MITSURU NAKAI

LEO SARIO

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Manifolds with Strong Harmonic Boundaries but without Green's Functions of Clamped Bodies (*).

MITSURU NAKAI (**) - LEO SARIO (***)

We are interested in costructing, on a given Riemannian manifold M, the biharmonic Green's function $\beta_M(x, y)$ of the clamped body, characterized by

$$\Delta^2\beta_M(\cdot, y) = \Delta(\Delta\beta_M(\cdot, y)) = \delta_y \tag{1}$$

on M, with δ_y the Dirac measure at y, and the « Dirichlet data

$$\beta_M(\cdot, y) = \frac{\partial}{\partial n}\beta_M(\cdot, y) = 0 \tag{2}$$

at the ideal boundary of M ». The function is the analogue of the Green's function of the clamped plate, significant in the theory of elasticity. The construction of the function and the nature of (2) have been discussed in Ralston-Sario [6] from the view point of a priori estimates and in Nakai-Sario [2]-[4] from the view point of kernel potentials. We denote by

$$O_\beta \tag{3}$$

the class of noncompact Riemannian manifolds M which do not carry β_M. Several complete characterizations of the class O_β were obtained in Nakai-Sario [5].

The natural question arises as to whether or not a harmonic nondegeneracy of the ideal boundary of M is sufficient to entail the existence of β_M. We denote by O_{HX} the class of Riemannian manifolds M which carry no nonconstant harmonic functions with the property $X = P$ (positive), B (bounded),

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(**) Nagoya Institute of Technology.

(***) University of California, Los Angeles.

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D (Dirichlet finite), or BD (B and D). Let O_G be the class of parabolic manifolds. The following strict inclusion relations are well known (Schiffer-Sario-Glasner [8], Hada-Sario-Wang [1], and, for a complete reference, Sario-Nakai [7]):

$$O_G < O_{HP} < O_{HB} < O_{HD} = O_{HBD}\,. \tag{4}$$

Thus the strongest harmonic nondegeneracy is $M \notin O_{HD}$, and we ask explicitly: does $M \notin O_{HD}$ assure that $M \notin O_\beta$? The purpose of the present paper is to show that the answer is in the *negative*, i.e.,

$$O_\beta - O_{HD} \neq \emptyset\,, \tag{5}$$

and this relation holds for *every* dimension $N \geqslant 2$ of the manifold. Since any subregion of the Euclidean N-space E^N for $N \geqslant 5$, and any subregion of E^N with an exterior point for $N = 2, 3, 4$ carries β (Nakai-Sario [4]), a manifold in (5) will be, by necessity, somewhat intricate.

1. – Consider the sets

$$\Sigma^N : |x| > 1,\ \Gamma^N : |x| = 1$$

in the Euclidean space E^N of dimension $N \geqslant 2$. Denote by

$$\hat{\Sigma}^N \tag{7}$$

the *double* of Σ^N with respect to Γ^N, i.e., the topological manifold $(\Sigma^N)_1 \cup \cup (\Sigma^N)_2 \cup \Gamma^N$ defined in an obvious manner, with the $(\Sigma^N)_i$ $(i = 1, 2)$ duplicates of Σ^N.

In terms of the polar coordinates $(r, \theta) = (r, \theta^1, \ldots, \theta^{N-1})$ of points $x = (x^1, \ldots, x^N)$, $|x| = r$, in $\Sigma^N \cup \Gamma^N$, the Euclidean metric (line element) $|dx|$ is given by

$$|dx|^2 = dr^2 + r^2 \sum_{j=1}^{N-1} \lambda_j(\theta)(d\theta^j)^2\,. \tag{8}$$

We introduce the Riemannian metric

$$ds^2 = \varphi(r)^2 dr^2 + \psi(r)^2 r^2 \sum_{j=1}^{N-1} \lambda_j(\theta)(d\theta^j)^2 \tag{9}$$

on $\Sigma^N \cup \Gamma^N$, with φ and ψ strictly positive C^∞ functions on $[1, \infty)$ such that (9) defines a C^∞ metric on $\hat{\Sigma}^N$ by symmetry. Denote by

$$\hat{\Sigma}^N_{\varphi,\psi} \tag{10}$$

the manifold $\hat{\Sigma}^N$ with metric (9). We could actually allow ds to be any smooth metric on (10) as long as it satisfies (9) on a neighborhood of each « point at infinity » δ_i $(i=1,2)$, i.e., duplicate in $(\Sigma^N)_i$ $(i=1,2)$ of the point at infinite δ of Σ^N and hence of E^N. The manifold $\hat{\Sigma}^N_{\varphi,\psi}$ can also be viewed as the *double of the Riemannian manifold* $\hat{\Sigma}^N_{\varphi,\psi}=(\Sigma^N, ds)$ with respect to Γ^N.

