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On a Free Boundary Value Problem Connected with a Non Steady Filtration Phenomenon.

ALESSANDRO TORELLI (*)

dedicated to Jean Leray

1. – Introduction.

The present paper, as the previous ones [20] and [21] (see also [19]), is devoted to the study of a free boundary problem of evolution type related to a *non-steady* flow of an *incompressible* fluid moving across a homogeneous porous dam which separates two reservoirs of fluid (cfr. the figure). More precisely, we assume that the dam is bounded by parallel vertical walls and by a horizontal base. We suppose also, that: $a > 0$ (resp. $b > 0$) is the width (resp. the height) of the dam; $[0, T]$, with $T > 0$, is the time interval during which we want to study the filtration process; $y_k(t)$ $(k = 0, a)$ are the levels of the reservoirs as function of time; $\varphi_0(x)$ represents the level of the « free boundary » for $t = 0$; and furthermore $l(x, t)$ is the rate of fluid moving across the base. Finally we assume that such functions are « sufficiently smooth » and that the following relations are verified $(k = 0, a)$:

$$(1.1) \qquad 0 \leqslant y_k(t) < b\,, \qquad \forall t \in [0, T]\,,$$

$$(1.2) \qquad 0 < \varphi_0(x) < b\,, \qquad \forall x \in [0, a]\,,$$

$$(1.3) \qquad y_k(0) \leqslant \varphi_0(k)\,.$$

Therefore the free boundary problem may be stated as follows (see for instance [7]):

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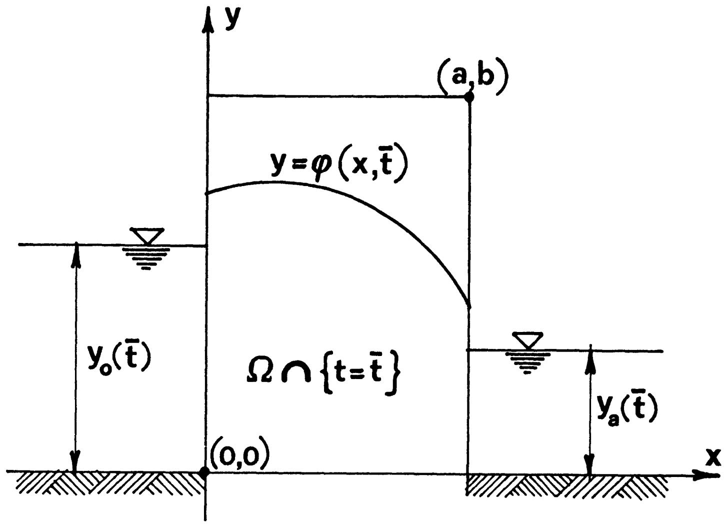

Figure at the time $t = \bar{t}$.

Problem 1.1. *We look for a triplet* $\{\varphi, \Omega, u\}$ *such that*:

i) φ *is a «regular» function defined in* $[0, a]\times[0, T]$, *satisfying*

(1.4) $$0 < \varphi(x, t) < b\,, \qquad \forall (x, t) \in [0, a]\times[0, T]\,,$$

(1.5) $$y_k(t) \leqslant \varphi(k, t)\,, \qquad \forall t \in [0, T],\ (k = 0, a)\,,$$

(1.6) $$\varphi(x, 0) = \varphi_0(x)\,, \qquad \forall x \in [0, a]\,;$$

ii) Ω *is defined by the relation*:

(1.7) $$\Omega = \{(x, y, t)\colon\ 0 < x < a,\ 0 < t < T,\ 0 < y < \varphi(x, t)\}\,;$$

iii) u *is a «regular» function defined in* $\bar{\Omega}$ *such that*:

(1.8) $$(D_{xx} + D_{yy})u = 0 \qquad \textit{in } \Omega\,,$$

(1.9) $$u(k, y, t) = y_k(t) \qquad \textit{if } y \leqslant y_k(t),\ (k = 0, a)\,,$$

(1.10) $$u(k, y, t) = y \qquad \textit{if } y_k(t) < y \leqslant \varphi(k, t),\ (k = 0, a)\,,$$

(1.11) $$u_y(x, 0, t) = -l(x, t)$$

(1.12) $$u = y \qquad \textit{on } \Gamma\,,$$

(1.13) $$u_t = u_x^2 + u_y^2 - u_y \qquad \textit{on } \Gamma\,,$$

where:

$$\Gamma = \{(x, y, t)\colon\ 0 < x < a,\ 0 < t < T,\ y = \varphi(x, t)\}\,. \tag{1.14}$$

A first study of problem 1.1 has been carried out in [20] (see also [19]), in the case where the initial datum arises from a steady state or arises from a final datum of a non steady process (see in particular Osservazione 2.2 of [20]). We shall eliminate such a restriction in the present paper.

In the *stationary* case (i.e., in the case where the levels of the reservoirs and all the other physical quantities are not time-dependent) C. Baiocchi [1] and [2], has introduced a transform which reduces such filtration process to variational or quasi-variational inequalities: Baiocchi's technique has been systematically developed and generalized at « Laboratorio di Analisi Numerica del C.N.R. » in Pavia, where several new results have been obtained (see for instance [3], [4], [5], [6], [8], [14], [17] and [18]).

In the *non-stationary* case, by adapting suitably Baiocchi's technique, we are able to reduce problem 1.1 to an equivalent (at least formally) non linear problem with an evolution boundary condition (see [19], [20] and no. 2 of the present paper). A first study of the transformed problem was carried out in [21] (see also [19]), where an existence and uniqueness theorem for the solution of the transformed problem is obtained; hence a uniqueness theorem for the solution of problem 1.1 is also obtained. It would be interesting, now, to prove an existence theorem in the physical formulation of problem 1.1 and not only in the weaker framework given by the transformed problem. In order to achieve this, it would be useful to have some regularity theorem for the solution of the transformed problem. A first result in this direction shall be presented in this paper.

In the first part of the present paper we introduce a transform (which is a variant of [19] and [20]): as we have mentioned above, by this transform we are able to study problem 1.1 with « any initial data ».

We note that problem 1.1 seems to be « well-posed » (according to the fact that the evolution condition appears only on the free boundary) under the only condition (1.6) (i.e. it is not necessary to require the value of $u(x, y, t)$ for $t = 0$).

In the second part, following a suggestion of J. L. Lions, we show that the transformed problem may be equivalently interpreted as an evolution variational inequality on $D \times]0, T[$ ($D =]0, a[\times]0, b[$ is the dam), where a time-dependent and non differentiable functional appears. By using the « regularization » technique (see for example [10] and [15]) we obtain a new existence and uniqueness theorem for the solution of the transformed problem. Such a new method, compared with the one considered in [21], requires more regularity on the data, but gives us the solution in a better functional space.

Some results of this paper have been announced in [19].

2. – Transformation of problem 1.1.

a) In the present section we assume that all the functions which appear in the formulation of problem 1.1 are « sufficiently smooth », so that the following considerations are correct (even if the hypotheses may be weakened, by proceeding similarly to [20]).

