ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

DAVID G. SCHAEFFER

Some examples of singularities in a free boundary

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 4, nº 1 (1977), p. 133-144
<http://www.numdam.org/item?id=ASNSP_1977_4_4_1_133_0>

Some Examples of Singularities in a Free Boundary (*).

DAVID G. SCHAEFFER (**)

dedicated to Hans Lewy

Let Ω be a domain in $\boldsymbol{R}^n$ with a smooth boundary and let ψ be a smooth function on Ω with $\psi < 0$ on the boundary. We consider the following constrained variational problem (the so-called obstacle problem): Minimize the Dirichlet integral $\int |\text{grad}\, u|^2 dx$ over the closed convex subset $\mathcal{K}$ of the Sobolev space $H_1(\Omega)$:

$$\mathcal{K} = \{u \in H_1(\Omega) \colon u = 0 \text{ on } \partial\Omega,\ u \geqslant \psi \text{ in } \Omega\}.$$

The existence of a unique minimizing function u is trivial, and it is known [1] that u is C^1 with Lipschitz continuous first derivatives. The most interesting questions here focus on the contact set

$$I = \{x \in \Omega \colon u(x) = \psi(x)\}.$$

In this paper we construct examples where ∂I is singular even though the obstacle function is super-harmonic and real analytic or C^∞. In the real analytic case (§ 1) ∂I has an isolated singularity of the form illustrated in Fig. 1 or 2. We think it noteworthy in Fig. 1 that at the double point ∂I consists of two tangent curves, not two curves intersecting at a non-zero angle as is the generic situation for the level sets of a smooth function at a saddle point, and in Fig. 2 that at the cusp point $x \sim y^{\frac{2}{5}}$, not the $\frac{2}{3}$ power one might have expected. In the C^∞ case (§ 2) ∂I may have more or less

(*) Research supported in part under NSF contract GP22927.
(**) Alfred P. Sloan fellow. M.I.T., Cambridge, Massachusetts and I.H.E.S., Bures-sur-Yvette, France.
Pervenuto alla Redazione il 5 Maggio 1976.

arbitrary behavior along a subspace of codimension 1; in particular I may contain an infinite number of components. To our mind these examples show the need for a generic theory.

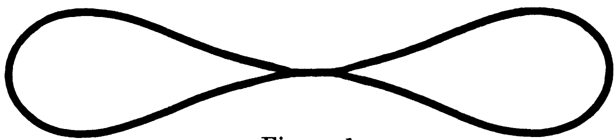

Figure 1

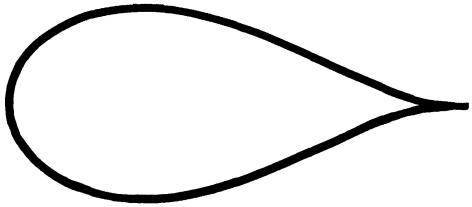

Figure 2

It is readily seen that the minimizing function u is harmonic in $\Omega \sim I$ and vanishes at $\partial\Omega$. Moreover $u \equiv \psi$ on I and is C^1 across ∂I so we have the free boundary conditions

$$u = \psi, \quad \operatorname{grad} u = \operatorname{grad} \psi \quad \text{on } \partial I. \tag{1}$$

Conversely, suppose that $I \subset \Omega$ is a closed set with a piecewise C^1 boundary and that u is a harmonic function on $\Omega \sim I$ which vanishes at $\partial\Omega$ and satisfies (1); we claim that the minimizing function is given by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for } x \in \Omega \sim I \\ \psi(x) & \text{for } x \in I, \end{cases}$$

provided $u \geqslant \psi$ on $\Omega \sim I$ and $\Delta\psi \leqslant 0$ on I. In the notation of variational inequalities (see [2]) the minimizing function may be uniquely characterized by the relation

$$-\Delta u + \beta(u - \psi) \ni 0, \tag{2}$$

where β is the maximal monotone graph

$$\beta(v) = \begin{cases} 0 & \text{if } v > 0 \\ (-\infty, 0] & \text{if } v = 0. \end{cases}$$

The reader may easily verify that $\tilde{u}$ is a solution of (2), thereby proving the claim. To construct our examples we introduce a set I with a singularity of the type desired and then determine a harmonic function satisfying (1).

