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The Class Number of Quadratic Fields and the Conjectures of Birch and Swinnerton-Dyer.

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1. – Introduction.

The value of the Dirichlet L-function

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)\,n^{-s},$$

formed with a real primitive character $\chi(\operatorname{mod} d)$, at the point $s=1$ has received considerable attention ever since the appearance of Dirichlet's class number formulae

$$L(1,\chi)=\begin{cases}\dfrac{2\pi h}{w\sqrt{d}} & \text{if } \chi(-1)=-1\,,\\[2ex] \dfrac{2h\log\varepsilon_0}{\sqrt{d}} & \text{if } \chi(-1)=+1\,,\end{cases}\tag{1}$$

where h is the class number, w the number of roots of unity, and $\varepsilon_0>1$ the fundamental unit of the quadratic field $Q\big(\sqrt{\chi(-1)d}\big)$. Siegel's basic inequality (see [18], [7])

$$L(1,\chi)>c(\varepsilon)\,d^{-\varepsilon}\qquad(\varepsilon>0)$$

is fundamental in this field, and has wide applications in the theory of numbers. The only disadvantage is that there is no known method to compute the constant $c(\varepsilon)>0$.

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In this connection, it is interesting to note, [6], [9], that if $L(1, \chi) = o(1/\log d)$ then $L(s, \chi)$ will have a real zero β (Siegel zero) near to $s = 1$ satisfying

$$1-\beta \sim \frac{6}{\pi^2} L(1, \chi) \Bigl(\sum_{(a,b,c)} \frac{1}{a} \Bigr)^{-1},$$

where the sum goes over all rational integers a, b, c such that $b^2 - 4ac = \chi(-1)d$, $-a < b \leqslant a < \frac{1}{4}\sqrt{d}$. This, of course, contradicts the Riemann hypothesis, and it is, therefore, likely that $L(1, \chi)\log d$ will never get too small.

Non trivial effective lower bounds for $L(1, \chi)$ seem to be very difficult to obtain. Heegner [13], Stark [19] and Baker [1] established that there are only 9 imaginary quadratic fields with class number one. Also, Stark and Baker, [20], [2] by using a transcendence theorem showed that there are exactly 18 imaginary quadratic fields with class number two. As a consequence, the lower bound

$$L(1, \chi) \geqslant 3\pi/\sqrt{d} \qquad (d > 427)$$

was obtained.

By developing a novel method, we shall prove

THEOREM 1. *Let E be an elliptic curve over Q with conductor N. If E has complex multiplication and the L-function associated to E has a zero of order g at $s = 1$, then for any real primitive Dirichlet character $\chi(\mathrm{mod}\, d)$ with $(d, N) = 1$ and $d > \exp\exp(c_1 N g^3)$,* we have

$$L(1, \chi) > \frac{c_2}{g^{4g} N^{13}} \frac{(\log d)^{g-\mu-1} \exp\bigl(-21 g^{\frac{1}{2}} (\log\log d)^{\frac{1}{2}}\bigr)}{\sqrt{d}},$$

where $\mu = 1$ or 2 is suitably chosen so that $\chi(-N) = (-1)^{g-\mu}$, and the constants c_1, $c_2 > 0$ can be effectively computed and are independent of g, N and d.

If the condition $(d, N) = 1$ is dropped, then Theorem 1 will still hold. In this case, however, the relation $\chi(-N) = (-1)^{g-\mu}$ will have to be replaced by a more complicated one.

Theorem 1 is also true for elliptic curves E without complex multiplication provided $L_E(s)$ comes from a cusp form of $\Gamma_0(N)$ as conjectured by Weil [23]. It can even be shown that if $L_E(s) = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}$ has a zero of order $g \geqslant 1$ at $s = 1$, and if $\prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1}$ has a zero of order ϱ ($\varrho \geqslant 0$ or $\varrho < 0$) at $s = 2$, then

$$L(1, \chi) > \frac{c_2}{g^{4g} N^{13}} \frac{(\log d)^{g-\mu-\varrho} \exp\left(-21 g^{\frac{1}{2}} (\log \log d)^{\frac{1}{2}}\right)}{\sqrt{d}},$$

with c_2 effectively computable. The value of ϱ is given in the conjectures of Tate [22] on the zeros and poles of L-functions associated to products of elliptic curves, and as shown by Ogg [17], $\varrho = 1$ assuming Weil's conjecture.

If the curve E may be taken in Weierstrass normal form

$$(2) \qquad E: y^2 = 4x^3 - g_2 x - g_3 , \qquad \Delta = g_2^3 - 27 g_3^2 \neq 0 ,$$

then the associated L-function is defined as

$$(3) \qquad \begin{aligned} L_E(s) &= \prod_{p \mid \Delta} (1 - t_p p^{-s})^{-1} \prod_{p \nmid \Delta} (1 - t_p p^{-s} + p^{1-2s})^{-1} \\ &= \sum_{n=1}^{\infty} E_n n^{-s} , \end{aligned}$$

where $t_p = p + 1 - N_p$, and N_p is just the number of solutions (including the point at infinity) of the congruence

$$y^2 \equiv 4x^3 - g_2 x - g_3 \pmod{p} .$$

If $p \nmid \Delta$ then t_p is the « trace of Frobenius », and otherwise $t_p = \pm 1$ or 0. Weil [23] has conjectured that $L_E(s)$ is entire and satisfies the functional equation

$$(4) \qquad \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_E(s) = \pm \left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s) L_E(2-s) ,$$

where N, a certain integer divisible only by primes $p \mid \Delta$, is the conductor of E.

If the group of rational points on E, which is finitely generated by the Mordell-Weil Theorem ([16], pp. 138-149), has g independent generators of infinite order, then Birch-Swinnerton-Dyer [4] have conjectured

CONJECTURE. *$L_E(s)$ has a zero of order g at $s = 1$.*

This conjecture has been confirmed in hundreds of cases (see [4], [21]) for which $g = 0, 1$, and 2. Stephens [21] has shown that the curve

$$E: y^2 = x^3 - 2^4 \cdot 3^7 \cdot 73^2$$

has rank $g = 3$, that $L_E(s)$ satisfies the functional equation (4) with $N = 3^3 \cdot 73^2$ and the minus sign, and that $L_E(s)$ has a zero of odd order $\geqslant 1$ at $s = 1$. It is a particular example of curves admitting complex multiplication by $\sqrt{-3}$. The constant $L_E'(1)$ was calculated to three decimal places and turned out to be 0.000, all in support of the Birch-Swinnerton-Dyer conjecture.

The only curve (*) that seems to be known with rank $g \geqslant 4$ and complex multiplication is the example given by Wiman [24]

$$E: y^2 = x^3 + (3\cdot 7\cdot 11\cdot 17\cdot 41)^2 x .$$

This curve has complex multiplication by $\sqrt{-1}$ and is 2-isogenous to the curve

$$y^2 = x^3 - (2\cdot 3\cdot 7\cdot 11\cdot 17\cdot 41)^2 x .$$

Using the results of [3], it can be shown that for this example $L_E(s)$ satisfies (4) with $N = 2^6\cdot(3\cdot 7\cdot 11\cdot 17\cdot 41)^2$ and the plus sign, and that $L_E(s)$ has a zero of even order $\geqslant 2$ at $s = 1$.

If in the last example one could prove that $L_E(s)$ has a zero of order 4 at $s = 1$, then $h(-d) \to +\infty$ with d in a constructive way and hence the class number problem $h(-d) = \text{const}$ is effectively solvable.

The proof of Theorem 1 is divided into three parts. First, $L_E(s)$ is « twisted » (in this connection see also [8]) by χ and Liouville's function λ where

$$\lambda(n) = \prod_{p^r \| n} (-1)^r , \qquad \frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} ,$$

the series

$$L_E(s, \chi) = \sum_{n=1}^{\infty} \chi(n) E_n n^{-s} , \qquad L_E(s, \lambda) = \sum_{n=1}^{\infty} \lambda(n) E_n n^{-s}$$

being obtained. The key functions

$$\varphi(s) = L_E(s + \tfrac{1}{2}) L_E(s + \tfrac{1}{2}, \chi) \qquad \text{and} \qquad \varphi_1(2s) = L_E(s + \tfrac{1}{2}) L_E(s + \tfrac{1}{2}, \lambda)$$

are then defined. Note that if $L_E(s + \frac{1}{2}) = \prod (1 - \delta_p p^{-s})^{-1} (1 - \bar{\delta}_p p^{-s})^{-1}$ then $\varphi_1(s) = \prod (1 - \delta_p^2 p^{-s})^{-1} (1 - \bar{\delta}_p^2 p^{-s})^{-1}$. We also let

$$\varphi(s) = G(s) \varphi_1(2s)$$

(*) For this example I am indebted to Professor A. Schinzel.

where it is clear that $G(s) \equiv 1$ under the absurd assumption that $\chi(p) = -1$ for all primes p. If $G(s) = \sum g_n n^{-s}$ and $G(s, x) = \sum_{n<x} g_n n^{-s}$, then $G(s, x)$ measures the deviation by which $\chi(n)$ differs from $\lambda(n)$ for $n < x$.

In the second part of the proof, by a careful analysis of $\zeta(s)L(s, \chi)/\zeta(2s)$, we show that $G(\frac{1}{2}, x)$ can be measured in terms of $L(1, \chi)$ and x. For example, if $Q(\sqrt{-d})$ has class number one, then $\chi(p) = +1$ if and only if $p = x^2 + xy + ((d+1)/4)y^2$ so that $\chi(p) = -1$ for all primes $p < (d+1)/4$. In this case, $G(s, x) \equiv 1$ for $x < (d+1)/4$. In general, if $L(1, \chi)$ is small, then $G(\frac{1}{2}, x) \to 1$ for suitable x.

In the final part, we prove that

$$\left(\frac{d}{ds}\right)^{\kappa}\left[A^s\Gamma^2\left(s+\frac{1}{2}\right)\varphi(s)\right]_{s=\frac{1}{2}} = \left(\frac{d}{ds}\right)^{\kappa}\left[\delta A^s\Gamma^2\left(s+\frac{1}{2}\right)G(s, U)\varphi_1(2s)\right]_{s=\frac{1}{2}} + \\ + O(g^{4g}NL(1, \chi)A(\log\log A)^{\kappa+6})$$

where $A = dN/4\pi^2$, $U = (\log d)^{8g}$, and $\delta = 1 + (-1)^{\kappa}\chi(-N)$. Assuming that $\varphi(s)$ has a zero of order g at $s = \frac{1}{2}$, this leads to Theorem 1 as long as $\kappa = g - \mu$ is suitably chosen so that $\delta \neq 0$.

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2. – Hecke L-functions with « Grössencharakter ».

Let K be an imaginary quadratic field of discriminant k, and $\mathfrak{f}$ an integral ideal in K. A complex valued, completely multiplicative function $\psi(\mathfrak{a})$ defined on the integral ideals $\mathfrak{a} \in K$ is a « Grössencharakter » if $\psi(\mathfrak{a}) = 0$ whenever $\mathfrak{a}$ and $\mathfrak{f}$ have a common factor; and if there exists a fixed, positive rational integer a such that

$$\psi((\alpha)) = \alpha^a \quad \text{if } \alpha \equiv 1 \pmod{\mathfrak{f}}$$

for any integer $\alpha \in K$. The ideal $\mathfrak{f}$ is called the conductor of ψ, and if there is no smaller conductor $\mathfrak{f}_1 | \mathfrak{f}$, then ψ is said to be primitive. The Hecke L-function (with primitive ψ)

$$L_K(s, \psi) = \sum_{\mathfrak{a} \in K} \psi(\mathfrak{a})(N(\mathfrak{a}))^{-s},$$

where the sum goes over all integral ideals $\mathfrak{a} \in K$ with norm $N(\mathfrak{a})$ satisfies the functional equation [10]

$$\left(\frac{\sqrt{kN(\mathfrak{f})}}{2\pi}\right)^s \Gamma(s)L_K(s, \psi) = w\left(\frac{\sqrt{kN(\mathfrak{f})}}{2\pi}\right)^{1+a-s} \cdot \Gamma(1+a-s)L_K(1+a-s, \bar{\psi}) \tag{5}$$

$$|w| = 1 .$$

In the case of an elliptic curve with complex multiplication, Deuring [5] has proved.

