solution of Problem I with initial data u_0 and suppose that*

$$u_0(x) = 0 \qquad \textit{for all } x \geqq x_0\,,$$

for some $x_0 \in (-\infty, \infty)$. Then if (45) holds there exists a point $x_1 \in [x_0, \infty)$ such that

$$u(x,t) = 0 \qquad \textit{for all } (x,t) \in \bar{S} \textit{ satisfying } x \geqq x_1\,.$$

PROOF. Set

$$M_0 = \sup_{\bar{S}} u(x,t) < \infty\,.$$

Assume—for the moment—that there exists a weak solution $U(x, t)$ of Problem III with the properties:

(a) there exists a point $X_1 \in (0, \infty)$ such that

$$U(x, t) = 0 \text{ for all } (x, t) \in \bar{H} \text{ satisfying } x \geqq X_1,$$

(b) $U(0, t) > M_0$ for all $t \in [0, T]$.

Then we may invoke an argument of Oleinik, Kalashnikov and Yui-Lin [15] and conclude that

$$U(x, t) \geqq u(x + x_0, t) \quad \text{for all } (x, t) \in \bar{H},$$

which proves the lemma.

It remains to find a weak solution $U(x, t)$ of Problem III with the required properties. However, for technical reasons—as we shall see—for arbitrary functions a and b in equation (1) this is not straightforward. We shall overcome this technicality by choosing a function $\tilde{a} \in C[0, \infty) \cap C^1(0, \infty)$ such that $\tilde{a}'$ is locally Hölder continuous on $(0, \infty)$ and

$$\tilde{a}(s) = a(s) \quad \text{for } s \in [0, M_0],$$

and such that we can find a weak solution $\tilde{U}(x, t)$ of Problem III for the equation

$$u_t = (\tilde{a}(u)u_x)_x + b(u)u_x \tag{46}$$

with properties (a) and (b). This suffices. For, since $u \leqq M_0$ in $\bar{S}$ it is a weak solution of Problem I for equation (46), and so by the above argument applied to equation (46) there holds

$$\tilde{U}(x, t) \geqq u(x + x_0, t) \quad \text{for all } (x, t) \in \bar{H}.$$

We show that if (45) holds we can choose a suitable function $\tilde{a}$ such that there is a weak solution $\tilde{U}(x, t)$ of Problem III for equation (46) with the required properties.

Since (4) holds, $b(0) = 0$ and therefore for every $c > 0$ we may define $\varrho = \varrho(c)$, $0 < \varrho \leqq \infty$, by

$$\varrho(c) = \sup\{s \in (0, \infty): cr + B(r) > 0 \text{ for all } r \in (0, s)\}.$$

Choose and fix c so large that

$$\varrho(c) > M_0 .$$

Because (45) holds we may then set

$$\lambda_1 = \lambda_1(c) = \int_0^{M_0} a(s)\{cs + B(s)\}^{-1} ds < \infty .$$

The function $\tilde{a} \in C[0, \infty) \cap C^1(0, \infty)$ with $\tilde{a}'$ locally Hölder continuous on $(0, \infty)$ and such that

$$\tilde{a}(s) = a(s) \quad \text{for } s \in [0, M_0]$$

is now chosen in such a way that if we set

$$\lambda_2 = \lambda_2(c) = \int_0^{\varrho(c)} \tilde{a}(s)\{cs + B(s)\}^{-1} ds , \tag{47}$$

$\lambda_1 < \lambda_2 \leqq \infty$, then there holds

$$\lambda_2(c) > \lambda_1(c) + cT . \tag{48}$$

Plainly this is feasible. We now determine $\tilde{U}$ from defining the function $f\colon [0, \varrho(c)) \to [0, \lambda_2)$ by

$$\xi = \int_0^{f(\xi)} \tilde{a}(s)\{cs + B(s)\}^{-1} ds ,$$

and for some real constant γ setting

$$\tilde{U}(x, t) = \begin{cases} f(\gamma + ct - x) & \text{if } x < \gamma + ct \\ 0 & \text{if } x \geqq \gamma + ct . \end{cases}$$

The function $\tilde{U}$ is defined on $\bar{H}$ if $f(\gamma + cT)$ is defined, i.e. if

$$\gamma + cT < \lambda_2(c) . \tag{49}$$

Subject to this constraint it is not hard to verify that $\tilde{U}$ is a weak solution of Problem III for equation (46). Clearly $\tilde{U}$ has property (a). Moreover,

$\tilde{U}$ has property (b) if $f(\gamma) > M_0$, i.e. if

$$\gamma > \lambda_1(c)\,. \tag{50}$$

By our choice of $\tilde{a}$ it follows from (48) that we can pick γ so that both (49) and (50) are satisfied. Thus $\tilde{U}$ is the weak solution of Problem III for equation (46) which we require.

