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Tadeusz Mostowski

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Some Properties of the Ring of Nash Functions.

TADEUSZ MOSTOWSKI (*)

1. – Introduction and statement of the results.

In this paper we shall study the ring $\mathcal{N}(U)$ of Nash functions defined on an open semialgebraic set $U \subset \boldsymbol{R}^n$. Recall that a real analytic function

$$f\colon U \to \boldsymbol{R}$$

is called a Nash function if it satisfies an equation

$$P(x, f(x)) = 0$$

identically in U, where $x = (x_1, \ldots, x_n)$ and $P(x, t)$ is a nonzero polynomial.

It is known that $\mathcal{N}(U)$ is noetherian ([2], [4], [7]).

This paper will be devoted to the proofs of the following statements.

PROPOSITION 1. *Suppose that U is semialgebraic and connected. Then, if $\mathfrak{p} \subset \mathcal{N}(U)$ is a prime ideal, then its set of zeros $V(\mathfrak{p})$ is connected.*

PROPOSITION 2 (Artin's Theorem for Nash functions). *If U is semialgebraic, connected, and $f \in \mathcal{N}(U)$ takes nonnegative values, then f is a sum of squares in the field of quotients of the ring $\mathcal{N}(U)$.*

PROPOSITION 3 (Nullstellensatz). *Let U be semialgebraic and let $\mathfrak{a} \subset \mathcal{N}(U)$ be an ideal. Then the following conditions are equivalent:*

a) any function in $\mathcal{N}(U)$ vanishing on $V(\mathfrak{a})$ is in $\mathfrak{a}$,

b) if $f_1^2 + \ldots + f_m^2 \in \mathfrak{a}$, $f_i \in \mathcal{N}(U)$, then $f_i \in \mathfrak{a}$.

(*) Mathematical Institute of University of Warsaw.
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2. – Separation of semialgebraic sets.

The proofs are based on the following lemma.

SEPARATION LEMMA. *Let A, $B \subset \mathbf{R}^n$ be closed, semialgebraic and disjoint. Then there exists a Nash function F on $\mathbf{R}^n$ which separates them, i.e.*

$$F > 0 \quad on\ A, \qquad F < 0 \quad on\ B.$$

Moreover, F can be chosen in the form

$$F = \sum_i P_i \sqrt{\sum_j Q_{ij}^2},$$

where P_i, Q_{ij} are polynomials and

$$\sum_j Q_{ij}^2 > 0 \quad for\ all\ i.$$

PROOF OF THE SEPARATION LEMMA. Our main reference is [5].

NOTATION. Let θ (possibly with indices) denote on of the following sets:

$$\{t \in \mathbf{R} : t > 0\}, \quad \{0\}, \quad \{t \in \mathbf{R} : t < 0\}.$$

Let $\mathcal{S}_i(\mathbf{R}^m)$ denote the family of all i-dimensional sets of the form

$$\{(y_1, \ldots, y_m) : y_j \in \theta_j, \text{ for all } j\},$$

and let $\tilde{\mathcal{S}}_i(\mathbf{R}^m)$ be the family of all unions of the elements of $\bigcup_{j \leq i} \mathcal{S}_j(\mathbf{R}^m)$.

Let $\{A_\alpha\}_{\alpha=1}^N$ be a finite family of disjoint subsets of $\mathbf{R}^n$. We shall say that a map

$$\Phi : \mathbf{R}^n \to \mathbf{R}^m$$

separates $\{A_\alpha\}$ if there exists a family $\{S_\alpha\}$, $S_\alpha \in \tilde{\mathcal{S}}_m(\mathbf{R}^m)$, S_α disjoint, such that

$$\Phi(A_\alpha) \subset S_\alpha, \quad \text{for all } \alpha.$$

LEMMA 1. *Let* $\{A_\alpha\}_{\alpha=1}^N$ *be a finite family of disjoint semialgebraic subsets of* $\boldsymbol{R}^n$. *Then there exists a polynomial map*

$$Q: \boldsymbol{R}^n \to \boldsymbol{R}^m$$

separating $\{A_\alpha\}$.

PROOF. We use induction on n. The lemma is trivial for $n=1$. Assume it is true for some n and we prove if for $n+1$. The points of $\boldsymbol{R}^{n+1}$ will be denoted by (x, t), where $x \in \boldsymbol{R}^n$, $t \in \boldsymbol{R}$.

1) It is clear that if $\{\tilde{A}_\beta\}$ is a refinement of $\{A_\alpha\}$ (i.e. any A_α is a sum of some $\tilde{A}_\beta$'s) and we can prove the lemma for the family $\{\tilde{A}_\beta\}$, then we can prove it for $\{A_\alpha\}$ (in fact, the same polynomial map may be used).

2) It suffices to prove the lemma under the following assumption:

$$p(A_1) = p(A_2) = \dots = p(A_N),$$

where $p: \boldsymbol{R}^{n+1} \to \boldsymbol{R}^n$ is the canonical projection.

In fact, assume we can prove our lemma for any family satisfying this extra condition, and let $\{A_\alpha\}_{\alpha=1}^N$ be an arbitrary family of disjoint semialgebraic subsets of $\boldsymbol{R}^{n+1}$. Put $A'_\alpha = p(A_\alpha)$. We claim that there exists a finite refinement $\{B'_\beta\}$ of $\{A'_\alpha\}$ consisting of disjoint semialgebraic sets.

The proof is by induction on the greatest number r such that there exist indices $\alpha_1, \dots, \alpha_r$ such that

$$A'_{\alpha_1} \cap \dots \cap A'_{\alpha_r} \neq \emptyset.$$

If $r=0$, then the A'_α's are disjoint. Now let $r>0$ and we proceed by induction on the number $s = s(\{A'_\alpha\})$ of sets of indices having this property. If $\{\alpha_1^0, \dots, \alpha_r^0\}$ is such a set, then we put

$$B'_1 = A'_{\alpha_1^0} \cap \dots \cap A'_{\alpha_r^0},$$
$$A'_{\alpha 1} = A'_\alpha \setminus B_1, \quad \alpha = 1, \dots, N.$$

We observe that $\{B'_1, A'_1, \dots, A'_{N1}\}$ is a refinement of $\{A'_\alpha\}$, its elements are semialgebraic and $s(\{B'_1, A'_{11}, \dots, A'_{N1}\}) < s(\{A'_\alpha\})$, which finishes the proof.

By the induction hypothesis the B'_β's can be separated by a polynomial map

$$Q': \boldsymbol{R}^n \to \boldsymbol{R}^m.$$

For a fixed β consider the family

$$\{A_{\alpha\beta}\}_\alpha, \qquad A_{\alpha\beta} = A_\alpha \cap p^{-1}(B'_\beta),$$

where α runs over the set of all indices such that $A_{\alpha\beta} \neq \emptyset$. It is clear that

$$p(A_{\alpha\beta}) = B'_\beta.$$

Thus if

$$Q_\beta\colon \boldsymbol{R}^{n+1} \to \boldsymbol{R}^m$$

separates $\{A_{\alpha\beta}\}_\alpha$, then the map

$$Q\colon \boldsymbol{R}^{n+1} \to \boldsymbol{R}^{m'+\Sigma m_\beta},$$
$$Q(x, t) = \big(Q'(x), Q_1(x, t), \ldots\big)$$

separates $\{A_\alpha\}$.

