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Classe di Scienze

FEDERICO GAETA

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On the Collineation Group of a Normal Projective Abelian Variety.

FEDERICO GAETA (*)

Introduction.

It is well known that the holomorphic θ map: $E/G \xrightarrow{\theta} \mathbf{P}^{t-1}$ ($t = \text{Pfaff } A$, cf. below and § 1) of a complex torus E/G ($E = \mathbf{C}^n$) into a suitable complex projective space $\mathbf{P}^{t-1}$, defined by a given complete linear system of positive divisors $|\mathfrak{D}|$ on E/G by means of the usual bases of Fourier theta series is not « canonical » because of two necessary choices:

1) The origin O of $E = H_1(\mathcal{A}; \mathbf{R})$ $\mathcal{A} = E/G$ is also the origin for the « lattice of periods » and since the fundamental group of $\mathcal{A}$ is Abelian G appears naturally identified with the fiber over O in the universal covering also map $E \to E/G$.

Since E/G is a homogeneous space there is no natural privilege for any point. However for the projective embedding $t_1 \geqslant 3$ $\mathcal{A}$ the fix-points of the symmetries $u \mapsto \omega - u$ leaving invariant $|\mathfrak{D}|$ are natural choices (there are finitely many of such points, cf. § 7).

2) Siegel [2] shows that a change of modular basis $\mathfrak{m}: \mathfrak{g} \mapsto \tilde{\mathfrak{g}}$ (cf. Def. 1.3) in the lattice of periods G leaves pointwise invariant the image projective Abelian variety $\mathcal{A}$ iff $\mathfrak{m}$ belongs to the corresponding *congruence subgroup* $\Delta(G)$ (cf. § 3, Th. 1) of the modular group $\mathfrak{M}(G)$ (cf. Def. 1.5). This lack of uniqueness is embarrassing in the constructions of varieties of moduli of various types. Since $\Delta(G)$ has a *finite index* $\nu = [\mathfrak{M}(G) : \Delta(G)]$ as an invariant subgroup of $\mathfrak{M}(G)$ the maximum « perfection » we can achieve is to construct canonically a set of ν projectively equivalent models $\mathcal{A}^{(k)}$ of $|\mathfrak{D}|$, $h = 1, 2, \ldots, \nu$. Then the diagonal D of the product $\prod \mathcal{A}^{(k)}$, [in the Segre way) is a canonical projective model of E/G, although it is not given by $|\mathfrak{D}|$ but by a

(*) SUNY at Buffalo, Department of Mathematics.
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certain multiple of $|\mathfrak{D}|$. This paper is devoted to the study of this construction (cf. § 11). It leads us to the study of the *reduced modular group* $\mathfrak{m}(G) = \mathfrak{M}(G)/\Delta(G)$. $\mathfrak{m}(G)$ has an algebraic and a geometric interpretation in $\Theta(E/G)$ that *can be recovered independently of G*. Cf. also Mumford [2]. The former one comes from the fact that $\mathfrak{m}(G)$ *is the full automorphism group of the quotient group* $\mathfrak{C} = \bar{G}/G$ (cf. Th. I, § 3) *endowed with an additional « reduced symplectic structure »* $\mathfrak{C}\times\mathfrak{C} \xrightarrow{\mathcal{R}} Q/\mathbb{Z}$, (cf. § 3). $\bar{G}$ is a certain over-lattice (cf. Def. 3.1) of G (called *the completion of G with respect to the non-singular antisymmetric form* $A|G\times G \to \mathbb{Z}$) intrinsically associated with $|\mathfrak{D}|$. The geometric interpretation arises from the fact that $\mathfrak{C}$ is *canonically isomorphic with the subgroup of the torus group E/G* leaving invariant $|\mathfrak{D}|$. Accordingly $\mathfrak{C}$ can be regarded as a subgroup of the full collineation group of the normal (¹) projective Abelian variety $\mathcal{A}$. This property was already established by Rosati [1, 2]; Morikawa and Weil [4] showed that the finiteness of the subgroup of E/G leaving invariant $|\mathfrak{D}|$ is characteristic of the « non degenerate Abelian varieties » (the only ones considered here). The alternating form A induces naturally $\mathbb{R}$ in $\mathfrak{C}\times\mathfrak{C}$ with values in $Q/\mathbb{Z}$. $\mathbb{R}$ can be lifted by the exponential map: $\exp(2\pi i)\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{C}_1^\times$ $(\mathbb{C}_1^\times = \{z \in \mathbb{C} \,|\, |z| = 1\})$; then $\exp[2\pi i\mathbb{R}]$ takes values in the cyclic group of roots of unity of index t_n (the last elementary divisor of $A|G\times G$). In the classical case $t_1 = t_n = 2$ this gives as a particular case the *syzigetic* and *azygetic* phenomena and a good understanding of *« the group of characteristics »* (cf. Igusa's [2], Ch. V, § 6, p. 209); the general case, however, shows these facts better ... We obtain $\mathfrak{C}$ and $\mathfrak{m}$ *ex-novo* in Th. I, § 3 together with several useful geometrical complements in a much simpler way, using the intrinsic approach to the theory of θ-functions, recalled briefly in § 2. Actually, this paper tries to push forward this intrinsic point of view. This enables a much simpler transition from E/G to its projective image $\Theta(E/G)$ (when $|\mathfrak{D}|$ is ample enough to define an embedding) than the old, non intrinsic treatment on the θ's. This paper was born from an attempt to unify some recent contributions to the classical problem of the « period relations » between Abelian integrals of the first kind (cf. Andreotti-Mayer and Rauch-Farkas). Both approaches are independent and apparently unrelated but, actually, both use the theta-constants and *both are geometrically related to the Kummer-Wirtinger variety*. I believe that there is no conflict of interest between the intrinsic