We suppose that

$$l(x, t) > -1 \quad in\ [0, a] \times [0, T]. \tag{2.1}$$

Let

$$D = \{(x, y):\ 0 < x < a,\ 0 < y < b\}, \tag{2.2}$$

$$Q = D \times]0, T[, \tag{2.3}$$

$$Q_1 = \{(x, y, t):\ 0 < x < a,\ 0 < t < T,\ 0 < y < b + T - t\}. \tag{2.4}$$

We have then (by (1.4) and (1.7)):

$$\Omega \subset Q \subset Q_1. \tag{2.5}$$

Let also:

$$\tilde{u}(x, y, t) = \begin{cases} u(x, y, t), & if\ (x, y, t) \in \bar{\Omega} \\ y, & if\ (x, y, t) \in \bar{Q}_1 - \bar{\Omega}. \end{cases} \tag{2.6}$$

Because of (1.12) it follows that:

$$\tilde{u} \in C^0(\bar{Q}_1). \tag{2.7}$$

The same arguments used in [20] may now be applied to prove that:

$$u(x, y, t) > y \qquad in\ \Omega, \tag{2.8}$$

$$\varphi_t(x, t) > -1 \qquad in\ ]0, a[\times [0, T], \tag{2.9}$$

$$(D_{xx} + D_{yy})\tilde{u} = (D_t - D_y)\chi(\Omega) \qquad (in\ the\ sense\ of\ \mathcal{D}'(Q)), \tag{2.10}$$

where $\chi(\Omega)$ is the characteristic function of Ω.

b) Let us now set

$$w(x, y, t) = \int_0^t [\tilde{u}(x, t + y - \tau, \tau) - (t + y - \tau)]\, d\tau, \tag{2.11}$$

where $(x, y, t) \in \bar{Q}$.

REMARK 2.1. The transform (2.11) differs from the one introduced in [20], in the fact that it does not require the knowledge of $u(x, y, t)$ when $t<0$; therefore, it allows us to study the problem with a « general » initial datum. The transform (2.11) (as well as the analogous one introduced in [20]) may be interpreted as a transform of Baiocchi's type made with respect to a mixed space-time direction. Some other free boundary problems of evolution, where Baiocchi's technique has proved very useful, are studied in [9], [11], [12] and [13]. We also define for $(y, t) \in [0, b]\times[0, T]$

$$\lambda_k(y, t) = \int_0^t [y_k(\tau) - (t + y - \tau)]^+ d\tau\,, \qquad (k = 0, a)\,, \tag{2.12}$$

$$Z = \{(x, y, t) \in Q : y + t < \varphi_0(x)\}\,, \tag{2.13}$$

where $v^+ = (|v| + v)/2$. Then we have (by (1.6), (1.7) and (2.9)):

$$\bar{Z} \cap Q \subset \Omega\,. \tag{2.14}$$

We obtain now:

THEOREM 2.1. *The function* w (*deduced from* $\{\varphi, \Omega, u\}$ *by* (2.6) *and* (2.11)) *has the following properties*:

$$w \in C^1(Q)\,, \tag{2.15}$$

$$w > 0 \quad in\ \Omega\,, \qquad w = 0 \quad in\ Q - \Omega\,, \tag{2.16}$$

$$(D_t - D_y) w = \tilde{u} - y \qquad in\ Q\,, \tag{2.17}$$

$$(D_{xx} + D_{yy}) w = \chi(\Omega) - \chi(Z) \qquad in\ Q\,, \tag{2.18}$$

$$w(x, y, 0) = 0\,, \tag{2.19}$$

$$w(x, b, t) = 0\,, \tag{2.20}$$

$$w(k, y, t) = \lambda_k(y, t)\,, \qquad (k = 0, a)\,, \tag{2.21}$$

$$D_t w_y(x, 0, t) + D_{xx} w(x, 0, t) = \alpha(x, t)\,, \tag{2.22}$$

where $((x, t) \in [0, a]\times[0, T])$:

$$\alpha(x, t) = \begin{cases} -l(x, t) & if\ t > \varphi_0(x)\,, \\ -l(x, t) - 1 & if\ t < \varphi_0(x)\,. \end{cases} \tag{2.23}$$

PROOF. By (2.7) and (2.11) we have obviously that w is a continuous function in $\bar{Q}$. Using (2.6), (2.8), (2.9) and (2.11), we deduce (2.16). We have also:

$$w_x(x, y, t) = \int_0^t \tilde{u}_x(x, t + y - \tau, \tau)\, d\tau , \tag{2.24}$$

$$w_y(x, y, t) = \int_0^t \tilde{u}_y(x, t + y - \tau, \tau) - 1]\, d\tau , \tag{2.25}$$

$$w_t(x, y, t) = \int_0^t [\tilde{u}_y(x, t + y - \tau, \tau) - 1]\, d\tau + \tilde{u}(x, y, t) - y . \tag{2.26}$$

(2.17) is then a consequence of (2.25) and (2.26). Let $\mathcal{T}(x, y, t)$ be the map which associates to each $(x, y, t) \in (\Omega - Z) \cup \Gamma$ the number $\mathcal{T}(x, y, t)$ verifying the relation:

$$\big(x, y + t - \mathcal{T}(x, y, t), \mathcal{T}(x, y, t)\big) \in \Gamma = \operatorname{graf}(\varphi) , \tag{2.27}$$

or equivalently

$$y + t - \mathcal{T}(x, y, t) - \varphi\big(x, \mathcal{T}(x, y, t)\big) = 0 . \tag{2.28}$$

By (2.9), we have that $\mathcal{T}(x, y, t)$ is a continuous function. We have also:

$$\mathcal{T}(x, y, t) = t \qquad \textit{if } (x, y, t) \in \Gamma , \tag{2.29}$$

$$\mathcal{T}(x, y, t) = 0 \qquad \textit{if } (x, y, t) \in \Omega \cap \partial Z . \tag{2.30}$$

By (2.6), (2.14), (2.24) and the second of (2.16), it follows:

$$w_x(x, y, t) = 0 \qquad \textit{if } (x, y, t) \in Q - \bar{\Omega} , \tag{2.31}$$

$$w_x(x, y, t) = \int_{\mathcal{T}(x,y,t)}^t \tilde{u}_x(x, t + y - \tau, \tau)\, d\tau \qquad \textit{if } (x, y, t) \in \Omega - \bar{Z} , \tag{2.32}$$

$$w_x(x, y, t) = \int_0^t u_x(x, t + y - \tau, \tau)\, d\tau \qquad \textit{if } (x, y, t) \in Z . \tag{2.33}$$

Because of (2.29) and (2.30) and since $\mathcal{T}(x, y, t)$ is a continuous function, it follows that w_x is continuous in Q. Analogously we obtain that w_y and w_t

are continuous functions in Q. Then (2.15) is verified. We have also that if $(x, y, t) \in Z$, then:

$$(D_{xx} + D_{yy}) w(x, y, t) = \int_0^t (u_{xx} + u_{yy})(x, t + y - \tau, \tau)\, d\tau$$

and therefore (by (1.8)):

$$(D_{xx} + D_{yy}) w = 0 \quad in\ Z. \tag{2.34}$$

Moreover we have (by (2.10) and (2.17)):

$$(D_t - D_y)\big((D_{xx} + D_{yy}) w - \chi(\Omega)\big) = 0 \quad in\ Q;$$

This result implies that

$$(D_{xx} + D_{yy}) w - \chi(\Omega)$$

is constant on all the straight lines that are parallel to the vector (0, 1, —1). Thanks to (1.4), (1.6), (1.7) and the second of (2.16) we have:

$$(D_{xx} + D_{yy}) w = \chi(\Omega) \quad in\ Q - Z. \tag{2.35}$$

By (2.14), (2.15), (2.34) and (2.35), it follows (2.18). (2.19), (2.20) and (2.21) are obvious. It remains only to prove (2.22). The relation (2.17) (after derivation with respect to y), shows that:

$$D_t D_y w - D_{yy} w = \tilde{u}_y - 1 \quad in\ Q,$$

then, thanks to (2.18), it follows

$$D_t D_y w + D_{xx} w - \chi(\Omega) + \chi(Z) = \tilde{u}_y - 1 \quad in\ Q.$$

Taking $t = 0$, we obtain (by (1.11)) the relation (2.22).

