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**A relative embedding theorem for Stein spaces**

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# A Relative Embedding Theorem for Stein Spaces (*).

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## Introduction.

In this work we prove the following theorems:

THEOREM 1. Let $X$ be a reduced Stein space, $n = \dim X < \infty$, $Y$ a closed subspace of $X$ and $\varphi\colon Y \to \boldsymbol{C}^l$ an embedding with $l \geqslant 2n + 1$.

Then the set of all maps $f\colon X \to \boldsymbol{C}^l$ which extend $\varphi$ to $X$ and are proper, one—one and regular at each regular point of $X$ is dense in the space of all maps extending $\varphi$ to $X$.

THEOREM 2. Let $(X, \mathcal{O}_X)$ be a Stein space of dimension $n$ and locally of type $N$, possibly non reduced. Let $(Y, \mathcal{O}_Y)$ be a closed subspace of $X$ and $\varphi\colon Y \to \boldsymbol{C}^l$ an embedding with $l \geqslant n + N$. Then the set of all maps $f\colon X \to \boldsymbol{C}^l$ which extend $\varphi$ to $X$ and are embeddings of $X$ is dense in the space of all maps extending $\varphi$ to $X$.

A similar result is obtained in the real case.

Theorems 1 and 2 were proved by R. Narasimhan in [2] and K. Wiegmann in [5] in the case $Y = \emptyset$.

## 1. – Convex open sets and admissible systems.

Let $(X, \mathcal{O}_X)$ be a Stein analytic complex space; if $\dim_{\boldsymbol{C}} X = n < \infty$ and $X$ is locally of finite type $N$, by [2], [5] we can always embed $(X, \mathcal{O}_X)$ as a closed subspace of $\boldsymbol{C}^m$ for suitable $m$. Let us consider a fixed embedding

$$\vartheta\colon (X, \mathcal{O}_X) \hookrightarrow (\boldsymbol{C}^m, \mathcal{O}_m)\,.$$

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In this way the Fréchet topology of $\Gamma(X, \mathcal{O}_X)$ can be considered as the quotient of the compact-open topology of $\Gamma(\mathbf{C}^m, \mathcal{O}_m)$.

DEFINITION 1. An open set $U \subset X$ is called $X$-convex if for each compact $K \subset U$, the set

$$\hat{K} = \{x \in U\colon |f(x)| \leqslant \sup_K |f| \text{ for all } f \in \Gamma(X, \mathcal{O}_X)\}$$

is compact.

REMARK 1. Let $X_{\mathrm{red}}$ be the reduced space associated with $X$; an obvious consequence of definition 1 is that an open set $U \subset X$ is $X$-convex if and only if $U$ is $X_{\mathrm{red}}$-convex.

REMARK 2. Let $\Omega \subset \mathbf{C}^m$ be a $\mathbf{C}^m$-convex open set (usually called a Runge-open set). Then $\vartheta^{-1}(\Omega) = \Omega \cap X$ is $X$-convex. In fact since $\Gamma(\mathbf{C}^m, \mathcal{O}_m) \to \Gamma(X, \mathcal{O}_X)$ is surjective, if $K \subset \Omega \cap X$ is a compact set, then

$$\hat{K} = \left(\{x \in \Omega\colon |f(x)| \leqslant \sup_K |f| \forall f \in \Gamma(\mathbf{C}^m, \mathcal{O}_m)\}\right) \cap X \quad \text{is compact}.$$

If $X$ is a reduced space and $U \subset X$ is a $X$-convex open set it is known that each section $s \in \Gamma(U, \mathcal{O}_X)$ can be approximated by global sections (see for instance [1]). We give now a generalization of this:

THEOREM 1. *Let $\Omega \subset \mathbf{C}^m$ be a $\mathbf{C}^m$-convex open set and $U = \Omega \cap X$. Then the restriction map*

$$r\colon \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$$

*has dense image.*

PROOF. Let $g \in \Gamma(U, \mathcal{O}_X)$. Since $U$ is a closed Stein subspace of the Stein manifold $\Omega$, there exists $G \in \Gamma(\Omega, \mathcal{O}_m)$ such that $G|_U = g$.

Since $\Omega$ is $\mathbf{C}^m$-convex, $G$ can be approximated by a sequence $\{F_n\}$ of holomorphic functions defined on $\mathbf{C}^m$. Let $\{f_n\} \subset \Gamma(X, \mathcal{O}_X)$ be the sequence induced by $\{F_n\}$. Since $\Gamma(U, \mathcal{O}_X)$ has the quotient topology of $\Gamma(\Omega, \mathcal{O}_m)$, the sequence $\{f_n|_U\}$ converges to $g$.

DEFINITION 2. A locally-finite family $\{U_i\}_{i\in I}$ of relatively compact open sets in $X$ is called an admissible system, if it verifies the following properties:

(1) $U = \bigcup_{i\in I} U_i$ is $X$-convex;

(2) $U_i \cap U_j = \emptyset$ if $i \neq j$;

(3) is given a sequence $\{B_n\}$ of open sets such that:

$$B_n \subset\subset B_{n+1}, \quad \bigcup_n B_n = X, \quad B_n \cup U \text{ is } X\text{-convex for each } n.$$

Moreover we can suppose $U_i \subset B_n$ if $B_n \cap U_i \neq \emptyset$ (see [2]). In this case the sequence $\{B_n\}$ is called associated to $\{U_i\}$.

REMARK. If $\{\Omega_i\}$ is an admissible system for $C^m$ and $\{A_n\}$ is an associated sequence then $\{\Omega_i \cap X\}$ and $\{A_n \cap X\}$ are respectively an admissible system for $X$ and an associated sequence.

Lemmas 1 and 2, theorems 1 and 2 of [2] still hold for admissible systems obtained as in the remark.

Only theorem 1 of [2] needs some care. We have to show that it is possible to find $2n+1$ admissible systems $\{\Omega_i^\lambda\}$ $\lambda = 1, \dots, 2n+1$ such that

$$X = \bigcup_{\lambda=1}^{2n+1} \Big( \bigcup_{i\in I} \Omega_i^\lambda \cap X \Big).$$

Following the proof of Theorem 1 of [2] it is enough to prove that there is an admissible system $\{\Omega_i\}$ in $C^m$ such that $A = X - \bigcup_i (\Omega_i \cap X)$ is a real analytic set which doesn't contain any point in a chosen countable subset of $X$.

Consider a family of open cubes (1) $\{Q_h\}$ in $C^m$, with sides parallel to the real axes, such that if $\bar{Q}_h \cap \bar{Q}_k \neq \emptyset$ then $\bar{Q}_h \cap \bar{Q}_k$ is exactly a common face of some dimension. Then $C^m - \bigcup_i \Omega_i$ is the union of a countable family of real hyperplanes. We can clearly choose them so that their equations are satisfied by none of the points of the chosen set.

Now let $(Y, \mathcal{O}_Y)$ be a closed subspace of $(X, \mathcal{O}_X)$ and let $\varphi\colon Y \to C^l$ be *a* fixed embedding. We have $\varphi = (\varphi_1, \dots, \varphi_l) \in \Gamma(Y, \mathcal{O}_Y)^l$ and we suppose $l \geqslant n + N$.