(¹) We use the word *normal* in the early meaning of the Italian school, i.e. the linear system of hyperplane sections of $\mathcal{A}$ is complete. This, in general does not imply arithmetical projective normality. ($\Leftrightarrow$ the projective coordinate ring integrally closed in its field of fractions.) In fact, Muhly proved that such a.p. normality is equivalent to the completeness of the system cut on the variety by hypersurfaces F^m of Σ^{t-1} for any $m \geqslant 1$.

approach and the aim to make everything *absolutely explicit* in treating the Schottky relations .... The intrinsic point of view helps to make an optimal choice of coordinate systems, necessary to simplify the explicit expressions of the relations themselves. With this in mind I make explicit the expressions of the intrinsic invariants attached to θ functions (cf. § 2) in terms of the classical terminology of Krazer and Krazer-Wirtinger following some friendly suggestions of Farkas and Rauch. For the same reasons I will emphasize explicitly the following achievements of this paper, summarized in five theorems in the text:

1) Our introduction of the reduced symplectic structure in $\mathfrak{C}$ (1) enable us to associate bijectively a distinguished projective coordinate system (cf. § 11) in $\mathbb{P}^{t-1}$ corresponding to every symplectic basis of $\mathfrak{C}$ (cf. Th. I, § 3). We associate in a natural way with the symplectic basis a direct sum decomposition $\mathfrak{C}_1 \oplus \mathfrak{C}_2 = \mathfrak{C}$ of $\mathfrak{C}$. This leads to the construction for an arbitrary normal projective Abelian variety of the generalization of the « *singuläre Koordinatensysteme* » of a normal elliptic curve $\mathcal{N}_n$ in $\mathbb{P}^{n-1}$ (studied by Klein for n odd and Hurwitz for n even). We offer a generalization of the *triangle of flexes* of the plane cubic $\mathcal{N}_3$. Since the decomposition of $\mathfrak{C}$ is invariant under $\Delta(G)$ we associate $\nu = |\mathfrak{m}(G)|$ such natural projective coordinate systems to the polarized complex torus E/G. We refer to these special theta basis and the associate projective coordinate systems as Hessian (cf. § 10) because we can extend to Chow's *zugeordnete Form* of $\mathcal{A}$ (cf. also Hodge-Pedoe, Van der Waerden (or Siegel's *Normalgleichung*)) the simple properties of the Hessian equation of $\mathcal{N}_3$ (cf. below and (10.6)). These Hessian bases are related to the usual θ bases by a certain non singular matrix M (depending on the n elementary divisors $t_1|t_2| \dots |t_n$ of A) independent of the moduli of $\mathcal{A}$. A certain non zero multiple of M is *unitary*. As a consequence, since the ordinary Fourier thetas can be normalized in order that $\langle \theta_a, \theta_l \rangle = \delta_{al}$ $(k, l = 1, 2, \dots, t)$ for the Hermitean scalar product induced by the Kähler structure (cf. Siegel's paper § 3, page 388), it turns out also that *the Hessian theta basis is also orthonotmal with respect to the natural Hermitean matrices*: (cf. Th. IV, § 11).