If we replace the function $\tilde{a}$ in (47) by a we cannot be certain, in general, that (48) holds for any $c > 0$ such that $\varrho(c) > M_0$. Hence it may be impossible to find a pair of positive constants c and γ satisfying (49) and (50). We circumvent this technical obstacle by introducing the function $\tilde{a}$.

We remark that the function $\tilde{U}(x, t)$ is differentiable at points $(x, t) \in H$ where $x \neq \gamma + ct$. Specifically,

$$\tilde{U}_x(x, t) = 0 \qquad \text{for } x > \gamma + ct$$

and

$$\tilde{U}_x(x, t) = -\{c\tilde{U} + B(\tilde{U})\}/\tilde{a}(U) \quad \text{for } x < \gamma + ct\,.$$

Thus if

$$\tilde{a}(s) = O(s) \quad \text{as } s \to 0+$$

the derivative $\tilde{U}_x$ is discontinuous in H and $\tilde{U}$ is not a classical solution of Problem III for equation (46).

We now turn to showing that if $u(x, t)$ is a nontrivial weak solution of Problem I then (45) is necessary for u to have compact support in $\bar{S}$. We shall do this in two steps. This is the first.

LEMMA 7. *Suppose that* (45) *does not hold, and let* $u(x, t)$ *be a weak solution of Problem I. Then, given any* $t \in (0, T]$ *either*:

(i) $u(x, t) > 0$ *for all* $x \in (-\infty, \infty)$;

or;

(ii) $u(x, t) = 0$ *for all* $x \in (-\infty, \infty)$.

PROOF. Suppose that the lemma is false. Thus, suppose that there exists $t_1 \in (0, T]$ and points x_0, $x_1 \in (-\infty, \infty)$ such that

$$u(x_0, t_1) > 0 \quad \text{and} \quad u(x_1, t_1) = 0\,.$$

Then, in view of the continuity of u in $\bar{S}$ there exists a $t_0 \in [0, t_1]$ and a positive constant μ such that

$$u(x_0, t) \geqq \mu (> 0) \quad \text{for all } t \in [t_0, t_1].$$

Without any loss of generality we shall take $x_0 = 0$, $t_0 = 0$, $t_1 = T$, and, recalling our preliminary observation, $x_1 > 0$. We seek a contradiction by using a modification of a technique due to Peletier [16].

Let $\{u_k\}_{k=1}^{\infty}$ denote the sequence of classical solutions of equation (1) from which u was constructed in Theorem 4 as a pointwise limit. We recall that each $u_k \in C^{2,1}(\bar{Q}_k)$, where $Q_k = (-k-1, k+1) \times (0, T]$, and that there exists a sequence of positive constants $\{\varepsilon_k\}_{k=1}^{\infty}$ and a fixed positive constant M such that

$$M \geqq u_k(x, t) \geqq \varepsilon_k \quad \text{for all } (x, t) \in \bar{Q}_k$$

and

$$u_k(k+1, t) = M \quad \text{for all } t \in [0, T]. \tag{51}$$

We also note that since $\{u_k\}$ is decreasing:

$$u_k(0, t) \geqq \mu \quad \text{for all } t \in [0, T] \tag{52}$$

and for all k.

Now, since $a(s)/s \notin L^1(0, 1)$, the problem

$$\begin{cases} (a(f)f')' + \frac{1}{2}\eta f' = 0, & 0 < \eta < \infty, \\ f(0) = \frac{1}{2}\mu, \quad f(\infty) = 0, \end{cases} \tag{53}$$

has a unique positive solution on $[0, \infty)$ [1]. Let us denote this function by $f(\eta)$. Then, $f'(\eta) < 0$ for all $\eta \in [0, \infty)$ [1], and for each $k \geqq 1$ we can choose a constant $\tau_k \in (0, \infty)$ such that

$$u_k(x, 0) \geqq f(x\tau_k^{-\frac{1}{2}}) \quad \text{for all } x \in [0, k+1] \tag{54}$$

[16].