1. – Examples with a real analytic obstacle.

Both examples of the present section are two dimensional, and their construction is greatly simplified by the use of complex notation, $z = x + iy$. Let ψ be an obstacle function on an open set Ω and let $I \subset \Omega$ be a closed set with a piecewise C^1 boundary. Suppose $f(z)$ is a (complex) analytic function on $\Omega \sim I$ such that

$$f = \psi_x - i\psi_y \quad \text{on } \partial I . \tag{1.1}$$

Consider the indefinite integral of f,

$$u(z) = Rl\int f(z)\, dz . \tag{1.2}$$

In principle u need not be single-valued, because $\Omega \sim I$ is not simply connected. However we observe that

$$Rl\int_{\partial I} f(z)\, dz = Rl\int_{\partial I} (\psi_x - i\psi_y)\, dz = 0 ,$$

as $Rl(\psi_x - i\psi_y)\, dz$ is the differential of ψ. Thus the integral of f around any closed curve is pure imaginary or zero, so u is well defined. Of course u is harmonic and $u_x - iu_y = f$. It follows from (1.1) that $\operatorname{grad} u = \operatorname{grad} \psi$ on ∂I, and by an appropriate choice of the constant of integration in (1.2) we may also arrange that $u = \psi$ on ∂I. In conclusion, given an analytic function on $\Omega \sim I$ satisfying (1.1), equation (1.2) defines a harmonic function there which satisfies (1).

In this paragraph we define the contact set I_j, $j = 1$ or 2, for our two examples. Consider the analytic function

$$\phi(z) = (z + z^{-1})/2 + \varepsilon P(z)(z - z^{-1})/2 , \tag{1.3}$$

where ε is a real parameter and $P(z)$ is chosen as follows:

In example 1, $\quad P_1(z) = z^2 + 2 + z^{-2}$,

In example 2, $\quad P_2(z) = (z - 2 + z^{-1})^2$.

If $|z|=1$ then $z^{-1}=\bar{z}$, so we have the relations

$$\left.\begin{aligned} (z+z^{-1})/2 &= Rl\, z \\ (z-z^{-1})/2 &= i\, \mathrm{Im}\, z \\ P_1(z) &= |z^2+1|^2 \\ P_2(z) &= |z\ -1|^4 \end{aligned}\right\} \quad |z|=1\,.$$

It follows from these relations that for example 1

$$\phi(e^{i\theta}) = \cos\theta + \varepsilon i |e^{2i\theta}+1|^2 \sin\theta\,.$$

Therefore if $\varepsilon>0$ then ϕ maps the unit circle S onto a curve C_1 of the form shown in Fig. 1, the inverse image of the double point being $\pm i$. It may be seen similarly that in example 2, the curve of Fig. 2 displays the qualitative features of the image C_2 of S under ϕ; the derivative of ϕ is non-zero on S except at $z=1$. For $\varepsilon=0$, observe that ϕ is a conformal map of $\{z\colon |z|>1\}$ onto the plane cut along the real axis from -1 to 1. By choosing $\varepsilon>0$ small, we may arrange that, in either example 1 or example 2, ϕ maps the circle $\{z\colon |z|=2\}$ onto a Jordan curve Γ_j which contains C_j in its interior. Let I_j be the (closed) region inside C_j and let Ω_j be the (open) region inside Γ_j. It follows from the principle of the argument that ϕ maps the annulus

$$A = \{z\colon 1<|z|<2\}$$

one-to-one onto $\Omega_j \sim I_j$, as we have the formula

$$\operatorname*{var\,arg}_{z\in\partial A}\,[f(z)-\zeta] = \begin{cases} 2\pi & \text{if } \zeta\in\Omega_j\sim I_j \\ 0 & \text{otherwise}, \end{cases}$$

the variation being computed with z moving counter-clockwise around the outer circle of ∂A and clockwise around the inner circle. (The variation around both circles vanishes if ζ is outside Ω_j, and the two contributions cancel for ζ in I_j, provided $\varepsilon>0$.) In our subsequent calculations we suppress the subscript j; I will denote one of the sets I_j, it not mattering for the argument which.