3) Let

$$p(A_1) = \ldots = p(A)_N = A'.$$

Let $P_i(x, t)$ $(i = 1, \ldots, I)$ be a set of polynomials which define the A_α's. Let

$$P_{ij}(x, t) = \frac{\partial^j P_i}{\partial t^j}(x, t),$$

and for any $\psi \subset \{1, \ldots, I\} \times \Theta$,

$$P_\psi(x, t) = \prod_{(i\,j)\in\psi} P_{ij}(x, t),$$
$$A_{\psi,k} = \{a \in A'\colon P_\psi(a', \cdot) \text{ has exactly } k \text{ complex roots}\}.$$

The $A'_{\psi,k}$'s are semialgebraic, so they have a finite number of (topological) components which are also semialgebraic ([5], [11]). Let $\mathfrak{A}'$ be the family of all intersections of the components of the $A'_{\psi,k}$'s, and let $\mathfrak{A}$ be the family of all minimal non-empty elements of $\mathfrak{A}'$.

It is shown in [5] that if $K \in \mathfrak{A}$, then any $P_\psi(a, \cdot)$ has a constant number of real roots for $a \in K$, and these roots are continuous functions of a. For any $K \in \mathfrak{A}$ let

$$\psi_K = \{(i, j)\colon P_{ij} \not\equiv 0 \text{ on } K \times \boldsymbol{R}\},$$
$$P_K = P_{\psi_K},$$
$$\zeta_1(a) < \ldots < \zeta_{\nu_K}(a) \text{ — real roots of } P_K(a, \cdot).$$

It is also shown in [5] that each $p^{-1}(K) \cap A_\alpha$ is a sum of sets of the form

$$(1) \qquad \begin{cases} \{(a, t) \in K \times \boldsymbol{R} : \zeta_\mu(a) < t < \zeta_{\mu+1}(a)\}, \\ \{(a, t) \in K \times \boldsymbol{R} : \zeta_\mu(a) = t\}. \end{cases}$$

We note (reasoning exactly as in 2)) that it suffices to prove the lemma for every family

$$\mathfrak{A}_K = \{p^{-1}(K) \cap A_\alpha\}, \qquad K \in \mathfrak{A},$$

and therefore (by 1)) for every family $\mathfrak{L}_K$ consisting of sets of the form (1).

It follows directly from Thom's lemma ([5], p. 69) that $\mathfrak{L}_K$ is separated by the map

$$\boldsymbol{R}^{n+1} \ni (x, t) \to \left(P_K(x, t), \frac{\partial}{\partial t} P_K(x, t), \frac{\partial^2}{\partial t^2} P_K(x, t), \ldots\right),$$

and thus lemma 1 is proved.

Lemma 2. 1) *If $A, B \subset \boldsymbol{R}^n$ are semialgebraic, closed and disjoint, then for some constants $C, N > 0$*

$$d(a, B) \geqslant C(1 + |\alpha|^2)^{-N} \quad \textit{for every } a \in A;$$

2) *If $F \in \mathcal{N}(\boldsymbol{R}^n)$, then for some constants $C, N > 0$*

$$F(x) \leqslant C(1 + |x|^2)^N.$$

Proof. 1) Clearly the set

$$S = \{(u, v) \in \boldsymbol{R}^2 : \exists a \in A,\ u(1 + |a|^2) = 1,\ v = d(a, B)\}$$

is semialgebraic, as a projection onto the (u, v)-plane of the semialgebraic set

$$\{(u, v, a, b) \in \boldsymbol{R}^2 \times \boldsymbol{R}^n \times \boldsymbol{R}^n : a \in A,\ b \in B,\ u(1 + |a|^2) = 1,\ v = d(a, b)\} \setminus \\ \setminus \{(u, v, a, b) \in \boldsymbol{R}^2 \times \boldsymbol{R}^n \times \boldsymbol{R}^n : d(a, b) < v\}.$$

A and B are closed and disjoint, so

$$S \cap \{(u, 0) : u \in \boldsymbol{R}\} = \emptyset.$$

Thus in some neighbourhood of the origin in $\boldsymbol{R}^2$

$$v \geqslant d\big((u, v), \{(u, 0): u \in \boldsymbol{R}\}\big) \geqslant C|u|^N$$

for $(u, v) \in S$ and for some constants $C, N > 0$. Therefore 1) holds if a is sufficiently big and thus holds everywhere (where C is replaced by a bigger constant, if necessary).

2) follows immediately from 1) and the following lemma (cf. [2]).

LEMMA 3. *If $U \subset \boldsymbol{R}^n$ is open and semialgebraic and $f\colon U \to \boldsymbol{R}$ is Nash, then*

$$\operatorname{graph}(f) = \{(x, f(x)) : x \in U\} \subset \boldsymbol{R}^{n+1}$$

is semialgebraic.

In fact, if we put

$$A = \{(x, 0): x \in \boldsymbol{R}^n\} \subset \boldsymbol{R}^{n+1},$$

$$B = \operatorname{graph}\left(\frac{1}{1+F^2}\right) \subset \boldsymbol{R}^{n+1}$$

and use 1), we get 2).

PROOF OF LEMMA 3. We can find a polynomial $P(x, t)$ such that

$$P(x, f(x)) = 0 \quad \text{identically in } U$$

and its discriminant $\delta_P(x) \neq 0$. Thus if $\delta_P(x) \neq 0$, all the roots of $P(x, \cdot)$ are simple. Therefore

$$A = \operatorname{graph}(f) \cap \{(x, t) \in \boldsymbol{R}^n \times \boldsymbol{R}\colon \delta_P(x) \neq 0\}$$

is a sum of components of the semialgebraic set

$$\{(x, t) \in U \times \boldsymbol{R}\colon \delta_P(x) \neq 0,\ P(x, t) = 0\}$$

and thus is semialgebraic. It is obvious that graph(f) is the closure of A and since the closure of a semialgebraic set is semialgebraic, we are finished.

LEMMA 4. *Let $f \in \mathcal{N}(\boldsymbol{R}^n)$. Let $A, B \subset \boldsymbol{R}^n$ be closed semialgebraic and suppose that*

$$f > 0 \quad \text{on } A; \qquad f < 0 \quad \text{on } B.$$

Then for some constants C, $\alpha > 0$

$$f > 0 \quad \text{on } U_A = \}x \in \boldsymbol{R}^n : d(x, A) < C(1 + |x|^2)^{-\alpha}\}$$
$$f < 0 \quad \text{on } U_B = \{x \in \boldsymbol{R}^n : d(x, B) < C(1 + |x|^2)^{-\alpha}\}.$$

PROOF. Since $\{(x, f(x)) : x \in A\}$ and $\boldsymbol{R}^n \times \{0\}$ are semialgebraic, closed and disjoint, we have, by lemma 2, 1)

$$|f(x)| = f(x) \geqslant d\big(\{(x, f(x)) : x \in A\}, \boldsymbol{R}^n \times \{0\}\big) \geqslant C_1(1 + |x|^2)^{-N_1}, \qquad \text{for } x \in A,$$

for some constants C_1, N_1. All the derivatives $\partial f/\partial x_i$ are Nash, so, by lemma 2, 2),

$$|\operatorname{grad} f(x)| \leqslant C_2(1 + |x|^2)^{N_2}, \qquad x \in A,$$

for some C_2, N_2. The lemma follows now from the formula

$$|f(y) - f(x)| \leqslant |y - x| \sup |\operatorname{grad} f(z)|,$$

the sup being taken over the interval joining x and y.