Choose

$$\nu > \sup_{s \in [0,\mu]} |b(s)|.$$

Next set

$$w_k(x, t) = f(\{x + \nu t\}\{t + \tau_k\}^{-\frac{1}{2}}), \quad (x, t) \in \bar{H}.$$

Observe that since f is strictly decreasing on $[0, \infty)$,

$$\tfrac{1}{2}\mu \geqq w_k(0, t) \quad \text{for all } t \in [0, T] \tag{55}$$

and also

$$(w_k)_x(x, t) < 0 \quad \text{in } \bar{H}\,.$$

Thus, by computation using (53), we find that

$$\{a(w_k)\{w_k\}_x\}_x + b(w_k)\{w_k\}_x - \{w_k\}_t = \{b(w_k) - \nu\}\{w_k\}_x > 0 \quad \text{in } \bar{H}\,.$$

It follows that w_k is a positive subsolution of (1) in $\bar{H}$. Therefore, in view of (51), (52), (54) and (55), by the maximum principle

$$u_k(x, t) \geqq w_k(x, t) \qquad \text{for all } (x, t) \in \overline{Q_k \cap H}\,.$$

Thus for all $k \geqq 1$ there holds

$$u_k(x, t) \geqq f(\{x + \nu t\}\{t + \tau_k\}^{-\frac{1}{2}}) \qquad \text{for all } (x, t) \in \overline{Q_k \cap H}\,.$$

Letting $k \to \infty$, and using the monotonicity of f, we obtain

$$u(x, t) \geqq f(\{x + \nu t\} t^{-\frac{1}{2}}) > 0 \quad \text{for all } (x, t) \in H\,.$$

Particularly this means that

$$u(x_1, t_1) > 0\,.$$

Hence we have a contradiction.

From Lemma 8 it follows that to show that (45) is necessary for a weak solution $u(x, t)$ with nontrivial initial data u_0 to have compact support it is enough to show that $u(x, t) \not\equiv 0$ for any $t \in (0, T]$. This follows from the following lemma. The lemma has already been proved for the special case of equation (6) [9] and extends to equation (1) without involving any extra difficulties. We shall therefore omit the proof.

LEMMA 8. *Let $u(x, t)$ be a weak solution of Problem I. Then for all $t \in (0, T]$:*

$$\int_{-\infty}^{\infty} u(x, t)\, dx = \int_{-\infty}^{\infty} u_0(x)\, dx\,.$$

Thus, combining Lemmas 6-8 we have proved the following proposition.

PROPOSITION 4. *Suppose that a' and b' exist and are locally Hölder continuous on* $(0, \infty)$ *and that* (4) *and* (5) *are satisfied. Let u denote a weak solution of Problem I with nontrivial initial data u_0, and suppose that $A(u_0)$ satisfies a Lipschitz condition on* $(-\infty, \infty)$ *and that u_0 has compact support. Then u has compact support in $\bar{S}$ if and only if* (45) *is satisfied.*

Similarly we can prove the following results.

PROPOSITION 5. *Suppose that a' and b' exist and are locally Hölder continuous on* $(0, \infty)$ *and that* (4) *and* (5) *are satisfied. Let u denote a weak solution of Problem II with data u_0, Ψ^-, Ψ^+ and suppose that $A(u_0)$ is locally Lipschitz continuous on* $(-1, 1)$ *and $A(\Psi^-)$ and $A(\Psi^+)$ are absolutely continuous on* $[0, T]$, *and that there exist points $\varrho_1, \varrho_2 \in (-1, 1)$, $\varrho_1 < \varrho_2$, such that*

$$u_0(x) = 0 \qquad \text{if } \varrho_1 \leqq x \leqq \varrho_2 ,$$

and

$$\Psi^-(t), \Psi^+(t) > 0 \qquad \text{for all } t \in [0, T] .$$

Then there exists an open subset of R in which u vanishes if and only if (45) *is satisfied.*

PROPOSITION 6. *Suppose that a' and b' exist and are locally Hölder continuous on* $(0, \infty)$ *and that* (4) *and* (5) *are satisfied. Let u denote a weak solution of Problem III with data u_0, Ψ and suppose that $A(u_0)$ satisfies a Lipschitz condition on any set of the form* (δ, ∞), $\delta > 0$, *and $A(\Psi)$ is absolutely continuous on* $[0, T]$, *and that u_0 has compact support and*

$$\Psi(t) > 0 \qquad \text{for all } t \in [0, T] .$$

Then u has compact support in $\bar{H}$ if and only if (45) *is satisfied.*

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