We define $\Gamma(X)_\varphi \subset \Gamma(X, \mathcal{O}_X)^l$ to be the set of the maps: $X \to C^l$ extending $\varphi$. Clearly, since $Y$ is a Stein subspace, $\Gamma(X)_\varphi$ is a closed non void subset of $\Gamma(X, \mathcal{O}_X)^l$.

(1) For a cube in $C^m$ we mean a subset of $C^m$ defined by

$$\{x \in C^m\colon a_i < |\operatorname{Re} z_i| < b_i,\ c_i < |\operatorname{Im} z_i| < d_i\},$$

where $z_1, \dots, z_m$ are coordinates in $C^m$.

THEOREM 2. *Let $\Omega \subset \mathbf{C}^m$ be a $\mathbf{C}^m$-convex open set and $U = \Omega \cap X$. Let $F \in \Gamma(U, \mathcal{O}_X)$ be such that*

$$F|_{U\cap Y} = g|_{U\cap Y}$$

*where $g \in \Gamma(Y, \mathcal{O}_Y)$.*

*For each neighborhood $A$ of $F|_{U\cap Y}$ in $\Gamma(U, \mathcal{O}_X)$, there exists a section $h \in \Gamma(X, \mathcal{O}_X)$ such that $h|_Y = g$ and $h|_U \in A$.*

PROOF. Since $Y$ is a Stein subspace of $X$, there is a section $G \in \Gamma(X, \mathcal{O}_X)$ such that $G|_Y = g$: therefore we need only to prove that it is possible to approximate the section $f = F - G \in \Gamma(U, \mathcal{O}_X)$, which vanishes on $U \cap Y$, with a section in $\Gamma(X, \mathfrak{J}_Y)$, where $\mathfrak{J}_Y$ denotes the ideal sheaf defining $Y$.

Let us firstly suppose $U$ to be relatively compact in $X$.

Because of theorem $A$, for each $y \in Y$ the stalk $\mathfrak{J}_{Y,y}$ is generated by a finite number of sections in $\Gamma(X, \mathfrak{J}_Y)$. Since $U$ is relatively compact we can choose $t_1, \ldots, t_q \in \Gamma(X, \mathfrak{J}_Y)$ generating $\mathfrak{J}_{Y,y}$ for each $y \in U$. These sections define a surjective homomorphism of sheaves:

$$\mathcal{O}^q_X|_U \xrightarrow{t_1,\ldots,t_q} \mathfrak{J}_Y|_U .$$

Because of theorem $B$ the restriction homomorphism between $\Gamma(U, \mathcal{O}_X)^q$ and $\Gamma(U, \mathfrak{J}_Y)^q$ is again surjective. In particular we get:

$$f = \sum_{i=1}^{q} \alpha_i t_i$$

where $\alpha_i \in \Gamma(U, \mathcal{O}_X|_U)$. For each open neighborhood $A$ of $f|_U$, we can obviously find neighborhoods $A_i$ of $\alpha_i$ in such a way that if $\beta_i \in A_i$ then $\sum \beta_i t_i \in A$. Because of theorem 1 for each $i$ there is $\gamma_i \in \Gamma(X, \mathcal{O}_X)$ such that $\gamma_i|_U \in A_i$; therefore $\tilde{f} = \sum_{i=1}^{q} \gamma_i t_i$ is such that $\tilde{f}|_U \in A$.

If $U$ is not relatively compact, consider any neighborhood $A$ of $f|_U$. Since $\Gamma(U, \mathcal{O}_X)$ has the quotient topology of the compact-open topology on $\Gamma(U, \mathcal{O}_m)$ for each extension $h \in \Gamma(\Omega, \mathcal{O}_m)$ of $f|_U$ there is a compact $K \subset \Omega$ and a constant $\varepsilon > 0$ such that if $s \in \Gamma(\mathbf{C}^m, \mathcal{O}_m)$ and $\|s - h\|_k < \varepsilon$ then $s|_U \in A$.

Let $\Omega' \subset\subset \Omega$ be a $\mathbf{C}^m$-convex open neighborhood of $K$ and let $t_1, \ldots, t_q \in \Gamma(\mathbf{C}^m, \mathfrak{J}_Y)$ be global generators of $\mathfrak{J}_{Y,y}$ for each $y \in \Omega'$. Then $h|_{\Omega'} = \sum_{i=1}^{q} \alpha_i t_i$. Since $\Omega'$ is convex we can approximate $\alpha_i$ on $K$ with $\beta_i \in \Gamma(\mathbf{C}^m, \mathcal{O}_m)$, in such a way that $\|\sum \beta_i t_i - \sum \alpha_i t_i\|_k < \varepsilon$. Define $\tilde{f} = \left(\sum_{i=n}^{q} \beta_i t_i\right)\Big|_X$ and, by construction, $\tilde{f}|_U \in A$.

REMARK. If $(X, \mathcal{O}_X)$ is a reduced space, theorem 2 holds for any $X$-convex open set $U$. In fact we can apply the proof of theorem 2 without considering $\Omega$ and $\Omega'$. Since in the reduced case the topology of $\Gamma(X, \mathcal{O}_X)$ is again the compact-open topology, it is sufficient to approximate $f$ in the topology of $\Gamma(V, \mathcal{O}_X)$ where $V$ is any relatively compact $X$-convex open set containing the compact set $K$ which defines the neighborhood $A$.

THEOREM 3. *Let $\{\Omega_i\}$ be an admissible system for $C^m$ and $U_i = \Omega_i \cap X$. Let $g$ be in $\Gamma(X, \mathcal{O}_X)$ and $f_i \in \Gamma(U_i, \mathcal{O}_X)$ be such that*

$$f_i|_{U\cap Y} = g|_{U\cap Y}.$$

*For any choice of open neighborhoods $A_i$ of $f_i$ in $\Gamma(U_i, \mathcal{O}_X)$ for each $i$, there is a section $f \in \Gamma(X, \mathcal{O}_X)$ such that:*

1) $f|_Y = g|_Y$
2) $f|_{U_i} \in A_i$ *for each* $i$.

PROOF. Suppose firstly that $(X, \mathcal{O}_X)$ is reduced (2) and the neighborhoods $A_i$ are given by compact sets $K_i$ and constants $\varepsilon_i$. Let $\{B_n\}_{n\in N}$ be associated to the admissible system $\{U_i\}$. Let $U_i \subset B_1$ for $i \leqslant i_1$, $U_i \subset B_n - B_{n-1}$ for $i_{n-1} < i \leqslant i_n$.