LEMMA 5. *Let $S \in \mathcal{S}_i(\boldsymbol{R}^m)$, $D, \beta > 0$. Then there exists a Nash function $\boldsymbol{R}^m \to \boldsymbol{R}$ such that*

$$f < 0 \quad \textit{on } \{y \in S : d(y, \partial S) \geqslant D(1 + |y|^2)^{-\beta}\}$$
$$f > 0 \quad \textit{on all } S' \in \mathcal{S}_i(\boldsymbol{R}^m), \qquad S' \neq S.$$

Moreover, f can be chosen to be of the same form as in the statement of the separation lemma.

PROOF. First we remark that it is enough to prove the lemma in the case $m = i$. For in the general case we identify $\boldsymbol{R}^i$ with the i-dimensional hyperplane in $\boldsymbol{R}^m$ spanned by S, construct the function f_1 for S considered as a subset of $\boldsymbol{R}^i$ and put $f = f_1 p$, where $p : \boldsymbol{R}^m \to \boldsymbol{R}^i$ is the orthogonal projection.

We may assume that S is described by the inequalities $x_1 > 0, \ldots, x_m > 0$. Thus the function

$$-\prod_{i=1}^{m} (x_i + |x_i|)$$

is zero outside of S and is negative in S. It is easy to check that or $\varepsilon > 0$ sufficiently small and N sufficiently big the function

$$f(x) = -\prod_{i=1}^{m} [(x_i + \sqrt{\xi(x) + x_i^2})/(1 + |x_i|^2)] + \sqrt{\xi(x)},$$

where

$$\xi(x) = \varepsilon(1 + |x|^2)^{-N},$$

satisfies all the requirements of the lemma.

Now we prove the separation lemma. We may assume that A, B are separated in the sense of lemma 1, for (in the notation of lemma) if $F > 0$ on $Q(A)$ and $F < 0$ on $Q(B)$, then $FQ > 0$ on A and $FQ < 0$ on B (clearly $Q(A)$ and $Q(B)$ are semialgebraic and disjoint, and, replacing $Q: \boldsymbol{R}^n \to \boldsymbol{R}^m$ by $Q \times id: \boldsymbol{R}^n \to \boldsymbol{R}^m \times \boldsymbol{R}^n$ if necessary, we may assume that they are closed).

Let

$$\boldsymbol{R}_i^m = \bigcup \tilde{S}_i(\boldsymbol{R}^m), \quad i \leqslant m,$$
$$A_i = A \cap \boldsymbol{R}_i^m, \quad B_i = B \cap \boldsymbol{R}_i^m.$$

For any i we shall construct a Nash function

$$F_i: \boldsymbol{R}^m \to \boldsymbol{R},$$

of the form as in the statement of the separation lemma, such that

$$F_i > 0 \quad \text{on } A_i, \qquad F_i < 0 \quad \text{on } B_i.$$

For $i = 0$ such a construction is trivial, so assume F_i is constructed.

First observe that $\boldsymbol{R}_i^m$ can be defined by a single polynomial equation: $P_i(x) = 0$, where

$$P_i(x) = \prod_{\substack{j_k \neq j_1 \\ \text{for } k \neq 1}} (x_{j_1}^2 + \ldots + x_{j_i}^2).$$

Clearly $P_i(x) \geqslant 0$ and $\operatorname{grad} P_i = 0$ on $\boldsymbol{R}_i^m$.

We claim that there are constants $C, \alpha > 0$ such that

$$(2) \qquad \begin{cases} F_i > 0 & \text{on } A \cap U, \\ F_i < 0 & \text{on } B \cap U, \end{cases}$$

where

$$U = \{x \in \boldsymbol{R}^m: d(x, \boldsymbol{R}_i^m) < C(1 + |x|^2)^{-\alpha}\}.$$

In fact, by lemma 4

$$F_i > 0 \quad \text{on } U_i = \{x: d(x, A_i) < C_1(1 + |x|^2)^{-\alpha_1}\},$$
$$F_i < 0 \quad \text{on } V_i = \{x: d(x, B_i) < C_1(1 + |x|^2)^{-\alpha_1}\}$$

for some C_1, $\alpha_1 > 0$. Now consider the sets $\overline{A\setminus U_i}$, $\overline{B\setminus V_i}$: Clearly they are semilagebraic and disjoint from $\boldsymbol{R}_i^m$. Thus, by Lemma 2, there are constants C_2, $\alpha_2 > 0$ such that

$$d(x, \boldsymbol{R}_i^m) \geqslant C_2(1 + |x|^2)^{-\alpha_2}, \qquad \text{for every } x \in \overline{A\setminus U_i} \cup \overline{B\setminus V_i},$$

which proves our claim.

By an appropriate choice of N we can find constants C_3, α_3, $\varepsilon > 0$ such that if

$$W = \{x: d(x, \boldsymbol{R}_i^m) < C_3(1 + |x|^2)^{-\alpha_3}\},$$

then

$$(1 + |x|^2)^P P_i(x) > 2\varepsilon \quad \text{for every } x \in \boldsymbol{R}^m\setminus W.$$

Put

$$Q_i(x) = \frac{(1 + x^2)^N P_i(x)}{\varepsilon}.$$

Q_i vanishes together with its first derivatives on $\boldsymbol{R}_i^m$. Thus for some constants L, C_4

$$Q_i(x) \leqslant C_4(1 + |x|^2)^L d^2(x, \boldsymbol{R}_i^m).$$

So for some C_5, $\alpha_5 > 0$, $C_5 < 1$, if

$$V = \{x: d(x, \boldsymbol{R}_i^m) < C_5(1 + |x|^2)^{-\alpha_5},\ d(x, \boldsymbol{R}_i^m) < 1\},$$

then

$$\boldsymbol{R}_i^m \subset V \subset W$$

and

(3) $$Q_i(x) < \tfrac{1}{2} d(x, \boldsymbol{R}_i^m) < \tfrac{1}{2}, \quad \text{for all } x \in V.$$

For any $\psi \in \mathcal{S}_{i+1}(\boldsymbol{R}^m)$ let f_S be the Nash function constructed in lemma 5, where we replace D by C_5 and β by α_5. Let

$$f = \prod_{\substack{S \in \mathcal{S}_{i+1}(\boldsymbol{R}^m) \\ S \cap B \neq \emptyset}} f.$$

We observe that

$$f > 0 \quad \text{on } A_{i+1} \setminus V ,$$
$$f < 0 \quad \text{on } B_{i+1} \setminus V .$$

F_{i+1} will be of the following form:

$$F_{i+1}(x) = F_i(x) + C_6 Q_i^{M_1}(x)(1 + |x|^2)^{M_2} f(x) .$$

First we note that, since F_i and f are both positive on $(W \setminus V) \cap A_{i+1}$ and negative on $(W \setminus V) \cap B_{i+1}$,

$$F_{i+1} > 0 \quad \text{on } (W \setminus V) \cap A_{i+1} ,$$
$$F_{i+1} < 0 \quad \text{on } (W \setminus V) \cap B_{i+1}$$

for any constants C_6, M_1, M_2.