THEOREM 2. *Let E be an elliptic curve over Q with complex multiplication, so that $K \sim \mathrm{End}(E) \otimes Q$ is an imaginary quadratic field. Then $L_E(s) = L_K(s, \psi)$ for some primitive « Grössencharakter » ψ of K.*

Assume E has complex multiplication by $\sqrt{-k}$. By comparing Euler products (here χ_k is a real primitive character mod k)

$$L_K(s, \psi) = \prod_{\substack{p=\mathfrak{p}^2 \\ p|k}} \left(1 - \frac{\psi(\mathfrak{p})}{p^s}\right)^{-1} \prod_{\substack{p=\mathfrak{p}\bar{\mathfrak{p}} \\ \chi_k(p)=+1}} \left(1 - \frac{\psi(\mathfrak{p})}{p^s}\right)^{-1} \left(1 - \frac{\psi(\bar{\mathfrak{p}})}{p^s}\right)^{-1} \cdot \prod_{\substack{p \\ \chi_k(p)=-1}} \left(1 - \frac{\psi(p)}{p^{2s}}\right)^{-1}, \tag{6}$$

$$L_E(s) = \prod_{p|\Delta} \left(1 - \frac{t_p}{p^s}\right)^{-1} \prod_{p \nmid \Delta} \left(1 - \frac{\alpha_p \sqrt{p}}{p^s}\right)^{-1} \left(1 - \frac{\bar{\alpha}_p \sqrt{p}}{p^s}\right)^{-1}, \tag{7}$$

where $|\alpha_p| = 1$ by the Riemann hypothesis for curves, and $\mathfrak{p}$ is a suitable prime ideal of $K = Q(\sqrt{-k})$ dividing the rational prime p, it follows from Theorem 2 that

$$\psi(\mathfrak{p}) = t_p = \pm 1 \quad \text{or} \quad 0$$

if and only if $p | kN(\mathfrak{f})$. Furthermore, the fact that $|\alpha_p| = 1$ implies that the integer a defining the « Grössencharakter » ψ must be equal to one. Since $L_E(s)$ is real for real values of s, we get from (5) that in the case of complex multiplication $L_E(s)$ satisfies the functional equation (4) with $N = kN(\mathfrak{f})$, it being clear from this discussion that N is divisible only by primes $p|\Delta$.

From now on let E and χ satisfy the conditions of Theorem 1. The twisted series

$$L_E(s, \chi) = \sum_{n=1}^{\infty} E_n \chi(n) n^{-s} = \sum_{\mathfrak{a} \in K} \psi(\mathfrak{a}) \chi(N(\mathfrak{a})((N(\mathfrak{a}))^{-s}$$

is again a Hecke series with « Grôssencharakter » having conductor $(d)\mathfrak{f}$. The function $L_E(s, \chi)$ satisfies the functional equation

$$\left(\frac{d\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_E(s, \chi) = \pm\chi(-N)\left(\frac{d\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s) L_E(2-s, \chi)$$

with the same sign as in (4). This is in accordance with Weil's principle [23]. Consequently, the function

$$\varphi(s) = L_E(s+\tfrac{1}{2}) L_E(s+\tfrac{1}{2}, \chi) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{8}$$

satisfies the functional equation

$$A^s \Gamma^2(s+\tfrac{1}{2})\varphi(s) = \chi(-N) A^{1-s}\Gamma^2(\tfrac{3}{2}-s)\varphi(1-s), \qquad A = dN/4\pi^2. \tag{9}$$

We shall make essential use of the fact that if $L(1, \chi)$ is too small, then $\chi(p) = -1$ for most primes $p \ll d$, so that $\chi(n)$ behaves like Liouville's function $\lambda(n)$. If

$$\varphi_1(2s) = L_E(s+\tfrac{1}{2}) L_E(s+\tfrac{1}{2}, \lambda),$$

then it is clear from (7) that

$$\varphi_1(s) = \prod_{p|\Delta}\left(1-\frac{t_p^2}{p}p^{-s}\right)^{-1} \prod_{p\nmid\Delta}(1-\alpha_p^2 p^{-s})^{-1}(1-\bar{\alpha}_p^2 p^{-s})^{-1}. \tag{10}$$

Now, write

$$\varphi(s) = G(s)\varphi_1(2s);$$

and note that if $\chi(p) = -1$ for all primes p then $G(s) \equiv 1$. So we expect $G(s)$ to be near to 1 if $L(1, \chi)$ is « small ». We also note that if

$$G(s) = \sum_{n=1}^{\infty} g_n n^{-s}, \qquad G(s, x) = \sum_{n<x} g_n n^{-s},$$

then

LEMMA 1. *For $n < x$, the n-th coefficient in the Dirichlet series expansion for $\varphi(s)$ agrees with the n-th coefficient in the Dirichlet series expansion of*

$G(s, x)\varphi_1(2s)$ where $G(s) = G_1(s)G_2(s)G_3(s)$, and

$$G_1(s) = \prod_{\substack{p|\Delta \\ \chi(p)=+1}} \left(1 - \frac{t_p}{\sqrt{p}} p^{-s}\right)^{-1} \left(1 + \frac{t_p}{\sqrt{p}} p^{-s}\right)^{-1},$$

$$G_2(s) = \prod_{\substack{p\nmid\Delta \\ \chi(p)=+1}} (1 - \alpha_p p^{-s})^{-1}(1 - \bar{\alpha}_p p^{-s})^{-1}(1 + \alpha_p p^{-s})(1 + \bar{\alpha}_p p^{-s}),$$

$$G_3(s) = \prod_{p|d} (1 + \alpha_p p^{-s})(1 + \bar{\alpha}_p p^{-s}).$$

$G(s)$ is majorized by $\big(\zeta(s)L(s, \chi)/\zeta(2s)\big)^2$.

LEMMA 2. *If $L_E(s) = L_K(s, \psi)$ as in Theorem (2), then*

$$\varphi_1(s) = L_K(s + 1, \psi^2)\frac{L(s, \chi_k)}{\zeta(s)} \prod_{p|k} (1 - p^{-s})^{-1}$$

where χ_k is a real, primitive, Dirichlet character (mod k)

PROOF. Upon comparing (6) and (7), we get

$$p|k \text{ if and only if } p|\Delta \text{ and } t_p = \psi(\mathfrak{p})$$

$$\alpha_p = \psi(\mathfrak{p})p^{-\frac{1}{2}} \neq \pm i \quad \text{when } \chi_k(p) = +1$$

$$\alpha_p = i, \quad \psi(p) = -p \quad \text{when } \chi_k(p) = -1.$$

The Lemma now follows from (10) on noting that

$$\frac{L(s, \chi_k)}{\zeta(s)} \prod_{p|k} (1 - p^{-s})^{-1} = \prod_{\substack{p \\ \chi_k(p)=-1}} (1 + p^{-s})^{-1}(1 - p^{-s}) \qquad \text{Q.E.D.}$$

3. – Zeta functions of quadratic fields.

Let

$$\frac{\zeta(s)L(s, \chi)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\nu_n}{n^s}.$$

In order to estimate sums of the type

$$\sum_{y \leqslant n \leqslant x} \nu_n n^{-\frac{1}{2}}, \qquad \sum_{y \leqslant n \leqslant x} n^{-\frac{1}{2}} \sum_{m|n} \nu_n \nu_{n/m},$$

it will be necessary to obtain an asymptotic expansion for $\zeta(s)L(s,\chi)$, the Dedekind zeta function of $Q(\sqrt{\chi(-1)d})$.

LEMMA 3. *Let α, β, γ be real numbers with $\alpha>0$ and $4\alpha\gamma-\beta^2=\Delta>0$. Then for any $x\geq 0$*

$$S(x)=\sum_{\substack{\alpha m^2+\beta mn+\gamma n^2\leq x\\ n\neq 0}} 1=\frac{2\pi x}{\sqrt{\Delta}}+4\theta\sqrt{\frac{\alpha x}{\Delta}}-4\varrho\sqrt{\frac{x}{\alpha}}$$

where the sum goes over rational integers m, n with $n\neq 0$; and $|\theta|\leq 1$ and $0\leq\varrho<1$ are real numbers.

PROOF. The argument is due to Iseki [14]. $S(x)$ is equal to the number of solutions of

$$(2\alpha m+\beta n)^2+\Delta n^2\leq 4\alpha x \qquad (n\neq 0)$$

which is equivalent to

$$-\sqrt{4\alpha x-\Delta n^2}-\beta n\leq 2\alpha m\leq\sqrt{4\alpha x-\Delta n^2}-\beta n\,,$$

$$0<|n|\leq\lambda=2\sqrt{\frac{\alpha x}{\Delta}}\,.$$

Therefore,

$$\begin{aligned} S(x)&=\frac{2}{\alpha}\sum_{0<n\leq\lambda}\sqrt{4\alpha x-\Delta n^2}+2\theta\sum_{n\leq\lambda}1\\ &=\frac{2\sqrt{\Delta}}{\alpha}\sum_{0<n\leq\lambda}\sqrt{\lambda^2-n^2}-2\theta\lambda\\ &=\frac{2\sqrt{\Delta}}{\alpha}\left(\frac{\pi}{4}\lambda^2-\varrho\lambda\right)+2\theta\lambda\,. \qquad \text{Q.E.D.}\end{aligned}$$

THEOREM 3. *Let $d>4$ and $\chi(-1)=-1$. Then for $s=\sigma+it$, $\sigma>\frac{1}{2}$*

$$\zeta(s)L(s,\chi)=\zeta(2s)\sum_{(a,b,c)}\frac{1}{a^s}+\frac{\pi}{\sqrt{d}}\frac{s}{s-1}\sum_{(a,b,c)}\left(\frac{d}{4a}\right)^{1-s}+R(s)\,,$$

$$|R(s)|\leq 4^{\sigma+\frac{1}{2}}\frac{|s|}{\sigma-\frac{1}{2}}\sum_{(a,b,c)}\left(\frac{a^{\sigma-1}}{d^{\sigma-\frac{1}{2}}}+\frac{a^\sigma}{d^\sigma}\right),$$

where the sum goes over the set of reduced forms $am^2+bmn+cn^2$ of discriminant $-d$. (That is to say $-a<b\leq a<c$ or $0\leq b\leq a=c$).

PROOF. Again following Iseki [14]

$$\zeta(s)L(s,\chi) = \frac{1}{2}\sum_{(a,b,c)}\sum_{\substack{m,n=-\infty \\ m,n)\neq((0,0)}}^{\infty}(am^2+bmn+cn^2)^{-s}$$

$$= \zeta(2s)\sum_{(a,b,c)}\frac{1}{a^s} + \sum_{(a,b,c)}\sum_{v=1}^{\infty}\frac{b_v}{v^s}$$

where b_v is equal to half the number of solutions of

$$v = am^2 + bmn + cn^2 \qquad (n \neq 0) .$$

Now,

$$\sum_{v=1}^{\infty} b_v v^{-s} = \frac{s}{2}\int_1^{\infty} S(u)u^{-s-1}\,du .$$

But $S(u) = 0$ for $u < d/4a$ since there are no solutions to

$$(2am + bn)^2 + dn^2 \leqslant 4au < d .$$

Therefore, by Lemma (3)

$$\sum_{v=1}^{\infty} b_v v^{-s} = s\int_{d/4a}^{\infty}\left(\frac{\pi u}{\sqrt{d}} + 2\theta\sqrt{\frac{au}{d}} - 2\varrho\sqrt{\frac{u}{a}}\right)u^{-s-1}\,du$$

$$= \frac{\pi}{\sqrt{d}}\,\frac{s}{s-1}\left(\frac{d}{4a}\right)^{1-s} + s\int_{d/4a}^{\infty}\left(2\theta\sqrt{\frac{a}{d}} - 2\varrho\sqrt{\frac{1}{a}}\right)u^{-s-\frac{1}{2}}\,du ,$$

where the last integral is regular for $\sigma > \frac{1}{2}$ and bounded by

$$\frac{|s|}{\sigma - \frac{1}{2}}\left(1 + \frac{\sqrt{d}}{a}\right)\left(\frac{d}{4a}\right)^{-\sigma}.$$

The Theorem is obtained by summing over all (a, b, c) and using the fact that $a < \sqrt{d}/3$ for a reduced form. Q.E.D.

An analogous Theorem for the zeta function of a real quadratic field does not seem to be in the literature. We, therefore, give complete details for what appears to be a new technique. The ideas go back to Hecke [11].

Let C be an ideal class in $F = Q(\sqrt{d})$, and let $\zeta_F(s, C)$ denote the zeta function of the class. If $\mathfrak{b} \in C^{-1}$, then the correspondence

$$\mathfrak{a} \mapsto \mathfrak{a}\mathfrak{b} = (\lambda)$$

is a bijection between ideals $\mathfrak{a} \in C$ and principal ideals (λ) with $\lambda \in \mathfrak{b}$. Two numbers λ_1 and λ_2 define the same principal ideal if and only if $\lambda_1 = \varepsilon\lambda_2$ for some unit ε of $U = \{\pm \varepsilon_0^n\}$, where U is the multiplicative group of units of F, generated by ± 1, and $\varepsilon_0 > 1$, the fundamental unit. Hence

$$\zeta_F(s, C) = N(\mathfrak{b})^s \sum_{\substack{\lambda \in \mathfrak{b}/U \\ \lambda \neq 0}} \frac{1}{|\lambda\lambda'|^s},$$

and in view of the well known correspondence between ideal classes and binary quadratic forms (see [12]), we can choose

$$\mathfrak{b} = \left[a, \frac{-b+\sqrt{d}}{2}\right], \qquad b + \sqrt{d} > 2|a| > -b + \sqrt{d} > 0,$$

$$\lambda = am + \frac{-b+\sqrt{d}}{2}\, n, \qquad \lambda' = am + \frac{-b-\sqrt{d}}{2}\, n,$$

$$N(\mathfrak{b}) = |a|,$$

for rational integers m, n.