2. – We consider the following two conditions on the functions φ and ψ:

$$\int_1^\infty \frac{\varphi(r)}{r^{N-1}\,\psi(r)^{N-1}}\,dr<\infty\,, \tag{11}$$

$$\int_1^\infty\left(\int_r^\infty \frac{\varphi(\varrho)}{\varrho^{N-1}\,\psi(\varrho)^{N-1}}\,d\varrho\right)^2 \varphi(r)\,\psi(r)^{N-1}\,r^{N-1}\,dr=\infty\,. \tag{12}$$

The existence of functions satisfying these conditions is obvious, the choice $\varphi(r)=r^2$ and $\psi(r)=r^{-(N-5)/(N-1)}$ for large r being an example. The purpose of the present paper is to prove:

THEOREM. *The manifold* $\hat{\Sigma}^N_{\varphi,\psi}$ $(N\geqslant 2)$ *carries nonconstant Dirichlet finite harmonic functions, yet does not admit the biharmonic Green's function* β *of the clamped body, i.e.,*

$$\Sigma^N_{\varphi,\psi}\in O_\beta - O_{HD}\qquad (N\geqslant 2)\,, \tag{13}$$

if and only if the functions φ *and* ψ *satisfy conditions* (11) *and* (12).

We will prove, in Nos. 3 and 4 below, two propositions, A) and B), from which the above theorem will follow at once.

3. – Denote by $W=W(\sigma)$ $(\sigma>1)$ the region $|x|>\sigma$ in Σ^N, and by $W_i=W(\sigma)_i$ its duplicates in $(\Sigma^N)_i$ $(i=1,2)$. On each $W(\sigma)_i$, the Laplace-Beltrami operator $\Delta=d\delta+\delta d$ for $\hat{\Sigma}^N_{\varphi,\psi}$ takes the form

$$\Delta u=-\frac{1}{\varphi(r)\,\psi(r)^{N-1}\,r^{N-1}}\frac{\partial}{\partial r}\left(\frac{r^{N-1}\,\psi(r)^{N-1}}{\varphi(r)}\frac{\partial}{\partial r}u\right) - \frac{1}{r^2\,\psi(r)^2\,\lambda(\theta)}\sum_{j=1}^{N-1}\frac{\partial}{\partial\theta^j}\left(\lambda(\theta)\,\lambda_j(\theta)^{-1}\frac{\partial}{\partial\theta^j}u\right), \tag{14}$$

with $\lambda(\theta)=\left(\prod_1^{N-1}\lambda_j(\theta)\right)^{\frac12}$. If (11) is satisfied, then

$$w(x;\sigma)=\left(\int_{|x|}^\infty \frac{\varphi(\varrho)}{\varrho^{N-1}\,\psi(\varrho)^{N-1}}\,d\varrho\right)\Big/\left(\int_1^\infty \frac{\varphi(\varrho)}{\varrho^{N-1}\,\psi(\varrho)^{N-1}}\,d\varrho\right) \tag{15}$$

is a harmonic function on $W(\sigma)_i$. This means that the *harmonic measure* $1 - w(x; \sigma)$ of δ_i with respect to $W(\sigma_i)$ is positive and, therefore,

$$(W(\sigma)_i, \partial W(\sigma)_i) \notin SO_{HD} \quad (i = 1, 2), \tag{16}$$

with SO_{HD} the class of those Riemannian manifolds M with compact or noncompact distinguished smooth boundaries γ which carry no nonconstant Dirichlet finite harmonic functions with boundary values zero on γ. By the *two region criterion* (cf., e.g., Sario-Nakai [7]), (16) implies that $\hat{\Sigma}^N_{\varphi,\psi} \notin O_{HD}$, if (11) is assumed.

