ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

LUCIA BERETTA
ALBERTO TOGNOLI

Some basic facts in algebraic geometry on a non algebraically closed field

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Some Basic Facts in Algebraic Geometry on a non Algebraically Closed Field. (*)

LUCIA BERETTA - ALBERTO TOGNOLI (**)

Introduction.

Some aspects of algebraic geometry on R are studied in [1]. One of the most useful tools to study algebraic (and analytic) geometry on R is the concept of complexification. In this work we introduce a generalization of the complexification which is called completion and is well defined for any field K. One of the diseagreable facts in algebraic geometry on R is that theorem B is false.

In [5] Lucia Beretta proves the following theorem: the functor Γ of the global sections is exact in the category of O_V A-coherent (***) modules if (V, O_V) is an affine variety on R.

Clearly this theorem gives the good condition that replaces the theorem B and makes it possible to work using sheaf theory.

In this work we prove that the functor Γ is exact on the category of O_V A-coherent modules where (V, O_V) is an affine variety on K and K is any field.

1. – Regular functions on an affine reduced variety.

Let K be a field, subfield of the field $\hat{K}$; in the following K^n is considered embedded in $\hat{K}^n$.

The algebraic closure of K shall be noted by $\bar{K}$.

DEFINITION 1. Let V be an affine variety of K^n, we shall call *closure* (or *completion*) of V in $\hat{K}$ the intersection $\hat{V}$ of all closed sets of $\hat{K}^n$ that contain V.

(*) Lavoro eseguito nell'ambito del G.N.S.A.G.A. del C.N.R.

(**) Università della Calabria.

(***) An O_V module is called A-coherent if there exists an exact sequence $O_V^n \to O_V^p \to \mathcal{F} \to 0$.

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REMARK 1. *The completion $\hat{V}$ of V depends on the embedding and if $K = \mathbf{R}$, $\hat{K} = \mathbf{C}$, $\hat{V}$ is the usual complexification of V.*

LEMMA 1. *Let K be an infinite field and $P \in K[X_1, \dots, X_n]$. If $P(x_1, \dots, x_n) = 0$ for any $(x_1, \dots, x_n) \in K^n$ then $P \equiv 0$.*

PROOF. If $n = 1$ the result is clear.

Suppose the lemma is proved for $n-1$ and let:

$$P(X_1, \dots, X_n) = X_n^s \alpha_s(X_1, \dots, X_{n-1}) + \dots + \alpha_0(X_1, \dots, X_{n-1}).$$

For any $(x_1, \dots, x_{n-1}) \in K^{n-1}$ and $x_n \in K$ we have: $P(x_1, \dots, x_n) = 0$ then, by induction $\alpha_i(X_1, \dots, X_{n-1}) \equiv 0$ and the lemma is proved.

Let K be a subfield of $\hat{K}$ and let $\{\alpha_i\}_{i\in I}$ be a basis of $\hat{K}$ as K module. We shall suppose $1_k = \alpha_{i_1} \in \{\alpha_i\}_{i\in I}$.

If $P = \sum a_i x^i \in \hat{K}[X_1, \dots, X_n]$ (we use the multiindex) we have:

$$a_i = \sum_j b_j^i \alpha_j, \qquad b_j^i \in K$$

$$P(x) = \sum_i a_i x^i = \sum_i \left(\sum_j b_j^i \alpha_j\right) x^i = \sum_j \alpha_j \sum_i b_j^i x^i = \sum_j \alpha_j P_j(x) \tag{1}$$

where

$$P_j(x) = \sum_i b_j^i x^i \in K[X_1, \dots, X_n].$$

DEFINITION 2. Let, as before, $P \in \hat{K}[X_1, \dots, X_n]$ and $\{\alpha_i\}_{i\in I}$ be a basis of $\hat{K}$ on K then the $P_j(x)$ of (1) are called the *components* of P in the basis $\{\alpha_j\}$.

$P_{i_1}(x)$ is called the *K-component* of P.

REMARK 2. *Let $P \in \hat{K}[X_1, \dots, X_n]$ and $x \in K^n$ then if $P(x) \in K$ we have $P(x) = P_{i_1}(x)$.*

By definition it results that $P(x) = P_{i_1}(x) + \sum_{i\neq i_1} P_i(x)\alpha_i$ but $P_i(x) \in K$ and $\{\alpha_i\}$ is a basis hence $\sum_{i\neq i_1} \alpha_i P_i(x) = 0$.

In particular we have $x \in K^n$ and $P(x) = 0 \Rightarrow P_i(x) = 0$, $\forall i \in I$.

LEMMA 2. *Let $\hat{K}$ be a field and K a subfield, $V \subset K^n$ an affine variety and $\hat{V}$ the completion in $\hat{K}$.*

Let I_V, $I_{\hat{V}}$ be the ideals of the elements of $K[X_1, \dots, X_n]$, $\hat{K}[X_1, \dots, X_n]$ which are zero on V, $\hat{V}$.

We have: $I_{\hat{V}}$ is generated as $\hat{K}$ module by I_V hence $\hat{V} \cap K^n = V$.

PROOF. Let $P_1, \ldots, P_q \in K[X_1, \ldots, X_n]$ be generators of I_V. Any P_i defines an element $\hat{P}_i$ of $\hat{K}[X_1, \ldots, X_n]$ and $\{x \in \hat{K}^n | \hat{P}_i(x) = 0\}$ contains V hence $\hat{P}_i \in I_{\hat{V}}$.

So we have proved: $I_{\hat{V}} \supset$ (ideal generated by P_i).

Let now $R \in I_{\hat{V}}$ and $R = \sum_i \alpha_i R_i$ be a decomposition of R associated to a basis $\{\alpha_i\}_{i \in I}$. By remark 2 any R_i is an element of I_V hence: $I_{\hat{V}} =$ (ideal generated by P_i).

REMARK 3. *Lemma 2 is equivalent to the following relation:* $I_{\hat{V}} \simeq I_V \otimes_K \hat{K}$ *and shows that* $\hat{K}^n$ *induces on* K^n *its own topology.*

LEMMA 3. *Let* $\hat{K}$ *be a field and* K *a subfield. Let* $V = \bigcup_{i=1}^{q} V_i$ *be an affine variety of* K^n *and* V_i *the irreducible components. If* $\hat{V} = \bigcup_{i=1}^{s} \hat{V}_i$ *is the completion of* V *in* $\hat{K}$ *and* $\hat{V}_i$ *are the irreducible components of* $\hat{V}$ *we have:* $q = s$ *and* $\hat{V}_i$ *is the completion of* V_i.

PROOF. If $\tilde{V}_i$ is the completion of V_i clearly $\bigcup_{i=1}^{q} \tilde{V}_i$ is the completion of V. We must prove that $\tilde{V}_i$ is irreducible on $\hat{K}$ and $\tilde{V}_i \not\subset \bigcup_{j \neq i} \tilde{V}_j$, $i = 1, \ldots, q$.

If $\tilde{V}_i = \tilde{V}_i' \cup \tilde{V}_i''$ is reducible we may suppose $\tilde{V}_i' \supset V_i$, $\tilde{V}_i' \not\supset \tilde{V}_i''$ (V_i is irreducible) and this is impossible because of the minimality of $\tilde{V}_i$.

We have proved that the $\tilde{V}_i$ are irreducible on $\hat{K}$.

If $\tilde{V}_i \subset \bigcup_{j \neq i} \tilde{V}_j$ then there exists $\bar{j}$ such that $\tilde{V}_{\bar{j}} \supset \tilde{V}_i$ ($\tilde{V}_i$ is irreducible) in this case $V_{\bar{j}} \supset V_i$ but this is impossible. The lemma is proved.

DEFINITION 3. Let $\hat{V} \subset \hat{K}^n$ be an affine variety, we shall say that $\hat{V}$ *is defined on the subfield* K *if the ideal* $I_{\hat{V}} \subset \hat{K}[X_1, \ldots, X_n]$ is generated by $P_1, \ldots, P_s \in K[X_1, \ldots, X_n]$.