Since,

$$\frac{1}{|\lambda\lambda'|^s} = 2\,\frac{\Gamma(s)}{\Gamma^2(s/2)} \int_{-\infty}^{\infty} \frac{d\varphi}{(\lambda^2 e^{\varphi} + (\lambda')^2 e^{-\varphi})^s}$$

it follows that

$$(11) \qquad \zeta_F(s, C) = 2\,\frac{\Gamma(s)}{\Gamma^2(s/2)}\, N(\mathfrak{b})^s \sum_{\substack{\lambda \in \mathfrak{b}/U \\ \lambda \neq 0}} \int_{-\infty}^{\infty} \frac{d\varphi}{(\lambda^2 e^{\varphi} + (\lambda')^2 e^{-\varphi})^s}$$

$$= \frac{\Gamma(s)}{\Gamma^2(s/2)}\, N(\mathfrak{b})^s \sum_{\substack{\lambda \in \mathfrak{b} \\ \lambda \neq 0}} \int_{\log\eta}^{2\log\varepsilon_0 + \log\eta} \frac{d\varphi}{(\lambda^2 e^{\varphi} + (\lambda')^2 e^{-\varphi})^s}$$

where $\eta > 0$ is arbitrary.

In (11) make the transformation $\varphi \to e^{\varphi}$ and sum over all classes. Then

$$(12) \qquad \zeta(s) L(s, \chi) = \sum_{(a,b,c)} \frac{\Gamma(s)}{\Gamma^2(s/2)} \int_{\eta}^{\eta\varepsilon_0^2} \left(\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (\alpha m^2 + \beta mn + \gamma n^2)^{-s} \right) \frac{d\varphi}{\varphi}$$

where

$$\beta^2 - 4\alpha\gamma = -4d$$

and

$$\alpha = |a|\left(\varphi + \frac{1}{\varphi}\right), \quad \beta = 2\alpha\left(\omega\varphi + \frac{\omega'}{\varphi}\right), \quad \gamma = \left(|a|\omega^2\varphi + \frac{(\omega')^2}{\varphi}\right),$$

$$\omega = \frac{-b + \sqrt{d}}{2a}, \quad \omega' = \frac{-b - \sqrt{d}}{2a}.$$

The Epstein zeta function occurring in (12) can be expressed

$$\sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (\alpha m^2 + \beta mn + \lambda n^2)^{-s} = d^{-s/2} f(z, s)$$

$$f(z, s) = y^s \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} |m + nz|^{-2s}$$

$$z = x + iy = \frac{\omega' + \omega\varphi^2}{\varphi^2 + 1} + i\frac{\sqrt{d}}{|a|}\frac{\varphi}{\varphi^2 + 1}.$$

Since $\omega^{-1} > 1$ and $0 > (\omega')^{-1} > -1$ (so that ω is reduced) it can be expanded into a continued fraction $\omega = [0, \overline{b_1, b_2, \ldots, b_k}]$ where the bar denotes the primitive period. The corresponding complete quotients

$$\omega_v = \frac{P_v + \sqrt{d}}{Q_v}, \qquad \omega_0 = \omega$$

form again a periodic sequence, where for all $v \geq 0$, $\omega \geq 0$, ω_v is reduced. Letting

$$\frac{A_v}{B_v} = [0, b_1, \ldots, b_v]$$

denote the v-th convergent to ω, it follows that for

$$B_n \leq y^{-\frac{1}{2}} < B_{n+1} \tag{13}$$

that

$$(B_n x - A_n)^2 + (B_n y)^2 \leq \left(B_n\omega - A_n - \frac{B_n y}{\varphi}\right)^2 + y \leq 5y$$

as long as $\varphi \geq 1$.

Now, $f(z, s)$ is invariant under the unimodular transformation

$$z \to z^* = \frac{rz + u}{B_n z - A_n}, \qquad \begin{pmatrix} r & u \\ B_n & -A_n \end{pmatrix} \in SL_2(Z),$$

$$y^* = \frac{y}{(B_n x - A_n)^2 + (B_n y)^2}.$$

Therefore

$$y^* \geqslant \frac{1}{5} \qquad (\text{for } \varphi \geqslant 1) . \tag{14}$$

The condition (13), subject to $\varphi \geqslant 1$, can be expressed

$$H'_n \leqslant \varphi < H'_{n+1} , \tag{15}$$

$$H'_n = \frac{1}{2}\left[\frac{\sqrt{d}}{|a|} B_n^2 + \left(\frac{d}{a^2} B_n^4 - 4\right)^{\frac{1}{2}}\right], \qquad \frac{d}{a^2} B_n^4 \geqslant 4 .$$

Letting $\|B_0\omega\| = 1$, $\|B_n\omega\| = |B_n\omega - A_n|$ for $n > 0$, it is easy to see that

$$y^* = \frac{\sqrt{d}}{a} \frac{\varphi}{\varphi^2\|B_n\omega\|^2 + \|B_n\omega'\|^2} ,$$

and after making the transformation

$$\varphi \to \varphi \frac{\|B_n\omega'\|}{\|B_n\omega\|} ,$$

it follows that

$$y^* = \frac{\sqrt{d}}{|a|\, \|B_n\omega\|\, \|B_n\omega'\|} \frac{\varphi}{\varphi^2 + 1} \tag{16}$$

for

$$\frac{\|B_n\omega\|}{\|B_n\omega'\|} H'_n \leqslant \varphi < \frac{\|B_n\omega\|}{\|B_n\omega'\|} H'_{n+1} .$$

Now, let

$$H_0 = \frac{|a|}{\sqrt{d}} , \qquad H_1 = \min\left(\frac{\sqrt{d}}{|a|} , \frac{|a|}{\sqrt{d}} \varepsilon_0^2\right),$$

$$H_n = \min\left(\frac{\|B_n\omega\|}{\|B_n\omega'\|} H'_n , \frac{|a|}{\sqrt{d}} \varepsilon_0^2\right) \qquad (n > 1) .$$

Also, let M denote the least integer n for which

$$H_n = \frac{|a|}{\sqrt{d}} \varepsilon_0^2 .$$

Choosing $\eta = |a|/\sqrt{d}$ in (12), we get

$$\zeta(s)L(s, \chi) = \frac{\Gamma(s)}{\Gamma^2(s/2)} \sum_{(a,b,c)} \sum_{n=1}^{M} \int_{H_{n-1}}^{H_n} d^{-s/2} f(z^*, s) \frac{d\varphi}{\varphi} , \tag{17}$$

where in the interval $H_0 \leqslant \varphi \leqslant H_1$ we take $z^* = z$.

Using the results of § 3 of [9]

$$(a', b', c') = (a, b, c)\begin{pmatrix} A_v & A_{v-1} \\ B_v & B_{v-1}\end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & (-1)^{v+1}\end{pmatrix},$$

$$t = (-1)^{v+1}\left[\frac{P_{v+1}}{Q_{v+1}} + \frac{1}{2}\right],$$

where the new form (a', b', c') satisfies

$$(18) \qquad -|a'| < |b'| \leqslant |a'|, \quad (b')^2 - 4a'c' = d,$$

and is uniquely determined by w_v. Moreover by formula (14) of [9]

$$(19) \qquad |a'| = |a| \cdot \|B_n\omega\| \cdot \|B_n\omega'\| = \tfrac{1}{2}Q_{n+1},$$

and every form satisfying (18) with $|a'| < \frac{1}{4}\sqrt{d}$ can be obtained by such a transformation.

Now,

$$H_{M-1} \leqslant \frac{|a|}{\sqrt{d}}\,\varepsilon_0^2 \leqslant H_M$$

which implies for $M > 2$

$$B_{M-1} - \frac{1}{4} \leqslant \frac{|a|}{\sqrt{d}}\,\varepsilon_0 \leqslant B_M,$$

and therefore by equation (17) of [9] we must have $M = [k, 2]$. This insures that there will be no repetitions among the forms (a', b', c') associated to the transformations (16) in the range $H_1 \leqslant \varphi \leqslant H_M$. It now follows from (12), (16) and (19) that

$$(20) \qquad \zeta(s)L(s, \chi) = \frac{\Gamma(s)}{\Gamma^2(s/2)} \sum_{(a,b,c)} \sum_{n=1}^{M} \int_{H_{n-1}}^{H_n} d^{-s/2} f(z^*, s)\,\frac{d\varphi}{\varphi},$$

$$\frac{1}{5} \leqslant y^* = \frac{\sqrt{d}}{|a'|}\frac{\varphi}{\varphi^2 + 1} \quad \text{and } a' = a \text{ for } H_0 \leqslant \varphi \leqslant H_1.$$

THEOREM 4. *Let $d > 1$, $\chi(-1) = +1$ and $\alpha^* = |a'|(\varphi + 1/\varphi) \leqslant 5\sqrt{d}$, where $|a'| = |a| \cdot \|B_n\omega\| \cdot \|B_n\omega'\|$ for $H_n \leqslant \varphi \leqslant H_{n+1}$. Then for $s = \sigma + it$ and $\sigma > \frac{1}{2}$*

$$\zeta(s)L(s, \chi) = \frac{\Gamma(s)}{\Gamma^2(s/2)}\left[\sum_{(a,b,c)} \sum_{n=1}^{M} 2\zeta(2s)\int_{H_{n-1}}^{H_n} (\alpha^*)^{-s}\frac{d\varphi}{\varphi} + \frac{\pi}{\sqrt{d}}\frac{s}{s-1}\int_{H_{n-1}}^{H_n}\left(\frac{d}{\alpha^*}\right)^{1-s}\frac{d\varphi}{\varphi} + R_1(s)\right],$$

$$|R_1(s)| \leqslant \frac{4|s|}{\sigma - \frac{1}{2}} \sum_{(a,b,c)} \sum_{n=1}^{M} \int_{H_{n-1}}^{H_n} \left(\frac{1}{2}\left(\frac{\alpha^*}{d}\right)^{\sigma} + \frac{(\alpha^*)^{\sigma-1}}{d^{\sigma-\frac{1}{2}}}\right)\frac{d\varphi}{\varphi},$$

where the outer sum goes over the set of reduced, inequivalent forms (a, b, c) of discriminant d.

PROOF. In equation (20)

$$d^{-s/2}f(z^*, s) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (\alpha^* m^2 + \beta^* mn + \gamma^* n^2)^{-s},$$

$$\beta^* - 4\alpha^*\gamma^* = -4d, \qquad \alpha^* = |a'|\left(\varphi + \frac{1}{\varphi}\right).$$

Let $\lambda_1 < \lambda_2 < \dots$ be the real numbers represented by the form $(\alpha^*, \beta^*, \gamma^*)$ and r_v the exact number of solutions of

$$\lambda_v = \alpha^* m^2 + \beta^* mn + \gamma^* n^2 \qquad (n \neq 0).$$

Now, $\lambda_1 \geqslant d/\alpha^*$ since there are no solutions to

$$(2\alpha^* m + \beta^* n)^2 + 4dn^2 = 4\alpha^* \lambda_1 < 4d.$$

By use of Lemma (3) it follows that

$$d^{-s/2}f(z^*, s) = 2\zeta(2s)\frac{1}{(\alpha^*)^s} + \sum_{v=1}^{\infty} r_v(\lambda_v)^{-s},$$

$$\sum_{v=1}^{\infty} r_v(\lambda_v)^{-s} = s\int_{d/\alpha^*}^{\infty} S(u)u^{-s-1}\,du =$$

$$= \frac{\pi}{\sqrt{d}}\frac{s}{s-1}\left(\frac{d}{a^*}\right)^{1-s} + 4s\int_{d/\alpha^*}^{\infty}\left(\theta\sqrt{\frac{\alpha^*}{4d}} - \varrho\sqrt{\frac{1}{\alpha^*}}\right)u^{-s-\frac{1}{2}}\,du,$$

where the last integral is regular for $\sigma > \frac{1}{2}$ and bounded by

$$\frac{4|s|}{\sigma - \frac{1}{2}}\left[\frac{1}{2}\left(\frac{\alpha^*}{d}\right)^{\sigma} + \frac{(\alpha^*)^{\sigma-1}}{d^{\sigma-\frac{1}{2}}}\right]. \qquad \text{Q.E.D.}$$

LEMMA 4. *For $d > 4$,*

$$\sum_{n<\frac{1}{4}\sqrt{d}} \nu_n < \frac{1}{4\log 2} L(1, \chi)\sqrt{d}.$$

PROOF. As in the proof of Lemma (1) of [9[, every ideal $\mathfrak{a}$ of $Q\sqrt{\chi(-1)d}$ can be uniquely represented in the form $\mathfrak{a} = u[a, \frac{1}{2}(b + \sqrt{\chi(-1)d})]$ where

u, a are positive integers such that

$$(*) \qquad b^2 \equiv \chi(-1)d(\text{mod } 4a)\,, \qquad -a < b \leqslant a\,.$$

Consequently, since $N(\mathfrak{a}) = u^2 a$

$$\frac{\zeta(s)L(s,\chi)}{\zeta(2s)} = \sum{}^* a^{-s}$$

where $\sum^*$ goes over all a, b satisfying $(*)$, and therefore

$$(21) \qquad \nu_n = \sum_{a=n}{}^* 1\,.$$

When $\chi(-1) = -1$, each solution of $(*)$ with $a < \frac{1}{4}\sqrt{d}$ corresponds to a reduced form. Hence

$$\sum_{n<\frac{1}{4}\sqrt{d}} \nu_n = \sum_{a<\frac{1}{4}\sqrt{d}}{}^* 1 \leqslant \frac{1}{\pi} L(1,\chi)\sqrt{d}$$

by Dirichlet's class number formula (1).