Outside of W we have

$$f > 0 , \qquad Q_i > 2 ;$$

therefore, by lemma 2, we can find constants C_6, M_2 such that

$$C_6 Q_i^{M_1}(x)(1 + |x|^2)^{M_2} |f(x)| > |F_i(x)| ,$$

for every $x \in \boldsymbol{R}^m W \setminus$, for any constant M_1. Therefore for any M_1, F has the desired property outside of W.

Now, using (2), p. 11, and (3), p. 12, and choosing a sufficiently large constant M_1, we have

$$|F_i(x)| > C_6 Q_i^{M_1}(x)(1 + |x|^2)^{M_2} |f(x)| ,$$

for every $x \in V$.

Thus $F_{i+1} > 0$ on A_{i+1}, $F_{i+1} < 0^i$ on B_{i+1}, which finishes the proof.

In the sequel we shall have to work with a class of functions $\mathfrak{R}(U)$ which is slightly larger than the polynomials. If $U \subset \boldsymbol{R}^n$ is open, we put

$$\mathfrak{R}_0(U) \;\; = \boldsymbol{R}[x_1, \ldots, x_n] ,$$

$\mathfrak{R}_{i+1}(U) =$ the set of all functions of the form

$$\sum_k \varphi_k \sqrt{\sum_l \psi_{kl}^2} ,$$

where φ_k, $\psi_{kl} \in \mathcal{R}_i(U)$ and $\sum_l \psi_{kl}^2 > 0$ in U, for all k,

$$\mathcal{R}(U) \quad = \bigcup_{i=0}^{\infty} \mathcal{R}_i(U) .$$

Clearly if $f \in \mathcal{R}(U)$, then $f|U \in \mathcal{N}(U)$.

COROLLARY. *If $U \subset \boldsymbol{R}^n$ is open, semialgebraic, and $A, B \subset U$ are semialgebraic, disjoint and closed in U, then there exists a function $f \in \mathcal{R}(U)$ such that*

$$f > 0 \quad on\ A , \qquad f < 0 \quad on\ B .$$

For the proof we need a lemma.

LEMMA 6. *If $C \subset \boldsymbol{R}^n$ is semialgebraic and closed, then there exists a function $f \in \mathcal{R}(\boldsymbol{R}^n \setminus C)$ such that*

$$f = 0 \quad on\ B , \qquad f > 0 \quad on\ \boldsymbol{R}^n \setminus C .$$

PROOF OF LEMMA 6. Clearly if $C = C_1 \cup C_2$ and we can construct such functions for C_1 and C_2, then their product will be good for C. Thus we may assume that C is defined by

$$p_0 = 0 \quad \& \quad p_1 \geqslant 0 \quad \& \ \dots \ \& \quad p_k \geqslant 0$$

(cf. [5]), where $p_0, \dots, p_k$ are polynomials. Now we use induction on k. If $k = 0$, then we may put $f = p_0^2$. Let C_1 be defined by

$$p_0^2 + p_k^2 = 0 \quad \& \quad p_1 \geqslant 0 \quad \& \ \dots \ \& \quad p_{k-1} \geqslant 0 ,$$

and C_2 by

$$p_0 = 0 \quad \& \quad p_1 \geqslant 0 \quad \& \ \dots \ \& \quad p_{k-1} \geqslant 0 .$$

Therefore C_i is the set of zeroes of a function $g_i \in \mathcal{R}(\boldsymbol{R}^n \setminus C_i)$, $i = 1, 2$. In $\boldsymbol{R}^n \times \boldsymbol{R}$ consider the sets

$$K \ = \{(x, t) : x \in C_2,\ g_1^2(x) t = 1\} ,$$
$$K_0 = \{(x, t) : x \in C,\ g_1^2(x) t = 1\} ;$$

They are both semialgebraic closed and $K \setminus K_0$ is also closed. Therefore, by the separation lemma, there exists a function $\varphi \in \mathcal{R}(\boldsymbol{R}^n \times \boldsymbol{R})$ such that

$$\varphi > 0 \quad \text{on } K_0 , \qquad \varphi < 0 \quad \text{on } K \setminus K_0 .$$

Thus

$$\sqrt{g_2^2(x) + \varphi^2(x, g_1^{-2}(x))} - \varphi(x, g_1^{-2}(x))$$

multiplied by an appropriate (positive) power of g_1^2 (to cancel the negative powers of g_1^2) satisfies all the requirements of the lemma.

PROOF OF THE COROLLARY. Since $\overline{U}\setminus U$ is closed semialgebraic, it is the set of zeroes of a function $g\in\mathfrak{R}(\boldsymbol{R}^n\setminus(\overline{U}\setminus U))$. The sets

$$A_1=\{(x,t)\in\boldsymbol{R}^n\times\boldsymbol{R}\colon x\in A,\ g^2(x)t=1\},$$
$$B_1=\{(x,t)\in\boldsymbol{R}^n\times\boldsymbol{R}\colon x\in B,\ g^2(x)t=1\}$$

are semialgebraic, disjoint and closed in $\boldsymbol{R}^n\times\boldsymbol{R}$, and thus there exists an $F_1\in\mathfrak{R}(\boldsymbol{R}^n\times\boldsymbol{R})$ such that

$$F_1>0 \quad \text{on } A_1, \qquad F_1<0 \quad \text{on } B_1.$$

It suffices to put

$$f(x)=F_1(x,g^{-2}(x))g^N(x),$$

where N is so big that the negative powers of g^{-2} cancel out.

REMARK. It is interesting to note that even if $A, B\subset\boldsymbol{R}^n$ are components of an algebraic set, then, in general, they cannot be separated by a polynomial. We give an example.

Consider the following set in $\boldsymbol{R}^2$:

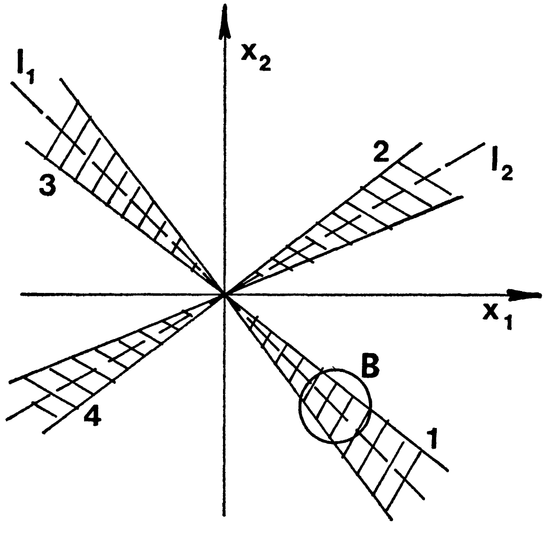

Figure 1

It is trivial to see that there is no polynomial V such that $V>0$ in 2, 3, 4, and $V<0$ in 1, in some neighbourhood of 0. In fact, supposing the contrary, let V_0 be the homogeneous part of V of lowest degree. Replacing the angles 1, 2, 3, 4 by smaller ones we see that V_0 would have the similar property. But either the (total) degree of V_0 is even, and then if $V_0<0$ in 1, then $V_0<0$ in 3, or its degree is odd, and then if $V_0>0$ in 2, then $V_0<0$ in 4.