REMARK 2.2. We observe that the knowledge of the function w allows us (by (2.16) and (2.17)) to come back to the triplet $\{\varphi, \Omega, u\}$ of the problem 1.1.

c) Let:

$$\Gamma_n = \{(x, y): 0 < x < a,\ y = 0\}, \tag{2.36}$$

$$\Gamma_d = \partial D - \Gamma_n, \tag{2.37}$$

$$H^1_\Delta(D) = \{v \in H^1(D): \Delta v \in L^2(D)\}, \tag{2.38}$$

with the graph norm (1). We denote by γ_0 (resp. γ_1) the linear and continuous operator defined between the spaces (cfr. [16]):

$$\gamma_0 \colon H^1(D) \to H^{\frac{1}{2}}(\Gamma_n) \qquad \big(\text{resp. } \gamma_1 \colon H^1_\Delta(D) \to (H^{\frac{1}{2}}_{00}(\Gamma_n))'\big)\,, \tag{2.39}$$

and such that to every $v \in H^1(D)$ (resp. $H^1_\Delta(D)$) there corresponds the trace of v on Γ_n (resp. the trace of $D_y v$ on Γ_n). Let also F, g, α, q be given functions defined, respectively, in Q, $\Gamma_d \times]0, T[$, $\Gamma_n \times]0, T[$, Γ_n, such that $(0 < \varepsilon < \frac{1}{2})$:

$$F \in L^\infty(0, T; L^2(D))\,, \qquad g \in L^2(0, T; H^{\frac{1}{2}+\varepsilon}(\Gamma_d))\,, \tag{2.40}$$

$$\alpha \in L^2(0, T; H^{-\frac{3}{2}}(\Gamma_n))\,, \qquad q \in H^{-1}(\Gamma_n)\,. \tag{2.41}$$

Let (which is meaningful a.e. in $t \in]0, T[$):

$$K(t) = \{v \in H^1(D) \colon v = g(t) \;\; on \;\; \Gamma_d\}\,. \tag{2.42}$$

We consider now:

PROBLEM 2.1. *Find a function w such that*

$$w \in L^2(0, T; H^1(D))\,, \tag{2.43}$$

$$\Delta w \in L^\infty(0, T; L^2(D))\,, \tag{2.44}$$

$$w(t) \in K(t)\,, \tag{2.45}$$

$$\int_D \operatorname{grad} w(t) \cdot \operatorname{grad}(v - w(t))\,dx\,dy + \int_D (v^+ - w(t)^+)\,dx\,dy + \\ + \langle \gamma_1 w(t), \gamma_0(v - w(t))\rangle \geq \int_D F(t)(v - w(t))\,dx\,dy\,, \qquad \forall v \in K(t)\,, \tag{2.46}$$

$$D_t \gamma_1 w + D_{xx} \gamma_0 w = \alpha\,, \qquad \big(\textit{in the sense of } \mathcal{D}'(0, T; (H^{\frac{3}{2}}_{00}(\Gamma_n))')\big)\,, \tag{2.47}$$

$$(\gamma_1 w)(0) = q\,, \tag{2.48}$$

(2.46) *being verified for every $t \in]0, T[$, except for a set of measure zero (independent of the « test function » v), the brackets denoting the duality between $(H^{\frac{3}{2}}_{00}(\Gamma_n))'$ and $H^{\frac{1}{2}}_{00}(\Gamma_n)$ and the gradient being only in the variables x and y.*

(1) For the functional spaces used in the present paper and their main properties, we refer to [16].

We need to justify the relations (2.47) and (2.48). In fact by (2.43) and (2.44) we have (see [16]):

$$D_{xx}\gamma_0 w \in L^2\big(0,\, T;\, (H^{\frac{3}{2}}_{00}(\Gamma_n))'\big)\,, \tag{2.49}$$

$$\gamma_1 w \in L^2\big(0,\, T;\, (H^{\frac{1}{2}}_{00}(\Gamma_n))'\big)\,; \tag{2.50}$$

therefore it is possible to read (2.47) (keeping in mind the first of (2.41)) as:

$$\mathcal{D}'\big(0,\, T;\, (H^{\frac{3}{2}}_{00}(\Gamma_n))'\big)\,.$$

A consequence of (2.41), (2.47) and (2.49) is then:

$$D_t\gamma_1 w \in L^2\big(0,\, T;\, (H^{\frac{3}{2}}_{00}(\Gamma_n))'\big)\,. \tag{2.51}$$

By (2.50) and (2.51) we have (cfr. [16]):

$$\gamma_1 w \in C^0\big([0,\, T],\, H^{-1}(\Gamma_n)\big) \tag{2.52}$$

and also (2.48) has a meaning.

If we set

$$F = \chi(Z)\,, \qquad q = 0\,, \tag{2.53}$$

$$g(k,\, y,\, t) = \lambda_k(y,\, t)\,, \qquad g(x,\, b,\, t) = 0\,, \qquad\qquad (k = 0,\, a)\,, \tag{2.54}$$

and α defined as in (2.23), then it follows that the function w of theorem 2.1 is a solution of problem 2.1. Moreover in [21] we have proved:

THEOREM 2.2. *Under the hypotheses* (2.40) *and* (2.41), *the problem* 2.1 *has one and only one solution.*

Thanks to remark 2.2, the problem 2.1 may be considered a weak formulation of problem 1.1. Moreover theorem 2.2 gives as a uniqueness theorem for problem 1.1.

3. – Reduction of problem 2.1 to a variational inequality.

a) Let

$$W = \{v \in H^1(D) : v = 0 \ \text{ on } \ \Gamma_d\}\,, \tag{3.1}$$

$$V = \{v \in W : \gamma_0 \ w \in H^1_0(\Gamma_n)\}\,, \tag{3.2}$$

with their graph norms. Since $W \subset L^2(D)$ (with continuous and dense inclusion), then we may identify $L^2(D)$ to a linear subspace of W', that is:

$$W \subset L^2(D) \subset W', \tag{3.3}$$

all the imbeddings being continuous.

Having in mind the demonstration of a regularity theorem for the problem 2.1, we assume that (see (2.40) and (2.41)):

$$F \in C^0([0, T]; L^2(D)), \qquad D_t F \in L^2(0, T; W'), \tag{3.4}$$

$$g \textit{ is a trace on } \Gamma_d \times]0, T[\textit{ of a function } G \in H^2(Q), \tag{3.5}$$

$$\alpha \in L^2\big(0, T; (H^{\frac{1}{2}}_{00}(\Gamma_n))'\big), \tag{3.6}$$

$$q \in L^2(\Gamma_n). \tag{3.7}$$

REMARK 3.1. Since the problem 2.1 is connected with the filtration phenomenon described in problem 1.1, it is necessary to check that (under the physical hypotheses of problem 1.1) the functions F and g, defined by the first of (2.53) and by (2.54), verify (3.4) and (3.5). Actually it is easy to prove (²) this fact under the additional hypothesis ($k = 0, a$):

$$y_k'(t) + 1 > 0, \qquad (t \in [0, T]). \tag{3.8}$$

Consider now:

PROBLEM 3.1. *Find a function w which solves problem* 2.1 *and such that:*

$$w \in L^\infty(0, T; H^{\frac{3}{2}}(D)), \tag{3.9}$$

$$D_t w \in L^2(0, T; H^1(D)). \tag{3.10}$$

The aim of the remaining part of the present paper is to prove the following regularity result:

THEOREM 3.1. *Under the hypotheses* (3.4), (3.5), (3.6) *and* (3.7), *the problem* 3.1 *has one and only one solution.*

(²) On the other hand, the assertion $D_t F = D_t \chi(Z) \in L^2(0, T; W')$ is formally anticipated, because $D_t \chi(Z) = D_y \chi(Z)$ (in the sense of $\mathfrak{D}'(Q)$).

b) Let:

$$a(u, v) = \int_D (u_x v_x + u_y v_y)\, dx\, dy\,, \tag{3.11}$$

$$b(u, v) = \int_{\Gamma_n} D_x\gamma_0 u \cdot D_x \gamma_0 v\, dx\,, \tag{3.12}$$

$a(u, v)$ (resp. $b(u, v)$) being defined for every $u, v \in H^1(D)$ (resp. for every $u, v \in H^1(D)$, such that $\gamma_0 u$, $\gamma_0 v \in H^1(\Gamma_n)$).