In the case that $\chi(-1) = +1$, every form (a, b, c) satisfying $(*)$ is equivalent to a reduced form (α, β, γ) with

$$\beta + \sqrt{d} > 2|\alpha| > -\beta + \sqrt{d} > 0\,,$$

$$\frac{1}{2\alpha}(-\beta + \sqrt{d}) = [0, \overline{b_1, b_2, \ldots, b_k}]\,.$$

It now follows from (21) and Lemma (3) of [9] that

$$\sum_{n<\frac{1}{4}\sqrt{d}} \nu_n \leqslant \frac{1}{2} \sum_{(\alpha,\beta,\gamma)} \sum_{\substack{v=2 \\ 2\leqslant b_v \leqslant \sqrt{d}}}^{[k,2]} 1 \leqslant \frac{1}{2\log 2} \sum_{(\alpha,\beta,\gamma)} \sum_{v=1}^{[k,2]} \log b_v\,,$$

and by equation (17) of [9]

$$\sum_{v=1}^{[k,2]} \log b_v < \log \varepsilon_0\,.$$

Hence, by Dirichlet's class number formula (1), we get

$$\sum_{n<\frac{1}{4}\sqrt{d}} \nu_n < \frac{1}{4\log 2} L(1,\chi)\sqrt{d}\,. \qquad \text{Q.E.D.}$$

LEMMA 5. *Let* $d>4$ *and* $\chi(-1)=-1$. *Then for* $0<10y<x$ *and* $0<\varepsilon<1/10$

$$\sum_{y\leqslant n\leqslant x} v_n n^{-\frac{1}{2}} < L(1,\chi)\left[\tfrac{5}{2}\sqrt{d}\,y^{-\frac{1}{2}} + 65\sqrt{x} + 8\varepsilon^{-1}x^{\frac{1}{2}-\varepsilon}d^{\varepsilon}\right].$$

PROOF. If $y\leqslant n\leqslant x$ and $x>10y>0$

$$\exp\left(-\frac{n}{x}\right)-\exp\left(-\frac{n}{y}\right)\geqslant \exp(-1)\left(1-\exp\left(-\frac{9}{10}\frac{n}{y}\right)\right)>\frac{1}{5}.$$

It follows that

$$\sum_{y\leqslant n\leqslant x} v_n n^{-\frac{1}{2}} < \frac{5}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta(s+\frac{1}{2})L(s+\frac{1}{2},\chi)}{\zeta(2s+1)}\Gamma(s)(x^s-y^s)\,ds\,.$$

Substituting the expression for $\zeta(s)L(s,\chi)$ as given in Theorem 3, the above integral is trasformed into a sum of 3 integrals. These are calculated as follows.

(I) $$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\sum_{(a,b,c)}a^{-s-\frac{1}{2}}\right)\Gamma(s+1)(x^s-y^s)\frac{ds}{s}\leqslant(1+10^{-\frac{1}{2}})\frac{L(1,\chi)\sqrt{d}}{\pi^2}y^{-\frac{1}{2}}\cdot$$
$$\cdot\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1}{2}+it\right)\right|dt<\frac{1}{2}L(1,\chi)\sqrt{d}\,y^{-\frac{1}{2}}\,,$$

after shifting the line of integration to $\sigma=-\frac{1}{2}$ and using

$$\sum_{(a,b,c)}1=\frac{1}{\pi}L(1,\chi)\sqrt{d}\,.$$

(II) $$\left|\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\pi}{\sqrt{d}}\frac{s+\frac{1}{2}}{s-\frac{1}{2}}\sum_{(a,b,c)}\left(\frac{d}{4a}\right)^{\frac{1}{2}-s}\frac{\Gamma(s)}{\zeta(2s+1)}(x^s-y^s)\,ds\right|\leqslant$$
$$\leqslant L(1,\chi)\left|\frac{\sqrt{\pi}}{\zeta(2)}(\sqrt{x}-\sqrt{y})+\frac{\varepsilon^{-1}}{2\pi}(1+10^{-\frac{1}{2}+\varepsilon})x^{\frac{1}{2}-\varepsilon}\left(\frac{d}{4}\right)^{\varepsilon}\frac{\zeta(2-2\varepsilon)}{\zeta(4-4\varepsilon)}\cdot\right.$$
$$\left.\cdot\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1}{2}-\varepsilon+it\right)\right|dt\right|<L(1,\chi)(1.08\sqrt{x}+1.6\,\varepsilon^{-1}x^{\frac{1}{2}-\varepsilon}d^{\varepsilon})\,,$$

after shifting the line of integration to $\sigma=\frac{1}{2}-\varepsilon$ with $0<\varepsilon<1/10$.

The extra term arises from the simple pole at $s = \frac{1}{2}$.

$$\text{(III)} \qquad \left|\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s)\frac{R(s+\frac{1}{2})}{\zeta(2s+1)}(x^s-y^s)\,ds\right| \leqslant$$

$$\leqslant \frac{16}{\pi^2}(1+10^{-\frac{1}{2}})\frac{\zeta(2)}{\zeta(4)}\sqrt{x}\,L(1,\chi)\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1}{2}+it\right)\right|\sqrt{1+t^2}\,dt$$

$$< 19.6L(1,\chi)\sqrt{x}\,. \qquad \text{Q.E.D.}$$

LEMMA 6. *Let $d>1$ and $\chi(-1)=+1$. Then for $0<10y<x, 0<\varepsilon<1/10$*

$$\sum_{y\leqslant n\leqslant x} v_n n^{-\frac{1}{2}} < L(1,\chi)\left[10\sqrt{d}\,y^{-\frac{1}{2}} + 370\sqrt{x} + 68\varepsilon^{-1}x^{\frac{1}{2}-\varepsilon}d^{\varepsilon}\right].$$

PROOF. For $c>0$, let

$$K(c) = \int_0^\infty \exp\left(-c\left(u+\frac{1}{u}\right)\right)\frac{du}{u}\,.$$

As a Mellin transform

$$2K\left(\frac{1}{x}\right) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^s\Gamma^2\left(\frac{s}{2}\right)ds \qquad \text{if } x>0\,.$$

Now, for $y\leqslant n\leqslant x$ and $x>10y>0$,

$$\exp\left(-\frac{n}{x}u+\frac{1}{u}\right)\right) - \exp\left(-\frac{n}{y}\left(u+\frac{1}{u}\right)\right) \geqslant \exp\left(-\left(u+\frac{1}{u}\right)\right)\cdot$$

$$\cdot\left(1-\exp\left(-\frac{9}{10}\frac{n}{y}\left(u+\frac{1}{u}\right)\right)\right) > 0.835\exp\left(-\left(u+\frac{1}{u}\right)\right).$$

Hence,

$$2K\left(\frac{n}{x}\right) - 2K\left(\frac{n}{y}\right) > 3.34\int_1^\infty \exp\left(-\left(u+\frac{1}{u}\right)\right)\frac{du}{u} > \frac{1}{5}\,.$$

It follows that

$$\sum_{y\leqslant n\leqslant x} v_n < \frac{5}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma^2\left(\frac{s}{2}\right)\frac{\zeta(s)L(s,\chi)}{\zeta(2s)}(x^s-y^s)\,ds\,.$$

Now, substitute the expression for $\zeta(s)L(s, \chi)$ as given in Theorem (4). The resulting integrals are calculated as follows.

$$\text{(I)}\qquad 2\sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\frac{1}{2\pi i}\left(\int_{2-i\infty}^{2+i\infty}(\alpha^*)^{-s}\Gamma(s)(x^s-y^s)\,ds\right)\frac{d\varphi}{\varphi}=$$

$$=2\sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H}\left(\exp\left(-\frac{\alpha^*}{x}\right)-\exp\left(-\frac{\alpha^*}{y}\right)\right)\frac{d\varphi}{\varphi}\leqslant 2L(1,\chi)\sqrt{d}\,.$$

Here, we have used Dirichlet's class number formula (1) in conjunction with the fact that for each of the h forms (a, b, c)

$$\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\frac{d\varphi}{\varphi}=2\log\varepsilon_0\,.$$

The second integral is

$$\text{(II)}\qquad \left|\frac{\pi}{\sqrt{d}}\sum_{(a,b,c)}\sum_{n=1}^{M}\int_{H_{n-1}}^{H_n}\left(\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{s}{1-s}\left(\frac{d}{\alpha^*}\right)^{1-s}\frac{\Gamma(s)}{\zeta(2s)}(x^s-y^s)\,ds\right)\frac{d\varphi}{\varphi}\right|\leqslant$$

$$\leqslant\frac{\pi L(1,\chi)(x-y)}{\zeta(2)}+\frac{(1+10^{-1+\varepsilon})}{2\varepsilon}\frac{\zeta(2-2\varepsilon)}{\zeta(4-4\varepsilon)}L(1,\chi)x^{1-\varepsilon}d^{\varepsilon}\int_{-\infty}^{\infty}|\Gamma(1-\varepsilon+it)|\,dt$$

$$<2L(1,\chi)x+6\varepsilon^{-1}L(1,\chi)x^{1-\varepsilon}d^{\varepsilon}\,,$$

after shifting the line of integration to $\sigma=1-\varepsilon$ with $0<\varepsilon<1/10$. Finally,

$$\text{(III)}\qquad \left|\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}R_1(s)\Gamma(s)(x^s-y^s)\frac{1}{\zeta(2s)}\,ds\right|\leqslant$$

$$\leqslant\frac{4}{\pi}\frac{\zeta(2)}{\zeta(4)}\left(\frac{5}{2}+1\right)\left(1+\frac{1}{10}\right)L(1,\chi)x\int_{-\infty}^{\infty}|\Gamma(1+it)|\sqrt{1+t^2}\,dt<35L(1,\chi)x\,.$$

Here, we have shifted the line of integration to $\sigma=1$ and used the bound for $|R_1(s)|$ given in Theorem 4 together with the upper bound $\alpha^*\leqslant 5\sqrt{d}$.

Combining these last three estimates

$$S(y,x)=\sum_{y\leqslant n\leqslant x}\nu_n<5L(1,\chi)\left(2\sqrt{d}+37x+6\varepsilon^{-1}x^{1-\varepsilon}d^{\varepsilon}\right).$$

Hence,

$$\sum_{y\leqslant n\leqslant x} \nu_n n^{-\frac{1}{2}} \leqslant x^{-\frac{1}{2}} S(y, x) + \frac{1}{2}\int_y^x S\left(\frac{y}{10}, u\right) u^{-\frac{3}{2}}\, du <$$

$$< L(1, \chi)(10\sqrt{d}\, y^{-\frac{1}{2}} + 370\sqrt{x} + 68\,\varepsilon^{-1} x^{\frac{1}{2}-\varepsilon} d^{\varepsilon}) \qquad \text{Q.E.D.}$$

LEMMA 7. *Let* $x > d > \frac{1}{4}\sqrt{d}$ *and* $10 < y < \frac{1}{4}\sqrt{d}$. *Then*

$$\sum_{y\leqslant n\leqslant x} n^{-\frac{1}{2}} \sum_{m|n} \nu_m \nu_{n/m} \leqq 1500\,(L(1, \chi)^2 x y^{-\frac{1}{2}} + L(1, \chi) x^{\frac{1}{2}})(\log y)^3 .$$

PROOF. For $x > d$ and $y < \frac{1}{4}\sqrt{d}$

$$\sum_{y\leqslant n\leqslant x} n^{-\frac{1}{2}} \sum_{m|n} \nu_m \nu_{n/m} \leqslant \Big(\sum_{y\leqslant n\leqslant x/y} \nu_n n^{-\frac{1}{2}}\Big)^2 + 2 \sum_{y\leqslant n\leqslant x} n^{-\frac{1}{2}} \sum_{\substack{m|n\\ m\leqslant y}} \nu_m \nu_{n/m} .$$