(1) Cf. some pertinent historical comments in the final « Acknowledgements » section of this Introduction.

(2) The vector space E can be identified with the tangent vector space at the origin. Then the Riemann form H can be transported uniquely to any other point of E/G (or $\mathcal{A}$) a torus translation defining obviously a global Hermitean form, which it turns out has the Kähler property. The Hermitean metrics introduced by Siegel on $\theta(\mathfrak{D})$ is the lifting of the metrics defined on $\mathfrak{D}$ by the Kähler structure. Cf. also Cartier's approach via the Fock representation.

In the general case *the triangle of flexes is replaced by a projective coordinate simplex*; each one of its hyperplanar faces is invariant by any collineation of $\mathfrak{C}_1$, while $\mathfrak{C}_2$ induces a finite Abelian permutation group of the faces. Because of $\mathfrak{C} = \mathfrak{C}_1 \oplus \mathfrak{C}_2$ these distinguished simplexes appear in pairs, as for the cubic. In the cubic the Hessian equation: $x_0^3 + x_1^3 + x_2^3 + 6ax_0x_1x_2 = 0$ illustrates very well the behavior of $\mathfrak{C}_1$, $\mathfrak{C}_2$ (which are then cyclic groups of order three). In the general case $\mathfrak{C}_1 \approx \mathfrak{C}_2$ is a finite Abelian of type $(t_1, t_2, \ldots, t_n)$. I abuse somewhat the terminology calling this *configuration of flexes*, because there are no flexes, but just « simplexes of flexes ». However, the fix-point set introduced in § 7 is closed to the « flexes » of the plane cubic.

2) The general group of characteristics (arbitrary $t_1|t_2| \ldots |t_n$) is explained also via the « configuration of flexes ».

3) We introduce intrinsic series expansions (cf. § 10) for the *ordinary reduced thetas* in Weil's sense (cf. § 2), also called *normalized* (Igusa [2], Ch. II, p. 51-85). (cf. Th. III). These reduced thetas differ from the ordinary Fourier thetas by an exponential factor. From this we deduce the Hessian basis in a purely intrinsic way. *We do not need to introduce any basis in E to define such expansions.* If $|\mathfrak{D}|$ has type (H, ψ) (cf. Def. 2.4) the generic terms of both series (the usual « reduced » and the Hessian ones) (cf. § 1) depend only on the positive definite Hermitean form H (cf. § 1) and the choice of a certain extension to the lattice $\Gamma_1 = p^{-1}\mathfrak{C}_1$ (p is the canonical projection $E \to E/G$, $A(\Gamma_1 \times \Gamma_1) \subseteq \mathbf{Z}$) of the semicharacter ψ, (cf. § 6). For the former ones, the summation is taken with respect to all the $g \in \lambda$ where λ is any class of Γ_1 mod G.

The fact that there are exactly t extensions of ψ from G to Γ_1 leads us to introduce the Hessian coordinate systems, cf. § 10.

4) *The intrinsic series expansions make crystal clear that such series are absolutely invariant by $\Delta(G)$* (no automorphy factors!), without any calculation (cf. Siegel [39], Satz 9, pages 400-403!). As ac onsequence *we simplify considerably the transformation theory of the theta functions.* (Cf. Siegel [2], Igusa [2], Ch. II, § 5, p. 78).

5) On our way, we find other results not directly related to the original problem; perhaps the most interesting is the following sharpening of Andreotti [1, 2] Igusa's [1] duality theorems (cf. Th. II, § 4). It is well known that $E/\bar{G}$ is the Picard variety of the polarized Abelian variety $E/G = \mathcal{A}$. If $\mathcal{A}$ has the *polarization type* $(e_1, e_2, \ldots, e_{n-1})$ (cf. Def. 1.1) *$E/\bar{G}$ has the dual type* $e^* = (e_{n-1}, e_{n-2}, \ldots, e_1)$.