Consider (for each $t \in]0, T[$) the map that to every $v \in W$, associates the number:

$$\langle L(t), v\rangle = -a\big(G(t), v\big) - \Big\langle D_x\gamma_0 v, D_x\gamma_0\Big(\int_0^t G(\tau)\, d\tau\Big)\Big\rangle + \\ + \int_D F(t) v\, dx\, dy - \Big\langle \int_0^t \alpha(\tau)\, d\tau, \gamma_0 v\Big\rangle - \int_{\Gamma_n} q\cdot\gamma_0 v\, dx\,, \tag{3.13}$$

the duality being between W' and W, between $H^{-\frac{1}{2}}(\Gamma_n)$ and $H^{\frac{1}{2}}(\Gamma_n)$, between $(H^{\frac{1}{2}}_{00}(\Gamma_n))'$ and $H^{\frac{1}{2}}_{00}(\Gamma_n)$, respectively. It is easy to prove that L (under the hypotheses (3.4), (3.5), (3.6) and (3.7)) verifies

$$L, D_t L \in L^2(0, T; W')\,; \tag{3.14}$$

PROBLEM 3.2. *Find a function* $z(x, y, t)$ *defined in* Q *such that*:

$$z, z' \in L^\infty(0, T; V)\,, \tag{3.15}$$

$$z'' \in L^2(0, T; W)\,, \tag{3.16}$$

$$z(x, y, 0) = 0\,, \tag{3.17}$$

$$a\big(z'(t), v - z'(t)\big) + b\big(z(t), v - z'(t)\big) + \\ + \int_D [(v + G(t))^+ - (z'(t) + G(t))^+]\, dx\, dy \geqslant \langle L(t), v - z'(t)\rangle\,, \qquad \forall v \in V\,, \tag{3.18}$$

where $G(t)$ *and* $L(t)$ *have been introduced in* (3.5) *and* (3.13) (*respectively*); *moreover* (3.18) *must hold for every* $t \in]0, T[$, *except for a set of measure zero* (*independent of the* « *test function* » v).

We have the following result:

THEOREM 3.2. i) *If* w *is a solution of problem* 3.1, *then*

$$z(x, y, t) = \int_0^t \big(w(x, y, \tau) - G(x, y, \tau)\big)\, d\tau \tag{3.19}$$

is a solution of problem 3.2.

ii) *Conversely if z is a solution of problem* 3.2, *then*

$$w(x, y, t) = z_t(x, y, t) + G(x, y, t) \tag{3.20}$$

is a solution of problem 3.1.

PROOF. i) We assume that w is a solution of problem 3.1 and that z is defined by (3.19). Let now:

$$\mathcal{H}^1(D) = \{v \in H^1(D) \colon v|_{\partial D} \in H^1(\partial D)\}\,. \tag{3.21}$$

Since $G \in H^2(Q)$, we have (cfr. [16])

$$G \in C^0([0, T]; H^{\frac{3}{2}}(D)) \subset L^\infty(0, T; \mathcal{H}^1(D))\,, \tag{3.22}$$

because $H^{\frac{3}{2}}(D) \subset \mathcal{H}^1(D)$. Therefore (3.15) and (3.16) are implied by (3.5), (3.9) and (3.10). Moreover (3.17) is a consequence of (3.19). Since $w(t) = z'(t) + G(t)$ (by (3.19)) and $G(t) \in K(t)$ (by (3.5)), the relation (2.46) implies

$$\begin{aligned} a\big(z'(t) + G(t), v - z'(t)\big) &+ \int\limits_D [(v + G(t))^+ - (z'(t) + G(t))^+]\, dx\, dy + \\ &+ \langle \gamma_1(z'(t) + G(t)), \gamma_0(v - z'(t))\rangle \geqslant \int\limits_D F(t)(v - z'(t))\, dx\, dy\,, \qquad \forall v \in V\,. \end{aligned} \tag{3.23}$$

Thanks to (2.47) and (2.48), in view of the properties (2.50), (2.51) and (2.52), we have:

$$\gamma_1 w(t) = -D_{xx}\int\limits_0^t (\gamma_0 w)(\tau)\, d\tau + \int\limits_0^t \alpha(\tau)\, d\tau + q\,, \tag{3.24}$$

then ($\forall v \in V$):

$$\langle \gamma_1 w(t), \gamma_0 v\rangle = b\Big(\int\limits_0^t w(\tau)\, d\tau, v\Big) + \Big\langle \int\limits_0^t \alpha(\tau)\, d\tau, \gamma_0 v\Big\rangle + \int\limits_{\Gamma_n} q \cdot \gamma_0 v\, dx \tag{3.25}$$

the dualities being, for example, between $H^{-1}(\Gamma_n)$ and $H_0^1(\Gamma_n)$.

Therefore, thanks to (3.23) and (3.24), it follows that (3.18) holds. The i) part of the theorem is verified.

ii) Assume that z is a solution of problem 3.2 and that w is given by (3.20). Since $V \subset \mathcal{H}^1(D)$, it follows (by (3.15)) that $z' \in L^\infty(0, T; \mathcal{H}^1(D))$, hence (by (3.20) and (3.22)):

$$w \in L^\infty(0, T; \mathcal{H}^1(D))\,. \tag{3.26}$$

We observe now that (3.10) is a consequence of (3.5) and (3.16). Thanks to (3.17), (3.18), (3.13) and (3.20), it follows ($\forall v \in V$)

$$a\big(w(t), v + G(t) - w(t)\big) + b\Big(\int_0^t w(\tau)\,d\tau, v + G(t) - w(t)\Big) +$$

$$+ \int_D [(v + G(t))^+ - w(t)^+]\,dx\,dy \geqslant -\Big\langle \int_0^t \alpha(\tau)\,d\tau, \gamma_0\big(v + G(t) - w(t)\big)\Big\rangle +$$

$$+ \int_D F(t)\big(v + G(t) - w(t)\big)\,dx\,dy - \int_{\Gamma_n} q\cdot\gamma_0\big(v + G(t) - w(t)\big)\,dx\,.$$

Let:

$$\tilde{K}(t) = \{v + G(t), v \in V\{\,. \tag{3.27}$$

Because of (3.5), we have:

$$\tilde{K}(t) \subset K(t)\,. \tag{3.28}$$

Then it follows ($\forall v \in \tilde{K}(t)$, a.e. in t):

$$a\big(w(t), v - w(t)\big) + b\Big(\int_0^t w(\tau)\,d\tau, v - w(t)\Big) + \int_D \big(v^+ - w(t)^+\big)\,dx\,dy \geqslant \tag{3.29}$$

$$\geqslant -\Big\langle \int_0^t \alpha \geqslant d\tau, \gamma_0\big(v - w(t)\big)\Big\rangle + \int_D F(t)\big(v - w(t)\big)\,dx\,dy - \int_{\Gamma_n} q\cdot\gamma_0\big(v - w(t)\big)\,dx\,dy\,.$$

Therefore (see (3.11)), we obtain ($\forall v \in \tilde{K}(t)$):

$$-\int_D \Delta w(t)\cdot\big(v - w(t)\big)\,dx\,dy + b\Big(\int_0^t w(\tau)\,d\tau, v - w(t)\big) + \tag{3.30}$$

$$+ \int_D [v^+ - w(t)^+]\,dx\,dy \geqslant \Big\langle -\int_0^t \alpha(\tau)\,d\tau, \gamma_0\big(v - w(t)\big)\Big\rangle +$$

$$+ \int_D F(t)\big(v - w(t)\big)\,dx\,dy + \langle \gamma_1 w(t) - q, \gamma_0\big(v - w(t)\big)\rangle\,.$$