Consider the obstacle function $\psi(x,y)=-(x^2+y^2)/2$, for which equation (1.1) takes the form

$$f(z) = -\bar{z} \quad \text{if } z\in\partial I\,.$$

Using the identification of $\Omega \sim I$ and A provided by ϕ, we may re-write this equation as

$$f \circ \phi(z) = -\overline{\phi(z)} \quad \text{if } |z| = 1 . \tag{1.4}$$

But $\overline{\phi(z)} = \phi(\bar{z})$, and for $|z| = 1$ we have $\bar{z} = 1/z$, so the right hand side of (1.4) equals $-\phi(1/z)$. Thus

$$f(z) = -\phi\big(1/\phi^{-1}(z)\big)$$

is analytic on $\Omega \sim I$ and satisfies (1.1), where $\phi^{-1}\colon \Omega \sim I \to A$ is the inverse function of $\phi | A$. Hence (1.2) defines a harmonic function u on $\Omega \sim I$ which verifies (1). We will prove below that

$$u > \psi \quad \text{on } (\Omega \sim I) \cup \partial\Omega . \tag{1.5}$$

Let v be the harmonic function on Ω (not just $\Omega \sim I$, the domain of u) such that $v = u$ on $\partial\Omega$. The obstacle function $\psi' = \psi - v$ is real analytic and super-harmonic, and by (1.5) $\psi' < 0$ on $\partial\Omega$. Moreover $u' = u - v$ is harmonic on $\Omega \sim I$, vanishes at $\partial\Omega$, verifies (1) for the obstacle ψ', and by (1.5) satisfies $u' > \psi'$ on $\Omega \sim I$. As discussed above, I is therefore the region of contact for the obstacle problem with ψ' as data.

It remains to prove (1.5). Using the change of variable formula for integrals,

$$\int_{\phi(\Gamma)} f(z)\, dz = \int_{\Gamma} f \circ \phi(z) \phi'(z)\, dz ,$$

we see that (1.5) is equivalent to

$$|\phi(z)|^2/2 - Rl\int \phi(1/z)\phi'(z)\, dz > 0 \quad \text{if } z \in A' , \tag{1.6}$$

where $A' = A \cup \{|z| = 2\}$. Note that the left hand side of (1.6), which we denote by $U(z)$, may be written

$$U(z) = Rl\int [\overline{\varphi(z)} - \varphi(1/z)]\varphi'(z)\, dz ; \tag{1.6a}$$

the contour here may be started anywhere on the unit circle as the integrand vanishes identically there. We first show that $U(z)$ is non-negative on ∂A. Of course $U(z)$ vanishes for $|z| = 1$, and we need only consider $|z| = 2$. We write the two terms in (1.3) as

$$\phi(z) = \phi_+(z) + \varepsilon\phi_-(z) ,$$

the notation being chosen because $\phi_{\pm}(1/z) = \pm\phi(z)$. Now $U(z)$ depends quadratically on ε, say

$$U(z) = a(z) + \varepsilon b(z) + \varepsilon^2 c(z) .$$

A simple calculation shows that

$$a(z) = \tfrac{1}{2}\{|\phi_+(z)|^2 - Rl\phi_+^2(z)\} = [\operatorname{Im}\phi_+(z)]^2$$

and hence

$$a(re^{i\theta}) = \tfrac{1}{4}(r - r^{-1})^2 \sin^2\theta . \tag{1.7}$$

In particular $a(z)$ is positive on $\{|z| = 2\}$ except at $z = \pm 2$. For the next term, we may derive from (1.6a) the formula

$$b(z) = 2\int_1^z \varphi_-(x)\varphi_+'(x)\,dx , \qquad \text{if } z \text{ is real and positive},$$

from which we conclude that $b(2) > 0$, as the integrand is non-negative. Similar analysis shows that $b(-2) > 0$. (This computation may also be reduced to an inspection of signs if the contour in (1.6a) is started at $z = -1$. As noted above, we are free to do so.) It follows that $U(z) > 0$ for $|z| = 2$, provided ε is chosen sufficiently small and positive.