Let $K'_n \subset B_n$ be a compact set such that $B_{n-1} \subset K'_n$ and $K_i \subset K'_n$ for $i \leqslant i_n$. Choose $\delta_n$ such that $\sum_{\mu=n+1}^{\infty} \delta_\mu < \frac{1}{2} \min_{i\leqslant i_n} \varepsilon_i$. Because of theorem 2, since $U = \bigcup_{i\in I} U_i$ is $X$-convex, there is $F_1 \in \Gamma(X, \mathcal{O}_X)$ such that $F_1|_Y = g|_Y$ and $\|F_1 - f_i\|_{K_i} < \frac{1}{2}\varepsilon_i$ for $i \leqslant i_1$. Since $B_1 \cup U$ is $X$-convex there is $F_2 \in \Gamma(X, \mathcal{O}_X)$ such that $F_2|_Y = g_Y$, $\|F_2 - F_1\|_{K_1} < \delta_1$ and $\|F_2 - f_i\|_{K_i} < \frac{1}{2}\varepsilon_i$ for $i_1 < i \leqslant i_2$ and so on.

Since the constructed sequence converges uniformely on compact sets, its limit $f = \lim_{n\to\infty} F_n$ is in $\Gamma(X, \mathcal{O}_X)$. It is clear that $\|f - f_i\|_{K_i} < \varepsilon_i$ and by construction $f|_Y = g|_Y$.

If $X$ is not reduced let $\tilde{g} \in \Gamma(C^m, \mathcal{O}_m)$ be an extension of $g$ and $\tilde{f}_i \in \Gamma(\Omega_i, \mathcal{O}_m)$ be extensions of $f_i$. Fix compact sets $K_i \subset \Omega_i$ and constants $\varepsilon_i$ in such a way that by the open map

$$r_i\colon \Gamma(\Omega_i, \mathcal{O}_m) \to \Gamma(U_i, \mathcal{O}_X)$$

we have $r_i(\{h \in \Gamma(\Omega_i, \mathcal{O}_m) \mid \|h - \tilde{f}_i\|_{K_i} < \varepsilon_i\}) \subset A_i$.

(2) In the reduced case our proof is the proof of theorem 2 of [2].

Applying the previous proof to $(\boldsymbol{C}^m, \mathcal{O}_m)$, $\{\Omega_i\}$, $\tilde{g}$ and $\tilde{f}_i$, we can find $\tilde{f} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ such that $\tilde{f}|_Y = g|_Y$ and $\|\tilde{f} - \tilde{f}_i\|_{K_i} < \varepsilon_i$ for each $i$. Defining $f = \tilde{f}|_X$ we complete the proof.

LEMMA 1. *Let $N$ be a fixed integer and $f \in \Gamma(Y, \mathcal{O}_Y)^N$. For each compact set $K \subset X$ let $A(K) \subset \Gamma(X, \mathcal{O}_X)^N$ be a family with the following properties:*

(1) *$A(K)$ is dense in $\Gamma(X, \mathcal{O}_X)^N$.*

(2) *$A(K) \cap \Gamma(X)_f$ is dense in $\Gamma(X)_f$* (3).

(3) *If $K \subset \mathring{K}'$ then $A(K) \supset A(K')$.*

(4) *If $K'$ is a compact neighborhood of $K$ in $\boldsymbol{C}^m$ and $\tilde{f} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ is such that $f = \tilde{f}|_X \in A(K)$, there is an $\varepsilon > 0$ such that if $\|\tilde{g} - \tilde{f}\|_{K'} < \varepsilon$ then $g = \tilde{g}|_X \in A(K)$. This implies that $A(K)$ is open in $\Gamma(X, \mathcal{O}_X)^N$.*

*Let $\{\Omega_i^\lambda\}$, $\lambda = 1, \ldots, N$, be admissible systems in $\boldsymbol{C}^m$, $U_i^\lambda = \Omega_i^\lambda \cap X$; fix $g \in \Gamma(X)_f$; for any choice of an open neighborhood $A$ of $g$ in $\Gamma(X, \mathcal{O}_X)^N$ and open neighborhoods $A_i^\lambda$ of $g_\lambda|_{U_i^\lambda}$ in $\Gamma(U_i^\lambda, \mathcal{O}_X)$, there is $\varphi \in \Gamma(X)_f$ such that $\varphi \in A$, $\varphi_\lambda|_{U_i^\lambda} \in A_i^\lambda$ for $\lambda = 1, \ldots, N$ and $\varphi \in A(K)$ for each compact set $K \subset X$.*

PROOF. Suppose firstly $(X, \mathcal{O}_X)$ is reduced (4). Let the open neighborhoods $A$, $A_i^\lambda$ be given by compact sets $C \subset X$, $K_i^\lambda \subset U_i^\lambda$ and constants $\varepsilon$, $\varepsilon_i > 0$.

Consider a sequence $\{B_n^\lambda\}_{n \in N}$ associated to the admissible system $\{U_i^\lambda\}$, $\lambda = 1, \ldots, N$. Let $'K_n^\lambda$ be a compact neighborhood of $B_{n-1}^\lambda$ in $B_n^\lambda$ such that if $K_i^\lambda \subset B_n^\lambda$ then $K_i^\lambda \subset {'K_n^\lambda}$. Let $C_n^\lambda$ be a compact neighborhood of $'K_n^\lambda$ in $B_n^\lambda$. Define $K_n' = \bigcap_{\lambda=1}^{N} {'K_n^\lambda}$, $C_n = \bigcap_{\lambda=1}^{N} C_n^\lambda$. Suppose $U_i^\lambda \subset B_n^\lambda - B_{n-1}^\lambda$ for $i_{n-1}^\lambda < i \leqslant i_n^\lambda$.

Clearly $K_n' \subset (\mathring{K}_{n+1}')$, $\bigcup_n K_n' = X$. If $f \in A(K_n')$ for all $n$ then, by property (3), $f \in A(K)$ for all compact subsets $K$ of $X$.

We may suppose $C \subset K_1'$ and $\varepsilon < \varepsilon_i$ for $i \leqslant \max_\lambda i_1^\lambda$.

$A(K_1')$ is dense; therefore we can find $f^1 = (f_1^1, \ldots, f_N^1) \in A(K_1') \cap \Gamma(X)_f$ such that $\|f^1 - g\|_{K_1'} < \frac{1}{2}\varepsilon$ and $\|f_\lambda^1 - g_\lambda\|_{K_i^\lambda} < \frac{1}{2}\varepsilon < \frac{1}{2}\varepsilon_i$ for $i \leqslant i_1^\lambda$. By property (4) there is a $\delta_1$ such that if $\|F - f^1\|_{C_1} < \delta_1$ then $F \in A(K_1')$. Since $B_1^\lambda \cup U^\lambda$ is $X$-convex (here $U^\lambda = \bigcup U_i^\lambda$) there is $'f^2 \in \Gamma(X)_f$ such that $\|'f_\lambda^2 - f_\lambda^1\|_{C_1^\lambda} < \frac{1}{4}\delta_1$ (therefore $'f^2 \in A(K_1')$) and $\|'f_\lambda^2 - g_\lambda\|_{K_i^\lambda} < \frac{1}{4}\varepsilon_i$ for $i_1^\lambda < i \leqslant i_2^\lambda$. Since $A(K_2')$ is dense there is $f^2 = (f_1^2, \ldots, f_N^2) \in \Gamma(X)_f \cap A(K_2')$ such that $\|f^2 - f^1\|_{C_1} < \frac{1}{2}\delta_1$, $\|f_\lambda^2 - g_\lambda\|_{K_i^\lambda} < \frac{1}{2}\varepsilon_i$ for $i \leqslant i_2^\lambda$. There is $\delta_2$, with $0 < \delta_2 < \frac{1}{2}\delta_1$, such that if

(3) We recall that $\Gamma(X)_f$ is the set of all extensions of $f$ to $X$.