Acknowledgements. The only case where I found a fairly complete transcendental treatment of the Hessian coordinate systems is, just for $n = 1$, the mentioned papers of Klein and Hurwitz. Mumford's starting point is also « to construct *canonical bases* of all linear systems on all Abelian varieties » (in a pure algebraic way over ground fields of any characteristic). He told me that his bases coincide with my « Hessian » (essentially due to Klein and Hurwitz (1)). I believe that my geometric interpretations of these basis is the most general extension of the « couples of conjugate triangles of flexes » on the non singular plane cubic is a very direct and elementary one, since, still today these cubics are the simplest « visual » examples (i.e. I disregard the double line with four branch points). For an arbitrary normal elliptic curve $\mathcal{N}_n$ in $\mathbf{P}^{n-1}$ each one of the hyperplanar sides of a Hessian system contains n « hyperosculating points », thus the analogy is still very close: It is not so close in the general case, but still the transcendental construction of the Hessian basis of theta function of type (H, ψ) attached to a canonical decomposition $\mathfrak{C} = \mathfrak{C}_1 \oplus \mathfrak{C}_2$ of the collineation groups in terms of the « prolongations of ψ to $p^{-1}\mathfrak{C}_1$ », mentioned before is *equally simple in the general* case as in the elliptic one (2).

The reduced symplectic structure appears also in Part I, page 293 of Mumford's paper « On the equations ... » in algebraic way. The interpretation using XIXth century mathematics, although not needed in the text can be the following. Let (M) be the equivalence class of a linear map in the vector space $\Theta(\mathfrak{D})$ defining $\mathbf{P}^{t-1}$. If (A), (B) belong to the « collineation group », since $(A)(B) = (B)(A)$ we have $ABA^{-1}B^{-1} = \lambda I$ $(\lambda \in k^*)$ and since $\det(ABA^{-1}B^{-1}) = 1$, $\lambda^t = 1$. The « reduced symplectic structure » can be defined algebraically by $(A, B) \mapsto \lambda$. That map is invariant by scalar multiplications $\lambda(A, B) = \lambda(aA, bB)$ $(a, b \in k^*)$. Thus, it depends only on the collineation group. In the case $k = \mathbf{C}$ we can replace λ by $(2\pi i)^{-1} \log \lambda$ (defined mod $\mathbf{Z}$). As a consequence, the commutator group $[\mathcal{M}, \mathcal{M}]$ of the

(1) I did not use the abused term canonical because Klein introduces another type of projective coordinates system for normal elliptic curves, related to the Weierstrass $\wp$ function, and I needed to make a choice. I do not see any interest in trying to extend this second type to the general Abelian case.

(2) As a classical geometer who likes to read modern Algebraic geometry perhaps I unconciously avoided Andreotti-Meyer's blame in their « motto »:

> In primo luogo non dovrà il Poeta moderno aver letti, nè legger mai gli Autori antichi Latini o Greci. Imperocchè nemeno gli antichi Greci o Latini hanno mai letti i moderni.
>
> B. Marcello: *Il Teatro alla Moda*

group $\mathcal{M}$ of matrices generating $\mathfrak{C}$ coincides with the center, i.e. have the sequence

$$\mathcal{M} \supset [\mathcal{M}, \mathcal{M}] \supset [\mathcal{M}, [\mathcal{M}, \mathcal{M}]] = 1$$

defining on $\mathcal{M}$ a structure of « two step nilpotent group » (cf. Cartier, Mumford, Satake). Besides this algebraic point many points in Mumford's paper are surprisingly « classical ». The modern interpretations are essentially the identification of $\Theta(\mathfrak{D})$ with the vector space of regular sections of the line bundle attached to $|\mathfrak{D}|$ or equivalently with the global sections $\Gamma(\mathcal{A}, L)$ where L is the invertible sheaf of germs of that line bundle, which are obviously meaningful in the general algebraic case. I did not use it in the paper because I did not need any cohomology.