Now let P a fixed polynomial such that its homogenous part is of degree $4k$, positive on 1, 2, 3, 4. Then there can be no polynomials V, W such that $V+W\sqrt{P}<0$ in 1 and $V+W\sqrt{P}>0$ in 2, 3, 4. In fact, supposing that such V and W do exist, let V_0 and W_0 be the homogenous parts of V, W respectively, of lowest degree. Let l_1, l_2 be straight lines with l_1 constained in the angles 1 and 3, and l_2 in 2 and 4, such that $V_0|l_i$, $W_0|l_i \not\equiv 0$, $i=1,2$. Clearly there is a polynomial Q, of even degree, such that

$$\sqrt{P/l_i}=Q|l_i+v(Q|l_i)\,, \qquad \text{as } x\to 0\,,\ i=1,2\,.$$

Now, approximating $V+W\sqrt{P}|l_i$ by $V_0+W_0Q|l_i$, we get a contradiction.

Suppose that P is negative outside the angles 1, 2, 3, 4 and let P' be a polynomial negative on the ball B and positive outside of it we can adjust P and P' such that their sum S is positive on the shaded set and negative outside of it. We can adjust P and P' such that their sum S positive on the sharded set

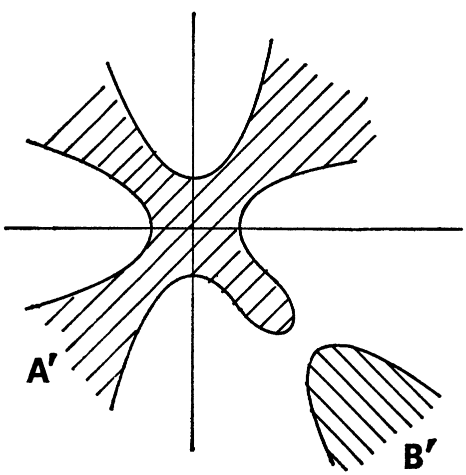

Figure 2

and negative outside of it. We can also assume that $|P'/P|\to 0$ as $|x|\to\infty$.

Now consider the set $C\subset \boldsymbol{R}^3$ described by

$$x_3^2 = S(x_1, x_2)\,.$$

Clearly it has two components A and B, projecting onto A' and B' respectively. We claim that A and B cannot be separated by a polynomial. For let $Q(x_1, x_2, x_3)$ be a polynomial such that $Q>0$ and A, $Q<0$ on B. Then the function

$$F(x_1, x_2) = Q\big(x_1, x_2, \sqrt{S(x_1, x_2)}\big)$$

would be >0 on A' and <0 on B'. But clearly

$$F(x_1, x_2) = V(x_1, x_2) + W(x_1, x_2)\sqrt{S(x_1, x_2)}\,,$$

for some polynomials V and W. Now we look at what happens near infinity. Put $z_i = x_i/|x|^2$, $i=1,2$. For big N the function $|z|^{2N}F(x_1, x_2)$ is of the form

$$T(z_1, z_2) + U(z_1, z_2)\sqrt{S(z_1, z_2)}\,,$$

where T, U, S are polynomials and the lowest homogenous part of S is of degree divisible by 4. But near infinity A and B look exactly like the angles inf fig. 1. Therefore we have a contradiction.

It is easy to prove that the separation by a polynomial is possible if one of the sets A and B is compact or is of (topological) dimension 1.

3. – Proofs.

Proposition 1 follows immediately (as observed by J.-J. Risler) from our corollary, p. 14.

Let $\mathfrak{p}\subset\mathcal{N}(U)$ be prime; since $\mathcal{N}(U)$ is noetherian, we can find a finite number of generators $f_1, \ldots, f_m$. By lemma 3, the set of zeroes $V(f)$ of any Nash function f on U is semialgebraic; in fact, it can be identified with

$$\operatorname{graph}(f)\cap\big(\boldsymbol{R}^n\times\{0\}\big)\subset\boldsymbol{R}^{n+1}\,.$$

Therefore $V(\mathfrak{p}) = V(f_1)\cap V(f_2)\cap\ldots\cap V(f_m)$ is semialgebraic.

Assume $V(\mathfrak{p})$ is not connected; thus

$$V(\mathfrak{p}) = A\cup B\,,$$

where A and B are closed, disjoint and semialgebraic (since the components of a semialgebraic set are semialgebraic, [5]). Let $f \in \mathcal{N}(U)$, $f > 0$ on A, $f < 0$ on B. Consider the functions

$$f_+ = \sqrt{f_1^2 + \ldots + f_m^2 + f^2} + f\,,$$
$$f_- = \sqrt{f_1^2 + \ldots + f_m^2 + f^2} - f\,.$$

Clearly they are Nash and $f_+ \neq 0$ on A, $f_- \neq 0$ on B. In particular, $f_+ \notin \mathfrak{p}$, $f_- \notin \mathfrak{p}$. But $f_+ f_- = f_1^2 + \ldots + f_m^2 \in \mathfrak{p}$, and we have a contradiction.

In the proofs of propositions 2 and 3 we shall use Tarski's theorem ([10]). It can be formulated as follows.

Let K be a real-closed field. Let us call a *polynomial relation in* K any formula which can be obtained using alternatives and negations from formulas of the form $f(x_1, \ldots, x_n) > 0$, where $f(X_1, \ldots, X_n) \in K[X_1, \ldots, X_n]$.

TARSKI'S THEOREM. *Let Q denote either $\forall$ or $\exists$ and let $F(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be a polynomial relation in K. Let us consider the formula*

$$Qx_1, \ldots, Qx_n F(x_1, \ldots, x_n, y_1, \ldots, y_m)\,.$$

Then there exists a polynomial relation $G(y_1, \ldots, y_m)$ in K such that for any real-closed extension K_1, of K

$$Qx_1, \ldots, Qx_n F(x_1, \ldots, x_n, y_1, \ldots, y_m) \Leftrightarrow G(y_1, \ldots, y_m)\,.$$

In particular, if $m = 0$ (i.e. $Qx_1, \ldots, Qx_n F(x_1, \ldots, x_n)$ is a sentence), and if $Qx_1, \ldots, Qx_n F(x_1, \ldots, x_n)$ holds in some real-closed extension of K, then it holds in any real-closed extension of K.

A quick proof of Tarski's theorem can be found in [1].

In the proof of proposition 2 we shall follow [8], pp. 214-225.

LEMMA 7 ([8]). *Let K be a field, and let C be a subset of K such that $0 \notin C$, $1 \in C$. In order that an element $a \neq 0$ of K be positive in all orderings of the field K in which all elements of C are positive it is necessary and sufficient that a can be represented in the form*

$$a = \sum c_i a_i^2\,, \qquad a_i \in K\,,$$

where the c_i are products of elements of C.

In particular, the necessary and sufficient condition for the existence of an order on K such that all elements of C are positive is that there is no relation of the form $\sum c_i a_i^2 = 0$, $a_i \in K$, c_i-products of elements of C, $a_i \neq 0$.

LEMMA 8. *Let K be an ordered field and $a \in K$, $a \neq 0$. Then the order on K can be extended to an order on $K(X)$ such that*

$$a^2 X > 1 .$$

The proof is a trivial application of lemma 7.