(4) The proof of the reduced case is, with slight differences, the proof of lemma 2 of [2].

$\|F - f^2\|_{C_2} < \delta_2$ then $F \in A(K_2')$. Iterating this process we get a sequence $\{f^n\}$ with the following properties: $f^n \in A(K_n') \cap \Gamma(X)_f$ for all $n$; $\|f^n - f^{n-1}\|_{C_{n-1}} < \frac{1}{2}\delta_{n-1}$; $\|f_\lambda^n - g_\lambda\|_{K_i^\lambda} < \frac{1}{2}\varepsilon_i$ for $i \leqslant i_n^\lambda$. Let us define $\varphi = \lim_{n\to\infty} f^n$; then $\varphi \in \Gamma(X)_f$. Since $\delta_n < \frac{1}{2}\delta_{n-1}$ we have $\sum_{\mu=n+1}^{\infty} \delta_\mu < \delta_n$; so we obtain $\|\varphi - f^n\|_{C_n} < \delta_n$ and therefore $\varphi \in A(K_n')$ for all $n$, i.e. $\varphi \in A(K)$ for every $K$. Moreover if $\delta_n$ is chosen sufficiently small, we have $\|\varphi - g\|_C < \varepsilon$ and $\|g_\lambda - \varphi_\lambda\|_{K_i^\lambda} < \varepsilon_i$.

If $X$ is not reduced, consider the restriction map $r$: $\Gamma(\boldsymbol{C}^m, \mathcal{O}_m)^N \to \Gamma(X, \mathcal{O}_X)^N$. Since $r$ is open $r^{-1}(A(K))$ is dense in $\Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ and $r^{-1}(A(K)) \cap \Gamma(\boldsymbol{C}^m)_f$ is dense in $\Gamma(\boldsymbol{C}^m)_f$. If $K \not\subset X$, $K$ compact subset of $\boldsymbol{C}^m$, let us define $A'(K) = r^{-1}(A(K \cap X))$. It is easy to verify properties (1), ..., (4) for the family $\{A'(K)\}$. If $g \in \Gamma(X)_f$, fix any extension $\tilde{g} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)^N$ of $g$. Applying lemma 1 for $\tilde{g}$ to the reduced space $\boldsymbol{C}^m$ we prove lemma 1 in the general case.

LEMMA 2. *Let $\varphi \in \Gamma(Y, \mathcal{O}_Y)^l$ be an embedding and $\phi \in \Gamma(X)_\varphi$. There is an open neighborhood $V$ of $Y$ in $X$ such that $\phi|_V$ is a proper map.*

PROOF. Let $\{P_n\}$ be a sequence of concentric polidiscs invading $\boldsymbol{C}^m$. Define $K_n = P_n \cap X$ and $H_n = K_n \cap Y$. Choose a subsequence $\{H_{n_i}\}$ such that $|\varphi_\lambda(x)| \geqslant i$ for each $x \in \bar{H}_{n_i} - H_{n_{i-1}}$. Each compact $\bar{H}_{n_i} - H_{n_{i-1}}$ has an open neighborhood $U_i$ in $X$ such that $\inf_{U_i} |\phi_\lambda(x)| \geqslant i - \frac{1}{2}$. Take $V_i = U_i \cap K_{n_{i+1}}$; $V_i$ is a relatively compact open neighborhood of $\bar{H}_{n_i} - H_{n_{i-1}}$ and clearly $\inf_{V_i} |\phi_\lambda(x)| \geqslant i - \frac{1}{2}$. Therefore $V = \bigcup V_i$ has the required property.

For an extension $\phi$ of an embedding $\varphi \in \Gamma(Y, \mathcal{O}_Y)^l$ we give the following

DEFINITION 3. An admissible system $\{U_i\}$ is called relative to $\phi$ if there is an open neighborhood $V$ of $Y$ in $X$ such that:

(1) $\phi|_V$ is a proper map;

(2) If $U_h \cap Y = \emptyset$ then $U_h \subset V$.

LEMMA 3. *Let $\phi \in \Gamma(X)_\varphi$; there is an admissible system $\{\Omega_i\}$ in $\boldsymbol{C}^m$ such that $\{U_i = \Omega_i \cap X\}$ is relative to $\phi$.*

PROOF. Let $V$ be an open neighborhood of $Y$ in $X$ such that $\varphi|_V$ is a proper map. We want to construct a countable family of open cubes $\{Q_h\}$ in $\boldsymbol{C}^m$, as done after the remark to definition 2, with the following properties:

(1) $\boldsymbol{C}^m = \bigcup_{h=0}^{\infty} \bar{Q}_h$.

(2) $Q_h \cap Q_{h'} = \emptyset$ if $h \neq h'$.

(3) If $\bar{Q}_h \cap \bar{Q}_{h'} = F \neq \emptyset$ then $F$ is exactly a face of one of them.

(4) There is a sequence $\{P_n\}$ of open cubes such that: $\boldsymbol{C}^m = \bigcup_{n=0}^{\infty} P_n$; $P_n \subset\subset P_{n+1}$; $\bar{P}_0 = \bigcup_{h=0}^{q_0} \bar{Q}_h; \ldots; \bar{P}_{k+1} = \bigcup_{h=q_k+1}^{q_{k+1}} Q_h; \ldots$.

For this purpose let us consider a sequence of open cubes $\{P_n\}$ invading $C^m$. If $K_n = P_n \cap X$ and $H_n = K_n \cap Y$, take a subsequence $\{H_{n_i}\}$ as in lemma 2. Divide $P_{n_0}$ into cubes $Q_h$, $h = 0, \ldots, q_0$ so small that if $Q_h \cap H_{n_0} \neq \phi$ then $Q_h \subset V$. Divide $P_{n_1} - \bar{P}_{n_0}$ in small cubes $Q_h$, $h = q_0 + 1, \ldots, q_1$, in such a way that property (3) holds and if $Q_h \cap H_{n_1} \neq \phi$ then $Q_h \subset V$; and so on.

Since $\{Q_h\}$ is an admissible system, $\{Q_h \cap X\}$ is an admissible system and by construction it is relative to $\phi$. Moreover it is easy to show, with the same argument used before, that $X$ can be covered by $2n+1$ admissible systems obtained in this way and relative to $\phi$.

## 2. – The first relative embedding theorem.

We want to prove the relative case of theorem 5 of [2]; in other words we try to find in $\Gamma(X)_\varphi$ all the maps of $X$ into $C^l$ which are one-one, proper and regular in each regular point of $X$.

Since a non reduced space might have no regular points, from now on $X$ will be supposed a reduced space.