Other cases where the collineating group appears explicitly, are the well-known Kummer-Wirtinger case (cf. Conforto, Wirtinger for arbitrary t_j). I know just the few cases recalled in Conforto's book page 89, 220: Traynard Bagnera-De Franchis, Enriques-Severi (hyperelliptic surfaces). For arbitrary t_j cf. Rosati [1, 2]. In the case $t_1 \geqslant 3$, θ is injective and $\theta(E/G)$ is an irreducible non singular algebraic variety of $\mathbb{P}^{t-1}$. (For proofs cf. Conforto, II Kap, § 18, page 210, Siegel's works, Band III), Weil [2], Igusa [2], Ch. III, § 7, p. 125; however, we do not want to assume $t_1 \geqslant 3$ because the Wirtinger case $t_1 = t_2 = \ldots = t_n = 2$ appears frequently in the literature (the Kummer-Wirtinger variety), (cf. Conforto and Wirtinger) and we need then to try to build a bridge between the Andreotti-Mayer and Rauch-Farkas approach to the Schottky relations problem. Rosati [1, 2] used to the classical basis of the Fourier thetas. Instead, Klein and Hurwitz used the Weierstrass sigma functions, because the transformation formulas with respect to $\mathcal{M}(T)$ (cf. Def. 1.5) are very simple and this makes it easier to interpret geometrically the analytic results for $n = 1$. On the contrary Siegel [2] deals with the general problem; the results are natural extensions of the case $n = 1$, but the behavior of the usual Fourier theta series under the modular group, studied by Hermite, Kloosterman and Siegel is much more complicated and the calculations have no apparent geometric meaning. We show in § 10 that, actually *the complications of the transformation theory for the ordinary Fourier thetas is due to the fact that they are not reduced, and the transformation formulas involve a change of* Φ, L (cf. Prop. 2.1, Def. 2.2) (not just the usual coordinate transformations of fixed Φ and L!) In other words if θ is a Fourier theta, then θ can be written as θ_0 times an exponential factor, where θ_0 is a reduced theta. *The complications of the transformation theory come from the exponential factor, not from* θ_0! Since the exponential factor has no geometric interest and it depends on the choice of a modular bases, the *normalized theta* which really matters, should have absolute priority in the transformation theory. Mumford called my attention to his simplification of the transformation formulae given in page 128, Part II (for the algebraic case, with a connection to the classical case treated in § 12 Part III. Obviously, I cannot go out of my way here, making a detour through the algebraic case. My treatment is transcendental, direct and aimed to simplify Siegel's « Satz 9 » by replacing his computational proof by a group theoretic and geometric study showing the role of $\Delta(G)$ via the intrinsic series expansions of the reduced thetas. I am grateful too to M. Fried for many valuable criticisms.

1. – Recall of preliminary definitions.

Most of them are known. They are recalled because we need to use various lattices simultaneously and such standard terminology as « the » modular group, etc. will be imprecise. Def. 1.1 will play a role in § 4.

This paper deals with an arbitrary Abelian variety, polarized by $|\mathfrak{D}|$ (cf. Def. 1.0 below) $|\mathfrak{D}|$ has the most general type $T = \operatorname{diag}\{t_1, t_2, \ldots, t_n\}$; $t = t_1 t_2 \ldots t_n$ equals the Pfaffian of A. $\dim_{\mathbb{C}} |\mathfrak{D}| = t - 1$.

The following structures will be fixed throughout the paper:

1) The structure of E as a $2n$-dimensional real vector space.
2) The lattice $G \subset E$ of maximal rank $= 2n$.
3) The non-singular, antisymmetric, real valued form

$$A: E \times E \to \mathbb{R} \tag{1.1}$$

with integral valued restriction to $G \times G$; precisely $A(G \times G) = (t_1) \subset \mathbb{Z}$ where (t_1) denotes the corresponding principal ideal of $\mathbb{Z}$.

On the contrary, the complex structure of E as an n-dimensional $\mathbb{C}$-vector space is fixed everywhere except in § 11 where we move the complex structure of E *in such a way that the necessary condition*

$$A(ix, iy) = A(x, y) \qquad \forall (x, y) \in E \times E \tag{1.2}$$

is always satisfied for our choice of $i \times E \to E$.

As a consequence of (1.2) the map $E \times E \xrightarrow{S} \mathbb{R}$ defined by

$$S(x, y) = -A(ix, y) = A(x, iy) \tag{1.3}$$

is bilinear symmetric and (1.2) implies $S(ix, iy) = S(x, y)$. Cf. Weil [2].

Any fixed divisor D on a non singular algebraic variety X defines two sets $\mathcal{T}_l(\mathcal{T}_a)$ of *polar divisors* where $A \in \mathcal{T}_l(\mathcal{T}_a)$ iff there is some pair of integers a, d such that $aA \sim dD \bmod \mathfrak{Z}_l$ (mod $\mathfrak{Z}_a$) (the groups of divisors linearly or algebraically equivalent to zero. (cf. Lang)).