We see from (1.6a) that

$$U_x - iU_y(z) = \{\bar{\phi}(z) - \phi(1/z)\}\phi'(z) . \tag{1.8}$$

Since ϕ is a conformal map of A, the second factor on the right in (1.8) is non-vanishing on A. It is readily computed that the first factor vanishes if and only if

$$\operatorname{Im}\phi_+(z) = 0 \qquad \text{and} \qquad \operatorname{Rl}\phi_-(z) = 0 .$$

By (1.7), $\operatorname{Im}\phi_+(z)$ is non-zero on the (open) annulus A except for z real, where $\operatorname{Rl}\phi_-(z)$ is non-zero. Thus the first factor in (1.8) is also non-vanishing. Since the gradient of U is non-zero on A, the minimum of $U(z)$ for $z \in \operatorname{Cl}(A)$ must be assumed for $z \in \partial A$. By the computation of the preceding paragraph, this minimum is zero, and (1.6) just expresses the fact that U cannot assume its minimum at an interior point. This completes the proof that I is a possible contact set for the obstacle problem.

The intuitive origin of these singularities is presumably the following. Let ψ_1 and ψ_2 be two obstacle functions whose regions of contact consist of one and two components respectively, the components being bounded by non-singular Jordan curves. Consider the one parameter family of obstacle functions $\{\alpha\psi_1 + (1-\alpha)\psi_2\}$ with contact sets $I(\alpha)$. As α varies from zero to one, it seems that either one of the two components of $I(0)$ must shrink to zero and disappear or else the two components must flow towards one another and unite. The singularity of Fig. 1 surely arises when two components join one another. As we mentioned above, at the double point ∂I consists of two tangent curves. It can be shown that two curves intersecting at a non-zero angle is not a possible singularity of ∂I if ψ is super-harmonic—the proof is by a localization of the conformal mapping argument of Lewy-Stampacchia [4] as is done for example in [3] or [3′].

A shorter proof of the existence of the singularity in Fig. 1 is available. Let C be an analytic curve with the qualitative features displayed in Fig. 1 (Formula (1.3) provides such a curve.), and let I be the region inside C. By the Cauchy-Kowalewsky theorem, the Cauchy problem $\Delta U = 1$ with homogeneous data on C may be solved in some neighborhood of C. That is, there is a neighborhood Ω of I and a solution u of the Cauchy problem on $\Omega \sim I$. We may assume, by shrinking Ω if necessary, that $u > 0$ on $\Omega \sim I$. Then u is the solution of a variational inequality

$$-\Delta u + \beta(u) \ni 1$$

with the constraint $u \geqslant 0$, and one may solve a Dirichlet problem as above to reduce to the case of an inhomogeneous constraint $u \geqslant \psi$ with a homogeneous equation as in (2). Indeed this is essentially the construction of § 2, although in that section the fact that C is only C^∞ necessitates some modifications. We have presented the example given above because it gives a singularity of the type illustrated in Fig. 1 on a domain Ω which is not itself pinched to a near figure eight.