To make the proof of proposition 2 clearer, we shall repeat the proof given in [8] of the classical theorem of Artin: if a polynomial $f \in \boldsymbol{R}[X_1, \ldots, X_n]$ takes only nonnegative values, then it is a sum of squares in $R(X_1, \ldots, X_n)$.

Assume it is not a sum of squares. Then, by lemma 7, there exists an order of $\boldsymbol{R}(X_1, \ldots, X_n)$ such that $f < 0$. Let K be a real closure of $\boldsymbol{R}(X_1, \ldots, X_n)$ and let $\overline{X}_1, \ldots, \overline{X}_n \in K$ be the images of the polynomials $X_1, \ldots, X_n$ under the natural imbedding $\boldsymbol{R}(X_1, \ldots X_n) \to K$. The phrase

$$\forall x_1, \ldots, x_n,\ f(x_1, \ldots, x_n) \geqslant 0$$

holds in $\boldsymbol{R}$, therefore, by Tarski's theorem, it holds also in K. But

$$f(\overline{X}_1, \ldots, \overline{X}_n) = \operatorname{im} f(X_1, \ldots, X_n) < 0$$

and we have a contradiction.

Now we pass to the proof of proposition 2.

Let $f \in \mathcal{N}(U)$, $f(x) \geqslant 0$ for all $x \in U$, and assume that f is not a sum of squares in the field $\mathcal{N}(U)^*$ of quotients of the ring $\mathcal{N}(U)$. Thus we can order $\mathcal{N}(U)^*$ so that $f < 0$.

We choose a polynomial $P(x, t)$ such that $P(x, f(x)) = 0$ and its discriminant $\delta(x) \not\equiv 0$.

Consider the sets

$$A = \{(x, y, t) \in \boldsymbol{R}^n \times \boldsymbol{R} \times \boldsymbol{R} \colon x \in U,\ t = f(x)\},$$
$$B = \{(x, y, t) \in \boldsymbol{R}^n \times \boldsymbol{R} \times \boldsymbol{R} \colon x \in \boldsymbol{R}^n \setminus (\overline{U} \setminus U),\ \delta^2(x) y \geqslant 1,\ P(x, t) = 0,\ t \neq f(x)\}.$$

Clearly they are semialgebraic, disjoint and closed in $(\boldsymbol{R}^n \setminus U) \times \boldsymbol{R} \times \boldsymbol{R}$. Thus there exists a function $\varphi \in \mathfrak{R}((\boldsymbol{R}^n \setminus U) \times \boldsymbol{R} \times \boldsymbol{R})$ such that $\varphi > 0$ on A $\varphi < 0$ on B. Therefore we have: if $P(x, t) = 0$, $\delta^2(x) y \geqslant 1$, $\varphi(x, y, t) > 0$, then $t \geqslant 0$.

By lemma 8 we can extend the order on $\mathcal{N}(U)^*$ to an order on $\mathcal{N}(U)^*(Y, T)$ such that $\delta^2(x) Y > 1$.

Now consider the field K of quotients of the ring $\mathcal{T}$ defined as follows. Put

$$\mathcal{T}_0 = \mathcal{N}(U)[Y, T] .$$

and consider it as a ring of functions on $U\times\boldsymbol{R}\times\boldsymbol{R}$. Let

$$\mathcal{T}_{i+1} = \text{the set of all functions of the form } \sum_k \varphi_k \sqrt{\sum_k \psi_{kl}^2}, \text{ where } \varphi_k, \psi_{kl}\in\mathcal{T}_i,$$
$$\text{and } \sum_l \psi_{kl}^2 > 0 \text{ on } U\times\boldsymbol{R}\times\boldsymbol{R}, \text{ for every } k,$$
$$\mathcal{T} = \bigcup_{i=0}\mathcal{T}_i .$$

Since $\mathcal{T}$ is obtained from $\mathcal{N}(U)[Y, T]$ by an inductive procedure of adding square roots of sums of squares, K can be ordered in such a way that it extends the order on $\mathcal{N}(U)^*(Y, T)$.

It is clear that for any $V\subset\boldsymbol{R}^m$, open and semialgebraic, and for any $\psi\in\mathcal{R}(V)$, the formulas $\psi(y)>0$, $\psi(y)=0$, can be translated in the obvious way into formulas obtained from polynomial relations by using quantificators; e.g. if $\psi=\sum_i P_i\sqrt{\sum_j Q_{ij}^2}$, where P_i, Q_{ij} are polynomials, then

$$\psi(y)>0 \Leftrightarrow \forall\eta_i\Big[\eta_i^4=\sum_j Q_{ij}^2(y)\Rightarrow\sum_i P_i(y)\eta_i^2>0\Big].$$

In the sequel the formulas $\psi(y)>0$, etc., will always mean the polynomial relations with quantificators described above.

Now consider the following formula:

$$\forall x_1,\dots,x_n, t[P(x,t)=0,\ \delta^2(x)y\geqslant 1,\ \varphi(x,y,t)>0\Rightarrow t\geqslant 0].$$

Clearly it holds if we interpret the variables as real numbers. We shall show that it does not hold in the real closure L of K, and thus, with the help of Tarski's theorem, we shall obtain a contradiction.

Let $\lambda\colon K\to L$ be the canonical imbedding. We shall show that the elements $\bar{x}_1=\lambda(x_1),\dots,\bar{x}_n=\lambda(x_n)$, $\bar{y}=\lambda(y)$, $\bar{f}=\lambda(f)$ (where x_i, y are the coordinate functions on $U\times\boldsymbol{R}\times\boldsymbol{R}$) do not satisfy our formula. It is obvious that $P(\bar{x},\bar{f})=\lambda(P(x,f))=0$, $\delta^2(\bar{x})\bar{y}=\lambda(\delta^2(x)y)\geqslant 1$, $\bar{f}=\lambda(f)<0$. Thus we have to check that $\varphi(\bar{x},\bar{y},\bar{f})>0$.

The last formula is by definition equivalent to a sentence of the following form:

$$\forall\bar{\eta}_1\quad \bar{\eta}_1^4=\sum_i\Phi_{1i}^2(\bar{x},\bar{y},\bar{f})\ \&$$
$$\forall\bar{\eta}_2\quad \bar{\eta}_2^4=\sum_j\Phi_{2j}^2(\bar{x},\bar{y},\bar{f},\bar{\eta}_1^2)\ \&$$
$$\cdots\cdots\cdots\cdots$$
$$\Phi(\bar{x},\bar{y},\bar{f},\bar{\eta}_1^2,\bar{\eta}_2^2,\dots)>0,$$

where $\Phi_{1i}, \Phi_{2j}, \ldots, \Phi$ are polynomials such that

$$\sum_i \Phi_{1i}^2(x, y, t) > 0 \qquad \text{for all } (x, y, t) \in U \times \boldsymbol{R} \times \boldsymbol{R},$$

$$\sum_j \Phi_{2j}^2(x, y, t, u) > 0 \qquad \text{for all } (x, y, t, u) \in U \times \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}.$$

Thus we may choose $\bar{\eta}_1$ to be the image in L of the function

$$\eta_1(x, y) = \sqrt[4]{\sum \Phi_{1i}^2(x, y, f(x))} \in \mathfrak{I},$$

η_2 to be the image of

$$\eta_2(x, y) = \sqrt[4]{\sum \Phi_{2j}^2(x, y, f(x), \eta_1^2(x, y))}, \qquad \text{etc.}.$$

Then $\Phi(\bar{x}, \bar{y}, \bar{f}, \bar{\eta}_1^2, \bar{\eta}_2^2, \ldots) = \lambda(\Phi(x, y, f, \eta_1^2, \eta_2^2, \ldots))$ and the condition $\Phi(\bar{x}, \bar{y}, \bar{f}, \bar{\eta}_1^2, \bar{\eta}_2^2, \ldots) > 0$ means that $\varphi(x, y, f) > 0$ in the order of K. But for any $x, y \in U \times \boldsymbol{R}$ $\varphi(x, y, f(x)) > 0$, so it is a square.