We want to prove the following theorem:

THEOREM 4. *Let $X$ be a reduced Stein space, $Y$ a closed subspace of $X$ and $\varphi\colon Y \to \boldsymbol{C}^l$ an embedding. Suppose $1 \geqslant 2n+1$, where $n = \dim_{\boldsymbol{C}} X$.*

*Then the set of all maps which are proper, one-one and regular in each regular point of $X$ is dense in $\Gamma(X)_\varphi$.*

*If $l < n+1$ the same result is obtained by adding to $(\varphi_1, \ldots, \varphi_l)$ $n+1-l$ arbitrary components.*

In order to prove this theorem let us give some definitions, following [2]. Let $S$ be the set of singular points of $X$. Then, if $f$ is any section in $\Gamma(X, \mathcal{O}_X)^k$, we define:

$$X(f, m) = \{x \in X - S \mid \text{rank of } f \text{ in } x \leqslant m\} \text{ for } m \leqslant n.$$

$M(f)$ = union of all the irreducible components different from the diagonal of $\{(x, y) \in X \times X \mid f(x) = f(y)\}$.

Here, if $x$ is a regular point of $X$, for rank of $f$ in $x$ we mean the rank of the jacobian matrix of $f$.

If $f \in \Gamma(X)_\varphi$ then, for each $x \in Y - S$, rank of $f$ in $x$ is greater or equal to the dimension of the Zarisky tangent space to $Y$ in $x$; in particular the equality holds if this dimension is equal to $\dim_x = X$. Moreover, since $f|_Y = \varphi$ is one-one, $M(f) \cap Y \times Y = \emptyset$.

We now need the following lemmas.

LEMMA 4. *Let $x_0, \ldots, x_p \in X$, $x_1, \ldots, x_p \notin Y$. Then the set of sections $f \in \Gamma(X, \mathfrak{J}_Y)$ separating $x_0, \ldots, x_p$ is dense in $\Gamma(X, \mathfrak{J}_Y)$.*

(Here $\mathfrak{J}_Y$ is the ideal sheaf of $\mathcal{O}_X$ which defines $Y$).

PROOF. Consider the exact sequence of coherent sheaves:

$$0 \to \mathfrak{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathfrak{J} \to 0$$

where $\mathfrak{J}$ is the sheaf of germs of holomorphic functions vanishing on $Y$ and in $x_0, \ldots, x_p$. Since $\mathfrak{J}$ is coherent, because of theorem $B$, the map:

$$\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X/\mathfrak{J})$$

is surjective. But $\Gamma(X, \mathcal{O}_X/\mathfrak{J}) = \Gamma(Y, \mathcal{O}_Y) \underbrace{\oplus \boldsymbol{C} \oplus \ldots \oplus \boldsymbol{C}}_{k \text{ times}}$, where $k = p$ if $x_0 \in Y$, $k = p + 1$ if $x_0 \notin Y$. Then there is a function $f$ such that $f(x_i) = i$, $i = 0, \ldots, p$. If $h$ is any section in $\Gamma(X, \mathfrak{J}_Y)$, then $h + \varepsilon f$ does not separate $x_0, \ldots, x_p$ only if $\varepsilon$ takes a finite number of values. If $\varepsilon$ is less than all these values then $h + \varepsilon f$ separates $x_0, \ldots, x_p$ and approximates $h$.

LEMMA 5. *Let $x_1, \ldots, x_p \in X - S$ be such that if $x_i \in Y$ then the dimension of the Zarisky tangent space $\mathcal{T}(Y)_{x_i}$ to $Y$ in $x_i$ is less than $n = \dim_{\boldsymbol{C}} X$. Let $f_1, \ldots, f_k$ be holomorphic functions on $X$ such that $f_i|_Y = \varphi_i$ for each $i \leqslant l$. Then the set of $f \in \Gamma(X, \mathcal{O}_X)$ such that:*

(1) $f|_Y = \varphi_{k+1}$ (if $k + 1 \leqslant l$).

(2) *rank of $(f_1, \ldots, f_k, f)$ in $x_i >$ rank of $(f_1, \ldots, f_k)$ in $x_i$,*

*is dense in $\Gamma(X)_{\varphi_{k+1}}$.*

PROOF. For each $x_i \notin Y$ we choose $n$ sections $z_1^i, \ldots, z_n^i \in \Gamma(X, \mathfrak{J}_Y)$ giving a system of local coordinates in a neighborhood of $x_i$. If $x_i \in Y$ we choose $z_1^i, \ldots, z_q^i \in \Gamma(X, \mathfrak{J}_Y)$ giving a minimal system of equations for $\mathcal{T}(Y)_{x_i}$ in a neighborhood of $x_i$. Then if $\xi$ is a suitable linear combination of all these sections

$$\operatorname{rank}(f_1, \ldots, f_k, \xi) \text{ in } x_i > \operatorname{rank}(f_1, \ldots, f_k) \text{ in } x_i .$$

(This is not true only for a finite number of values of the coefficients of $\xi$). If $h$ is any section in $\Gamma(X)_{\varphi_{k+1}}$, consider $h + \varepsilon\xi$. Then

$$\operatorname{rank}(f_1, \ldots, f_k, h + \varepsilon\xi) \text{ in } x_i = \operatorname{rank}(f_1, \ldots, f_k) \text{ in } x_i$$

only for a finite number of values. Taking $\varepsilon$ less than their minimum we get the thesis.

LEMMA 6. *Let* $(f_1, \ldots, f_k) \in \Gamma(X, \mathcal{O}_X)^k$, $k \geqslant 0$, *be a holomorphic map verifying the following conditions*:

1) $f_i|_Y = \varphi_i$ *for each* $i \leqslant l$.

$a_k$) $\dim_{\mathbf{C}} M(f_1, \ldots, f_k) \leqslant 2n - k$.

$b_k$) $\dim_{\mathbf{C}} X(f_1, \ldots, f_k, m) \leqslant n - k + m$ *for each* $0 \leqslant m < n$.

*Let* $h \in \Gamma(X)_{\varphi_{k+1}}$ *be a holomorphic function. For any choice of an admissible system* $\{U_i\}$ *relative to* $h$, *compact sets* $K_i \subset U_i$, *a compact set* $C \subset X$ *and constants* $\varepsilon_i$, $\varepsilon > 0$ *there is a function* $f \in \Gamma(X, \mathcal{O}_X)$ *such that* $f|_Y = h|_Y$, $\|f - h\|_{K_i} < \varepsilon_i$, $\|f - h\|_C < \varepsilon$ *and moreover the map* $(f_1, \ldots, f_k, f)$ *verifies* $a_{k+1}$ *and* $b_{k+1}$.

PROOF. We can suppose that $\varphi_1, \ldots, \varphi_l$ are ordered in such a way that $(\varphi_1, \ldots, \varphi_k)$ verifies $a_k$ and $b_k$ relatively to $Y$ for each $k \leqslant l$. Obviously in this case, if $m =$ rank of $(\varphi_1, \ldots, \varphi_k)$ in $y \in Y$, then $0 \leqslant m \leqslant \dim \mathcal{T}(Y)_y$. If $y \in Y$ is a regular point of $X$, then $\dim \mathcal{T}(Y)_y \leqslant n$ and so $0 \leqslant m \leqslant n$.