Our understanding of the singularity in Fig. 2 is as follows. Let ψ_j, $j = 1$ or 2, be chosen as above; that is, the contact set for ψ_j has j non-singular components, and as ψ_2 is deformed continuously to ψ_1 the two components of I flow together and unite. Choose a third function ψ_0 whose contact set has one non-singular component and such that as ψ_2 is deformed continuously to ψ_0, one of the two components of I shrinks to zero and disappears. Consider the two parameter family of convex combinations of ψ_0, ψ_1, ψ_2. Among the contact sets for this family there will be some which consist of a very small component joining a much larger one. In particular we interpret Fig. 2 as a component of diameter zero joining a component of finite size. This interpretation suggests that in any neighborhood of the given obstacle there will be obstacles which produce two separate components.

We recall the somewhat surprising $\frac{2}{5}$ power that characterized the cusp point in Fig. 2. It can be proved that a $\frac{2}{3}$ power cusp cannot occur as a singularity of ∂I, assuming that ψ is super-harmonic. Suppose one attempts to imitate the construction of the present paper by defining a proposed contact set I analogous to Fig. 2 but with a $\frac{2}{3}$ power cusp. Although it is possible to find a harmonic function u on $\Omega \sim I$ which satisfies (1), a simple power series calculation shows that the condition $u \geqslant \psi$ is violated in any neighborhood of the cusp point.

2. – Examples with a C^∞ obstacle.

Let Ω be a convex subset of $\mathbf{R}^n$ with smooth boundary such that Ω is symmetric with respect to the reflection $(x', x_n) \mapsto (x', -x_n)$. Here we use the standard notation $x = (x', x_n)$ for $x \in \mathbf{R}^n$, and we will identify $\mathbf{R}^{n-1}$ with the subspace $\{(x', 0): x' \in \mathbf{R}^{n-1}\}$. In the following theorem, the main result of this section, we show that, given any subsets E and F of $\mathbf{R}^{n-1}$ with E open, F closed, and

$$E \subset F \subset \Omega\,, \tag{2.1}$$

there is a smooth, super-harmonic obstacle ψ for which the region of contact I satisfies

$$E = \operatorname{Int}(I) \cap \mathbf{R}^{n-1}\,, \qquad F = I \cap \mathbf{R}^{n-1}\,. \tag{2.2}$$

In the theorem we use the notation $E = \bigcup E_j$ for the decomposition of E into connected components, with similar notation for F, I, and J, where J is the interior of I.

THEOREM 2.1. – *If E and F are open and closed, respectively, and satisfy* (2.1), *then there is a C^∞, super-harmonic obstacle ψ whose contact set I has the following property*: *The components of I (resp. J) may be put in one-to-one correspondence with the components of F (resp. E) and*

$$F_j = I_j \cap \mathbf{R}^{n-1} \qquad (\text{resp. } E_j = J_j \cap \mathbf{R}^{n-1})\,. \tag{2.3}$$

LEMMA 2.2. *For any open set $\mathcal{O} \subset \mathbf{R}^n$ there is a non-negative C^∞ function $\alpha(x)$ such that*

$$\mathcal{O} = \{x: \alpha(x) > 0\}\,. \tag{2.4}$$

PROOF. Let $\{B_i\}$ be a countable basis for the topology of $\boldsymbol{R}^n$, where B_i is an open ball of radius r_i and center x_i. Let $\zeta(x)$ be a non-negative C^∞ function on $\boldsymbol{R}^n$ such that $\zeta(x) > 0$ if and only if $|x| < 1$. Consider a series of the form

$$\sum_i c_i \zeta\big((x - x_i)/r_i\big)$$

where $c_i \geqslant 0$ and $c_i > 0$ if and only if $B_i \subset \mathcal{O}$. We may arrange that the coefficients $\{c_i\}$ tend to zero so rapidly that this series and all its derived series are uniformly convergent. The limit function $a(x)$ satisfies (2.4), and the proof is complete.