This completes the proof of proposition 2.

The proof of proposition 3 follows similar lines, but is slightly more involved. We shall need the following lemma, which is a rough version of Łojasiewicz's «normal partitions» and «normal systems of distinguished polynomials».

LEMMA 9. *Let $U \subset \boldsymbol{R}^n$ be open semialgebraic, $f_0, f_1, \ldots, f_m \in \mathcal{N}(U)$. Then there exist a finite number of (finite) systems of polynomial equations and inequalities in $\boldsymbol{R}^n$, $\mathfrak{P}_1, \ldots, \mathfrak{P}_k$, and polynomials $P_{ij}(x, t)$ $(i = 1, \ldots, k, j = 0, \ldots, m)$ such that:*

1) *every $x \in U$ satisfies exactly one $\mathfrak{P}_i$ (written $\mathfrak{P}_i(x)$),*
2) *if $\mathfrak{P}_i(x)$, then $P_{ij}(x, f_j(x)) = 0$ and $\delta_{ij}(x) \neq 0$, $\delta_{ij}(x)$ being the discriminant of P_{ij}.*

The proof is an easy adaptation of Łojasiewicz's construction, [5], pp. 60-62.

Before proving proposition 3, we give a simple proof, based on Tarski's theorem, of the following Nullstellensatz for polynomials [6]. Let $\mathfrak{a} \subset \boldsymbol{R}[X_1, \ldots, X_n]$ be an ideal; then the following conditions are equivalent:

a) if $f \in \boldsymbol{R}[X_1, \ldots, X_n]$ vanishes on $V(\mathfrak{a})$, then $f \in \mathfrak{a}$;

b) if $\sum f_i^2 \in \mathfrak{a}$, then $f_i \in \mathfrak{a}$.

Of course if sufficies to prove that *b*) implies *a*).

We can assume $\mathfrak{a}$ is prime. In fact, any ideal in $\boldsymbol{R}[X_1, \ldots, X_n]$ is an intersection of primary ideals; since $\mathfrak{a}$ satisfies b), it is a radical and thus intersection of prime ideals: $\mathfrak{a} = \bigcap \mathfrak{p}_i$. It is easy to check that every $\mathfrak{p}_i$ satisfies b). Let f vanish on $V(\mathfrak{a})$. Then it vanishes on $V(\mathfrak{p}_i)$ and, assuming the Nullstellensatz for prime ideals, we get $f \in \mathfrak{p}_i$ for all i, i.e. $f \in \mathfrak{a}$.

Now let $f_1, \ldots, f_m$ generate $\mathfrak{a}$ and let f vanish on $V(\mathfrak{a})$. It follows from b) that the field of quotients of the (integral) ring $\boldsymbol{R}[X_1, \ldots, X_n]/\mathfrak{a}$ is real. Let K be its real closure.

The phrase

$$\forall x_1, \ldots, x_n\{[f_i(x_1, \ldots, x_n) = 0 \text{ for } i = 1. \ldots, m] \Rightarrow f(x_1, \ldots, x_n) = 0\}$$

holds in $\boldsymbol{R}$, so it holds also in K. Let $\bar{x}_1, \ldots, \bar{x}_n$ be the images in K of $X_1, \ldots, X_n$ under the natural homomorphism $\boldsymbol{R}[X_1, \ldots, X_n] \to K$. Clearly

$$f_i(\bar{x}_1, \ldots, \bar{x}_n) = \operatorname{im} f_i(X_1, \ldots, X_n) = 0 ,$$

so $f(\bar{x}_1, \ldots, \bar{x}_n) = 0$, which implies that $f \in \mathfrak{a}$.

We return to the proof of proposition 3. Again it suffices to prove the implication $b) \Rightarrow a)$. We can also assume that $\mathfrak{a}$ is prime and U is connected. Thus $\mathcal{N}(U)/\mathfrak{a}$ is integral and its field of quotients $(\mathcal{N}(U)/\mathfrak{a})^*$ is real. We order it arbitrarily.

Let $f_1, \ldots, f_m$ generate $\mathfrak{a}$. Let $f \in \mathcal{N}(U)$ vanish on $V(\mathfrak{a})$. Put $f_0 = f$ and let $\mathfrak{P}_i$, P_{ij} have the same meaning as in lemma 9.

Fix a set of polynomials in $\boldsymbol{R}[x_1, \ldots, x_n]$ which define U. Thus the formula $x \in U$ can be interpreted as a polynomial relation, and therefore makes sense in any real-closed extension of $\boldsymbol{R}$.

If $\varphi \in \mathcal{N}(U)$, then $[\varphi]$ will denote its class in $(\mathcal{N}(U)/\mathfrak{a})^*$.

From here the proof will be obtained via three steps.

Step 1. $([x_1], \ldots, [x_n]) \in U$.

Let K be the real closure of $(\mathcal{N}(U)/\mathfrak{a})^*$. For $\varphi \in \mathcal{N}(U)$ we shall denote by $\bar{\varphi}$ its image under the composition of the natural maps

$$\mathcal{N}(U) \to (\mathcal{N}(U)/\mathfrak{a})^* \to K .$$

It suffices to prove that $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in U$.

By lemma 6 and corollary, p. 14, there exist functions $\varphi_1, \varphi_2 \in \mathfrak{R}(\boldsymbol{R}^n \setminus (:\Gamma \setminus U))$ such that

$$\varphi_1 \geqslant 0 , \quad \varphi_1^{-1}(0) = \overline{U} \setminus U ,$$

$$\varphi_2 > 0 \quad \text{on } U , \qquad \varphi_2 < 0 \quad \text{on } \boldsymbol{R}^n \setminus \overline{U} .$$

Let $\varphi = \varphi_1\varphi_2$. Thus, for any $x \in \boldsymbol{R}^n$,

$$x \in U \Leftrightarrow \varphi(x) > 0 \ .$$

Now the formula $\varphi(x) > 0$ is equivalent to a formula of the following form:

$$\begin{aligned} &\forall \eta_1 \quad \eta_1^4 = \sum_i \Phi_{1i}^2(x) \ \& \\ &\forall \eta_2 \quad \eta_2^4 = \sum_j \Phi_{2j}^2(x, \eta_1^2) \ \& \\ &\cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ &\Phi(x, \eta_1^2, \eta_2^2, \ldots) > 0 \ , \end{aligned}$$

where $\Phi_{1i}, \Phi_{2j}, \ldots, \Phi$ are polynomials and

$$\begin{aligned} &\sum_i \Phi_{1i}^2(x) > 0 \qquad \text{for all } x \in U \ , \\ &\sum_j \Phi_{2j}^2(x, y) > 0 \quad \text{for all } (x, y) \in U \times \boldsymbol{R}, \text{ etc.} \end{aligned}$$