LEMMA 4. *Let* $V \subset K^n$ *be an affine variety and* $\hat{V}$ *the completion in* $\hat{K}$. *The variety* $\hat{V}$ *is the intersection of a finite number of hypersurfaces defined on* K.

PROOF. Let $P_1 = \ldots = P_s = 0$ be a system of generators of $I_{\hat{V}}$ and $\{\alpha_i\}_{i \in I}$ a basis of $\hat{K}$ on K.

In this hypothesis the components P_i^j of the P_i generate $I_{\hat{V}}$ and we have:

$$\hat{V} = \bigcap_{i,j} \{P_i^j(x) = 0\}.$$

The lemma is proved.

DEFINITION 4. Let $V \subset K^n$ be an affine variety, $U \subset V$ an open set and $f\colon U \to K$ a function. f is called *regular* in $x_0 \in U$ iff there exists an open set $U' \ni x_0$ such that $f_{|U'} = P/Q$, $P, Q \in K[X_1, \ldots, X_n]$, $Q(x) \neq 0$, if $x \in U'$. f is called *regular* if it is regular at any point.

PROPOSITION 1. *Let $V \subset K^n$ be an affine variety and $\hat{V} \subset \hat{K}^n$ the completion of V in $\hat{K}$, $\hat{K} \supset K$.*

Any regular function $f\colon V \to K$ is the restriction of a regular function $\hat{f}\colon (\hat{V} - \hat{S}) \to \hat{K}$, where $\hat{S}$ is an affine subvariety of $\hat{V}$ defined on K, such that: $\hat{S} \cap V = \emptyset$.

Two extensions $\hat{f}$ and $\hat{f}'$ of f coincide where both are defined and the extension $\hat{f}$ is defined on K (i.d. *locally $\hat{f}$ is the restriction of a regular element of $K(X_1, \ldots, X_n)$*).

PROOF. For any $x_0 \in V$ there exists a neighbourhood U_{x_0} in V and P, $Q \in K[X_1, \ldots, X_n]$ such that $f_{|U_{x_0}} = (P/Q)_{|U_{x_0}}$, $Q(x) \neq 0$, $x \in U_{x_0}$. Let $\hat{P}$, $\hat{Q}$ ... be the elements of $\hat{K}[X_1, \ldots, X_n]$ defined by P, Q ....

We can construct a covering $\{\hat{U}_{x_i}\}_{i=1\ldots s}$ of V using open sets $\hat{U}_{x_i} = \hat{V} - \{\hat{Q}_i = 0\}$ of $\hat{V}$ such that on any $\hat{U}_{x_i}$ an extension $\hat{P}_i/\hat{Q}_i$ of f is defined.

Now we prove the proposition in the case: V is irreducible.

We have $\hat{P}_i\hat{Q}_k - \hat{P}_k\hat{Q}_{i|V} \equiv 0$, $\forall i, k$ then $\hat{P}_i\hat{Q}_k - \hat{P}_k\hat{Q}_{i|\hat{V}} \equiv 0$ and the rational functions $\hat{P}_i/\hat{Q}_i$ define a regular function $\hat{f}\colon (\hat{V} - \hat{S}) \to \hat{K}$, where $\hat{S} = \cap \{\hat{Q}_i = 0\}$. In the general case let $V = \bigcup_{i=1}^{a} V_i$, $\hat{V} = \bigcup_{i=1}^{a} \hat{V}_i$ be the decomposition of V, $\hat{V}$ into irreducible components (see lemma 3).

We remark that if U is open in $\hat{V}$ and $U \cap \hat{V}_i \neq \emptyset$ then $U \cap V_i \neq \emptyset$ ($\hat{S} = \hat{V} - U$ is closed, hence if $\hat{S}$ contains V_i, contains also $\hat{V}_i$). If $\hat{U}_{x_i}$ and $\hat{U}_{x_k}$ intersect $\hat{V}_j$ then they intersect V_j hence $(\hat{P}_i/\hat{Q}_i)|_{\hat{V}_j}$ and $(\hat{P}_k/\hat{Q}_k)|_{\hat{V}_j}$ are extensions of $f_{|V_i}$ and hence coincide. So we have proved the first part of the proposition.

The unicity of f is proved in the first part of the proposition.

REMARK 4. *Proposition* 1 *is true* (same proof), *if V is an open set of an affine variety and $\hat{V}$ a neighbourhood of V in the completion.*

LEMMA 5. *Let $\hat{K}$ be a field algebraic on the subfield K and suppose $\hat{K} \neq K$. Let $P_1, \ldots, P_s \in K[X_1, \ldots, X_n]$ and $\hat{S} = \{x \in \hat{K}^n\colon P_1(x) = \ldots = P_s(x) = 0\}$. There exists $P \in K[X_1, \ldots, X_n]$ such that $\{x \in K^n\colon P(x) = 0\} = \hat{S} \cap K^n$ and $\{x \in \hat{K}^n / P(x) = 0\} \supset \hat{S}$.*

PROOF. Let $K \subset K_i \subset \hat{K}$ where any element of K_i is separable on K and any element of $\hat{K} - K_i$ is purely inseparable on K_i. Let $\omega \in K_i - K$, we have

$$\{x \in \hat{K}^n | (P_1 + \omega P_2)(x) = 0\} \cap K^n = \{P_1 = 0\} \cap \{P_2 = 0\} \cap K^n .$$

Let $P_{12} = \prod_{\alpha \in G} \alpha(P_1 + \omega P_2)$ where G is the Galois group of the extension $K(\omega)$. We have that P_{12} is G-invariant, hence $P_{12} \in K[X_1, \ldots, X_n]$ and clearly:

$$\{P_{12} = 0\} \cap K^n = \{P_1 = 0\} \cap \{P_2 = 0\} \cap K^n \tag{1}$$

and

$$\{x \in \hat{K}^n | P_{12}(x) = 0\} \supset \{P_1 = 0\} \cap \{P_2 = 0\} . \tag{2}$$

Starting from $P_{12} + \omega P_3$ in a similar way we define $P_{123} \in K[X_1, \ldots, X_n]$ and so on. Clearly $P = P_{12\ldots s}$ satisfies the conditions of the lemma. If $K_i = K$, then there exists $\omega \in \hat{K} - K_i$ and ω satisfies an equation $x^{p^n} - \beta = 0$, where $p =$ characteristic of K. Let now $P_{12} = (P_1 + \omega P_2)^{p^n}$; clearly P_{12} satisfies (1) and (2). Starting from P_{12} and P_3 we may construct P_{123} and so on .... The lemma is now proved.

PROPOSITION 2. *Let K be a field, $V \subset K^n$ an affine variety and $f\colon V \to K$ a regular function.*

There exists R, $P \in K[X_1, \ldots, X_n]$ such that:

$$P(x) \neq 0 , \qquad \forall x \in K^n , \qquad (R/P)|_V = f .$$

PROOF. If K is algebraically closed the result is well known ([2]). Let $\bar{K}$ be the algebraic closure of K and $\bar{V}$ the completion of V. By proposition 1 there exists a regular function $\bar{f}\colon (\bar{V} - \bar{S}) \to \bar{K}$ that extends f, $\bar{S}$ closed set such that $\bar{S} \cap K^n = \emptyset$.

We may suppose that the equations $Q_1 = 0$, $Q_2 = 0, \ldots, Q_s = 0$ of $\bar{S}$ are in $K[X_1, \ldots, X_n]$ (see proposition 1).

By the lemma 5 we know that there exists $P \in K[X_1, \ldots, X_n]$ such that $\{P = 0\} \supset \bar{S}$ and $P(x) \neq 0$ if $x \in K^n$.