Let $M_q$ be one of the irreducible components of $M(f_1, \ldots, f_k)$ whose dimension is $2n - k$. It cannot be $M_q \subset Y \times Y$, since in this case we should have

$$\dim M_q \leqslant 2 \dim Y - k < 2n - k .$$

Therefore we can choose a point $(x_q, y_q) \in M - \Delta - Y \times Y$.

Let $X_m$ be the union of those irreducible components $X_m^v$ of $X(f_1, \ldots, f_k, m)$ whose dimension is $n - k + m$. We choose a point $x_m^v \in X_m^v$ in such a way that $x_m^v \notin S \cup Y$. This is possible since if $X_m^v - S \subset Y$ then

$$\dim X_m^v \leqslant \dim Y + m - k < n + m - k .$$

Both the set $\{(x_q, y_q)\} \subset X \times X$ and the set $\{x_m^v\} \subset X$ are discrete.

For each compact set $K \subset X$ we define $A(K)$ as follows:

$$g \in A(K) \text{ if and only if } \begin{cases} 1)\ g(x_q) \neq g(y_q) & \text{for each } (x_q, y_q) \in K \times K \\ 2)\ \text{rank of } (f_1, \ldots, f_k, g) \text{ in } x_m^p = m+1 \\ & \text{for each } x_m^p \in K. \end{cases}$$

By the results of [2], $A(K)$ is dense in $\Gamma(X, \mathcal{O}_X)$; it is dense in $\Gamma(X)_{\varphi_{k+1}}$ by lemmas 1 and 2; all the other hypothesis of lemma 1 are satisfied.

Therefore we can find $g \in \Gamma(X)_{\varphi_{k+1}}$ such that $\|g - h\|_{K_i} < \varepsilon_i$, $\|g - h\|_C < \varepsilon$ and $g \in A(K)$ for each compact set $K \subset X$. Therefore $g(x_q) \neq g(y_q)$ for any $q$ and $\operatorname{rank}(f_1, \ldots, f_k, g)$ in $x_m^p$ is equal to $m+1$ for any $p$ and $m$. Then we get:

1) $\dim M(f_1, \ldots, f_k, g) < \dim M(f_1, \ldots, f_k)$ since every component of $M(f_1, \ldots, f_k, g)$ different from $\Delta$ must be contained in some component of $M(f_1, \ldots, f_k)$ and cannot be one of those of greatest dimension.

2) $\dim X(f_1, \ldots, f_k, g, m) < \dim X(f_1, \ldots, f_k, m)$ since every component of the first one is contained in some component of the second one and cannot be one $X_m^p$ because $(f_1, \ldots, f_k, g)$ has rank $m+1$ in $x_m^p \in X_m^p$.

Then $(f_1, \ldots, f_k, g)$ satisfies $a_{k+1}$ and $b_{k+1}$ and the lemma is proved.

PROOF OF THEOREM 4. Let us fix an element $\phi \in \Gamma(X)_\varphi$, $\phi = (\phi_1, \ldots, \phi_l)$. Suppose $l \geqslant 2n+1$ (if not it is sufficient to add arbitrarily to $(\phi_1, \ldots, \phi_l)$ $2n+1-l$ components). Choose $l$ admissible systems $\{U_i^\lambda\}$ $\lambda = 1, \ldots, l$ relative to $\phi$. Choose compact sets $K_i^\lambda \subset U_i^\lambda$ in such a way that $X = \bigcup_{i,\lambda} K_i^\lambda$. Because of theorem 3 there is $h = (h_1, \ldots, h_l) \in \Gamma(X)_\varphi$ such that:

1) $\|h_\lambda - \varphi_\lambda\|_C < \frac{1}{2}\varepsilon$.

2) If $K_i^\lambda \cap Y \neq \emptyset$ then $\|h_\lambda - \phi_\lambda\|_{K_i^\lambda} < \frac{1}{2}$.

3) If $K_i^\lambda \cap C = \emptyset$, $K_i^\lambda \cap Y = \emptyset$ then $\|h_\lambda\|_{K_i^\lambda} \geqslant i+1$.

Since conditions $a_0$ and $b_0$ are void, a function $f_1 \in \Gamma(X, \mathcal{O}_X)_{\varphi_1}$ exists such that $\dim M(f_1) \leqslant 2n-1$, $\dim X(f_1, m) \leqslant n-1+m$ for $0 \leqslant m < n$. We can successively choose functions $f_2 \in \Gamma(X)_{\varphi_2}, \ldots, f_l \in \Gamma(X)_{\varphi_l}$ in such a way that: $\dim M(f_1, \ldots, f_k) \leqslant 2n-k$; $\dim X(f_1, \ldots, f_k, m) \leqslant n-k+m$ for $0 \leqslant m < n$; $\|f_\lambda - h_\lambda\|_{K_i^\lambda} < \frac{1}{2}$; $\|f_\lambda - h_\lambda\|_C < \frac{1}{2}\varepsilon$.

Therefore the map $F = (f_1, \ldots, f_l)$ verifies the following properties:

1) $\|\phi - F\|_C < \varepsilon$;

2) Since $\dim M(F) \leqslant 2n-1 \leqslant -1$ and $\dim X(F, m) \leqslant n-1+m \leqslant -1$, the map $F$ is one-one and regular on the regular points of $X$;

3) It is a proper map: in fact if $\{x_n\} \subset X$ is a divergent sequence, then either $\{x_n\}$ is definitively in $\bigcup_{K_i^\lambda \cap Y \neq \phi} K_i^\lambda$ or it is definitively outside; in the first case $\{F(x_n)\}$ diverges, since $\{\phi(x_n)\}$ does so and $\|F_\lambda - \phi_\lambda\|_{K_i^\lambda} < 1$, in the second case $\{F(x_n)\}$ diverges because $\|F_\lambda\|_{K_i^\lambda} > i$.

## 3. – The second relative embedding theorem.

From now on the Stein space $(X, \mathcal{O}_X)$ will not be supposed reduced. We want to show the following theorem:

THEOREM 5. *Let $(X, \mathcal{O}_X)$ be a Stein space of dimension $n$ and locally of type $N$. Let $(Y, \mathcal{O}_Y)$ be a closed subspace and $\varphi = (\varphi_1, \ldots, \varphi_l) \in \Gamma(Y, \mathcal{O}_Y)^l$ a fixed embedding; we may suppose $l \geqslant n + N$ (5). Then the set of all maps of $X$ into $\boldsymbol{C}^l$ which are embeddings is dense in the space $\Gamma(X)_\varphi = \{f \in \Gamma(X, \mathcal{O}_X)^l | f|_Y = \varphi\}$.*

Let us recall the following

DEFINITION 4. A map $f: X \to \boldsymbol{C}^l$ is called regular at the point $x \in X$ if $f_x$ can be extended to a map $\tilde{f}_x$ defined in an open neighborhood of $x$ in the Zarisky tangent space $\mathcal{T}(X)_x$ such that the jacobian matrix of $\tilde{f}_x$, which is a morphism between two complex manifolds, has rank at $x$ equal to $\dim \mathcal{T}(X)_x$.