Thus $\varphi(\bar{x}) > 0$ means

$$\begin{aligned} &\forall \bar{\eta}_1 \quad \bar{\eta}_1^4 = \sum_i \Phi_{1i}^2(\bar{x}) \ \& \\ &\forall \bar{\eta}_2 \quad \bar{\eta}_2^4 = \sum_j \Phi_{2j}^2(\bar{x}, \bar{\eta}\text{-}) \ \& \\ &\cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ &\Phi(\bar{x}, \bar{\eta}_1^2, \bar{\eta}_2^2, \ldots) > 0 \ . \end{aligned}$$

We may choose $\bar{\eta}_1$ to be the image in K of the function

$$\eta_1(x) = \sqrt[4]{\sum_i \Phi_{1i}^2 (x)} \in \mathcal{N}(U) \ ,$$

$\bar{\eta}_2$ to be the image of

$$\eta_2(x) = \sqrt[4]{\sum_j \Phi_{2j}^2(x, \eta_1^2(x))} \ , \qquad \text{etc.}\,.$$

Then $\Phi(\bar{x}, \bar{\eta}_1^2, \bar{\eta}_2^2, \ldots) > 0$ if and only if the class $[\varphi]$ of $\varphi(x) \in \mathcal{N}(U)$ in $(\mathcal{N}(U)/\mathfrak{a})^*$ is positive. But φ is positive on U, so $\sqrt{\varphi} \in \mathcal{N}(U)$; therefore $[\varphi]$ is a square.

Step 2. We shall show that there exists exactly one s_0 such that $\mathfrak{P}_{s_0 i}(\bar{x})$; moreover, $\delta_{s_0 i}(\bar{x}) \neq 0$ for $i = 0, \ldots, m$.

In fact, the formula

$$\forall x [x \in U \Rightarrow \exists ! s \, \mathfrak{P}_s(x)]$$

holds in $\boldsymbol{R}$, so it holds also in K (the symbol $\exists ! s\, \mathfrak{P}_s(x)$ can be easily translated into usual quantificators: $\exists s\, \mathfrak{P}_s(x)\, \&\, \forall s_1, s_2[\mathfrak{P}_{s_1}(x)\, \&\, \mathfrak{P}_{s_2}(x) \Rightarrow s_1 = s_2]$). In particular, $\bar{x}$ satisfies exactly one $\mathfrak{P}_{s_0}$, since $\bar{x} \in U$.

To prove that $\delta_{s_0 i}(\bar{x}) \neq 0$ we consider the formula

$$\forall x\{[x \in U\, \&\, \mathfrak{P}_{s_0}(x)] \Rightarrow \delta_{s_0 i}(x) \neq 0\}$$

and argue in the same way.

In the sequel we shall write $\mathfrak{P}$ for $\mathfrak{P}_{s_0}$, P_i for $P_{s_0 i}$ and δ_i for $\delta_{s_0 i}$. Let $\delta = \prod_{i=0}^{m} \delta_i$.

Step 3. The generic point in $\boldsymbol{R}^{m+1}$ will be written as $t = (t_0, \ldots, t_m)$ and let

$$A = \{(x, t, y) \in U \times \boldsymbol{R}^{m+1} \times \boldsymbol{R}\colon t_i = f_i(x) \text{ for } i = 0, \ldots, m,\},$$
$$B = \{(x, t, y) \in U \times \boldsymbol{R}^{m+1} \times \boldsymbol{R}\colon P_i(x, t_i) = 0 \text{ for } i = 0, \ldots, m,\ t_i \neq f_i(x) \text{ for some } i,\ \delta^2(x) y \geqslant 1\}.$$

A and B are semialgebraic, disjoint and closed in $U \times \boldsymbol{R}^{m+1} \times \boldsymbol{R}$; let $\varphi \in \mathfrak{R}(U \times \boldsymbol{R}^{m+1} \times \boldsymbol{R})$ be such that

$$\varphi > 0 \quad \text{on } A, \qquad \varphi < 0 \quad \text{on } B.$$

Let L be the field of quotients of the ring $\mathfrak{T}$ defined as follows: Put $\tilde{\mathfrak{T}}_0 = \mathcal{N}(U)[y]$ and consider it as a ring of functions on $U \times \boldsymbol{R}$. Let

$\tilde{\mathfrak{T}}_{i+1} =$ the set of all functions of the form $\sum_k \varphi_k \sqrt{\sum_l \psi_{kl}^2}$, where $\varphi_k,\ \psi_{kl} \in \mathfrak{T}_i$ and $\sum_l \psi_{kl}^2 > 0$ on $U \times \boldsymbol{R}$, for all k,

$$\tilde{\mathfrak{T}} = \bigcup_{i=0}^{\infty} \tilde{\mathfrak{T}}_i,$$
$$\mathfrak{T} = \tilde{\mathfrak{T}}/\mathfrak{a}\cdot\tilde{\mathfrak{T}}.$$

By lemma 8, the field $(\mathcal{N}(U)/\mathfrak{a})^*(y)$ has an order extending the (arbitrarily chosen) order on $(\mathcal{N}(U)/\mathfrak{a})^*$ such that $\delta^2(x) y \geqslant 1$. It is easily seen that L can be obtained from $(\mathcal{N}(U)/\mathfrak{a})^*(y)$ by an inductive procedure of adding square roots sums of squares. Therefore we can extend the order on $(\mathcal{N}(U)/\mathfrak{a})^*$ to an order on L such that $\delta^2(x) y \geqslant 1$.

Consider the formula

$$\forall (x, t_0, y) \in U \times \boldsymbol{R} \times \boldsymbol{R}\colon \Big[P_0(x, t_0) = 0,\ P_i(x, 0) = 0 \text{ for } i = 1, \ldots, m,\ \delta^2(x) y \geqslant 1,\ \varphi(x, t_0, \underbrace{0, \ldots, 0}_{m}, y) > 0\Big] \Rightarrow t_0 = 0.$$

Let $\bar{L}$ be the real closure of L. If $\varphi \in \mathcal{N}(U)[y]$, then $\bar{\varphi}$ will denote its image under the composition

$$\mathcal{N}(U)[y] \to (\mathcal{N}(U)/\mathfrak{a})^*(y) \to L \to \bar{L}\,.$$

Our formula holds in $\bar{L}$ since it holds in $\boldsymbol{R}$.

It is obvious that $(\bar{x}, \bar{f}_0, \bar{y})$ satisfy:

$$P_0(\bar{x}, \bar{f}_0) = 0\,,$$

$$P_i(\bar{x}, 0) = P_i(\bar{x}, \bar{f}_i) = 0, \text{ since } \bar{f}_i = 0 \text{ for } i = 1, \ldots, m\,,$$

$$\delta^2(\bar{x})\bar{y} \geqslant 1\,.$$

The argument that $\varphi(\bar{x}, \bar{f}_0, 0, \ldots, 0, \bar{y}) > 0$ is exactly the same as the last part of the proof of proposition 2, so it will be omitted. Therefore $\bar{f}_0 = 0$ which implies $f_0 \in \mathfrak{a}$.

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