DEF. 1.0. The pair $(X, \mathcal{T})$ is called a *polarized variety.*

REMARK. Both polarizations $\mathcal{T}_l$ or $\mathcal{T}_a$ of E/G defined by $|\mathfrak{D}|$ will be fixed (except in § 11). Its existence insures, that E/G is really an algebraic variety embeddable in some projective space (cf. Chow).

The integral valued form $G \times G \to Z$ attached to any D of $\mathcal{T}_a$ has always the form qA (for some $q \in Q$).

DEF. 1.1. The ordered set $e = \{e_1, e_2, ..., e_{n-1}\}$ of $n-1$ positive integers well defined by:

$$\tau_{j+1} = e_j \tau_j \qquad j = 1, 2, ..., n-1 \tag{1.4}$$

in terms of the elementary divisors τ_j of any allowable $qA|G \times G$ is obviously independent of q. It is called *the polarization type of* $\mathcal{T}_a(E/G)$. The first elementary divisor τ_1 is called *the degree of* $\alpha = A|G \times G$. α is called *primitive* iff $\tau_1 = 1$.

The type $e^* = \{e_{n-1}, e_{n-2}, ..., e_2, e_1\}$ *is called the dual of* e. Obviously, $e^{**} = e$. The map $H = S + iA: E \times E \to \mathbf{C}$ *is a Hermitean form.* We assume that the diagonal restriction of H always satisfies

$$u \mapsto H(u, u) = S(u, u) \geqslant 0 \qquad \forall u \in E \tag{1.5}$$

and

$$H(u, u) = 0 \Leftrightarrow u = 0 \tag{1.6}$$

i.e. *H is a non-degenerate Riemann form* (cf. Igusa's book, Ch. II, p. 66). I.e. (1.5) (or (1.6)) defines *a positive definite Hermitean* (*real quadratic*) *form.* A is uniquely determined by the integral cohomology class of $\mathfrak{D}$ in E/G (or, equivalently, by the intersection number $\mathfrak{D} \cdot P$ of the real, oriented, $(2n-2)$-dimensional cycle $\mathfrak{D}$ with the oriented parallelogram $P(g_1, g_2)$ determined by any ordered pair g_1, g_2 of vectors of the « period lattice » G).

A can be represented by the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{1.7}$$

on a suitable basis.

DEF. 1.2. An $\mathbb{R}$-basis of E: $\mathfrak{r} = (\mathfrak{r}_1 \mathfrak{r}_2) = (r_1 r_2 \dots r_{2n})$ (where $\mathfrak{r}_1$, $\mathfrak{r}_2$ represent thetwo « halves » $r_1 r_2 \dots r_n$ and $r_{n+1} \dots r_{2n}$ is called *symplectic* if A can be represented by (1.7), i.e. iff

$$(1.8) \qquad A(r_j, r_{j+n}) = -A(r_{j+n}, r_j) = 1 \qquad j = 1, 2, \dots, n$$

and $A(r_\alpha, r_\beta) = 0$ otherwise.

If A is integral valued on $G \times G$ it is natural to choose a basis in G but then, we cannot always obtain the basis (1.8). The best result (of Frobenius) allows us to reduce the matrix of $A|G \times G$ to the form

$$(1.9) \qquad \mathfrak{L} = \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}$$

where T is again the uniquely determined diagonal matrix of elementary divisors of A, $T = \operatorname{diag}(t_1, t_2, \dots, t_n)$, where

$$(1.10) \qquad t_1 | t_2 | t_3 | \dots | t_n \qquad t_j \in \mathbb{Z},\ j = 1, 2, \dots, n\,.$$

DEF. 1.3. A $\mathbb{Z}$-basis of G $\mathfrak{g} = (\mathfrak{g}_1; \mathfrak{g}_2)$ is called a *modular basis* iff the matrix of $A|G \times G$ has the form $\mathfrak{L}$ of (1.9) i.e. iff

$$A(g_j, g_{j+n}) = -A(g_{j+n}, g_j) = t_j \qquad j = 1, 2, \dots, n \text{ and } A(g_\alpha, g_\beta) = 0$$

otherwise; a modular basis is also symplectic iff G has the unit type: $T = I_n$.