By Lemma (4), (5), (6) and the simple bound

$$\sum_{m\leqslant y} \frac{\nu_m}{m} \leqslant \sum_{m\leqslant y} \frac{1}{m} \sum_{d|m} 1 \leqslant \sum_{d\leqslant y} \frac{1}{d}\left(1 + \int_1^{y/d} \frac{du}{u}\right) \leqslant (1 + \log y)^2 ,$$

we get for any $0 < \varepsilon < 1/10$

$$\Big(\sum_{y\leqslant n\leqslant x/y} \nu_n n^{-\frac{1}{2}}\Big)^2 < L(1, \chi)^2 y^{-1}\left[380 x^{\frac{1}{2}} + 68 x^{\frac{1}{2}} \frac{y^{\varepsilon}}{\varepsilon}\right]^2 ,$$

$$\sum_{y\leqslant n\leqslant x} n^{-\frac{1}{2}} \sum_{\substack{m|n\\ m\leqslant y}} \nu_m \nu_{n/m} = \sum_{m\leqslant y} \nu_m m^{-\frac{1}{2}} \sum_{y/m\leqslant n\leqslant x/m} \nu_n n^{-\frac{1}{2}} \leqslant$$

$$\leqslant L(1, \chi) \sum_{m\leqslant y} \nu_m m^{-\frac{1}{2}} \left[10\sqrt{d}\left(\frac{y}{m}\right)^{-\frac{1}{2}} + 370\left(\frac{x}{m}\right)^{\frac{1}{2}} + \frac{68}{\varepsilon}\left(\frac{x}{m}\right)^{\frac{1}{2}-\varepsilon} d_{\varepsilon}\right] \leqslant$$

$$\leqslant \frac{5}{2\log 2} L(1, \chi)^2 d y^{-\frac{1}{2}} + L(1, \chi) x^{\frac{1}{2}} (1 + \log y)^2 \left(370 + \frac{68}{\varepsilon} y^{\varepsilon}\right) .$$

The Lemma follows on choosing

$$\varepsilon = \frac{\log 10}{10\log y} < \frac{1}{10} \qquad \text{for } y > 10 . \qquad \text{Q.E.D.}$$

LEMMA 8. *Let* $x < d$ *and* $10 < y < \min\left(\frac{1}{4}\sqrt{d}, x/10\right)$. *Then*

$$\sum_{y\leqslant n\leqslant x} n^{-\frac{1}{2}} \sum_{m|n} \nu_m \nu_{n/m} \ll (L(1, \chi)^2 d y^{-\frac{1}{2}} + L(1, \chi) x^{2/5} d^{1/10})(\log y)^3 .$$

PROOF. The proof is almost identical to the proof of Lemma 7. It is only necessary to note the inequality $x^{\frac{1}{2}} \leqslant x^{\frac{1}{2}-\varepsilon} d^{\varepsilon} \leqslant x^{2/5} d^{1/10} \leqslant d^{\frac{1}{2}}$ for $0 < \varepsilon < 1/10$. Q.E.D.

4. – Proof of theorem 1.

Actually, we prove the stronger

THEOREM 5. *Under the conditions of Theorem* 1

$$L(1,\chi) \gg \frac{g^{-4g} N^{-13} (\log d)^{g-\mu-1}}{\sqrt{d} (\log\log d)^{g-\mu+6}} \prod_{\substack{\chi(p)\neq -1\\ p<(\log d)^{8g}}} (1+p^{-\frac{1}{2}})^{-4}$$

where the implied constant can be effectively computed and is independent of g, N and d.

In order to deduce Theorem 1 from Theorem 5, we appeal to the following simple result.

LEMMA 9. *Let* $d > \exp(500\, g^3)$. *Then either*

$$L(1,\chi) > (\log d)^{g-\mu-1} d^{-\frac{1}{2}}$$

or

$$P = \prod_{\substack{\chi(p)\neq -1\\ p<(\log d)^{8g}}} (1+p^{-\frac{1}{2}})^{-4} < \exp\left(20 g^{\frac{1}{2}} (\log\log d)^{\frac{1}{2}}\right).$$

PROOF. We can assume that

$$L(1,\chi) \leqslant (\log d)^{g-\mu-1} d^{-\frac{1}{2}} \qquad \left(d > \exp(500\, g^3)\right).$$

It follows from Lemma 4 that

$$\sum_{n<\frac{1}{4}\sqrt{d}} \nu_n \leqslant \frac{1}{4\log 2} (\log d)^{g-\mu-1} \tag{22}$$

where ν_n is the n-th coefficient in the Dirichlet series expansion of $\zeta(s)L(s,\chi)/\zeta(2s)$. On examination of the Euler product

$$\zeta(s)L(s,\chi)/\zeta(2s) = \prod_{\chi(p)=0} (1+p^{-s}) \prod_{\chi(p)=+1} (1-p^{-s})^{-1}(1+p^{-s})$$

it can easily be seen that $\nu_n \geqslant 1$ if n is squarefree and divisible only by primes p for which $\chi(p) \neq -1$.

Now, let $\mathscr{P}$ denote the set of primes $p < (\log d)^{8g}$ for which $\chi(p) \neq -1$, and $\mathscr{N}$ the set of squarefree integers $< \frac{1}{4}\sqrt{d}$ divisible only by primes $p \in \mathscr{P}$. We also take $1 \in \mathscr{N}$. As usual, $|\mathscr{P}|$, $|\mathscr{N}|$ denote the cardinalities of $\mathscr{P}$ and $\mathscr{N}$, respectively. If $p_1, p_2, \ldots, p_k \in \mathscr{P}$, then it is clear that $\prod_{1}^{k} p_i \in \mathscr{N}$ provided

$$(\log d)^{8gk} < \tfrac{1}{4}\sqrt{d}\,.$$

Consequently, if

$$|\mathscr{P}| < \log\left(\tfrac{1}{4}\sqrt{d}\right)/8g \log\log d\,,$$

the number of integers contained in $\mathscr{N}$ having exactly k prime factors is $\binom{|\mathscr{P}|}{k}$. Hence

$$|\mathscr{N}| = \sum_{k=0}^{|\mathscr{N}|} \binom{|\mathscr{P}|}{k} = 2^{|\mathscr{P}|}\,.$$

But $\nu_n \geqslant 1$ for $n \in \mathscr{N}$. It, therefore, follows from (22) that

$$2^{|\mathscr{P}|} \leqslant \frac{1}{4\log 2} (\log d)^{g-\mu-1}\,.$$

We get

$$|\log P| < 8 \sum_{p \in \mathscr{P}} p^{-\frac{1}{2}} \leqslant 20 g^{\frac{1}{2}} (\log\log d)^{\frac{1}{2}}\,. \qquad \text{Q.E.D.}$$

In order to prove Theorem 5, we expand

$$\varphi(s) = L_E(s + \tfrac{1}{2}) L_E(s + \tfrac{1}{2}, \chi)$$

into a rapidly converging series whose main contribution comes from the terms $n \ll A$. It will be seen that the success of the method lies in the fact that A is of order d. We make use of an idea due to Lavrik [15].

Let $s = \sigma + it$ with $0 < \sigma < 1$. On using the functional equation for $\varphi(s)$, we have

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \chi(-N) A^{1-s+z} \Gamma^2\left(\frac{3}{2} - s + z\right) \varphi(1-s+z) \frac{dz}{z} =$$

$$= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} A^{s-z} \Gamma^2\left((s - z + \frac{1}{2}\right) \varphi(s-z) \frac{dz}{z} =$$

$$= A^s \Gamma^2\left(s + \frac{1}{2}\right)\varphi(s) + \frac{1}{2\pi i} \int_{-2-i\infty}^{-2+i\infty} A^{s-z} \Gamma^2\left(s - z + \frac{1}{2}\right) \varphi(s-z) \frac{dz}{z} =$$

$$= A^s \Gamma^2\left(s + \frac{1}{2}\right)\varphi(s) - \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} A^{s+z} \Gamma^2\left(s + z + \frac{1}{2}\right) \varphi(s+z) \frac{dz}{z}\,.$$

Therefore

$$(23)\qquad A^s\Gamma^2\left(s+\frac{1}{2}\right)\varphi(s)=\sum_{n=1}^{\infty}a_n\frac{A^s}{n^s}\cdot\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma^2\left(s+z+\frac{1}{2}\right)\frac{(A/n)^z}{z}\,dz+$$
$$+\chi(-N)\sum_{n=1}^{\infty}a_n\frac{A^{1-s}}{n^{1-s}}\cdot\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma^2\left(\frac{3}{2}-s+z\right)\frac{(A/n)^z}{z}\,dz\,.$$

The integrals occurring in (23) are calculated as follows

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma^2\left(s+z+\frac{1}{2}\right)\frac{(A/n)^z}{z}\,dz=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\int_0^{\infty}\exp(-u)\,u^{s+z-\frac{1}{2}}\,du\right)^2\frac{(A/n)^z}{z}\,dz=$$
$$=\int_{u_1=0}^{\infty}\int_{u_2=0}^{\infty}\exp\big(-(u_1+u_2)\big)(u_1u_2)^{s-\frac{1}{2}}\cdot\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{u_1u_2}{n}\right)^z\frac{dz}{z}=$$
$$=\int_{u_1=0}^{\infty}\int_{u_2=n/Au_1}^{\infty}\exp\big(-(u_1+u_2)\big)(u_1u_2)^{s-\frac{1}{2}}\,du_1\,du_2\,.$$

Similarly,

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma^2\left(\frac{3}{2}-s+z\right)\frac{(A/n)^z}{z}\,dz=\int_{u_1=0}^{\infty}\int_{u_2=n/Au_1}^{\infty}\exp\big(-(u_1+u_2)\big)(u_1u_2)^{\frac{1}{2}-s}\,du_1\,du_2\,.$$

Substituting these expressions in (23) and differentiating both sides κ times at $s=\frac{1}{2}$ yields

$$(24)\qquad\left(\frac{d}{ds}\right)^{\kappa}\left[A^s\Gamma^2\left(s+\frac{1}{2}\right)\varphi(s)\right]_{s=\frac{1}{2}}=\delta\sum_{r=0}^{\kappa}\binom{\kappa}{r}\sum_{n=1}^{\infty}a_n\sqrt{A/n}(\log A/n)^{\kappa-r}I_r(n/A)\,,$$

where we have written

$$\delta=1+(-1)^{\kappa}\chi(-N)\,,$$
$$I_r(M)=\int_{u_1=0}^{\infty}\int_{u_2=M/u_1}^{\infty}\exp\big(-(u_1+u_2)\big)(\log u_1u_2)^r\,du_1\,du_2\qquad(M\geqslant 0)\,.$$

A simple estimate using integration by parts gives the bounds

$$(25)\qquad |I_r(M)|\leqslant\begin{cases}2^{r+1}r! & M\geqslant 0\\ 2^{r+1}(r+1)!\,(\log M)^r\exp\left(-\sqrt{M}\right) & M\geqslant 3\,.\end{cases}$$

We now divide the right hand side of (24) into two sums as follows

$$(26)\qquad \left(\frac{d}{ds}\right)^{\kappa}\left[A^{s}\Gamma^{2}\left(s+\frac{1}{2}\right)\varphi(s)\right]_{s=\frac{1}{2}}=\delta\sum_{r=0}^{\kappa}\binom{\kappa}{r}\left(\sum_{n\leqslant A_1}\cdots+\sum_{n>A_1}\cdots\right)$$
$$=T_1+T_2\,,$$

where we have put

$$A_1=A\big((8+2\kappa)\log A\big)^2\,.$$

Evaluation of T_2.

LEMMA 10. *If $\kappa\leqslant g$ and $d>\exp(500g^3)$, then*

$$|T_2|\leqslant 1\,.$$

PROOF. Since the Dirichlet series expansion of $\varphi(s)$ is majorised by that of $\zeta(s)^4$, we have

$$|a_n|<n^4\,.$$

Hence, by (26)

$$|T_2|\leqslant 4^{\kappa+1}(\kappa+1)!A^4\sum_{n>A_1}(n/A)^{\frac{7}{4}}(\log n/A)^{\kappa}e^{-\sqrt{n/A}}$$
$$\leqslant 4^{\kappa+1}(\kappa+1)!A^4\int_{A_1/4A}^{\infty}u^{\frac{7}{4}}(\log u)^{\kappa}e^{-\sqrt{u}}\,du$$
$$=8^{\kappa+1}(\kappa+1)!A^4\int_{x}^{\infty}u^{8}(\log u)^{\kappa}e^{-u}\,du\,,\qquad \big(x=(4+\kappa)\log A\big)\,.$$

Furthermore, since $x>2(8+\kappa)$

$$\int_{x}^{\infty}u^{8}(\log u)^{\kappa}e^{-u}\,du\leqslant x^{8}(\log u)^{\kappa}e^{-x}+\frac{8+\kappa}{x}\int_{x}^{\infty}u^{8}(\log u)^{\kappa}e^{-u}\,du$$
$$\leqslant 2x^{8}(\log x)^{\kappa}e^{-x}\,,$$

after integrating by parts.

Consequently, since $\kappa\leqslant g$ and $A>1/4\pi^2\exp(500g^3)$, we see that $|T_2|\leqslant 1$. Q.E.D.

Evaluation of T_1.

LEMMA 11. *Let $d>\exp(500g^3)$, and $\kappa\leqslant g$. Then either*

$$L(1,\chi)>(\log d)^{\varrho-\mu-1}d^{-\frac{1}{2}}$$

or

$$T_1 = \left(\frac{d}{ds}\right)^{\kappa}\left[\delta A^s \Gamma^2\left(s+\frac{1}{2}\right)G(s, U)\varphi_1(2s)\right]_{s=\frac{1}{2}} + O\left(g^{4g} N L(1, \chi) A (\log\log A)^{\kappa+6}\right)$$

where $U = (\log d)^{8g}$, *and the constant implied by the O-symbol is effectively computable and independent of* g, κ, N, d *and* A.