The function $\bar{f}$ is defined in the affine (see [2]), closed subset $\bar{V}_j - \{P = 0\}$ of $\bar{K}^n - \{P = 0\}$, hence it is the restriction of a regular function $\bar{f}'$ defined on $\bar{K}^n - \{P = 0\}$ (see [2]). It is known (see [2]) that $\bar{f}'$ is of the form $\bar{f}' = R'/P^h$. Let now R be the K-component of R' with respect to a basis of $\bar{K}$ on K, clearly $(R/P^h)|_V = f$ and the proposition is proved.

COROLLARY 1. *Let K be a non algebraically closed field then for any $n \in \mathbf{N}$ there exists a homogeneous polynomial $P \in K[X_1, \dots, X_n]$ such that $P(X) = 0$ if and only if $X = (0, \dots, 0)$.*

PROOF. We apply the construction of the lemma 5 to the polynomials $X_1, \dots, X_n$.

$$P_{12} = \prod_{\alpha \in G} \alpha(X_1 + \omega X_2) \quad \text{is homogeneous, say of degree } q, \text{ let}$$

$$P_{123} = \prod_{\alpha \in G} \alpha(P_{12} + \omega X_3^q) \quad \text{and so on}.$$

A similar construction is possible if $\hat{K}$ is purely inseparable on K.

In the following proposition we use the euclidean topology of $\mathbf{R}^n$.

PROPOSITION 3. *Let $V \subset \mathbf{R}^n$ be an affine, quasi-regular, coherent (*) compact affine variety. $U \supset V$ an open set of $\mathbf{R}^n$ and $f: U \to \mathbf{R}$ a C^∞ function such that $f|_V$ is the restriction of a regular function.*

For any compact set $H \subset U$, $\varepsilon > 0$, $q \in \mathbf{N}$ there exist $P, Q \in \mathbf{R}[X_1, \dots, X_n]$ such that:

i) $Q(x) \neq 0, \ \forall x \in \mathbf{R}^n$;

ii) $P/Q_{|V} = f_{|V}$;

iii) $\left| \dfrac{\partial^\alpha (f - P/Q)(x)}{\partial x_1^{i_1}, \dots, \partial x_n^{i_n}} \right| < \varepsilon, \ \forall x \in H, \ \alpha \leqslant q.$

PROOF. By proposition 2 there exist $P', Q' \in \mathbf{R}[X_1, \dots, X_n]$ such that $f - P'/Q'|_V \equiv 0$, $Q'(x) \neq 0$, $\forall x \in \mathbf{R}^n$.

By theorem 1 of [3] there exist $P'' \in \mathbf{R}[X_1, \dots, X_n]$, $P''_{|V} \equiv 0$ such that if $P/Q = (P''Q' + P')/Q'$ i), ii) and iii) are satisfied.

2. – Properties of coherence.

Let K be a field and O_{K^n} the sheaf of the germs of the regular functions on K^n.

A (non reduced) *affine variety* on K is a ringed space $(V, O_V = O_{K^n/\mathfrak{J}_V})$ where $\mathfrak{J}_V$ is a coherent ideal subsheaf of O_{K^n} and: $V =$ support O_V.

(V, O_V) is called *reduced* iff $\mathfrak{J}_V =$ sheaf of all germs zero on V.

(*) V is quasi regular in x if the ideal of analytic functions zero on V_x is generated by I_V, V is quasi regular if it is so at any point. V is coherent iff the associated analytic space is coherent.

Following [2] we call *algebraic prevariety* on K a ringed space (X, O_X) locally isomorphic to an affine variety defined on K.

Finally an algebraic prevariety (X, O_X) is called a *variety* iff the diagonal Δ_X of $X \times X$ is closed in $X \times X$.

Let (X, O_X) be an algebraic prevariety on K and $\mathfrak{J}$ a coherent ideal subsheaf of O_X, the ringed space ($V =$ support $O_X/\mathfrak{J}$, $O_X/\mathfrak{J}$) is called an algebraic subprevariety of X.

As usual we can define the locally closed subprevariety of (X, O_X) and it is easy to verify that any locally closed subprevariety of (X, O_X) has a natural structure of algebraic prevariety on K.

It is easy to verify that any subprevariety of an algebraic variety on K is an algebraic variety on K.

In particular we shall call projective variety any closed subvariety of $P_n(K) =$ projective space on K of dimension n.

Let K be an algebraically closed field, and (V, O_V) an affine variety. An O_V module $\mathcal{F}$ is coherent if, and only if, there exists an exact sequence:

$$O_V^p \to O_V^q \to \mathcal{F} \to 0 . \tag{1}$$

For any K we give the following definition: an O_V module $\mathcal{F}$ is called *A-coherent* if, and only if, there exists a resolution of type (1).

We have the following

THEOREM 1. *Let $(V, O_V = O_{K^n}/\mathfrak{J}_V)$ be an affine reduced variety on the field K, then O_{K^n}, O_V, $\mathfrak{J}_V$ are A-coherent O_{K^n} modules and O_V is an A-coherent O_V module.*

PROOF. Using the arguments of [2] we prove all the coherence conditions (the arguments do not depend on the fact that K is algebraically closed).

We have the exact sequence

$$0 \to \mathfrak{J}_V \to O_{K^n} \to O_V \to 0$$

hence to prove that O_V is A-coherent we must verify that there is a surjection $O_{K^n}^p \xrightarrow{\alpha} \mathfrak{J}_V \to 0$ and this is proved in [2].

To prove that $\mathfrak{J}_V$ is A-coherent we must verify that there is a surjection $O_{K^n}^q \to Ker\ \alpha \to 0$.

Let $\delta_i = (0, \dots 1, \dots 0)$, $i = 1, \dots, p$ be the generators of $\Gamma(O_{K^n}^p)$ and let $\alpha(\delta_i) = \beta_i$ be the images in $\Gamma(\mathfrak{J}_V)$.

Let now suppose K be infinite, otherwise K^n is discrete and the theorem is trivial.

Let $\gamma=\sum_{i=1}^{p}(a_i/b_i)\beta_i=0,\ a_i,\ b_i\in K[X_1,\dots,X_n]$ be a local relation. The function γ is zero on an open non empty set, hence $(\prod b_i)\left(\sum_{i=1}^{p}(a_i/b_i)\beta_i\right)$ is identically zero and $\sum^{p}(\prod b_i)(a_i/b_i)\beta_i$ gives a global relation that generates the local one. So we have proved that there exists an exact sequence $O^q_{K^n}\to Ker\ \alpha\to 0$ and $\mathfrak{J}_V$ is A-coherent.

The fact that O_V is an A-coherent O_V module is proved by the following.

LEMMA 2. *In the hypothesis of theorem* 1, *a sheaf* $\mathcal{F}$ *of* O_V *modules is* A-*coherent as* O_V *module if, and only if, it is* A-*coherent as* O_{K^n} *module.*

PROOF. Using the arguments of [2] we prove the conditions of coherence.

Let us suppose we have the exact sequence

$$O^p_V\to O^q_V\to\mathcal{F}\to 0$$

then we can construct the commutative diagram.

$$\begin{array}{ccccc} O^p_V & \xrightarrow{\alpha} & O^q_V & \xrightarrow{\eta} & \mathcal{F}\to 0 \\ \uparrow{\scriptstyle\pi'} & & \uparrow{\scriptstyle\pi''} & & \\ O^p_{K^n} & \xrightarrow{\sigma} & O^q_{K^n} & & \end{array}$$

where π' and π'' are defined using the canonical projection and σ is defined in the following way: let $\delta_i=(0,\dots,1,\dots,0)$, $i=1,\dots,p$ be the generators of $\Gamma(O^p_{K^n})$ and $\beta_i=\alpha\circ\pi'(\delta_i)\in\Gamma(O^q_V)$.

We remark that the maps $\pi':\Gamma(O^p_{K^n})\to\Gamma(O^p_V)$, $\pi'':\Gamma(O^q_{K^n})\to\Gamma(O^q_V)$ are surjective (see proposition 2 of § 1).

Let $\pi''(\theta_i)=\beta_i$, then σ is defined by $\sigma(\delta_i)=\theta_i$.