Let E be an open subset of $\boldsymbol{R}^{n-1}$. Choose $\alpha(x')$, a non-negative C^∞ function on $\boldsymbol{R}^{n-1}$ such that $E = \{x' : \alpha(x') > 0\}$. Let

$$\Gamma_+ = \{\big(x', \alpha(x')\big) : x' \in \Omega \cap \boldsymbol{R}^{n-1}\}$$

and let Γ_- be the image of Γ_+ under the reflection $(x', x_n) \mapsto (x', -x_n)$. By scaling $\alpha(x')$ if necessary we may arrange that $\Gamma_\pm \subset \Omega$. The C^∞ surfaces Γ_+ and Γ_- divide Ω into three regions Ω_+, I, Ω_-, a point $(x', x_n) \in \Omega$ belonging to one of these sets according as

$$x_n > \alpha(x')\,, \qquad -\alpha(x') \leqslant x_n \leqslant \alpha(x')\,, \qquad x_n < -\alpha(x')\,.$$

Note that $E = \operatorname{Int}(I) \cap \boldsymbol{R}^{n-1}$.

Consider the Cauchy problem

$$\Delta v = 1 \quad \text{in } \Omega_+ \tag{2.5}$$

$$v = 0\,, \quad \partial v/\partial x_n = 0 \qquad \text{on } \Gamma_+\,. \tag{2.6}$$

Of course the Cauchy problem for an elliptic equation is ill-posed, but the following approximate solution of (2.5), (2.6) will be sufficient for our purposes.

LEMMA 2.3. *There is a function* $v \in C^\infty(\Omega_+)$ *satisfying* (2.6) *such that* $\Delta v - 1$ *vanishes to infinite order along* Γ_+. *Moreover we may arrange that*

$$v > 0 \quad \textit{and} \quad \Delta v > 0 \qquad \textit{in } \Omega_+\,. \tag{2.7}$$

PROOF. By repeated differentiation we may determine all derivatives $(\partial/\partial x_n)^k v$ on Γ_+ from the data in (2.5), (2.6). Of course substitution of these derivatives into a power series would yield a divergent series in general.

Instead we interpret the result of this computation as a C^∞ Whitney function v_0 on the closed set Γ_+. (See [5] for definitions and for the extension theorem.) If v is a C^∞ extension of v_0 to Ω_+, then $\Delta v - 1$ vanishes to infinite order on Γ_+.

To show that (2.7) may be satisfied we recall the proof of the extension theorem, appropriately simplified for the case at hand. If $v_k(x')$ is the k-th derivative of v with respect to x_n on Γ_+, as discussed above, one defines

$$v(x', x_n) = \sum_{k=2}^{\infty} v_k(x')\, \zeta\{c_k(x_n - \alpha(x'))\} \frac{(x_n - \alpha(x'))^k}{k!}. \tag{2.8}$$

This differs from a power series in $x_n - \alpha(x')$ by the presence of the cut-off factors involving ζ. Here ζ is a C^∞ function of one variable with compact support which is identically one in some neighborhood of the origin. If $\{c_k\}$ tends to ∞ sufficiently rapidly, (2.8) will converge in the C^∞ topology.

Let us suppress the convergence factor in the first term T_2 of (2.8), so that we have

$$T_2 = v_2(x')(x_n - \alpha(x'))^2/2 .$$

A simple calculation shows that $v_2 = \{1 + (\nabla\alpha)^2\}^{-1}$. Observe that T_2 is positive on Ω_+. By increasing c_k in the remaining terms in (2.8) we may arrange that

$$|v - T_2| \leqslant \tfrac{1}{2} T_2$$

and in this way satisfy the first inequality in (2.7). For the second inequality of (2.7) we re-write T_2 in the form

$$T_2 = \frac{x_n^2}{2} - \frac{x_n^2(\nabla\alpha)^2 + 2x_n\alpha - \alpha^2}{2(1 + (\nabla\alpha)^2)} .$$

The Laplacian of the first term here is 1 while that of the second can be made small by scaling α by a small positive constant. We may therefore arrange by such a scaling that $\Delta T_2 > 0$ in Ω_+. We then argue as above that T_2 is the dominant term in (2.8), or can be made so by the choice of c_k. In this way we may arrange that $\Delta v > 0$ in Ω_+. The proof is complete.