PROOF. We henceforth assume that $L(1, \chi) \leqslant (\log d)^{g-\mu-1} d^{-\frac{1}{2}}$ with $d > \exp(500 g^3)$ and $\kappa \leqslant g$.

Let $F(s)$ be an analytic function satisfying

$$|F(s)| < c_1(|s| + 2)^{c_2}$$

for two fixed positive constants c_1 and c_2 in the half-plane $\sigma \geqslant 2$.

We define the transform

$$T(F(s)) = \left(\frac{d}{ds}\right)^{\kappa}\left[\frac{\delta}{2\pi i}\int_{2-i\infty}^{2+i\infty} A^{s+z}\Gamma^2\left(s+z+\frac{1}{2}\right)F(s+z)\varphi_1(2s+2z)\frac{dz}{z}\right]_{s=\frac{1}{2}}.$$

The growth condition on $|F(s)|$ ensures that the integral converges. Recall that

$$\varphi(s) = L_E(s+\tfrac{1}{2})L_E(s+\tfrac{1}{2}, \chi) = \sum_{n=1}^{\infty} a_n n^{-s}$$

$$\varphi_1(s) = L_E(s+\tfrac{1}{2})L_E(s+\tfrac{1}{2}, \lambda)$$

$$\varphi(s) = G(s)\varphi_1(2s)$$

where $G(s) \equiv 1$ if $\chi(p) = -1$ for all primes p.

If $G(s) = \sum g_n n^{-s}$ and $G(s, x) = \sum_{n<x} g_n n^{-s}$, then as in Lemma 1, a_n is identical with the n-th coefficient in the Dirichlet series expansion of $G(s, x)\varphi_1(2s)$ for $n < x$.

It follows that

$$T_1 = T(G(s, A_1)) + O(1). \tag{27}$$

The 0-term comes from the terms $n > A_1$, and is estimated precisely as in Lemma (10). The constant will not exceed one since $\kappa \leqslant g$ and $A > 1/4\pi^2 \cdot \exp(500 g^3)$.

Now, let $0 < A_0 < A_1$, and define

$$G(s, A_1)\varphi_1(2s) = G(s, A_0)\varphi_1(2s) + \sum_{n=1}^{\infty} b_n n^{-s}. \tag{28}$$

If $P(y)$ denotes the product of all prime powers $p^v \leqslant y$ for which $\chi(p) \neq -1$, then it is clear that $b_n = 0$ for $n \leqslant A_1$ unless

$$n = mk^2, \qquad m|P(A_1), \qquad A_0 \leqslant m \leqslant A_1,$$

and in this case

$$|b_n| \leqslant \sum_{n=mk^2} d(k) \sum_{f|m} v_f v_{m/f}, \qquad d(k) = \sum_{f|k} 1, \tag{29}$$

the bound (29) being obtained by noting that the k-th coefficient of $\varphi_1(2s)$ is bounded by the number of divisors of k while the m-th coefficient of $G(s, A_1)$ is bounded by $\sum_{f|m} v_f v_{m/f}$.

It follows from (27) and (28) that

$$\begin{cases} T_1 = T\big(G(s, A_0)\big) + T_3 + O(1), \\ T_3 = \delta \sum_{r=0}^{K} \binom{K}{r} \sum_{n \leqslant A_1} b_n \sqrt{A/n} (\log A/n)^{K-r} I_r(n/A), \end{cases} \tag{30}$$

with the contribution of all terms $n > A_1$ being absorbed in the 0-constant.

Now choose

$$A = A(\log A)^{-20\varrho}, \qquad J = A(\kappa + 6) \log\log A)^2.$$

Since $b_n = 0$ for $n < A_0$, we may write

$$|T_3| \leqslant 2 \sum_{r=0}^{K} \binom{K}{r} \Big(\sum_{A_0 \leqslant n \leqslant J} \cdots + \sum_{J \leqslant n \leqslant A_1} \cdots \Big) = S_1 + S_2. \tag{31}$$

On using the bound

$$|I_r(M)| \leqslant 2^{r+1}(r+1)!(\log M)^r \exp\big(-\sqrt{M}\big) \qquad (M \geqslant 3)$$

as given in (25), it follows that

$$S_2 \leqslant 4^{K+1}(\kappa + 1)!(\log A_1/A)^K \exp\big(-(\kappa+6)\big) \log\log A\big) \sum_{J \leqslant n \leqslant A_1} |b_n| n^{-\frac{1}{2}}. \tag{32}$$

By (29) and Lemma 7, we get for any $10 < y < \frac{1}{4}\sqrt{d} < J$

$$\begin{aligned} \sum_{J \leqslant n \leqslant A_1} |b_n| n^{-\frac{1}{2}} &\leqslant \sum_{k^2 \leqslant A_1/J} d(k) k^{-1} \sum_{J \leqslant m \leqslant A_1/k^2} m^{-\frac{1}{2}} \sum_{f|m} v_f v_{f|m} \\ &\ll (\log A_1/J)^2 \big[\big(L(1, \chi)^2 A_1 y^{-\frac{1}{2}} + L(1, \chi) A_1^{\frac{1}{2}}\big)(\log y)^3\big]. \end{aligned} \tag{33}$$

Now, recall that $A_1 = A\big((8+2\kappa)\log A\big)^2$, and choose

$$\begin{aligned} y &= L(1,\chi)^2 A_1 \\ &\leqslant (\log d)^{2g-2\mu-2} d^{-1} A\big((8+2\kappa)\log A\big)^2 \\ &\leqslant \frac{N}{4\pi^2}(8+2g)^2(\log A)^{2g}. \end{aligned}$$

Consequently, it follows from (33) that

$$\sum_{J\leqslant n\leqslant A_1} |b_n| n^{-\frac{1}{2}} \ll g(\log gN)^3 L(1,\chi)\sqrt{A}(\log A)(\log\log A)^5. \tag{34}$$

The implied constant is absolute.

On combining (32) and (34), we get

$$S_2 \ll g^{4g}(\log N)^3 L(1,\chi) A, \tag{35}$$

where the implied constant can be effectively computed.

In the estimate of S_1, we use the bound

$$I_r(n/A) \leqslant 2^{r+1} r! \qquad (n/A \geqslant 0),$$

as given in (25). We get

$$S_1 \leqslant 4^{\kappa+1}\kappa!(\log A/A_0)^{\kappa}\sqrt{A}\sum |b_n| n^{-\frac{1}{2}}.$$

Let $10 < y < \min(A_0, \frac14\sqrt{d})$. It follows from (29) and Lemma 7 that

$$\begin{aligned} \sum_{A_0\leqslant n\leqslant J} |b_n| n^{-\frac{1}{2}} &\leqslant \sum_{k^2\leqslant J/A_0} d(k)k^{-1}\sum_{A_0\leqslant m\leqslant J/k^2} m^{-\frac{1}{2}}\sum_{f\mid m}\nu_f\nu_{m/f} \\ &\ll (\log J/A_0)^2\big[\big(L(1,\chi)^2 J y^{-\frac{1}{2}} + L(1,\chi)J^{\frac{1}{2}}\big)(\log y)^3\big]. \end{aligned} \tag{36}$$

Now, choose

$$\begin{aligned} y &= L(1,\chi)^2 J \\ &\leqslant \frac{N}{4\pi^2}(g+6)^2(\log A)^{2g}. \end{aligned}$$

Substituting into (36) gives

$$\sum_{A_0\leqslant n\leqslant J} |b_n| n^{-\frac{1}{2}} \ll g^3(\log gN)^3 L(1,\chi) A(\log\log A)^{\kappa+6}.$$

Finally, we have

$$S_1 \ll g^{4g}(\log N)^3 L(1, \chi) A(\log\log A)^{\kappa+6}. \tag{37}$$

On combining (30), (31), (35) and (37) we get

$$T_1 = T\big(G(s, A_0)\big) + O\big(g^{4g}(\log N)^3 L(1, \chi) A(\log\log A)^{\kappa+6}\big). \tag{38}$$

The constant implied by the O-symbol can be effectively computed and is independent of g, κ, N, d and A.

The estimation of $T\big(G(s, A_0)\big)$.

The evaluation of T_1 has now been reduced to the determination of the transform $T\big(G(s, A_0)\big)$ (cf. (38)). Accordingly, we now turn out attention to this transform. Let

$$T\big(G(s, A_0)\big) = T\big(G(s, U)\big) + T\big(g(s)\big), \tag{39}$$

where we have put

$$G(s, A_0) = G(s, U) + g(s)$$

with

$$U = (\log d)^{8g}.$$

It is clear that the function $g(s)$ may be expanded into a Dirichlet series whose n-th coefficient is bounded by

$$\sum_{f|n} \nu_f \nu_{n/f}.$$

It now follows from Lemma (8) that on the line $\sigma = \frac{1}{2} + \varepsilon$ with $\varepsilon > 0$ (recall that $A_0 = A(\log A)^{-20g}$)

$$\begin{aligned} |g(s)| &\leqslant \sum_{U \leqslant n \leqslant A_0} n^{-\frac{1}{2}} \sum_{f|n} \nu_f \nu_{n/f} \\ &\ll \big(L(1, \chi)^2 A_0 u^{-\frac{1}{2}} + L(1, \chi) A_0^{2/5} d^{1/10}\big)(\log U)^3 \\ &\ll g^3 N(\log d)^{-2g-2}. \end{aligned} \tag{40}$$

In order to estimate $T\big(g(s)\big)$, we shall need to know the growth of $|\varphi_1(s)|$ just to the right of $\sigma = 1$. This is easily accomplished since on the line $\sigma = 1 + \varepsilon$ with $0 < \varepsilon < \frac{1}{4}$, it is easy to see that

$$|\varphi_1(s)| \leqslant \zeta^2(1 + \varepsilon) < (1 + \varepsilon^{-1})^2 < 2\varepsilon^{-2}. \tag{41}$$

Now, let C be the circle of radius ε centered at $s=\frac{1}{2}$. By Cauchy's Theorem and the bounds (40) and (41), we get

$$T(g(s))=\frac{\kappa!}{2\pi i}\int_C\left(s-\frac{1}{2}\right)^{-\kappa-1}\left(\frac{1}{2\pi i}\int_{2\varepsilon-i\infty}^{2\varepsilon+i\infty}A^{s+z}\Gamma^2\left(s+z+\frac{1}{2}\right)g(s+z)\varphi_1(2s+2z)\frac{dz}{z}\right)ds$$
$$\ll\kappa!\,g^3N\varepsilon^{-\kappa-3}A^{\frac{1}{2}+3\varepsilon}(\log d)^{-2g-2}\,.$$

Here, we have shifted the inner integral to $\mathrm{Re}(z)=2\varepsilon$, so that

$$\tfrac{1}{2}+\varepsilon\leqslant\mathrm{Re}(s+z)\leqslant\tfrac{1}{2}+3\varepsilon\qquad(\text{for } s \text{ on } C)\,.$$

Choosing

$$\varepsilon=(\log d)^{-1}\,,$$

it follows that

$$|T(g(s))|\ll g^{2g}N\sqrt{A}(\log d)^{-g}\,. \tag{42}$$

Consequently, by (38), (39) and (42)

$$T_1=T(G(s,U))+O\left(g^{4g}NL(1,\chi)A(\log\log A)^{\kappa+6}\right). \tag{43}$$

The constant implied by the O-symbol can be effectively computed and is independent of g, κ, N, d, and A.

The estimation of $T(G(s,U))$.

In view of (43), the final step in the proof of Lemma 11 is to evaluate $T(G(s,U))$. It will be demonstrated that

$$T(G(s,U))=\left(\frac{d}{ds}\right)^{\kappa}\left[\delta A^s\Gamma^2\left(s+\frac{1}{2}\right)G(s,U)\varphi_1(2s)\right]_{s=\frac{1}{2}}+\theta\sqrt{A}$$

where $|\theta|\leqslant 1$. This clearly establishes Lemma 11.

Let C be the circle of radius $\frac{1}{2}\varepsilon>0$ centered at $s=\frac{1}{2}$ for some ε to be chosen later. By Cauchy's Theorem

$$T(G(s,U))=\frac{\kappa!}{2\pi i}\int_C\left(s-\frac{1}{2}\right)^{-\kappa-1}\cdot \tag{44}$$
$$\cdot\left(\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\delta A^{s+z}\Gamma^2\left(s+z+\frac{1}{2}\right)G(s+z,U)\varphi_1(2s+2z)\frac{dz}{z}\right)ds\,.$$

We evaluate $T(G(s, U))$ by replacing the line integral in (44) by a suitable contour integral and computing the residue at $z=0$. This residue is just

$$R = \left(\frac{d}{ds}\right)^{\kappa}\left[\delta A^{s}\Gamma^{2}\left(s+\frac{1}{2}\right)G(s, U)\varphi_1(2s)\right]_{s=\frac{1}{2}}$$

To be precise, let

$$T(G(s, U)) = \frac{\kappa!}{2\pi i}\int_{C}\left(s-\frac{1}{2}\right)^{-\kappa-1}\sum_{r=1}^{5} I_r(s)\,ds + R \tag{45}$$

where

$$I_1 = \int_{\frac{1}{2}+iM}^{\frac{1}{2}+i\infty}, \qquad I_2 = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}-iM}, \qquad I_3 = \int_{\frac{1}{2}+iM}^{-\varepsilon+iM}, \qquad I_4 = \int_{\frac{1}{2}-iM}^{-\varepsilon-iM}, \qquad I_5 = \int_{-\varepsilon-iM}^{-\varepsilon+iM},$$

and M is a large number to be determined later.