The exact sequence $O^p_{K^n}\xrightarrow{\sigma}O^q_{K^n}\xrightarrow{\eta\circ\pi''}\mathcal{F}\to 0$ proves the A-coherence of $\mathcal{F}$ as O_{K^n} module.

Let now be defined the exact sequence

$$O^p_{K^n}\to O^q_{K^n}\to\mathcal{F}\to 0\,.$$

We deduce the commutative diagram:

$$\begin{array}{ccccc} O^p_{K^n} & \to & O^q_{K^n} & \xrightarrow{\eta} & \mathcal{F}\to 0 \\ \downarrow{\scriptstyle\pi'} & & \downarrow{\scriptstyle\pi''} & \nearrow{\scriptstyle j} & \\ O^p_V & \xrightarrow{\sigma} & O^q_V & & \end{array}$$

where σ is defined using the above arguments and j can be defined because $\mathcal{F}$ is an O_V module and hence if $\theta\in\Gamma(\mathfrak{J}^q)$ we have $\eta(\theta)=0$. From the diagram we deduce the exact sequence $O^p_V\xrightarrow{\sigma}O^q_V\xrightarrow{j}\mathcal{F}\to 0$ that ends the proof.

REMARK 1. There exist locally free sheaves on $\boldsymbol{R}$ that are not A-coherent.

Let $\varphi = x^2(x-1)^2 + y^2 + z^2 \in \boldsymbol{R}[x, y, z]$; φ is irreducible (see [1]) and $\{\varphi = 0\} = (0, 0, 0) \cup (1, 0, 0)$.

Let $F \to \boldsymbol{R}^3$ be the line bundle defined by the cocycle $1/\varphi$ given in the covering $U_1 = \boldsymbol{R}^3 - (0, 0, 0)$, $U_2 = \boldsymbol{R}^3 - (1, 0, 0)$.

Let $\mathcal{F}$ be the sheaf of the germs of algebraic sections of F.

It's easy to see that any section on U_1 that can be extended to a global section can be divided by φ and hence it is zero on $(0, 0, 0)$. From this fact it follows that $\mathcal{F}$ does'nt satisfy theorem A, and hence it is not A-coherent. We remark that there exist coherent subsheaves of $O_{\boldsymbol{R}^n}$ that do not satisfy theorem A.

Let $\mathcal{F}$ be the sheaf above constructed and γ the section that coincides with φ on $\boldsymbol{R}^3 - (0, 0, 0)$ and 1 on the second chart.

Let $\mathcal{F}'$ be the subsheaf of $\mathcal{F}$ generated by γ, we have clearly $\mathcal{F}' \simeq O_{\boldsymbol{R}^3}$. $\mathcal{F}'$ contains the subsheaf φ. $\mathcal{F} = \mathcal{F}''$ and $\mathcal{F}'' \simeq \mathcal{F}$; hence $\mathcal{F}''$ does'nt satisfy theorem A.

PROPOSITION 1. *Let K be a field, (V, O_V) an affine subvariety of K^n and $P \in K[X_1, \ldots, X_n]$. In this hypothesis $\left(V_p = \{x \in V : P(x) \neq 0\},\ O_{V|V_p}\right)$ is an affine variety. If K is not algebraically closed any open set of V is an affine variety.*

PROOF. It is not difficult to verify (see [2]), that $(V, O_{V|V_p})$ is isomorphic to the affine subvariety of K^{n+1} defined by $g_1 = \ldots = g_q = 0$, $1 - P\, X_{n+1} = 0$ where $g_1 = \ldots = g_q = 0$ are generators of $\Gamma(\mathfrak{J}_V)$.

If K is not algebraically closed the lemma 5 of § 1 proves that any closed set of V is just the locus of zero of some $P \in K[X_1, \ldots, X_n]$ intersected with V and the proposition is proved.

PROPOSITION 2. *Let K be a non algebraically closed field then the projective space $P_n(K)$ is isomorphic to an affine variety.*

PROOF. Let $R \in K[X_0, \ldots, X_n]$ be an homogeneous polynomial of degree q such that $R(X) = 0 \Leftrightarrow X = (0, \ldots, 0)$ (see corollary 1 of § 1). Let $v_q : P_n(K) \to P_N(K)$ the Veronese map of degree q and $w = v_q \times R : P_n(K) \Rightarrow P_{N+1}(K)$ the map obtained adding R to the set of all monomial functions of degree q.

Clearly $w(P_n(K)) \cap \{R = 0\} = \emptyset$ hence $w(P_n(K))$ is an affine variety of K^{N+1} isomorphic to $P_n(K)$.

3. – The completion of a prevariety.

In this paragraph we shall define the completion of an affine variety (non necessarly reduced), and of an algebraic prevariety.

DEFINITION 1. Let K be a field, subfield of the field $\hat{K}$, and $\mathcal{F} \to K^n$ an A-coherent $\mathcal{O}_{K^n}$ module. A sheaf $\hat{\mathcal{F}} \to U$, defined on a neighbourhood U of K^n in $\hat{K}^n$ is called a *completion* of $\mathcal{F}$ (in $\hat{K}$) if there exist two exact sequences

$$O^p_{K^n} \xrightarrow{\alpha} O^q_{K^n} \to \mathcal{F} \to 0$$

$$O^p_{\hat{K}^n|U} \xrightarrow{\hat{\alpha}} O^q_{\hat{K}^n|U} \to \hat{\mathcal{F}} \to 0$$

such that $\hat{\alpha}$ is an extension of α. A completion $\hat{\mathcal{F}} \to U$ is called a *strong completion* iff for any $x \in K^n$ we have: $\hat{\mathcal{F}}_x \simeq \mathcal{F}_x \otimes_K \hat{K}$.

In the following, we shall consider $\mathcal{F}$ canonically embedded in $\hat{\mathcal{F}}_{|K^n}$ and it is easy to verify that, from the fact that $\mathcal{O}_{|K^n}$ is a subsheaf of $\mathcal{O}_{\hat{K}^n}$ it follows that $\mathcal{F}$ is a subsheaf of $\hat{\mathcal{F}}_{|K^n}$.

REMARK 1. *If $\hat{K}$ is algebraic on K, then $O_{\hat{K}^n}$ is a strong completion of $\mathcal{O}_{K^n}$. If $\hat{K}$ is not algebraic on K, in general $\mathcal{O}_{\hat{K}^n,x} \neq \mathcal{O}_{K^n,x} \otimes_K \hat{K}$.*

PROOF. Let $\{\alpha_i\}_{i\in I}$ be a basis of $\hat{K}$ on K, and $\alpha_{i_1} = 1_K$. We need the following

LEMMA 1. *If $\hat{K}$ is algebraic on K, $x_0 \in K^n$ and $P, Q \in \hat{K}[X_1, \dots, X_n]$ such that $Q(x_0) \neq 0$, there exist P_i, $Q_i \in K[X_1, \dots, X_n]$ such that*

$$\frac{P}{Q} = \sum_{i=1}^{q} \alpha_i \frac{P_i}{Q_i}, \qquad Q_i(x_0) \neq 0 .$$

Using lemma 1, we shall prove remark 1.

For any $x \in K^n$, let

$N_x = \{P \in K[X_1, \dots, X_n] | P(x) \neq 0\}$ and $\hat{N}_x = \{P \in \hat{K}[X_1, \dots, X_n] | P(x) \neq 0\}$; we have:

$$\mathcal{O}_{K^n,x} = \big(K[X_1, \dots, X_n]\big)_{N_x},$$

$$\mathcal{O}_{\hat{K}^n,x} = \big(\hat{K}[X_1, \dots, X_n]\big)_{\hat{N}_x}$$

where A_B = ring of fractions with respect to B.