Suppose now that (11) is not valid. The function

$$e(x; \sigma) = \int_1^{|x|} \frac{\varphi(\varrho)}{\varrho^{N-1}\,\psi(\varrho)^{N-1}}\, d\varrho \tag{17}$$

is also harmonic on $W(\sigma)$, but $\lim_{x\to\delta_i} e(x; \sigma) = \infty$ $(i = 1, 2)$. Hence, $\hat{\Sigma}^N_{\varphi,\psi} \in O_g$ and, a fortiori, $\hat{\Sigma}^N_{\varphi,\psi} \in O_{HD}$.

We have shown:

A) The manifold $\hat{\Sigma}^N_{\varphi,\psi} \notin O_{HD}$ if and only if the functions φ and ψ satisfy (11).

4. - The surface element of Γ^N considered in E^N is $d\theta = \lambda(\theta)\, d\theta^1 \ldots d\theta^{N-1}$, with finite total measure

$$\omega_N = \int_{\Gamma^N} d\theta = 2\pi^{N/2}/\Gamma(N/2)\,.$$

Therefore, the surface element of $|x| = r > \sigma$ in $\hat{\Sigma}^N_{\varphi,\psi}$ is $r^{N-1}\psi(r)^{N-1}\, d\theta$ and the volume element of $W(\sigma)_i$ in $\hat{\Sigma}^N_{\varphi,\psi}$ is

$$dV = \varphi(r)\,\psi(r)^{N-1} r^{N-1}\, dr\, d\theta\,.$$

Under assumption (11), the condition (12) is thus equivalent to

$$\int_{W(\sigma)_1 \cup W(\sigma)_2} w(x; \sigma)^2\, dV_x = \infty\,. \tag{18}$$

If we assume that (12), or equivalently (18), does not hold, then by the Ralston-Sario theorem [6] or by the Nakai-Sario theorem [5], we conclude that $\hat{\Sigma}^N_{\varphi,\psi} \notin O_\beta$.

Conversely assume, under assumption (11), that $\hat{\Sigma}^N_{\varphi,\psi} \notin O_\beta$. We wish to deduce the invalidity of (12), or equivalently, of (18). The proof is by contradiction. Suppose that (12), or equivalently (18), holds. Fix an arbitrary $y \in \hat{\Sigma}^N_{\varphi,\psi}$ and then a $\sigma > 1$ such that $y \notin \overline{W(\sigma)_1 \cup W(\sigma)_2}$. Let $H_M(\cdot, y) = \Delta\beta_M(\cdot, y)$ be the β-density. Then, by Nakai-Sario [2]-[5], we have

$$(19) \qquad \int\limits_{W(\sigma)_1 \cap W(\sigma)_2} H_M(x, y)^2 \, dV_x < \infty .$$

The function $h = H_M(\cdot, y)$ has, for any fixed $r \geqslant \sigma$, the « Fourier expansion »

$$(20) \qquad h(r, \theta) = \omega_N^{-\frac{1}{2}} h_0(r) + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} h_{nm}(r) S_{nm}(\theta) ,$$

where $\{S_{nm}(\theta)\}$ $(m = 1, \ldots, m_n)$ is a complete orthonormal system of spherical harmonics of degree $n \geqslant 1$, and thus $\{\omega_N^{-\frac{1}{2}}\} \cup \{S_{nm}(\theta)\}$ $(n = 1, 2, \ldots; m = 1, \ldots, m_n)$ is a complete orthonormal system in $L_2(\Gamma^N; d\theta)$. By the Parseval identity,

$$(21) \qquad f(r) \equiv \int\limits_{\Gamma^N} h(r, \theta)^2 \, d\theta = h_0(r)^2 + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} h_{nm}(r)^2$$

for every $r \in [\sigma, \infty)$. Since $h_0(r)$ satisfies $\Delta h_0 = 0$ and $h_{nm}(r)$ satisfies the « P-harmonic equation »

$$\Delta u = Pu , \qquad P = n(n + N - 2) r^{-2} \psi(r)^{-2} > 0 ,$$

we see that h_0^2 and h_{nm}^2 are all subharmonic on $W(\sigma)_1 \cup W(\sigma)_2$ and consequently $f(r)$ is subharmonic on $W(\sigma)_1 \cup W(\sigma)_2$. Therefore, $f(r)$ takes on its maximum in $[a, b] \subset [\sigma, \infty)$ at a or b.