In order to estimate the above integrals, it is necessary to know the growth conditions of the functions occurring in the integrand. We shall, therefore, deal with each of these functions separately. By Stirling's asymptotic expansion

$$\log\Gamma(s) = \left(s-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + J(s)\,,$$

$$J(s) = \frac{1}{\pi}\int_{0}^{\infty} -\log\left(1-\exp\left(-2\pi u\right)\right)\frac{s}{s^2+u^2}\,du\,.$$

Consequently

$$|\Gamma(s)| = \sqrt{2\pi}\,|s|^{\sigma-\frac{1}{2}}\exp\left(-\sigma - t\operatorname{artg}(t/\sigma) + \operatorname{Re} J(s)\right)\,.$$

Since

$$\frac{\pi}{2}-\frac{1}{u} \leqslant \operatorname{artg} u \leqslant \frac{\pi}{2} \qquad (u>0)\,,$$

it follows for $\sigma>0$ that

$$|\Gamma(s)| \leqslant \sqrt{2\pi}\exp\left(1/12\sigma\right)|s|^{\sigma-\frac{1}{2}}\begin{cases}\exp\left(-\sigma\right) & |\sigma/t| \geqslant \pi/2\,,\\ \exp\left(-\dfrac{\pi}{2}|t|\right) & |\sigma/t| \leqslant \pi/2\,.\end{cases} \tag{46}$$

Now, for $\operatorname{Re}(s+z) \geqslant 0$

$$\begin{aligned}|G(s+z, U)| &\leqslant \sum_{n\leqslant U}\sum_{f|n} \nu_f \nu_{n/f}\\ &\leqslant \sum_{n\leqslant U} n^3\\ &\leqslant (\log d)^{32\sigma}\,.\end{aligned} \tag{47}$$

To estimate $|\varphi_1(s)|$, we recall Lemma 2

$$\varphi_1(s) = L_K(s+1, \psi^2)\frac{L(s, \chi_k)}{\zeta(s)}\prod_{p|k}(1-p^{-s})^{-1}\,. \tag{48}$$

We shall consider each function on the right-hand side of (48) separately.

For $\mathrm{Re}(s) \geqslant 0$

$$\prod_{p|k}(1-p^{-s})^{-1} \leqslant 2^{\omega(k)} \leqslant k \qquad \Big(\omega(k) = \sum_{p|k} 1\Big)\,.$$

For $\mathrm{Re}(s) \geqslant \frac{1}{2}$

$$|L(s, \chi_k)| \ll k(|t|+2)$$

where the implied constant is absolute and can be effectively computed (*).

We also need the known zero-free region for $\zeta(s)$ (**)

$$|\zeta(s)^{-1}| \leqslant c_3 \begin{cases} |t|^{C_4}\,, & 1-\dfrac{c_5}{\log|t|} < \sigma \leqslant 2\,, \qquad |t| \geqslant 2\,, \\ 1/|\sigma-1|\,, & \frac{3}{4} \leqslant \sigma \leqslant 2\,, \qquad |t| \leqslant 2\,, \end{cases}$$

where the constants c_3, c_4, c_5 can be effectively computed and are independent of s.

Now, suppose $f(s)$ is regular and of finite order for $\beta_1 \leqslant \sigma \leqslant \beta_2$. Then if $|f(s)| \leqslant B$ on the lines $\sigma = \beta_1$ and $\sigma = \beta_2$, it follows that $|f(s)| \leqslant B$ in the strip $\beta_1 \leqslant \sigma \leqslant \beta_2$ (***). This convexity principle will be applied in determining the growth of $L_K(s, \psi^2)$. If ψ^2 is not primitive, we let ψ' be the primitive « Grössencharketer » which induces it. Then

$$L_K(s, \psi^2) = P(s)L_K(s, \psi')$$

where

$$P(s) = \prod_{p|N}\left(1-\frac{\gamma_p}{p^s}\right)^{-1}\left(1-\frac{\bar{\gamma}_p}{p^s}\right)^{-1}\,, \qquad (|\gamma| \leqslant p)$$

is a finite product going over the bad primes. If $\mathfrak{f}_1|\mathfrak{f}$ is the conductor of ψ',

(*) K. Prachar, *Primzahlverteilung*, Springer (1957).

(**) E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Bd. 1, Leipzig (1909).

(***) G. H. Hardy - M. Riesz, *The general Theory of Dirichlet series*, Cambridge (1952).

then by (5)

$$\left(\frac{\sqrt{kN(\mathfrak{f}_1)}}{2\pi}\right)^s \Gamma(s) L_K(s, \psi') = W' \left(\frac{\sqrt{kN(\mathfrak{f}_1)}}{2\pi}\right)^{3-s} \Gamma(3-s) L_K(3-s, \psi'), \quad W' = \pm 1.$$

Since ψ' is not principal it follows that $L_K(s, \psi')$ is entire and its Dirichlet series expansion will be absolutely convergent for $\sigma > 2$. It is easily seen that

$$\left|L_K\left(\frac{5}{2} + it, \psi'\right)\right| \leqslant \zeta\left(\frac{3}{2}\right)^2 < 9 .$$

Consequently

$$\left|L_K\left(\frac{1}{2} + it, \psi'\right)\right| < 9 \frac{kN(\mathfrak{f}_1)}{4\pi^2} \left|\frac{\Gamma(\frac{5}{2} - it)}{\Gamma(\frac{1}{2} + it)}\right| < 10 \frac{N}{4\pi^2} \left|\frac{5}{2} + it\right|^2 .$$

Hence, the function

$$f(s) = L_K(s, \psi')(s+2)^{-2}$$

is bounded by

$$B = 10 \frac{N}{4\pi^2}$$

on the lines $\sigma = \frac{1}{2}$ and $\sigma = \frac{5}{2}$. By the aforementioned convexity principle, this implies that

$$|L_K(s, \psi')| \leqslant \frac{10N}{4\pi^2} |s+2|^2 \qquad \left(\frac{1}{2} \leqslant \sigma \leqslant \frac{2}{5}\right).$$

Since

$$|P(s)| \leqslant 2^{2\omega(N)} \leqslant N^2 \qquad (1 \leqslant \mathrm{Re}(s)),$$

it follows that

$$|L_K(s, \psi^2)| \leqslant \frac{10N^3}{4\pi^2} |s+2|^2 \qquad \left(1 \leqslant \sigma \leqslant \frac{5}{2}\right).$$

Combining all the previous estimates, we arrive at

$$(49) \qquad |\varphi_1(s)| \ll N^5 \begin{cases} |t|^{3+c_4}, & 1 - \dfrac{c_5}{\log|t|} < \sigma \leqslant \dfrac{3}{2}, \qquad |t| \geqslant 2, \\ \dfrac{1}{|\sigma - 1|}, & \dfrac{3}{4} \leqslant \sigma \leqslant \dfrac{3}{2}, \qquad |t| \leqslant 2 . \end{cases}$$

We now return to the estimation of the integrals I_r, $r=1, 2, 3, 4, 5$. Recall that s lies on the circle of radius $\frac{1}{2}\varepsilon$ centered at the point $\frac{1}{2}$.

First of all,

$$I_1=\frac{1}{2\pi i}\int_{\frac18+iM}^{\frac18+i\infty}\delta A^{s+z}\Gamma^2\left(s+z+\frac{1}{2}\right)G(s+z,\, U)\varphi_1(2s+2z)\frac{dz}{z}.$$

It follows from (46), (47), and (49) that

$$|I_1|\ll N^5(\log d)^{32g}A^{5/8+\varepsilon/2}e^{-M}.$$

Similarly

$$|I_2|\ll N^5(\log d)^{32g}A^{5/8+\varepsilon/2}e^{-M}.$$

We next consider

$$I_3=\frac{1}{2\pi i}\int_{\frac18+iM}^{-\varepsilon+iM}\delta A^{s+z}\Gamma^2\left(s+z+\frac{1}{2}\right)G(s+z,\, U)\varphi_1(2s+2z)\frac{dz}{z}.$$

In order to be able to apply (49), it is necessary that

$$\operatorname{Re}(2s+2z)>1-\frac{c_5}{\log 3M}\qquad\left(\varepsilon<\frac{1}{2}\right),$$

which implies that

$$1-3\varepsilon>1-\frac{c_5}{\log 3M}.$$

We can therefore choose

$$\varepsilon=\frac{c_6}{\log M}$$

with an effectively computable constant. We get

$$|I_3|\ll N^5(\log d)^{32g}A^{\frac58+\varepsilon/2}e^{-M}$$

$$|I_4|\ll N^5(\log d)^{32g}A^{\frac58+\varepsilon/2}e^{-M}.$$

It only remains to estimate

$$I_5=\frac{1}{2\pi i}\int_{-\varepsilon-iM}^{-\varepsilon+iM}\delta A^{s+z}\Gamma^2\left(s+z+\frac{1}{2}\right)G(s+z,\, U)\varphi_1(2s+2z)\frac{dz}{z}.$$

Again using (46), (47), (49), we find that

$$|I_5| \ll N^5(\log d)^{32g} A^{\frac{1}{2}(1-\varepsilon)}\varepsilon^{-2}.$$

It now follows from (45) that

$$|T(G(s, U)) - R| \ll 2^K \kappa!\, N^5(\log d)^{32g} \varepsilon^{-K}[A^{\frac{3}{4}+\varepsilon/2} e^{-M} + A^{\frac{1}{2}(1-\varepsilon)}\varepsilon^{-2}].$$

If we choose

$$M = c_7 gN \log A, \qquad \varepsilon = \frac{c_8}{\log M},$$

for a sufficiently large c_7, then

$$|T(G(s, U)) - R| \leqslant \sqrt{A}. \qquad \text{Q.E.D.}$$

5. – Proof of theorem 5.

It follows from (26), Lemma 10 and Lemma 11 that for $d > \exp(500\, g^3)$ and $\kappa \leqslant g$ that either Theorem 5 is true or else

$$(50)\quad \left(\frac{d}{ds}\right)^{\kappa}\left[A^s\Gamma^2\left(s+\frac{1}{2}\right)\varphi(s)\right]_{s=\frac{1}{2}} = \left(\frac{d}{ds}\right)^{\kappa}\left[\delta A^s\Gamma^2\left(s+\frac{1}{2}\right)G(s, U)\varphi_1(2s)\right]_{s=\frac{1}{2}} + O\big(g^{4g} NL(1, \chi) A (\log\log A)^{\kappa+6}\big),$$

where

$$U = (\log d)^{8g}, \qquad \delta = 1 + (-1)^{\kappa}\chi(-N).$$

Now, if $L_E(s)$ has a zero of order g at $s = 1$, then $\varphi(s)$ also has a zero of order g at $s = \frac{1}{2}$. This implies that

$$(51)\quad \left(\frac{d}{ds}\right)^{g-\mu}\left[A^s\Gamma^2\left(s+\frac{1}{2}\right)\varphi(s)\right]_{s=\frac{1}{2}} = 0$$

for $1 \leqslant \mu \leqslant g$.

Let us now choose

$$\kappa = g - \mu, \qquad \chi(-N) = (-1)^{g-\mu}, \qquad \mu = 1 \text{ or } 2.$$

This ensures that

$$\delta \neq 0.$$

We, therefore, get from (50) and (51) that

$$(52)\qquad \left(\frac{d}{ds}\right)^{g-\mu}\left[A^s\Gamma^2\left(s+\frac{1}{2}\right)G(s,\,U)\varphi_1(2s)\right]_{s=\frac{1}{2}} \ll g^{4g}NL(1,\,\chi)A(\log\log A)^{g-\mu+6}.$$

If H denotes the expression on the left-hand side of (52), we see on applying Leibniz' rule that

$$H=\sqrt{A}\sum_{r=1}^{g-\mu}\binom{g-\mu}{r}(\log A)^{g-\mu-r}\left(\frac{d}{ds}\right)^r\left[\Gamma^2\left(s+\frac{1}{2}\right)G(s,\,U)\varphi_1(2s)\right]_{s=\frac{1}{2}}$$

since $\varphi_1(2s)$ has a simple zero at $s=\frac{1}{2}$. Hence

$$(53)\qquad H=2(g-\mu)\sqrt{A}(\log A)^{g-\mu-1}G\left(\frac{1}{2},\,U\right)\varphi_1'(1)+$$
$$+\sqrt{A}\sum_{r=2}^{g-\mu}\binom{g-\mu}{r}(\log A)^{g-\mu-r}\left(\frac{d}{ds}\right)^r\left[\Gamma^2\left(s+\frac{1}{2}\right)G(s,\,U)\varphi_1(2s)\right]_{s=\frac{1}{2}}.$$

Again, by Leibniz' rule

$$\left(\frac{d}{ds}\right)^r\left[\Gamma^2\left(s+\frac{1}{2}\right)G(s,\,U)\varphi_1(2s)\right]_{s=\frac{1}{2}}=\sum_{h=0}^{r}\binom{r}{h}\left(\frac{d}{ds}\right)^{r-h}\left[\Gamma^2\left(s+\frac{1}{2}\right)\varphi_1(2s)\right]_{s=\frac{1}{2}}\cdot$$
$$\cdot\left(\frac{d}{ds}\right)^h[G(,\,U)]_{s=\frac{1}{2}}.$$

The derivatives of $G(s,\,U)$ at $s=\frac{1}{2}$ can be computed by Cauchy's Theorem as follows. Let C be the circle of radius $\frac{1}{4}$ centered at $s=\frac{1}{2}$. We have

$$\left(\frac{d}{ds}\right)^h[G(s,\,U)]_{s=\frac{1}{2}}=\frac{h!}{2\pi i}\int_C\left(s-\frac{1}{2}\right)^{-h-1}G(s,\,U)\,ds<2^{2h}h!\max_{s\text{ on }C}|G(s,\,U)|.$$

Moreover,

$$\max_{s\text{ on }C}|G(s,\,U)|<\prod_{\substack{\chi(p)\neq-1\\ p<(\log d)^{8g}}}(1-p^{-\frac{1}{4}})^{-4}$$
$$<\exp\left(100g^{3/4}(\log\log d)^{3/4}\right)$$

since there are at most $2g(\log\log d)$ primes $p<(\log d)^{8g}$ for which $\chi(p)\neq-1$, as we have seen earlier in the proof of Lemma 9.