DEF. 1.4. The *symplectic group* $S_p(A)$ *relative to* A is characterized by

$$(1.11) \qquad S_p(A) = \{\sigma \in GL(E; \mathbb{R}) | A(\sigma x, \sigma y) = A(x, y),\ \forall (x, y) \in E \times E\}$$

where $GL(E; \mathbb{R})$ is the full-linear group of E.

DEF. 1.5. *The modular group* $\mathfrak{M}(G)$ *of the lattice* G is the subgroup of $S_p(A)$ characterized by

$$(1.12) \qquad \mathfrak{M}(G) = \{\sigma \in S_p(A) | \sigma G = G\}\,.$$

The isomorphism class of $\mathfrak{M}(G)$ depends only on the matrix T of elementary divisors, and it is called the *modular group of level* T in Siegel. ([2], III, page 109).

DEF. 1.6. Let $\mathfrak{c} = (c_1, c_2, \dots, c_n)$ be a fixed $\mathbb{C}$-basis of E, $c_j \in E$, $j = 1, 2, \dots, n$; let $\mathfrak{g} = (\mathfrak{g}_1, \mathfrak{g}_2)$ be a $\mathbb{Z}$-basis of G; the $n \times (2n)$ coordinate matrix (CD) of $\mathfrak{g}$ with respect to $\mathfrak{c}$ is called the Riemann matrix of $\mathfrak{g}$ with respect to $\mathfrak{c}$ is called the Riemann matrix of $\mathfrak{g}$ with respect to $\mathfrak{c}$; (C, D $n\times n$ complex matrices representing $\mathfrak{g}_1, \mathfrak{g}_2$).

It is well known that if $\mathfrak{g}$ is a modular basis of G (cf. Def. 1.3) both C and D are non-singular ($\Leftrightarrow \mathfrak{g}_1, \mathfrak{g}_2$ are $\mathbb{C}$-basis of E); then for $T = l_n$ we can choose $\mathfrak{c} = \mathfrak{g}_1$; thus $D = Z \in \mathfrak{S}_n$. ($\mathfrak{S}_n$ is Siegel's upper half-plane). In the general case the choice of $\mathfrak{c}$ uniquely determined by $\mathfrak{g}_1 = \mathfrak{c}T$ reduces the Riemann matrix to the form (TZ) where (IZ) represents $(\mathfrak{c}\mathfrak{g}_2)$ with respect to $\mathfrak{c}$. The lattice $\mathbb{Z}$-generated by $(\mathfrak{c}\mathfrak{g}_2)$ contains properly G iff $T \neq I_n$. A unimodular linear substitution with rational coefficients can replace the $\mathbb{R}$-basis $\mathfrak{g}$ of E by another such that Z and T appear «separated» in various ways (for a list of the more frequent «canonical forms» of the Riemann matrix of level T appearing in the literature, cf. Conforto's book, page 90). In order to simplify computations with his modular group $\mathcal{M}(T)$ of level T, Siegel [2] chooses the canonical form (ZT); this is equivalent to introducing implicitly a change $\mathfrak{g}_2 = \mathfrak{c}_2 T$ and then to permute $\mathfrak{g}_1, \mathfrak{g}_2$. Precisely:

DEF. 1.7. Let A be represented by $\mathfrak{L}$ (cf. 1.9) in the $\mathbb{R}$-basis $\mathfrak{g} = (\mathfrak{g}_1, \mathfrak{g}_2)$. *Let* $\mathfrak{g}_j = \mathfrak{c}_j T$ $(j = 1, 2)$. We call *Siegel basis of E attached to* $\mathfrak{g}$ the $\mathbb{R}$-basis $\mathfrak{z} = (\mathfrak{z}_1 \mathfrak{z}_2)$ of E defined by

$$\mathfrak{z}_1 = \mathfrak{c}_2 = \mathfrak{g}_2 T \qquad \mathfrak{z}_2 = \mathfrak{c}_1 = \mathfrak{g}_1 T . \tag{1.13}$$

The change $\mathfrak{g}_j \mapsto \mathfrak{c}_j$ $(j = 1, 2)$ replaces T by T^{-1} in the matrix $\mathfrak{L}$ of (1.9). The transposition changes the signs, thus *with respect to the Siegel basis* $\mathfrak{z}$ *is* $\mathfrak{L}^{-1}$, i.e. *we have*

$$A(s_j, s_{j+n}) = -A(s_{j+n}, s_j) = -t_j^{-1}, \qquad j = 1, 2, \ldots, n \tag{1.14}$$

and $A(s_p, s_q) = 0$ otherwise, where $\mathfrak{z} = (\mathfrak{z}_1 \mathfrak{z}_2) = (s_1 s_2 \ldots s_{2n})$.