Hence

$$0 \leqslant \Delta w(t) + F \leqslant 1 . \tag{3.31}$$

By (3.4), it follows (2.44). We remark also that (2.44) and (3.26) implie (3.9). Thanks to (3.4) and (3.30) and to the fact that $a^+ - b^+ \leqslant (a-b)^+$, it follows that there exists $c > 0$ (independent of t) such that $(\forall v \in \tilde{K}(t))$:

$$\langle \gamma_1 w(t) - q, \gamma_0(v - w(t))\rangle - \Big\langle \int_0^t \alpha(\tau)\, d\tau, \gamma_0(v - w(t))\Big\rangle - b\Big(\int_0^t w(\tau)\, d\tau, v - w(t)\Big) \leqslant$$
$$\leqslant c\|v - w(t)\|_{L^1(D)} .$$

For every $\psi \in V$, if we set $v = w(t) \pm \psi$, it follows:

$$\Big|\Big\langle \gamma_1 w(t) - q - \int_0^t \alpha(\tau)\, d\tau, \gamma_0 \psi\Big\rangle - b\Big(\int_0^t w(\tau)\, d\tau, \psi\Big)\Big| \leqslant c\|\psi\|_{L^1(D)} . \tag{3.32}$$

Since ψ is arbitrary, we have $(\forall v \in V)$:

$$\Big\langle \gamma_1 w(t) - q - \int_0^t \alpha(\tau)\, d\tau, \gamma_0 \psi\Big\rangle - b\Big(\int_0^t w(\tau)\, d\tau, \psi\Big) = 0 . \tag{3.33}$$

Because of (3.29) and (3.33) and to the fact that V is dense in W, we obtain (2.46). The relation (3.33) implies also that:

$$\gamma_1 w(t) - q - \int_0^t \alpha(\tau)\, d\tau + D_{xx} \int_0^t (\gamma_0 w)(\tau)\, d\tau = 0 ,$$

hence, taking $t = 0$, it follows (2.48). By deriving with respect to t, we obtain (2.47). The theorem is completely proved.

REMARK 3.2. Because of theorem 3.2, the proof of theorem 3.1 is reduced to verifying an existence and uniqueness theorem for the problem 3.2. On the other hand, the uniqueness theorem is already a consequence of theorem 2.2. Nevertheless it is possible to give a direct proof of uniqueness of theorem 3.1: in fact, if z_1 and z_2 are two solutions of problem 3.2, writing the inequality (3.18) for z_1 (resp. z_2), taking $v = z_2'$ (resp. $v = z_1'$) and adding the two inequalities, it follows easily that $z_1 = z_2$.

4. – Approximation of problem 3.2.

For every $m \in \boldsymbol{N}$, consider the following functions:

$$(4.1)\qquad j_m(\lambda) = \begin{cases} 0 & \text{if } \lambda \leqslant -\dfrac{1}{2m}, \\ \dfrac{1}{2}\left(m\lambda^2 + \lambda + \dfrac{1}{4m}\right) & \text{if } -\dfrac{1}{2m} < \lambda < \dfrac{1}{2m}, \\ \lambda & \text{if } \dfrac{1}{2m} \leqslant \lambda; \end{cases}$$

$$(4.2)\qquad H_m(\lambda) = \begin{cases} 0 & \text{if } \lambda \leqslant -\dfrac{1}{2m}, \\ m\lambda + \dfrac{1}{2} & \text{if } -\dfrac{1}{2m} < \lambda < \dfrac{1}{2m}, \\ 1 & \text{if } \dfrac{1}{2m} \leqslant \lambda. \end{cases}$$

We have obviously that:

$$(4.3)\qquad j_{m'}(\lambda) = H_m(\lambda), \qquad \forall \lambda \in \boldsymbol{R},$$

and that $j_m(\lambda)$ is a sequence such that $j_m(\lambda) \to \lambda^+$ and $H_m(\lambda)$ is a sequence such that $H_m(\lambda) \to H(\lambda)$ (where H is the Heaviside function). For the moment, instead of problem 3.2, we consider the following approximate problem:

PROBLEM 4.1. *$\forall m \in \boldsymbol{N}$, find z_m such that*

$$(4.4)\qquad z_m, z_m' \in L^\infty(0, T; V), \qquad z_m'' \in L^2(0, T; W),$$

$$(4.5)\qquad z_m(x, y, 0) = 0$$

$$(4.6)\qquad a\big(z_m'(t), v - z_m'(t)\big) + b\big(z_m(t), v - z_m'(t)\big) + \int_D [j_m\big(v + G(t)\big) - j_m\big(z_m'(t) + G(t)\big)]\,dx\,dy \geqslant \langle L(t), v - z_m'(t)\rangle, \qquad \forall v \in V, \text{ a.e. } in\ t.$$

We consider also:

PROBLEM 4.2. *$\forall m \in \boldsymbol{N}$, find z_m verifying* (4.4) *and* (4.5) *and*

$$(4.7)\qquad a\big(z_m'(t), v\big) + b\big(z_m(t), v\big) + \int_D H_m\big(z_m'(t) + G(t)\big)v\,dx\,dy = \langle L(t), v\rangle$$

for all $v \in V$ (a.e. *in t*).

Since $j_m(\lambda)$ is differentiable, it follows:

PROPOSITION 4.1. *The problems* 4.1 *and* 4.2 *are equivalent.*
Moreover we have the result:

THEOREM 4.1. $\forall m \in \boldsymbol{N}$, *problem* 4.1 (*or equivalently problem* 4.2) *has at least one solution* z_m. *We have also*:

(4.8) *the functions* z_m *and* z'_m *belong to a bounded set of* $L^\infty(0, T; V)$

(4.9) *the functions* z''_m *belong to a bounded set of* $L^2(0, T; W)$.

Before we prove theorem 4.1, consider the following lemmas:

LEMMA 4.1. *Let* w_0 *be the solution of problem*:

$$\Delta w_0 \in H(w_0) - F(0) \quad \text{in } D, \quad w_0 = g(0) \quad \text{on } \Gamma_d, \quad \gamma_1 w_0 = q \quad \text{on } \Gamma_n. \tag{4.10}$$

Then we have $w_0 \in H^{\frac{3}{2}}(D)$.

PROOF. Let D_1 be a « regular » open set such that $D \subset D_1$ and $\Gamma_n \subset \partial D_1$. Let also $\tilde{q} \in L^2(\partial D_1)$, such that (cfr. (3.7)):

$$\tilde{q}|_{\Gamma_n} = q. \tag{4.11}$$

Consider now the following problem ($\boldsymbol{n}$ = interior normal)

$$-\Delta h + h = 0 \quad \text{in } D_1, \qquad \frac{\partial h}{\partial \boldsymbol{n}} = q \quad \text{on } \partial D_1. \tag{4.12}$$

This is a « *regular elliptic* » problem (in the sense of [16], cap. II), therefore $h \in H^{\frac{3}{2}}(D_1)$. Let:

$$\bar{h} = h|_D;$$

then we have $\bar{h} \in H^{\frac{3}{2}}(D)$. By (3.4), (3.5), (4.10), (4.11) and (4.12), it follows:

$$\Delta(w_0 - \bar{h}) \in L^2(D), \qquad (w_0 - \bar{h})|_{\Gamma_d} \in H^1(\Gamma_d), \qquad \gamma_1(w_0 - \bar{h}) = 0.$$

We have then (by using a « symmetry argument » across the line $y = 0$), $w_0 - \bar{h} \in H^{\frac{3}{2}}(D)$. Since $\bar{h} \in H^{\frac{3}{2}}(D)$, the lemma is completely proved.