In this case we shall say that $f$ has maximal rank at $x$.

Now for each compact set $K \subset X$ let us define $A(K) \subset \Gamma(X, \mathcal{O}_X)^l$ to be the set of all maps which are one-one and regular on $K$. The following lemmas show that the family $A(K)$ verifies the hypothesis of lemma 1.

LEMMA 7. *Let $K$ be a compact subset of $X$ and suppose $f \in A(K)$. Let $\tilde{f} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ be an extension of $f$. For each compact neighborhood $K'$ of $K$ in $\boldsymbol{C}^m$ there is an $\varepsilon > 0$ such that if $\tilde{g} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ and $\|\tilde{g} - \tilde{f}\|_{K'} < \varepsilon$ then $g = \tilde{g}|_X \subset A(K)$.*

PROOF. If $(X, \mathcal{O}_X)$ is a reduced space this is lemma 5 of [2]. Since the property of being one-one is a topological property we need only to prove that the set

$$\{\tilde{g} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m) \, | \tilde{g}|_X \text{ is regular at each point of } K\}$$

(5) Otherwise it is sufficient to add to $(\varphi_1, \ldots, \varphi_l)$ exactly $n + N - l$ arbitrary components.

is open in $\Gamma(C^m, \mathcal{O}_m)^l$; but this is clear from the definition of regularity, since by small changes the rank of a map does not decrease.

LEMMA 8. *$A(K)$ is dense in $\Gamma(X, \mathcal{O}_X)^l$ and in $\Gamma(X)_\varphi$.*

PROOF. Choose $g \in \Gamma(X, \mathcal{O}_X)^p$. Define $M(g)$ to be the union of all irreducible components of the analytic set $\{(x, y) \in X \times X \; g(x) = g(y)\}$ which are not contained in the diagonal. Define $V(g, m)$ as follows: a point $x$ is an element of $V(g, m)$ if and only if any extension $\tilde{g}: C^m \to C^p$ of $g$ has rank$\leqslant m$ at $x$. So $V(g, m)$ is defined in $X$ to be the set of common zeros to a family of holomorphic functions over $C^m$; hence $V(g, m)$ is an analytic subset of $X$.

We begin by proving the following:

Let $x$ be a point of $X$: if $x \in Y$ suppose $\dim \mathcal{T}(Y)_x < N$. Suppose $g \in \Gamma(X, \mathcal{O}_X)^p$ has an extension $\tilde{g} \in \Gamma(C^m, \mathcal{O}_m)^p$ of rank $r < N$ at $x$. Then the set of all $h \in \Gamma(X, \mathcal{O}_X)$ such that the following properties hold:

(1) $h|_Y = \varphi_{p+1}$ (if $p + 1 \leqslant l$)

(2) $(g, h)$ has an extension in $\Gamma(C^m, \mathcal{O}_m)^{p+1}$ of rank $r + 1$ at $x$ is open and dense in $\Gamma(X)_{\varphi_{p+1}}$.

In order to prove this let $\xi \in \Gamma(C^m, \mathfrak{J}_Y)$ [6] be fixed as follows: if $x \notin Y$ it is a linear combination of elements in $\Gamma(C^m, \mathfrak{J}_Y)$ giving a system of local coordinates at $x$: if $x \in Y$ it is a linear combination of elements in $\Gamma(C^m, \mathfrak{J}_Y)$ giving a minimal system of equations for the Zarisky tangent space $\mathcal{T}(Y)_x$, as a submanifold of $C^m$.

In both cases $(\tilde{g}, \xi)$ has rank $r + 1$ if the coefficients are suitably chosen (there is only a finite number of possibilities that this does not happen). Now if $\tilde{\varphi}_{p+1}$ is any extension to $C^m$ of $\varphi_{p+1}$ for small $\varepsilon$, $\tilde{\varphi}_{p+1} + \varepsilon\xi$ approximates $\tilde{\varphi}_{p+1}$ and is such that $(\tilde{g}, \tilde{\varphi}_{p+1} + \varepsilon\xi)$ has rank $r + 1$ at $x$.

This argument shows that the set we speak about is non void and dense since its inverse image in $\Gamma(C^m, \mathcal{O}_m)$ and in $\Gamma(C^m, \mathcal{O}_m)_{\varphi_{p+1}}$ is non void and dense; but this set is clearly open (same argument as lemma 7), so our statement is proved.

Now let $g = (g_1, \ldots, g_p) \in \Gamma(X, \mathcal{O}_X)^p$ have the following properties:

(1) $g_i|_Y = \varphi_i$ for each $i \leqslant l$;

($\alpha_p$) all the irreducible components of $M(g)$ meeting $K \times K$ have dimension $\leqslant 2n - p$;

($\beta_p$) all the irreducible components of $V(g, m)$ meeting $K$ have dimension $\leqslant n - p + m$ for $0 \leqslant m < n$;

[6] Here $\mathfrak{J}_Y$ is the ideal sheaf defining $(Y, \mathcal{O}_Y)$ as a closed subspace in $C^m$.

then the set of those $h \in \Gamma(X, \mathcal{O}_X)$ such that

(1) $h|_Y = \varphi_{p+1}$ if $p+1 \leqslant l$;

(2) $(g, h) \in \Gamma(X, \mathcal{O})^{p+1}$ satisfies $(\alpha_{p+1})$ and $(\beta_{p+1})$;

is dense in $\Gamma(X)_{\varphi_{p+1}}$.

In order to prove this statement in each component of $M(g)$ of dimension $2n-p$ meeting $K \times K$ choose a point $(x_q, y_q) \notin Y \times Y$; in each component of $V(g, m)$ of dimension $n-p+m$ meeting $K$ choose a point $x_i^m$ in such a way that if $x_i^m \in Y$ then $\dim \mathcal{C}(Y)_{x_i^m} < \dim \mathcal{C}(X)_{x_i^m}$.

This is possible because $\varphi$ is one-one and regular; therefore it separates the points of $Y$ and if $x \in Y$ any extension of $\varphi$ has rank at $x$ at least equal to the dimension of $\mathcal{C}(Y)_x$. In this way we have two finite sets $\{(x_q, y_q)\} \subset \subset K \times K$ and $\{x_i^m\} \subset K$. Moreover by hypothesis $g$ admits an extension of rank $m$ at $x_i^m$.

The same argument used in lemma 6 shows that condition (1) and (2) are equivalent to the following:

(*a*) $h(x_q) \neq h(y_q)$ for each $q$;

(*b*) $(g, h)$ admits an extension to $\boldsymbol{C}^m$ of rank $m+1$ at $x_i^m$ for each $r$ and each $m$.