We have a natural isomorphism $\hat{K}[X_1, \dots, X_n] \simeq K[X_1, \dots, X_n] \otimes_K \hat{K}$ and we define an homomorphism $j\colon \mathcal{O}_{K^n,x} \otimes_K \hat{K} \to \mathcal{O}_{\hat{K}^n,x}$

$$j\left(\sum_i \frac{P_i}{Q_i} \otimes h_i\right) = \sum_i h_i \frac{P_i}{Q_i} \in \mathcal{O}_{\hat{K}^n,x}, \qquad h_i \in \hat{K} .$$

We wish to prove that j is bijective.

If $h_i = \sum l^i_j \alpha_{i_j}$, $l^i_j \in K$, we have:

$$j\left(\sum_i \frac{P_i}{Q_i} \otimes \sum_j l^i_j \alpha_{i_j}\right) = j\left(\sum_{i,j} l^i_j \frac{P_i}{Q_i} \otimes \alpha_{i_j}\right) = \sum_{i,j} l^i_j \alpha_{i_j} \frac{P_i}{Q_i}.$$

But the last term is zero iff the coefficients of the α_{i_j} are zeros and in this case

$$\sum_{i,j} l^i_j \frac{P_i}{Q_i} \otimes \alpha_{i_j} = 0 ,$$

hence j is injective.

Lemma 1 proves that j is surjective.

Let now $K = Q =$ field of rational numbers, $\hat{K} = Q(\pi)$; we wish to prove $Q[X]_{N_0} \otimes_Q Q(\pi) \neq Q(\pi)[X]_{\hat{N}_0}$.

In fact $1/(X-\pi) \in Q(\pi)[X]_{\hat{N}_0}$, but any element of $Q[X]_{N_0} \otimes_Q Q(\pi)$ is a well defined function on π hence $1/(X-\pi) \notin Q[X]_{N_0} \otimes_Q Q(\pi)$.

Proof of Lemma 1. We remark that $\hat{K}$ is integer on K, hence $O_{\hat{K}^n,x}$ is integer on $O_{K^n,x}$. We have a natural injection $j\colon O_{K^n,x} \otimes_K \hat{K} \to O_{\hat{K}^n,x}$ so we can deduce that $A \stackrel{\text{def}}{=} O_{K^n,x} \otimes_K \hat{K}$ is integer on $O_{K^n,x}$. Let $\mathcal{M}_x$ be the maximal ideal of $O_{K^n,x}$, clearly $\mathcal{M}_x \otimes_K \hat{K}$ is maximal in A and A is a local ring (in general A has finite numbers $I_1, \ldots, I_p$ of maximal ideals over $\mathcal{M}_x$, but any I_j contains $\mathcal{M}_x \otimes_K \hat{K}$, hence A is local). It is now clear that the maximal ideal of $O_{\hat{K}^n,x}$ is isomorphic to $\mathcal{M}_x \otimes_K \hat{K}$, hence j is an isomorphism.

Remark 2. *Let K be a field, subfield of the field $\hat{K}$. For any A-coherent sheaf $\mathcal{F} \to K^n$ there exists a completion $\hat{\mathcal{F}} \to U$ and, if $\hat{K}$ is algebraic on K, then there exists a strong completion. In any case we have: $\hat{\mathcal{F}}_x \simeq \mathcal{F}_x \otimes_{O_{K^n}} O_{\hat{K}^n,x}$, $\forall x \in K^n$ and hence, if $\hat{K}$ is algebraic on K;*

$$\mathcal{F}_x \otimes_K \hat{K} \simeq \mathcal{F}_x \otimes_{O_{K^n}} O_{\hat{K}^n,x}.$$

Proof. There exists a resolution:

$$O^p_{K^n} \xrightarrow{\alpha} O^q_{K^n} \to \mathcal{F} \to 0 .$$

Let $\gamma_1, \ldots, \gamma_p \in \Gamma(O^p_{K^n})$, $\eta_1, \ldots, \eta_q \in \Gamma(O^q_{K^n})$ be the generators $(0, \ldots, 1, \ldots, 0) \ldots$.

To fix α is equivalent to the definition of $(a_{ij}) \in \Gamma(O_{K^n})^{pq}$ such that $\alpha_*(\gamma_i) = \sum_j a_{ij}\eta_j$, where $\alpha_*\colon \Gamma(O^p_{K^n}) \to \Gamma(O^q_{K^n})$ is induced by α.

There exists (see proposition 1 of § 1) an open set U of $\hat{K}^n$ such that $U \supset K^n$ and the a_{ij} are the restrictions of regular functions $\hat{a}_{ij}$ defined on U (moreover the $\hat{a}_{ij}$ are defined on K and $U = \hat{K}^n - S$ where S is closed and defined on K).

The matrix $(\hat{a}_{ij})$ defines $O_U^p \xrightarrow{\hat{\alpha}} O_U^q$ and the sheaf coker $\hat{\alpha} = \hat{\mathcal{F}}$ is a completion of $\mathcal{F}$.

If $\hat{K}$ is algebraic on K, then, by remark 1, we have for any $x \in K^n$:

$$O_{U,x}^p \simeq O_{K^n,x}^p \underset{K}{\otimes} \hat{K} \xrightarrow{\hat{\alpha}_x} O_{U,x}^q \simeq O_{K^n,x}^q \underset{K}{\otimes} \hat{K} \to \hat{\mathcal{F}}_x \to 0 . \tag{1}$$

From (1) it follows $\hat{\mathcal{F}}_x \simeq \mathcal{F}_x \underset{K}{\otimes} \hat{K}$ and the first part of the remark is proved.

In the general case we have $O_{\hat{K}^n}^s \simeq O_{K^n}^s \underset{O_{K^n}}{\otimes} O_{\hat{K}^n}$, and the sequence:

$$O_{\hat{K}^n|K^n}^p \xrightarrow{\hat{\alpha}} O_{\hat{K}^n|K^n}^q \to \hat{\mathcal{F}}_{|K^n} \to 0$$

may be written:

$$O_{K^n}^p \underset{O_{K^n}}{\otimes} O_{\hat{K}^n|K^n} \xrightarrow{\hat{\alpha}} O_{K^n}^q \underset{O_{K^n}}{\otimes} O_{\hat{K}^n|K^n} \to \mathcal{F} \underset{O_{K^n}}{\otimes} O_{\hat{K}^n|K^n} \to 0$$

and the remark is proved.

REMARK 3. *Let K be a field, subfield of $\hat{K}$, (V, O_V) an affine variety on K, and $\mathcal{F} \to V$ an A-coherent O_V module. Let $\hat{\mathcal{F}} \to U$ be a completion of $\mathcal{F}$ and $\gamma \in \Gamma_V(\mathcal{F})$, then there exists $\hat{\gamma} \in \Gamma_{U'}(\hat{\mathcal{F}})$, U' open in $\hat{V}$, $U' \supset V$, such that $\hat{\gamma}$ extends γ.*

Let $\alpha \colon \mathcal{F} \to \mathcal{G}$ be a homomorphism of coherent O_V modules, and $\hat{\mathcal{F}}$, $\hat{\mathcal{G}}$ two completions, then there exists an extension $\hat{\alpha} \colon \hat{\mathcal{F}}_{|U} \to \hat{\mathcal{G}}_{|U}$ of α where U is a neighbourhood of V in $\hat{V}$.

PROOF. Using the trivial extension of $\mathcal{F}$ to K^n, it is enough to prove the remark when $V = K^n$.

We have an exact sequence

$$O_{K^n}^p \xrightarrow{\alpha} O_{K^n}^q \to \mathcal{F} \to 0$$

that can be extended to:

$$O_{\hat{K}^n|U}^p \xrightarrow{\hat{\alpha}} O_{\hat{K}^n|U}^q \xrightarrow{\hat{\eta}} \mathcal{F} \to 0$$

(see remark 2).