LEMMA 4.2. *$\forall m \in N$, let w_m be the solution of the following problem:*

$$(4.13) \qquad a(w_m, v) + \int_D H_m(w_m)\cdot v\,dx\,dy = \int_D F(0)\,v\,dx\,dy - \int_\Gamma q\cdot\gamma_0 v\,dx\,dy\,, \quad \forall v \in W\,,$$

$$(4.14) \qquad w_m = g(0) \quad on\ \Gamma_d\,.$$

Then we have:

(4.15) *the functions w_m belong to a bounded set of $H^{\frac{3}{2}}(D)$.*

PROOF. Let us put $\overline{w}_m = w_m - w_0$ (for the definition of w_0 see (4.10)). We have then

$$\Delta\overline{w}_m \ \textit{belong to a bounded set of}\ L^2(D)\,,$$

$$\overline{w}_m = 0 \quad on\ \Gamma_d\,, \qquad \gamma_1\overline{w}_m = 0 \quad on\ \Gamma_n\,.$$

We have that Δ is an isomorphism of:

$$\{v \in H^2(D)\colon v = 0 \ on\ \Gamma_d,\ \gamma_1 v = 0\ on\ \Gamma_n\}$$

onto $L^2(D)$ (3). Therefore

(4.16) $\overline{w}_m$ *belongs to a bounded set of $H^2(D)$,*

and also (see lemma 4.1)

(4.17) w_m *belongs to a bounded set of $H^{\frac{3}{2}}(D)$.*

PROOF OF THEOREM 4.1. i) $\forall m \in N$, we introduce a sequence λ_1^m, $\lambda_2^m, \ldots, \lambda_k^m, \ldots$ such that (cfr. lemma 4.2):

$$(4.18) \qquad \begin{cases} \lambda_k^m \in V,\ \forall k\,; \quad \lambda_1^m = w_m - G(0)\,; \\ \forall k,\ \lambda_1^m, \lambda_2^m, \ldots, \lambda_k^m \quad \textit{are linearly indipendent}\,; \\ \textit{the set of finite linear combinations of } \lambda_k^m \textit{ is dense in } V\,. \end{cases}$$

(3) To show this isomorphism it is sufficient to prove that Δ is surjective. Really if $f \in L^2(D)$ and $u \in H^1(D)$ is the solution of the problem

$$\Delta u = f \ \ in\ D\,, \qquad u = 0\ \ on\ \Gamma_d\,, \qquad \gamma_1 u = 0\ \ on\ \Gamma_n\,,$$

then $u \in H^2(D)$, by a «symmetry argument» across the line $y = 0$.

Given $m \in \boldsymbol{N}$, we look for an « approximate solution » of problem 4.2, which is a function of the type:

$$z_{mk}(t) = \sum_{i=1}^{k} g_{ik}^{m}(t)\,\lambda_i^m \tag{4.19}$$

such that

$$z_{mk}(0) = 0\,, \tag{4.20}$$

$$a(z'_{mk}(t), \lambda_j^m) + b(z_{mk}(t), \lambda_j^m) + \int_D H_m(z'_{mk}(t) + G(t))\,\lambda_j^m\,dx\,dy = \langle L(t), \lambda_j^m\rangle\,, \quad 1 \leqslant j \leqslant k\,. \tag{4.21}$$

The problem (4.20) and (4.21) is a Cauchy problem for a system of ordinary differential equations in the unknown functions $g_{ik}^m(t)$. For every $m, k \in \boldsymbol{N}$ this problem has a solution in $[0, t_{mk}]$, with $t_{mk} > 0$. The following estimates shall prove that $t_{mk} = T$.

ii) *First estimate.* Multiplying (4.21) by $D_t(g_{jk}^m(t))$ and adding (with respect to j) between 1 and k, we obtain:

$$a(z'_{mk}(t), z'_{mk}(t)) + b(z_{mk}(t), z'_{mk}(t)) + \int_D H_m(z'_{mk}(t) + G(t))\,z'_{mk}(t)\,dx\,dy = \langle L(t), z'_{mk}(t)\rangle\,;$$

then by integration on $[0, t]$ (see also (4.20)):

$$\int_0^t a(z'_{mk}(\tau), z'_{mk}(\tau))\,d\tau + \tfrac{1}{2} b(z_{mk}(t), z_{mk}(t)) + \int_0^t \Big[H_m(z'_{mk}(\tau) + G(\tau))\cdot z'_{mk}(\tau)\,dx\,dy\Big]\,d\tau = \int_0^t \langle L(\tau), z'_{mk}(\tau)\rangle\,d\tau\,. \tag{4.22}$$

Since H_m is a bounded function (uniformly in m) and since $L \in L^2(0, T; W')$ (cfr. (3.14)), then we have

$$\int_0^t a(z'_{mk}(\tau), z'_{mk}(\tau))\,d\tau + \tfrac{1}{2} b(z_{mk}(t), z_{mk}(t)) \leqslant c_1\,, \tag{4.23}$$

where c_1 is a constant independent from m and k.

Thanks to (4.20) and (4.23), there exist two constants c_1' and c_1'' (independent from m and k) such that

$$\|z_{mk}\|_{L^\infty(0;T;V)} \leqslant c_1' , \tag{4.24}$$

$$\|z'_{mk}\|_{L^2(0,T;W)} \leqslant c_1'' . \tag{4.25}$$

iii) *Second estimate.* Differentiating (4.21) with respect to t, it follows:

$$a\big(z''_{mk}(t), \lambda_j^m\big) + b\big(z'_{mk}(t), \lambda_j^m\big) + \int_D \big(H_m(z'_{mk}(t) + G(t))\big)' \lambda_j^m \, dx\, dy = \tag{4.26}$$
$$= \langle L'(t), \lambda_j^m \rangle , \qquad 1 \leqslant j \leqslant k .$$

Multiplying the relations (4.26) by $D_{tt}\big(g_{jk}^m(t)\big)$ and adding (with respect to j) between 1 and k, we have

$$a\big(z''_{mk}(t), z''_{mk}(t)\big) + b\big(z'_{mk}(t), z''_{mk}(t)\big) +$$
$$+ \int_D \big(H_m(z'_{mk}(t) + G(t))\big)' z''_{mk}(t)\, dx\, dy = \langle L'(t), z''_{mk}(t) \rangle ,$$

then by integration on $[0, t]$

$$\int_0^t a\big(z''_{mk}(\tau), z''_{mk}(\tau)\big)\, d\tau + \tfrac{1}{2}\big[b\big(z'_{mk}(t), z'_{mk}(t)\big) - b\big(z'_{mk}(0), z'_{mk}(0)\big)\big] + \tag{4.27}$$
$$+ \int_0^t \Big[\int_D \big(H_m(z'_{mk}(\tau) + G(\tau))\big)' z''_{mk}(\tau)\, dx\, dy\Big]\, d\tau = \int_0^t \langle L'(\tau), z''_{mk}(\tau) \rangle\, d\tau .$$

By (3.14), we have ($c_2 > 0$ and $\varepsilon < 1$ independent from m and k);

$$\int_0^t \langle L'(\tau), z''_{mk}(\tau) \rangle\, d\tau \leqslant c_2 + \varepsilon \int_0^t \|z''_{mk}(\tau)\|_W^2\, d\tau .$$

Then it follows (by (4.27))

$$(1-\varepsilon)\int_0^t a\big(z''_{mk}(\tau), z''_{mk}(\tau)\big)\, d\tau + \tfrac{1}{2} b\big(z'_{mk}(t), z'_{mk}(t)\big) + A \leqslant \tag{4.28}$$
$$\leqslant B + c_2 + \tfrac{1}{2} b\big(z'_{mk}(0), z'_{mk}(0)\big) ,$$

where

$$(4.29)\qquad A=\int_0^t\Big[\int_D (H_m(z'_{mk}(\tau)+G(\tau)))'(z'_{mk}(\tau)+G(\tau))'\,dx\,dy\Big]d\tau\,,$$

$$(4.30)\qquad B=\int_0^t\Big[\int_D (H_m(z'_{mk}(\tau)+G(\tau)))'G'(\tau)\,dx\,dy\Big]d\tau\,.$$

We have that

$$A=\int_0^t\Big[\int_D H'_m(z'_{mk}(\tau)+G(\tau))\cdot\big((z'_{mk}(\tau)+G(\tau))'\big)^2\,dx\,dy\Big]d\tau,$$

then (since H_m is a non decreasing function):

$$(4.31)\qquad A\geqslant 0\,.$$

We have also (by integration by parts):

$$(4.32)\qquad B=\Big[\int_D H_m(z'_{mk}(\tau)+G(\tau))\cdot G'(\tau)\,dx\,dy\Big]_0^t-$$

$$-\int_0^t\Big[\int_D H_m(z'_{mk}(\tau)+G(\tau))\cdot G''(\tau)\,dx\,dy\Big]d\tau\,.$$

By (3.5), it follows:

$$G'\in L^2(0,T;H^1(D)),\qquad G''\in L^2\big(0,T;L^2(D)\big),$$

then we have (cfr. [16])

$$G'\in C^0\big([0,T];H^{\frac12}(D)\big)\subset L^\infty\big(0,T;L^1(D)\big)\,.$$

Since H_m is (uniformly in m) bounded, then the first term which appear in the second member of (4.32) is bounded. The second term is also bounded, since $G''\in L^2(Q)$.