It now follows from (53) that for $g-\mu-1\geq 0$

$$(54)\qquad H = 2(g-\mu)\sqrt{A}(\log A)^{g-\mu-1}G\left(\frac{1}{2}, U\right)\varphi_1'(1) + \\ + O\Big(g^{4g}\sqrt{A}(\log A)^{g-\mu-2}\exp\left(100g^{3/4}(\log\log d)^{3/4}\right)\Big)\max_{1\leq r\leq g-\mu}\cdot \\ \cdot\left(\left(\frac{d}{ds}\right)^r\left[\Gamma^2\left(s+\frac{1}{2}\right)\varphi_1(2s)\right]_{s=\frac{1}{2}}\right),$$

where the constant implied by the O-symbol can be effectively computed and is independent of g, d, and A.

In order to bring the O-term in (54) in its final form, we will obtain bounds for the derivatives of $\Gamma^2(s+\frac{1}{2})\varphi_1(2s)$ at $s=\frac{1}{2}$. This can be easily be accomplished by employing Cauchy's Theorem.

Let C be the circle of radius $\frac{1}{8}$ centered at $s=\frac{1}{2}$. By Cauchy's Theorem and (46) and (49) we have

$$\left(\frac{d}{ds}\right)^r\left[\Gamma^2\left(s+\frac{1}{2}\right)\varphi_1(2s)\right]_{s=\frac{1}{2}} = \frac{r!}{2\pi i}\int_C\left(s-\frac{1}{2}\right)^{-r-1}\Gamma_2\left(s+\frac{1}{2}\right)\varphi_1(2s)\,ds \\ < 2^{3r}r!\max_{s\text{ on }C}\left|\Gamma^2\left(s+\frac{1}{2}\right)\varphi_1(2s)\right| \\ \ll 2^{3r}r!N^5\,.$$

Consequently

$$H = 2(g-\mu)\sqrt{A}(\log A)^{g-\mu-1}G(\tfrac{1}{2}, U)\varphi_1'(1) \\ + O\Big(g^{8g}N^5\sqrt{A}(\log A)^{g-\mu-2}\exp\left(100g^{3/4}(\log\log d)^{3/4}\right)\Big)$$

with an effectively computable absolute constant.

Now, by Lemma 2

$$\varphi_1'(1) = L(2, \psi^2)L(1, \chi_k)\prod_{p|k}(1-p^{-1})\,.$$

We also need the lower bound

$$(55)\qquad G(\tfrac{1}{2}, U) \gg \prod_{\substack{\chi(p)\neq -1\\ p<(\log d)^{8g}}}(1+p^{-\frac{1}{2}})^{-4}\,.$$

In order to prove (55) we denote by $P(s, U)$ the partial Euler product of $G(s)$ for primes $p\leq U$; we also write

$$G(s, U) = P(s, U) - R(s, U)\,.$$

Since

$$|P(\tfrac{1}{2}, U)| \geqq \prod_{\substack{\chi(p)\neq -1 \\ p<U}} (1+p^{-\frac{1}{2}})^{-4}$$

it is only necessary to show that $R(\frac{1}{2}, U)$ is sufficiently small. If

$$\mathcal{N}_U = \{n \text{ such that } p|n \Rightarrow p < U\}$$

then

$$R(s, U) = \sum_{\substack{n>U \\ n\in\mathcal{N}_U}} g_n n^{-s}.$$

We have

$$|R(\tfrac{1}{2}, U)| \leqq \sum_{U<n\leqq\frac{1}{4}\sqrt{d}} |g_n| n^{-\frac{1}{2}} + \sum_{\substack{\frac{1}{4}\sqrt{d}<n \\ n\in\mathcal{N}_U}} |g_n| n^{-\frac{1}{2}} = R_1 + R_2 .$$

Since $|g_n|$ is bounded by the n-th coefficient of $\big(\zeta(s)L(s,\chi)/\zeta(2s)\big)^2$ it follows from Lemma 4 that

$$R_1 \leqq U^{-\frac{1}{2}}\Big(\sum_{n\leqq\frac{1}{4}\sqrt{d}} \nu_n\Big)^2$$

$$\ll U^{-\frac{1}{2}}(\log d)^{2g} = (\log d)^{-2g} .$$

To estimate R_2 we note that

$$R_2 \leqq \lim_{N\to\infty} \sum_{\substack{n\leqq N \\ n\in\mathcal{N}_U}} |g_n| n^{-\frac{1}{2}}\left(1-\frac{n}{N}\right) - \sum_{\substack{n\leqq\frac{1}{4}\sqrt{d} \\ n\in\mathcal{N}_U}} |g_n| n^{-\frac{1}{2}}\left(1-\frac{4n}{\sqrt{d}}\right)$$

and since $R(s, U)$ is majorized by

$$P_1(s, U) = \prod_{\substack{\chi(p)\neq -1 \\ p<U}} (1-p^{-s})^{-4}$$

it follows that

$$R_2 \leqq \lim_{N\to\infty} \int_{2-i\infty}^{2+i\infty} P_1\left(\frac{1}{2}+z, U\right) \frac{N^z - (\frac{1}{4}\sqrt{d})^z}{z(z+1)}\, dz$$

$$\ll \lim P_1(0, U)(N^{-\frac{1}{2}} + d^{-\frac{1}{4}})$$

$$\ll d^{-\frac{1}{4}}(\log d)^{4g}$$

by the estimate

$$2^{|\mathscr{P}|} \leqq \frac{1}{4 \log 2} (\log d)^{g-\mu-1}$$

obtained in the course of proof of Lemma 9. Hence

$$|G(\tfrac{1}{2}, U)| \geqq \prod_{\substack{\chi(p) \neq -1 \\ p < U}} (1 + p^{-\frac{1}{2}})^{-4} + O\big((\log d)^{-2g}\big)$$

and since by Lemma 9 the product must be larger than

$$\exp\big(-20 g^{\frac{1}{2}} (\log \log d)^{\frac{1}{2}}\big)$$

the lower bound (55) easily follows.
It follows that

$$(56) \qquad H \gg 2(g-\mu)\sqrt{A}(\log A)^{g-\mu-1} |L(2, \psi^2)| L(1, \chi_k) \prod_{p|k} (1-p^{-1}) \prod_{\substack{\chi(p)\neq -1 \\ p < (\log d)^{8g}}} (1 + p^{-\frac{1}{2}})^{-4}$$
$$- c_8 g^{8g} N^5 \sqrt{A} (\log A)^{g-\mu-2} \exp\big(100 g^{3/4} (\log \log d)^{3/4}\big)$$

where c_8 is an effectively computable absolute constant. To get a lower bound for H, we use the following facts:

$$(57) \qquad L(1, \chi_k) \geqslant k^{-\frac{1}{2}},$$

$$(58) \qquad \prod_{p|k} (1 - p^{-1}) \geqslant 2^{-\omega(k)} \geqslant k^{-1}, \qquad \Big(\omega(k) = \sum_{p|k} 1\Big)$$

$$(59) \qquad |L(2, \psi^2)| \geqslant N^{-2} L(2, \psi'),$$

where ψ' is primitive and induces ψ^2.

LEMMA 12. *Let ψ be a primitive « Grössencharakter » with conductor $\mathfrak{f}$ of $K = Q(\sqrt{-k})$, satisfying $\psi((\alpha)) = \alpha^a$, $\alpha \equiv 1 \pmod{\mathfrak{f}}$. Then if $L_K(s, \psi)$ is entire and has real coefficients*

$$L_K\Big(1 + \frac{a}{2}, \psi\Big) \ll \big(k^2 N(\mathfrak{f})\big)^{-(2+a)} (\log k)^{-1}$$

and the constant implied by the $\ll$-symbol is effectively computable and independent of K and ψ.

PROOF. We sketch the argument. First of all, on examination of the Euler product, it is easy to see that

$$F(s) = \zeta(s) L(s, \chi_k) L_K\left(s + \frac{a}{2}, \psi\right) = \sum_{n=1}^{\infty} b_n n^{-s}$$

where

$$b_1 = 1, \qquad b_n \geqslant 0 \qquad (\text{for } n > 1).$$

Since $\Gamma(s) F(s - a/2)$ is entire except for a simple pole at $s = 1 - a/2$, we see that

$$(60) \qquad 1 \ll \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s) F\left(s - \frac{a}{2}\right) x^s \, ds =$$

$$= L(1, \chi_k) L_K\left(1 + \frac{a}{2}, \psi\right) x + \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s) F\left(s - \frac{a}{2}\right) x^s \, ds .$$

To estimate the last integral above, we need to know the growth of $F(s - a/2)$ on the line $\sigma = -\frac{1}{2}$. This is easily achieved by using functional equations. We have

$$|L(s, \chi_k)| = \left(\frac{k}{\pi}\right)^{\frac{1}{2}-s} \left|\frac{\Gamma(1 - s/2)}{\Gamma(\frac{1}{2} + s/2)}\right| \cdot |L(1 - s, \chi_k)| ,$$

$$|L_K(s, \psi)| = \left(\frac{\sqrt{kN(\mathfrak{f})}}{2\pi}\right)^{1+a-2s} \left|\frac{\Gamma(1 + a - s)}{\Gamma(s)}\right| \cdot |L_K(1 + a - s, \psi)| .$$

A tedious calculation gives

$$\left|\frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s) F\left(s - \frac{a}{2}\right) x^s \, ds\right| \ll x^{-\frac{1}{2}} k^{1+a/2} (kN(\mathfrak{f}))^{1+a/2} .$$

Choosing $x = c_9 (k^2 N(\mathfrak{f}))^{2+a}$ for a sufficiently large constant c_9, it follows from (60) that

$$\left|L_K\left(1 + \frac{a}{2}, \psi\right)\right| \ll (k^2 N(\mathfrak{f}))^{-(2+a)} (\log k)^{-1} ,$$

with an absolute effectively computable constant. The $(\log k)^{-1}$ term comes from the easily proved upper bound (*)

$$L(1, \chi_k) \ll \log k . \qquad \text{Q.E.D.}$$

We now get from (56), (57), (58), (59) and Lemma 12 that for $d > \exp\exp(cNg^3)$ and c sufficiently large

$$H \gg gN^{-12+\frac{1}{2}}(\log d)^{g-\mu-1} \prod_{\substack{\chi(p)\neq -1 \\ p<(\log d)^{8g}}} (1 + p^{-\frac{1}{2}})^{-4} .$$

This together with (52) gives

$$L(1, \chi) \gg \frac{g^{-4g}N^{-13}(\log d)^{g-\mu-1}}{\sqrt{d}(\log\log d)^{g-\mu+6}} \prod_{\substack{\chi(p)\neq -1 \\ p<(\log d)^{8g}}} (1 + p^{-\frac{1}{2}})^{-4} ,$$

which is precisely Theorem 5. Q.E.D.

An alternative proof of Theorems 1 and 5 can be obtained by direct use of the Kronecker limit formula for $\varphi(s)$. If E admits complex multiplication by $\sqrt{-k}$, then $\varphi(s)$ will be a Hecke L-function of the biquadratic field $Q\sqrt{-k}, \sqrt{\chi(-1)\,d}$. The assumption that $L(1, \chi)$ is small implies that most of the inequivalent binary quadratic forms $ax^2 + bxy + cy^2$ of discriminant $b^2 - 4ac = \chi(-1)\,d$ with coefficients $a, b, c \in Q(\sqrt{-k})$ actually have coefficients $a, b, c \in Q$. It is noteworthy to remark that the limit formula (in Hecke's notation) occurs at the point $s = \frac{1}{2}$, that is to say, at the center of the critical strip of $\varphi(s)$, instead of at $s = 1$, which is the natural point for Hecke L-functions with ray class characters.

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