Let $\gamma \in \Gamma(\mathcal{F})$, then there exists a family $\{U_i\}_{i=1\dots,s}$ of open sets (of U) and sections $\beta_i \in \Gamma_{U_i}(O^q_{\hat{K}^n})$ such that: $\bigcup\limits_i U_i = U'$ is an affine open set of $\hat{K}^n$ containing K^n and $\hat{\eta}(\beta_i)_{|V} = \gamma_{|U_i \cap V}$

The cocycle $g_{ik} = \beta_i - \beta_k$ is defined on the covering $\{U_i\}$ and it has values in Ker $\hat{\eta}$, hence (by theorem B) is trivial. From this it follows that there exists $\gamma' \in \Gamma_{U'}(O^q_{\hat{K}^n})$ such that $\hat{\eta}(\gamma')_{|V} = \gamma$. Clearly $\hat{\eta}(\gamma')$ extends γ and the first part of the remark is proved.

To prove the second part we construct the commutative diagram:

$$\begin{array}{ccc} O^p_{K^n} & \xrightarrow{\gamma''} & O^t_{K^n} \\ \downarrow & & \downarrow \\ O^q_{K^n} & \xrightarrow{\gamma'} & O^s_{K^n} \\ \pi\downarrow & & \downarrow\pi' \\ \mathcal{F} & \xrightarrow{\gamma} & \mathcal{G} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

To define γ' and γ'' we make the following remark: all sheaves and vertical morphism can be extended to Spec $\Gamma(O_{K^n})$ (see lemma 2 of § 4); hence the functor of the global section is exact on the vertical sequences (we use the first part of lemma 3, § 4). Let $\delta_i = (0, \dots, 1, \dots, 0) \in \Gamma(O^q_{K^n})$ be the canonical generators and $\beta_i = \gamma \circ \pi(\delta_i)$.

As we have just remarked there exists $\theta_i \in \Gamma(O^s_{K^n})$ such that $\pi'(\theta_i) = \beta_i$; γ' is defined by $\gamma'(\delta_i) = \theta_i$.

In a similar way we define γ''.

The morphism γ', γ'' can be extended (see remark 2) and hence also γ has an extension to the completions and the remark is proved.

DEFINITION 2. Let K be a field, subfield of $\hat{K}$, and (V, O_V) an affine variety defined by the ideal subsheaf $\mathfrak{J}_V$ of O_{K^n}. A (strong) *completion* of (V, O_V) in $\hat{K}$, is any affine variety $(\hat{V}, O_{\hat{V}})$ such that the sheaf $O_{\hat{V}}$ is a (strong) completion of O_V.

REMARK 4. *Let $V \subset K^n$ be an affine variety and suppose $P_1, \dots, P_q \in K[X_1, \dots, X_n]$ generate $\mathfrak{J}_V$ in any point of V. (By the remark contained in* theorem 1 of § 2 $\mathfrak{J}_V$ is generated by a finite number of polynomials).

Let $\hat{P}_i$ be the elements of $\hat{K}[X_1, \dots, X_n]$ defined by P_i, and $\hat{\mathfrak{J}}_V \subset O_{\hat{K}^n}$ the sheaf of ideals generated by $\hat{P}_1, \dots, \hat{P}_q$. If $\hat{K}$ is algebraic on K, we have:

$$(\hat{V}, O_{\hat{V}} = O_{\hat{K}^n}/\hat{\mathfrak{J}}_V) \quad \textit{is a strong completion of } (V, O_V)$$

hence the completion defined in § 1 *coincides with the completion of definition* 2. *Moreover if* $V \subset K^n$ *then there exists a completion* $\hat{V}$ *closed in* $\hat{K}^n$.

PROOF. By definitions and remark 1 we have the exact sequences:

$$(1) \qquad O \to \mathfrak{J}_V \to O_{K^n} \to O_V \to 0$$

$$(2) \qquad O \to \mathfrak{J}_V \underset{K}{\otimes} \hat{K} \to O_{\hat{K}^n|K^n} \to O_V \underset{K}{\otimes} \hat{K} \to 0 .$$

Clearly $\hat{\mathfrak{J}}_{V|K^n} \simeq \hat{\mathfrak{J}}_V \otimes \hat{K}$ then from (2) we deduce the exact sequence:

$$(3) \qquad O \to \hat{\mathfrak{J}}_{V|K^n} \to O_{\hat{K}^n|K^n} \to O_V \underset{K}{\otimes} \hat{K} \to 0$$

and (3) proves that $(\hat{V}, O_{K^n/\hat{\mathfrak{J}}_V})$ is a strong completion of (V, O_V).

REMARK 5. *Let* (V, O_V) *be an affine variety on* K *and*

$$(1) \qquad \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}''$$

an exact sequence of A-*coherent* O_V *modules.*

If $(\hat{V}, O_{\hat{V}})$ *is a completion of* V *in the field* $\hat{K}$ *there exists an open set* U' *of* $\hat{V}$ *such that*: $U' \supset V$ *and the homomorphisms* α, β *extend to an exact sequence*:

$$(2) \qquad \hat{\mathcal{F}}' \xrightarrow{\hat{\alpha}} \hat{\mathcal{F}} \xrightarrow{\hat{\beta}} \hat{\mathcal{F}}''$$

where $\hat{\mathcal{F}}', \hat{\mathcal{F}}, \hat{\mathcal{F}}''$ *are completions of* $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ *defined on* U' *and* $\hat{\alpha}, \hat{\beta}$ *extend* α, β.

PROOF. The existence of (2) is proved in remark 3. For any $x \in V$ we have:

$$(3) \qquad \hat{\alpha}(\hat{\mathcal{F}}'_x) = \operatorname{Ker} \hat{\beta}_{|\hat{\mathcal{F}}_x} .$$

The sheaves $\hat{\alpha}(\mathcal{F}')$ and $\operatorname{Ker}\hat{\beta}$ are coherent hence (3) is true in a neighbourhood of x and the remark is proved.

DEFINITION 3. Let K be a field, subfield of $\hat{K}$ and (X, O_X) a prevariety defined on K.

We shall say that the prevariety $(\hat{X}, O_{\hat{X}})$ defined on $\hat{K}$ is a *completion* of (X, O_X) iff $X \subset \hat{X}$ and for any $x \in X$ there exist two affine neighbourhoods U_x in X and $\hat{U}_x$ in $\hat{X}$ such that $(\hat{U}_x, O_{\hat{X}|\hat{U}_x})$ is a completion of $(U_x, O_{X|U_x})$.

We have the following:

THEOREM 1. *Let K be a subfield of the field $\hat{K}$ and (X, O_X) a prevariety defined on K.*

Then there exists a completion $(\hat{X}, O_{\hat{X}})$ of (X, O_X) and if (X, O_X) is a variety, $(\hat{X}, O_{\hat{X}})$ can be chosen to be a variety.

Let $\varphi: (V, O_V) \to (X, O_X)$ be a morphism of K-prevariety and $(\hat{V}, O_{\hat{V}})$, $(\hat{X}, O_{\hat{X}})$ two completions; then there exists an extension $\hat{\varphi}$ of φ defined on a neighbourhood of V in $\hat{V}$ (hence the completion is unique near V).

We need the following:

LEMMA 2. *Let K be a subfield of $\hat{K}$, (V, O_V) be a prevariety on K and $(\hat{V}, O_{\hat{V}})$ a completion of V.*

Let (X, O_X) be a prevariety on K, $(\hat{X}, O_{\hat{X}})$ a completion on $\hat{K}$ and $\varphi: (V, O_V) \to \to (X, O_X)$ be a morphism, then there exists an extension $\hat{\varphi}: (\hat{V}', O_{\hat{V}'}) \to (\hat{X}, O_{\hat{X}})$ of φ defined on a neighbourhood $\hat{V}'$ of V in $\hat{V}$ and two such extensions coincide on a neighbourhood of V in $\hat{V}$.

PROOF. For any $x \in V$ there exists, in V, an open set $V_x \ni x$ such that: $\varphi(V_x)$ is contained in an affine open set W_x of X and W_x has a completion $\hat{W}_x$ that is an affine open set of $\hat{X}$.