Then we have (c_3 independent from m and k):

$$(4.33)\qquad B\leqslant c_3\,.$$

By writing (4.21) with $t=0$ (keeping in mind (3.13) and (4.20)), we have $1\leqslant j\leqslant k$

$$a(z'_{mk}(0),\lambda_j^m)+\int_D H_m(z'_{mk}(0)+G(0))\lambda_j^m\,dx\,dy=$$

$$=-a(G(0),\lambda_j^m)+\int_D F(0)\lambda_j^m\,dx\,dy-\int_{\Gamma_n} q\cdot\gamma_0\lambda_j^m\,dx\,dy\,.$$

If we put (for the definition of w_m, cfr. lemma 4.2):

$$\mu_m = w_m - G(0) \tag{4.34}$$

then we have $(1 \leqslant j \leqslant k)$:

$$a(\mu_m, \lambda_j^m) + \int_D H_m(\mu_m + G(0))\lambda_j^m \,dx\,dy =$$
$$= -a(G(0), \lambda_j^m) + \int_D F(0)\lambda_j^m \,dx\,dy - \int_{\Gamma_n} q\cdot\gamma_0\lambda_j^m \,dx$$

and therefore (cfr. also (4.18)):

$$\mu_m = z'_{mk}(0) = w_m - G(0)\,. \tag{4.35}$$

By lemma 4.2 and (3.5), it follows that functions $z'_{mk}(0)$ belong to a bounded set of $H^{\frac{3}{2}}(D)$ and then (c_4 independent from m and k):

$$b(z'_{mk}(0), z'_{mk}(0)) \leqslant c_4\,. \tag{4.36}$$

By (4.28), (4.31), (4.33) and (4.36), it follows that:

$$\|z'_{mk}\|_{L^\infty(0,T;V)} \leqslant c'_5\,, \tag{4.37}$$

$$\|z''_{mk}\|_{L^2(0,T;W)} \leqslant c''_5\,, \tag{4.38}$$

where c'_5 and c''_5 are two constants independent from m and k.

iv) By (4.24), (4.37) and (4.38) and by the fact that H_m is (uniformly in m) bounded, we have that there exist z_m and χ_m ($m \in \mathbf{N}$), such that:

(4.39) z_m and z'_m *belongs to a bounded set of* $L^\infty(0, T; V)$,

(4.40) z''_m *belongs to a bounded set of* $L^2(0, T; W)$,

(4.41) χ_m *belongs to a bounded set of* $L^\infty(Q)$

and such that there exists a subsequence z_{mr} of z_{mk} verifying:

(4.42) $z_{mr} \to z_m$ *in* $L^\infty(0, T; V)$ *weak**,

(4.43) $z'_{mr} \to z'_m$ *in* $L^\infty(0, T; V)$ *weak**,

(4.44) $z'_{mr} \to z''_m$ *in* $L^2(0, T; W)$ *weak*,

(4.45) $H_m(z'_{mr} + G) \to \chi_m$ *in* $L^\infty(Q)$ *weak**;

(4.20), (4.42) and (4.43) show us that:

$$z_m(0) = 0\,. \tag{4.46}$$

Passing to the limit in (4.21), taking $k = r$ and j fixed, it follows $(\forall j \in \mathbf{N})$

$$a(z'_m(t), \lambda_j^m) + b(z_m(t), \lambda_j^m) + \int_D \chi_m(t)\,\lambda_j^m\,dx\,dy = \langle L(t), \lambda_j^m\rangle \tag{4.47}$$

and then (by (4.18)):

$$a(z'_m(t), v) + b(z_m(t), v) + \int_D \chi_m(t)\,v\,dx\,dy = \langle L(t), v\rangle\,, \qquad \forall v \in V\,. \tag{4.48}$$

To complete the proof of theorem 4.1, it remains only to verify:

$$\chi_m = H_m(z'_m + G)\,. \tag{4.49}$$

At this aim, by denoting with (,) the scalar product in $L^2(D)$ and by setting $(\psi \in H^1(Q))$

$$X_{mr} = \int_0^T \big(H_m(z'_{mr}(t) + G(t)) - H_m(\psi(t))\,, \quad z'_{mr}(t) + G(t) - \psi(t)\big)\,dt\,, \tag{4.50}$$

we have (by the fact that H_m is a monotone function):

$$X_{mr} \geqslant 0\,. \tag{4.51}$$

By (4.22) we have also:

$$X_{mr} = -\int_0^T [a(z'_{mr}(t), z'_{mr}(t)) - \langle L(t), z'_{mr}(t)\rangle]\,dt - \tfrac{1}{2} b(z_{mr}(T), z_{mr}(T)) + \tag{4.52}$$
$$+ \int_0^T (H_m(z'_{mr}(t) + G(t)), G(t) - \psi(t))\,dt - \int_0^T (H_m(\psi(t)), z'_{mr}(t) + G(t) - \psi(t))\,dt\,.$$

By (4.42) and (4.43), we have:

$$z_{mr} \to z_m \qquad \textit{in } C^0([0, T]; V) \quad \textit{weak}\,,$$

hence

$$\gamma_0 z_{mr}(T) \to \gamma_0 z_m(T) \qquad \textit{in } H^1(\Gamma_n) \qquad \textit{weak}\,;$$

therefore

$$\limsup \big(-b(z_{mr}(T), z_{mr}(T)\big) \leqslant -b\big(z_m(T), z_m(T)\big)\,. \tag{4.53}$$

By (4.51), (4.52) and (4.53), it follows:

$$0 \leqslant \limsup X_{mr} \leqslant -\int_0^T \big[a\big(z_m'(t), z_m'(t)\big) - \langle L(t), z_m'(t)\rangle\big]\,dt - \tag{4.54}$$

$$-\tfrac{1}{2}b\big(z_m(T), z_m(T)\big) + \int_0^T \big(\chi_m(t), G(t) - \psi(t)\big)\,dt - \int_0^T \big(H_m(\psi(t)), z_m'(t) + G(t) - \psi(t)\big)\,dt\,.$$

On the other hand, by integrating (4.48) in $[0, T]$ and by taking $v = z_m'(t)$, it follows that:

$$\int_0^T \big[a\big(z_m'(t), z_m'(t)\big) - \langle L(t), z_m'(t)\rangle\big]\,dt + \tfrac{1}{2}b\big(z_m(T), z_m(T)\big) + \tag{4.55}$$

$$+ \int_0^T \big(\chi_m(t), z_m'(t)\big)\,dt = 0\,.$$

A consequence of (4.54) and (4.55) is then

$$\int_0^T \big(\chi_m - H_m(\psi(t)), z_m'(t) + G(t) - \psi(t)\,dt \geqslant 0\,. \tag{4.56}$$

Let now ϑ be an arbitrary function belonging to $H^1(Q)$. By taking $\psi = z_m' + G - \lambda\vartheta$ $(\lambda > 0)$ in (4.56) and by dividing with respect to λ, it follows:

$$\int_0^T \big(\chi_m(t) - H_m(z_m'(t) + G(t) - \lambda\vartheta(t), \vartheta(t)\big)\,dt \geqslant 0\,;$$

hence (as $\lambda \to 0$):

$$\int_0^T \big(\chi_m(t) - H_m(z_m'(t) + G(t)), \vartheta(t)\big)\,dt \geqslant 0\,, \qquad \forall \vartheta \in H^1(Q)\,.$$

Then (4.49) holds and then theorem 4.1 is completely proved.