To prove (*a*) it is sufficient to show that the set of extensions of $\varphi_{p+1}$ separating a given pair $(x, y)$, where $x \notin Y$, is open and dense in $\Gamma(X)_{\varphi_{p+1}}$. To prove this consider the exact sequence of coherent sheaves

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to 0$$

where $\mathcal{J}$ is the coherent ideal sheaf defining the subspace $Y \cup \{x\}$. By theorem $B$ the map

$$\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X/\mathcal{J}) = \Gamma(Y, \mathcal{O}_Y) \oplus \boldsymbol{C}$$

is surjective. Hence there is a global section $s \in \Gamma(X, \mathcal{O}_X/\mathcal{J})$ such that $s(x) \neq 0$, $s(y) = 0$. If $h \in \Gamma(X)_{\varphi_{p+1}}$ we extend $h$ to an $\tilde{h} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ and $s$ to $\tilde{s} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$; if $\varepsilon$ is sufficiently small $\tilde{h} + \varepsilon\tilde{s}$ approximates $\tilde{h}$, therefore $h + \varepsilon s$ approximates $h$. On the other hand it is clear that our set is open.

We have already proved that the set of $h$ satisfying (*b*) is open and dense, as intersection of a finite number of dense open sets.

So we obtain the thesis by intersecting the dense open set relative to condition (*a*) with the dense open set relative to condition (*b*).

Now, since conditions $\alpha_0$, $\beta_0$ are void, what we have shown proves that the set of all $h \in \Gamma(X, \mathcal{O}_X)^l$ such that

1) $h_i|_Y = \varphi_i$ for $i = 1, \ldots, l \geqslant n + N$;
2) $\dim\big(M(h) \cap K \times K\big) \leqslant 2n - l \leqslant -1$ (i.e. $h$ is one-one on $K$);
3) $\dim\big(V(h, m) \cap K\big) \leqslant n - l + m \leqslant -1$ (i.e. $h$ is regular on $K$);

is dense in $\Gamma(X)_\varphi$.

The same argument, except for the condition of extending $\varphi$, proves that $A(K)$ is dense in $\Gamma(X, \mathcal{O}_X)^l$.

PROOF OF THEOREM 5. For each compact set $K \subset X$ let us define as before

$$A(K) = \{f \in \Gamma(X, \mathcal{O}_X)^l | f \text{ is one-one and regular on } K\}.$$

Because of lemmas 7 and 8 all the hypothesis of lemma 1 hold. Given $g = (g_1, \ldots, g_l) \in \Gamma(X)_\varphi$, choose $l$ admissible systems $\{\Omega_i^\lambda\}$, $\lambda = 1, \ldots, l$, such that $\{U_i^\lambda = \Omega_i^\lambda \cap X\}$ is relative to $g$ for each $\lambda$ and compact sets $K_i^\lambda \subset \Omega_i^\lambda$ such that $X \subset \bigcup_{i,\lambda} K_i^\lambda$.

Choose any open neighborhood $A$ of $g$ in $\Gamma(X, \mathcal{O}_X)^l$ and any extension $\tilde{g}$ of $g$ to $\mathbf{C}^m$; let $C$ be a compact in $\mathbf{C}^m$ and $\varepsilon$ be a constant such that if $\tilde{f} \in \Gamma(\mathbf{C}^m, \mathcal{O}_m)$ and $\|\tilde{f} - \tilde{g}\|_C < \varepsilon$ then $\tilde{f}|_X \in A$.

By theorem 3 there are holomorphic functions $\tilde{h}_1, \ldots, \tilde{h}_l$ in $\Gamma(\mathbf{C}^m, \mathcal{O}_m)$ such that:

(1) $\tilde{h}_i|_Y = \varphi_i$, $i = 1, \ldots, l$;
(2) $\|\tilde{h}_i - \tilde{g}_i\|_C < \varepsilon/2$, $i = 1, \ldots, l$;
(3) if $i_0(C) = \sup\{i : K_i^\lambda \cap C \neq \emptyset\}$, then $\|h_\lambda\|_{K_i^\lambda} > i + 1$ for $i > i_0(C)$ and $K_i^\lambda \cap Y = \emptyset$;
(4) $\|h_\lambda - g_\lambda\|_{K_i^\lambda} < \frac{1}{2}$ for $K_i^\lambda \cap Y \neq \emptyset$.

By lemma 1 there is $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_l) \in \Gamma(\mathbf{C}^m, \mathcal{O}_m)$ with the following properties:

(1) $\tilde{f}_\lambda|_Y = \varphi_\lambda$, $\lambda = 1, \ldots, l$;
(2) $\|\tilde{f} - \tilde{h}\|_C < \varepsilon/2$;
(3) $\|\tilde{f}_\lambda - \tilde{h}_\lambda\|_{K_i^\lambda} < \frac{1}{2}$ for $K_i^\lambda \cap Y \neq \emptyset$;
(4) $\|\tilde{f}_\lambda - \tilde{h}_\lambda\|_{K_i^\lambda} < \frac{1}{2}$ for $i > i_0(C)$ and $K_i^\lambda \cap Y = \emptyset$;
(5) $\tilde{f} \in r^{-1}\big(A(K \cap X)\big)$ for each compact set $K \subset \mathbf{C}^m$.

Let $f$ be $\tilde{f}|_X$. Conditions (3) and (4) show that $f$ is a proper map, since the set of points $x$ such that $|f_\lambda(x)| < i$, $\lambda = 1, \ldots, l$, is contained in the compact set

$$\Big\{C \cup \Big(\bigcup_{j \leqslant \max(i_0(C), i)} \Big(\bigcup_{\lambda=1}^{l} K_i^\lambda\Big)\Big) \cup H\Big\} \cap X$$

where $H$ is the compact set where the proper map $g' = g|_V$ is less than $i+1$ ($V$ is the neighborhood of $Y$ defined in lemma 2).

Condition 5 says that $f$ is one-one and regular on all of $X$ and therefore theorem 5 is proved.

## 4. – The real case.

We can deduce from theorems 4 and 5 the statements for real analytic spaces (7). More precisely we obtain the following theorems.

THEOREM 6. *Let $(X, \mathcal{O}_X)$ be a real analytic space of dimension $n$ and $(Y, \mathcal{O}_Y)$ a closed subspace. Let $\varphi\colon Y \to \boldsymbol{R}^l$ be an embedding, $l \geqslant 2n+1$. The set of all analytic maps of $X$ into $\boldsymbol{R}^l$ which are proper, one-one, of maximal rank in each regular point of $X$ and extend $\varphi$ is dense in*

$$\Gamma(X)_\varphi = \{f \in \Gamma(X, \mathcal{O}_X)^l \colon f|_Y = \varphi\} .$$

THEOREM 7. *Let $(X, \mathcal{O}_X)$ be a real analytic space of dimension $n$ and locally of type $N$. Let $(Y, \mathcal{O}_Y)$ be a closed subspace and $\varphi\colon Y \to \boldsymbol{R}^l$ an embedding, $l \geqslant n + N$. Then the set of all maps which are embeddings of $X$ into $\boldsymbol{R}^l$ and extend $\varphi$ is dense in $\Gamma(X)_\varphi$.*

PROOF. The proof is reduced (as in [3]) to the existence of suitable admissible systems. Such systems can be constructed as in [4].

(7) Any real analytic space $(X, \mathcal{O}_X)$ is coherent, but it may be non reduced.

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