We may suppose moreover that V_x is contained in an open set $\hat{V}_x$ of $\hat{V}$ such that $\hat{V}_x$ is a completion of V_x and for any $\hat{\gamma} \in \Gamma_{\hat{V}_x}(O_{\hat{V}})$ we have $\hat{\gamma} = 0 \Leftrightarrow \hat{\gamma}_{|V_x} = 0$.

Using proposition 1 of § 1 we may construct an open (in $\hat{V}$) covering $\hat{V}_{x_1}, \dots, \hat{V}_{x_j}$ of $\hat{V}$ and some extensions $\hat{\varphi}_i: \hat{V}_{x_i} \to \hat{X}$ of $\varphi_{|V_{x_i}}$.

Let $x \in V$, using the argument of proposition 1 of § 1 we deduce that there exists an open set $\hat{B}_x \ni x$ of $\hat{V}$ such that all the $\hat{\varphi}_{i|\hat{B}_x}$ coincide (where defined). (If $\hat{B}_x = \cup \hat{B}_i$ is the decomposition of $\hat{B}_x$ into irreducible components we require that $\hat{B}_i \cap V = B_i$ has $\hat{B}_i$ as completion).

So *gluing together* the $\hat{\varphi}_i$ we define the extension of φ, the unicity of the extension is proved by the same argument.

We remark that the lemma 2 implies that if φ is an isomorphism we may extend φ to an isomorphism $\hat{\varphi}': (\hat{V}', O_{\hat{V}}) \to (\hat{X}', O_{\hat{X}'})$ (it is enough to extend φ and φ^{-1} and to verify that $\varphi \cdot \hat{\varphi}^{-1} = id \Rightarrow \hat{\varphi} \circ \hat{\varphi}^{-1} = id$).

PROOF OF THEOREM 1. Let $\mathfrak{U} = \{U_i\}_{i=1,\dots,a}$ be an open affine covering of X and $U_i' \subset K^{n_i}$ some realisation of U_i.

We shall denote by $\hat{U}_i'$ the completions of U_i' in K^{n_i} and by $\varrho_i: U_i' \to U_i$ the natural isomorphisms.

Let $U_{12}' = \varrho_1^{-1}(U_1 \cap U_2)$ and $\varrho_2^{-1} \circ \varrho_1 = \varrho_{12}: U_{12}' \to U_2'$.

Using the lemma 2 we may extend ϱ_{12} to a morphism $\hat{\varrho}_{12}$ defined on an open set $\hat{U}_{12}$ of $\hat{U}_1'$ and we may suppose $\hat{\varrho}_{12}: \hat{U}_{12} \to \hat{\varrho}_{12}(\hat{U}_{12})$ is an isomorphism.

Using $\hat{\varrho}_{12}$ we may *glue* together $\hat{U}'_1$ with $\hat{U}'_2$ and construct a completion of $U_1 \cup U_2$. After q constructions we have the completion of X. We wish to prove that if X is a variety then $\hat{X}$ can be constructed to be a variety.

It is enough to prove that if $X = X_1 \cup X_2$ is the decomposition of X into open sets, and $\hat{X}_i$ are two varieties, completions of X_i, then we may *glue* together two open sets $\hat{X}'_i \subset \hat{X}_i$ of $\hat{X}_i$ and obtain a variety completion of X.

Let $X_{12} = X_1 \cap X_2 \subset X_1$ and $\varrho_{12}: X_{12} \to X_2$ be the identity map. Let $\hat{\varrho}_{12}: \hat{X}_{12} \to \hat{X}_2$ be an extension of ϱ_{12} $\hat{X}^{12} = \hat{X}_1 \cup \hat{X}_2/\mathfrak{R}$, (where $x \overset{\mathfrak{R}}{\sim} y \Leftrightarrow y = \hat{\varrho}_{12}(x)$), be the quotient and $\pi: \hat{X}_1 \coprod \hat{X}_2 \to \hat{X}_{12}$ be the natural projection. The diagonal $\hat{\Delta}^{12}$ of $\hat{X}^{12} \times \hat{X}^{12}$ is the image (under $\pi \times \pi$) of $\hat{\Delta}_{22} \cup \hat{\Delta}_{11} \cup \hat{\Delta}_{12} \cup \hat{\Delta}_{21}$ where $\hat{\Delta}_{ii} =$ diagonal of $\hat{X}_i \times \hat{X}_i$ and $\hat{\Delta}_{ik} =$ graph of $\hat{\varrho}_{ik}$. It is easy to see that $\hat{X}^{12}$ is a variety iff the $\hat{\Delta}_{ik}$ are closed subsets of $\hat{X}_i \times \hat{X}_k$: Then $\hat{X}^{12}$ is a variety iff $\hat{\Delta}_{ik}$, $i \neq k$, are closed in $\hat{X}_i \times \hat{X}_k$.

$X = X_1 \cup X_2$ is a variety then Δ_{12} and Δ_{21} are closed in $X_1 \times X_2$ and $X_2 \times X_1$.

Let $\hat{\Delta}'_{ik}$ be a closed completion of Δ_{ik} in $\hat{X}_i \times \hat{X}_x$. For any $x \in X_{12}$, Δ_{12} is the graph of the map ϱ_{12}, hence in a neighbourhood $\hat{U}_x$ of x in $\hat{X}_1$, $\hat{\Delta}'_{12}$ is the graph of a map $\hat{\varrho}_{12}$ that extends ϱ_{12}. Then there exists an open neighbourhood $\hat{X}'_{12}$ of X_{12} in $\hat{X}_1$ such that $\hat{\Delta}'_{12}$ is the graph of a map $\hat{\varrho}'_{12}: \hat{X}'_{12} \to \hat{X}_2$.

Using the same argument for Δ_{21} we find two open sets $\hat{X}'_i \supset X_i$, $\hat{X}'_i \subset \hat{X}_i$, such that $\hat{\Delta}'_{ik}$ defines an extension of ϱ_{ik} on $\hat{X}'_i$ and clearly $\hat{\Delta}'_{ik}$ is closed in $\hat{X}'_i \times \hat{X}'_k$.

The theorem is now proved.

4. – The exactness of the functor Γ.

We wish to prove the following

THEOREM 1. *Let (V, O_V) be an affine variety defined on the field K. Then the functor $\mathcal{F} \xrightarrow{\Gamma} \Gamma(\mathcal{F})$ of the global sections is exact on the category of the A-coherent O_V modules.*

We begin with a definition.

DEFINITION 1. Let (V, O_V) be a (reduced) affine variety defined on the field K, we shall call *associated scheme* the affine scheme $(\operatorname{Spec}\Gamma_V, O_{SV})$ where O_{SV} is the usual structural sheaf associated to $\operatorname{Spec}\Gamma_V$ and $\Gamma_V = \Gamma(O_V)$.

LEMMA 1. *Let (V, O_V) be an affine variety and* $(\operatorname{Spec}\Gamma_V, O_{SV})$ *the associated scheme, then there exists a natural injection $j: V \to \operatorname{Spec}\Gamma_V$ such that $O_{SV|j(V)} = \mathfrak{J}_*(O_V)$. Moreover $j(V) =$* {set of the closed points of $\operatorname{Spec}(\Gamma_V)$}.

PROOF. Let V be realized in K^n, then to any point $x = (\alpha_1, \ldots, \alpha_n) \in V$ we associate the maximal ideal $j(x)$ generated by the classes $\{X_i - \alpha_i\}$ $i = 1, \ldots, n$.

Now we wish to prove that any $\gamma \in \Gamma_U(O_{SV})$ defines an element $\gamma_{U \cap j(V)}$ of $\Gamma_{U \cap j(V)}(j_*(O_V))$.

We may suppose $U = \{x \in \operatorname{Spec} \Gamma_V / f \notin x$ for a fixed $f \in \Gamma_V\}$ and by definition we have that any element of $\Gamma_U(O_{SV})$ is of the form P/f^p, $P \in \Gamma_V$.

If $x \in j(V)$ then $f \notin x \Leftrightarrow f(j^{-1}(x)) \neq 0$ and clearly P/f^p is a regular function on $j^{-1}(U \cap j(V))$.