5. – Proof of theorem 3.1.

a) The uniqueness for the theorem 3.1 has been already proved (cfr. remarks 3.2). As regards the existence, we have shown that this is reduced (cfr. theorem 3.2) to the existence of a solution for problem 3.2. Before obtaining this result, we prove

LEMMA 5.1. *If* v_m *is a sequence such that*

$$v_m \to v \qquad \text{in } L^2(0, T; H^1(D)) \quad (\text{weak}), \tag{5.1}$$

$$D_t v_m \to D_t v \qquad \text{in } L^2(0, T; H^1(D)) \quad (\text{weak}), \tag{5.2}$$

then (*for the definition of* $j_m(\lambda)$, cfr. (4.1))

$$\lim_{m\to\infty} \int_Q j_m(v_m)\, dx\, dy\, dt = \int_Q v^+\, dx\, dy\, dt\,. \tag{5.3}$$

PROOF. It is easy to verify that:

$$|\lambda^+ - j_m(\lambda)| \leqslant 1/m\,, \qquad \forall \lambda \in \boldsymbol{R} \tag{5.4}$$

By (5.1) and (5.2) we have:

$$v_m \to v \qquad \text{in } L^2(Q) \quad (\text{strong})\,. \tag{5.5}$$

We have also:

$$\left|\int_Q \big(v^+ - j_m(v_m)\big)\, dx\, dy\, dt\right| \leqslant \int_Q |v^+ - v_m^+|\, dx\, dy\, dt + \int_Q |v_m^+ - j_m(v_m)|\, dx\, dy\, dt\,. \tag{5.6}$$

The first term appearing in the second member of (5.6) converges to zero (as $m \to \infty$) by (5.5), and the second term is $\leqslant \mu(Q)/m$ by (5.4), where $\mu(Q)$ is the volume of Q.

b) According to theorem 4.1, we may consider a subsequence of the sequence z_m (we shall denote, for simplicity, this subsequence also with z_m) such that there exists z verifying:

$$z,\, z' \in L^\infty(0, T;\, V)\,, \qquad z'' \in L^2(0, T;\, W) \tag{5.7}$$

and such that

(5.8) $z_m \to z$ *in* $L^\infty(0, T; V)$ (*weak**),

(5.9) $z'_m \to z'$ *in* $L^\infty(0, T; V)$ (*weak**),

(5.10) $z''_m \to z''$ *in* $L^2(0, T; W)$ (*weak*),

(5.11) $z(0) = 0$

A consequence of (4.6) is then $\big(\forall v \in L^2(0, T; V)\big)$:

$$\int_0^T \Big[a\big(z'_m(t), v(t)\big) + b\big(z_m(t), v(t)\big) + \int_D j_m\big(v(t) + G(t)\big)\,dx\,dy - \langle L(t), v(t)\rangle\Big]\,dt \geqslant$$
$$\geqslant \int_0^T \Big[a\big(z'_m(t), z'_m(t)\big) + \int_D j_m\big(z'_m(t) + G(t)\big)\,dx\,dy - \langle L(t), z'_m(t)\rangle\Big]\,dt + \tfrac{1}{2} b\big(z_m(T), z_m(T)\big).$$

By passing to the limit (as $m \to \infty$) and by using (5.8), (5.9) and lemma 5.1, we have

$$\int_0^T \Big[a\big(z'(t), v(t)\big) + b\big(z(t), v(t)\big) + \int_D \big(v(t) + G(t)\big)^+\,dx\,dy - \langle L(t), v(t)\rangle\Big]\,dt \geqslant$$
$$\geqslant \int_0^T \Big[a\big(z'(t), z'(t)\big) + \int_D \big(z'(t) + G(t)\big)^+\,dx\,dy - \langle L(t), z'(t)\rangle\Big]\,dt + \tfrac{1}{2} b\big(z(T), z(T)\big).$$

Since $\tfrac{1}{2} b\big(z(T), z(T)\big) = \int_0^T b(z, z')\,dt$, we have $\big(\forall v \in L^2(0, T; V)\big)$:

(5.12)
$$\int_0^T \Big[a(z', v - z') + b(z, v - z') + \int_D \big[(v + G)^+ - (z' + G)^+\big]\,dx\,dy - \langle L, v - z'\rangle\Big]\,dt \geqslant 0.$$

c) Since V is separable, there exists a sequence

(5.13) $v_1, v_2, \ldots, v_r, \ldots \quad (v_r \in V,\ \forall r \in \boldsymbol{N})$,

such that $\{v_1, v_2, \ldots, v_r, \ldots\}$ is dense in V. It is well-known that if $\lambda \in L^1(0, T; B)$ (where B is a Banach space), then $(h \in \boldsymbol{N})$

(5.14)
$$\lim_{h\to\infty} \frac{1}{2h} \int_{t-1/h}^{t+1/h} \lambda(s)\,ds = \lambda(t) \qquad (\text{a.e. } in\ t)$$

The points where (5.14) is verified are usually called Lebesgue points.

Let E_r be the set of the Lebesgue points common to the following functions:

$$t \to a\big(z'(t),\, v_r - z'(t)\big)\,, \tag{5.15}$$

$$t \to b\big(z(t),\, v_r - z'(t)\big)\,, \tag{5.16}$$

$$t \to \int_D \big(v_r + G(t)\big)^+ dx\, dy\,, \tag{5.17}$$

$$t \to \int_D \big(z'(t) + G(t)\big)^+ dx\, dy\,, \tag{5.18}$$

$$t \to \langle L(t),\, v_r \rangle\,. \tag{5.19}$$

We obtain now that $m\big(]0, T[- E_r\big) = 0$, where m is the Lebesgue measure. If we set $E = \bigcap\limits_{r=1}^{\infty} E_r$, then we have

$$m\big(]0, T[- E\big) = 0\,. \tag{5.20}$$

Let now $(\forall t \in E)$:

$$\psi_r(t) = \begin{cases} v_r & \textit{if } t \in \left[t - \dfrac{1}{h},\, t + \dfrac{1}{h}\right] \\[2ex] z'(t) & \textit{if } t \notin \left[t - \dfrac{1}{h},\, t + \dfrac{1}{h}\right]. \end{cases} \tag{5.21}$$

We have then, taking $v = \psi_r(t)$ in (5.12),

$$\int\limits_{t+1/h}^{t+1/h} \Big\{a\big(z'(s),\, v_r - z'(s)\big) + b\big(z(s),\, v_r - z'(s)\big) + \\ + \int_D \big[\big(v_r + G(t)\big)^+ - \big(z'(t) + G(t)\big)^+\big]\, dx\, dy\Big\}\, ds \geq \int\limits_{t-1/h}^{t-1/h} \langle L(s),\, v_r - z'(s)\rangle\, ds\,;$$

multiplying by $1/2h$ and passing to the limit, it follows $(\forall t \in E)$:

$$a\big(z'(t),\, v_r - z'(t)\big) + b\big(z(t),\, v_r - z'(t)\big) + \\ + \int_D \big[\big(v_r + G(t)\big)^+ - \big(z'(t) + G(t)\big)^+\big]\, dx\, dy \geq \langle L(t),\, v_r - z'(t)\rangle\,, \qquad \forall r \in \boldsymbol{N}\,.$$

Since the set (5.13) is dense in V, we obtain the relation (3.18). The properties (3.15), (3.16), (3.17) has been already proved. Then theorem 3.1 is completely verified.

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