We extend the domain of v from Ω_+ to Ω by defining

$$v(x) = \begin{cases} v(x', -x_n) & \text{for } x \in \Omega_- \\ 0 & \text{for } x \in I . \end{cases}$$

If f is defined by

$$f = \begin{cases} \Delta v & \text{on } \Omega_+ \cup \Omega_- \\ 1 & \text{on } I, \end{cases}$$

then by Lemma 2.3 we see that $f \in C^\infty(\Omega)$ and $f > 0$. Let ψ be the solution of the Dirichlet problem

$$\begin{cases} \Delta\psi = -f & \text{in } \Omega \\ \psi = -v & \text{on } \partial\Omega . \end{cases} \tag{2.9}$$

Observe that $u = v + \psi$ is harmonic on $\Omega \sim I$, vanishes at $\partial\Omega$, and satisfies (1) at ∂I. As discussed in the introduction, it follows that u is the solution of the obstacle problem with data ψ. We have already arranged that $E = \mathrm{Int}(I) \cap \boldsymbol{R}^{n-1}$, and to satisfy the second half of (2.2) we modify the data very slightly as follows.

Let $\tilde{\alpha}(x')$ be a non-negative C^∞ function such that

$$F = \{x' : \tilde{\alpha}(x') = 0\} ; \tag{2.10}$$

such a function exists by Lemma 2.2. We claim that u defined above is also the solution of the obstacle problem with data $\psi' = \psi - \tilde{\alpha}$; indeed $\{u = \psi\}$ and $\{u = \psi'\}$ differ only by a set of measure zero, namely $(\boldsymbol{R}^{n-1} \cap \Omega) \sim F$, and hence (2) will continue to hold almost everywhere. Moreover it follows from Lemma 2.3 and (2.9) that $\psi \leqslant 0$ on $\partial\Omega$ with equality only on $\partial\Omega \cap \boldsymbol{R}^{n-1}$ and it follows from (2.1) and (2.10) that $\tilde{\alpha} > 0$ on $\partial\Omega \cap R^{n-1}$; hence $\psi' < 0$ on $\partial\Omega$. Finally, since $\Delta\psi \leqslant -\varepsilon < 0$ for some ε, we may arrange by scaling $\tilde{\alpha}$, if necessary, that $\Delta\psi' < 0$. The relations (2.3) are readily verified and the proof of Theorem 2.1 is complete.

It is instructive to consider these examples in the light of the regularity results of Caffarelli and Rivière [3]. Let us call a point $p \in \partial I$ exceptional if any neighborhood of p intersects an infinite number of components of I. One conclusion of these authors is that any component of I contains at most a finite number of exceptional points. (More precisely, one should replace I by the closure of its interior.) In our examples, if $n = 2$, the exceptional points all lie along the real axis with at most two exceptional points per component. One's appreciation of the delicacy of the arguments of [3] may be enhanced by the observation that in our examples the exceptional points from *different* components may have limit points, and that these limit points may themselves have limit points, etc. The examples also indicate the problems in extending these results to higher dimensions.

REFERENCES

[1] H. Brézis - D. Kinderlehrer, *The smoothness of solutions to non-linear variational inequalities*, Indiana Math. J., **23** (1974), pp. 831-844.
[2] H. Brézis, *Opérateurs Maximaux Monotones*, North Holland, Amsterdam, 1973.
[3] L. Caffarelli - N. Rivière, *Smoothness and analyticity of free boundaries in variational inequalities*, Ann. Scuola Norm. Sup. Pisa, (4), **3** (1967), pp. 289-310.
[3′] D. Kinderlehrer, *The free boundary determined by the solution to a differential equation*, Indiana Math. J., **25** (1976), pp. 195-208.
[4] H. Lewy - G. Stampacchia, *On the regularity of the solution of a variational inequality*, Comm. Pure Appl. Math., **22** (1969), pp. 155-188.
[5] B. Malgrange, *Ideals of Differentiable Functions*, Oxford University Press, London, 1965.