Let now $\gamma \in \Gamma_U(O_V)$, $x \in U$, then locally γ is the image of a rational function hence it is the restriction of a section of O_{SV} defined on an open set of $\operatorname{Spec} \Gamma_V$ containing $j(x)$. So we have proved $O_{SV|j(V)} = j_*(O_V)$.

We wish now to prove: $j(V) \supset \{$set of the closed points of $\operatorname{Spec}(\Gamma_V)\}$, (the other inclusion is clear).

Let α be a maximal ideal of Γ_V and $\overline{V} \subset \overline{K}^n$ be the completion of V into the algebraic closure $\overline{K}$ of K.

Let $\bar{\alpha}$ be the ideal of $\Gamma_{\overline{V}}$ generated by α and S the locus of zeros of $\bar{\alpha}$, let us suppose $S \cap V = \emptyset$.

Any $P \in K[X_1, \ldots, X_n]$ such that $P_{|S} \equiv 0$ is, by the maximality of α, in α.

By lemma 5 of § 1 there exists $P \in K[X_1, \ldots, X_n]$ such that $P(x) \neq 0$ if $x \in K^n$, $P_{|S} \equiv 0$ hence $(1/P) \in \Gamma_V$ and this is impossible. So we must suppose $S \cap V \neq \emptyset$ hence, again by maximality: $\alpha \in j(V)$ and the lemma is proved.

REMARK. *In the following we shall often identify V and $j(V)$, O_V and $O_{SV|j(V)}$ ....*

DEFINITION 2. Let (V, O_V) be an affine variety defined on K and $\mathcal{F} \to V$ a coherent sheaf of O_V module; a coherent sheaf $\hat{\mathcal{F}} \to \operatorname{Spec}(\Gamma_V)$ of O_{SV} module is called an extension of $\mathcal{F}$ iff there exist two exact sequences:

$$O_V^n \xrightarrow{\alpha} O_V^p \to \mathcal{F} \to 0\,, \qquad O_{SV}^n \xrightarrow{\tilde{\alpha}} O_{SV}^p \to \hat{\mathcal{F}} \to 0$$

where $\tilde{\alpha}$ is an extension of α.

LEMMA 2. *Any A-coherent sheaf $\mathcal{F} \to V$ defined on the affine variety V has an extension $\tilde{\mathcal{F}} \to \operatorname{Spec}(\Gamma_V)$.*

PROOF. We have an exact sequence

$$O_V^n \to O_V^p \to \mathcal{F} \to 0$$

and α is given by $\alpha(\tau_i) = \sum_k \alpha_{ik}\gamma_k$ where $\tau_i = (0, \ldots, 1, \ldots, 0) \in \Gamma(O_V^n)$, $\gamma_k = (0, \ldots, 1, \ldots, 0) \in \Gamma(O_V^p)$ and $\alpha_{ik} \in \Gamma_V$. Let $\tilde{\alpha}_{ik}$ be the extension of α_{ik} to $\mathrm{Spec}(\Gamma_V)$ and $O_{SV}^n \xrightarrow{\tilde{\alpha}} O_{SV}^p$ the morphism defined by $(\tilde{\alpha}_{ik})_{i,k}$.

Clearly coker $\tilde{\alpha}$ is a coherent sheaf and is an extension of $\mathcal{F}$.

LEMMA 3. *Let (V, O_V) be an affine variety defined on the field K and $\mathcal{F} \to V$ an A-coherent sheaf of O_V module. Let $\tilde{\mathcal{F}} \to \mathrm{Spec}(\Gamma_V)$ be an extension of $\mathcal{F}$ and $\gamma \in \Gamma(\mathcal{F})$ then there exists $\tilde{\gamma} \in \Gamma(\tilde{\mathcal{F}})$ such that $j_*(\gamma) = \tilde{\gamma}_{|j(V)}$.*

If $\mathcal{F}$, $\mathcal{G}$ are two coherent sheaves on V and $\tilde{\mathcal{F}}$, $\tilde{\mathcal{G}}$ two extensions, any morphism $\alpha: \mathcal{F} \to \mathcal{G}$ can be extended to a morphism $\tilde{\alpha}: \tilde{\mathcal{F}} \to \hat{\mathcal{G}}$.

PROOF. Is the same as the proof of the remark 3 of § 3.

COROLLARY 1. *Let (V, O_V) be an affine variety defined on the field K and $\mathcal{F} \to V$ an A-coherent sheaf. Any two extensions $\tilde{\mathcal{F}}_i \to \mathrm{Spec}(\Gamma_V)$, $i = 1, 2$, of $\mathcal{F}$ are canonically isomorphic.*

PROOF. The identity section of $\mathrm{Hom}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)_{|V}$ can be extended to a global section $\tilde{\gamma}$; $\tilde{\gamma}: \tilde{\mathcal{F}}_1 \to \tilde{\mathcal{F}}_2$ is an isomorphism for any $x \in j(V)$ hence, by the argument used in lemma 3, $\tilde{\gamma}$ is an isomorphism.

LEMMA 4. *Let (V, O_V) be an affine variety defined on the field K and $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ an exact sequence of A-coherent O_V modules. The sequence of extended morphisms:*

$$0 \to \tilde{\mathcal{F}}' \to \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}'' \to 0$$

is still exact.

PROOF. The relation $\mathrm{Ker}\,\tilde{\beta}_x = \mathrm{im}\,\tilde{\alpha}_x$ is true for any $x \in j(V)$ hence for any $x \in \mathrm{Spec}\,\Gamma_V$ (see the argument of lemma 3).

REMARK. *It is possible to associate to any algebraic prevariety (V, O_V) on the field K a prescheme in the following way: if $\bar{K}$ is the algebraic closure of K, to (V, O_V) we may associate one completion $(\bar{V}, O_{\bar{V}})$,* (see § 3), *and to $(\bar{V}, O_{\bar{V}})$ we may associate in a unique way a prescheme* (see D. Munford, « An introduction to algebraic geometry »).

PROOF OF THEOREM 1. First we remark that it is enough to prove that if the sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \tag{1}$$

is exact, then the sequence

$$0 \to \Gamma(\mathcal{F}') \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'') \to 0 \tag{2}$$

is exact.

In general we know that if (1) is exact, then:

$$0 \to \Gamma(\mathcal{F}') \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'')$$

is exact.

It is enough to prove that if

$$\mathcal{F} \to \mathcal{F}'' \to 0$$

is exact, then

$$\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'') \to 0 \tag{3}$$

is exact.

Let $\tilde{\mathcal{F}} \xrightarrow{\tilde{\alpha}} \tilde{\mathcal{F}}'' \to 0$ be an extension of α; the coherent sheaves $\tilde{\mathcal{F}}$, $\mathcal{F}''$ are defined on $\operatorname{Spec}(\Gamma_V)$ hence we have:

$$\tilde{\alpha}_*\colon \Gamma(\tilde{\mathcal{F}}) \to \Gamma(\tilde{\mathcal{F}}'') \quad \text{is surjective (by theorem } B). \tag{4}$$

Let now $\gamma \in \Gamma(\mathcal{F}'')$; by lemma 3 there exists an extension $\tilde{\gamma} \in \Gamma(\tilde{\mathcal{F}}'')$ of γ, by (4) there exists $\tilde{\eta} \in \Gamma(\tilde{\mathcal{F}})$ such that $\tilde{\alpha}_*(\tilde{\eta}) = \tilde{\gamma}$ hence $\gamma = \alpha_*(\tilde{\eta}_{|j(V)})$ and the theorem is proved.

REMARK. Theorem 1 *can be proved also using the completion of V instead of the extension* (see [5] for the proof in the real case).

As a consequence of theorem 1 we have:

THEOREM 2. *Let K be a field and (V, O_V) an affine variety on K. In this hypothesis the category of A-coherent O_V modules is isomorphic to the category of finitely generated $\Gamma(O_V)$ modules.*

PROOF. *The same as in* [2] *and* [5].

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