By (19),

$$(22) \qquad \int_\sigma^\infty f(r)\varphi(r)\psi(r)^{N-1} r^{N-1} \, dr = \int_\sigma^\infty \Big(\int\limits_{\Gamma^N} h(r, \theta)^2 \, d\theta \Big) \varphi(r)\psi(r)^{N-1} r^{N-1} \, dr$$
$$= \int_\sigma^\infty h(x)^2 \, dV_x = \tfrac{1}{2} \int\limits_{W(\sigma)_1 \cup W(\sigma)_2} H_M(x, y)^2 \, dV_x < \infty .$$

As a direct consequence of (12), we see that

$$(23) \qquad \int_\sigma^\infty \varphi(r)\psi(r)^{N-1} r^{N-1} \, dr = \infty .$$

In view of (22) and (23), we must have $\liminf_{r\to\infty} f(r) = 0$, i.e., there exists an increasing divergent sequence $\{r_n\}$ in (σ, ∞) such that $f(r_n) \to 0$ $(n \to \infty)$. Since

$$\max_{r_n \leqslant r \leqslant r_{n+1}} f(r) = \max\big(f(r_n), f(r_{n+1})\big)$$

for every $n = 1, 2, \ldots$, we have $\lim_{r\to\infty} f(r) = 0$, i.e.,

$$\lim_{x\to\sigma_i} H_M(x, y) = 0 \qquad (i = 1, 2)\,. \tag{24}$$

This with $\lim_{x\to y} H_M(x, y) = \infty$ and the maximum principle yields

$$H_M(\cdot\,, y) > 0 \tag{25}$$

on $\hat{\Sigma}^N_{\varphi,\psi}$. Again by the maximum principle, $w(\cdot\,, \sigma) \leqslant cH_M(\cdot\,, y)$ on $W(\sigma)_1 \cup \cup\, W(\sigma)_2$, with $c = (\inf_{|x|=\sigma} H_M(x, y))^{-1} < \infty$. Hence

$$\int\limits_{W(\sigma)_1 \cup W(\sigma)_2} w(x; \sigma)^2\, dV_x \leqslant c^2 \int\limits_{W(\sigma)_1 \cup W(\sigma)_2} H_M(x, y)^2\, dV_x\,. \tag{26}$$

However, this is impossible in view of (18) and (19). Thus $\hat{\Sigma}^N_{\varphi,\psi} \notin O_\beta$ implies that (12) is not satisfied.

B) Under assumption (11), *the manifold* $\hat{\Sigma}^N_{\varphi,\psi} \in O_\beta$ *if and only if the functions* φ *and* ψ *satisfy* (12).

REFERENCES

[1] D. Hada - L. Sario - C. Wang, *N-manifolds carrying bounded but no Dirichlet finite harmonic functions*, Nagoya Math. J., **54** (1974), pp. 1-6.

[2] M. Nakai - L. Sario, *Parabolic Riemannian planes carrying biharmonic Green's functions of the clamped plate*, J. Analyse Math. (to appear).

[3] M. Nakai - L. Sario, *A strict inclusion related to biharmonic Green's functions of clamped and simply supported bodies*, Ann. Acad. Sci. Fenn. (to appear).

[4] M. Nakai - L. Sario, *Existence of biharmonic Green's functions* (to appear).

[5] M. Nakai - L. Sario, *A nonexistence test for biharmonic Green's functions of clamped bodies*, Math. Scand. (to appear).

[6] J. Ralston - L. Sario, *A relation between biharmonic Green's functions of simply supported and clamped bodies*, Nagoya Math. J. (to appear).

[7] L. Sario - M. Nakai, *Classification Theory of Riemann Surfaces*, Grundlehren vol. **164**, Springer-Verlag, Berlin-Heidelberg-New York (1970), 446 pp.

[8] L. Sario - M. Schiffer - M. Glasner, *The span and principal functions on Riemannian spaces*, J. Analyse Math., **15** (1965), pp. 115-134.