Let us take as $\mathbb{C}$-basis associated to $\mathfrak{z}$, $\mathfrak{c} = \mathfrak{z}_2$. Then *the Riemann matrix of the canonical basis* $\mathfrak{g}$ *with respect to* $\mathfrak{z}$ *is* (ZT).

REMARK. Any basic vector $s_j \in \mathfrak{z}$ has the property $A(s_j, g) \in \mathbb{Z}$ for any $g \in G$ thus any $\bar{g} = \sum_{j=1}^{2n} m_j s_j$ $(m_j \in Z,\ j = 1, 2, \ldots, 2n)$ has the property $A(\bar{g}, g) \in \mathbb{Z}$ for every $g \in G$. In § 3 we shall consider this again from an intrinsic point of view; *the vectors of a Siegel bases did not need to be periods* ($\Leftrightarrow \in G$), but they generate an over-lattice $\bar{G}$ (cf. Def. 3.1) $\subset G$.

Other remarkable bases deduced from $\mathfrak{g}$ are $(\mathfrak{c}_1 \mathfrak{g}_2)$ or $(\mathfrak{g}_1, -\mathfrak{c}_2)$. They are *symplectic* (cf. Def. 1.2).

2. – Intrinsic approach to the theta functions.

Classical geometers knew a long time ago that an irreducible algebraic hypersurface in the torus E/G can be characterized by the vanishing of an irreducible theta function. This is essentially the statement of the theorem of Appell-Humbert-Lefschetz, (cf. quoted papers). It is a natural idea to develop a systematic study of the theta functions starting from an arbitrary divisor $\mathfrak{D}$ of E/G and its lifting $\mathfrak{D}\circ p$ to E as done in H. Cartan's Seminar and then by Weil [1], [2], (cf. also Igusa's book, for a didactical exposition, Ch. II, p. 51). An improved revision of such constructions using the well-known duality between divisors and line bundles (cf. Weil [3]) and the cohomology of G can be found in the recent book of Mumford [1]. We summarize here just the main results needed later, referring to the bibliography for proofs. Most of the next Prop. are useful remarks stated for further reference.

PROP. 2.1. *A meromorphic function* θ (*not identically zero*) *defined in* E *represents a divisor* $\mathfrak{D}\circ p$ (*not necessarily positive*) *iff there are four «invariants»* (H, ψ, Φ, L) (defined below) *such that for any* $u \in E$ *and any period* $g \in G$ *we have*:

$$\text{(2.1)}\qquad \theta(u+g) = \\ = \theta(u)\,\psi(g)\exp\pi[H(g,u) + \tfrac{1}{2}H(g,g) + \Phi(g,u) + \tfrac{1}{2}\Phi(g,g) + 2iL(g)]$$

H) $H\colon E\times E \to \mathbf{C}$ is the Hermitean form (linear in the second variable, antilinear in the first one) introduced in § 1.

ψ) $\psi\colon G \to \mathbf{C}_1^\times$ is a mapping of the lattice G into the unit circle $\mathbf{C}_1^\times$ of the complex plane satisfying the functional equation

$$\text{(2.2)}\qquad \psi(a+b) = \psi(a)\psi(b)\exp\pi i A(a,b)\,, \qquad \forall (a,b) \in G\times G\,.$$

DEF. 2.1. The previous map ψ satisfying (2.2) is called a *semi-character* of G attached to A (or to H), Weil [2] ($\Leftrightarrow$ Mumford's [1] *multiplicator*).

Φ) $\Phi\colon E\times E \to C$ is $\mathbf{C}$-valued, symmetric, bilinear form.

L) $L\colon E \to \mathbf{C}$ is a $\mathbf{C}$-valued $\mathbf{C}$-linear form on E.

Furthermore, H, Φ, L *and* ψ *are uniquely determined by* (2.1) *and* $\operatorname{Im} H = A$ (*with* $A(G\times G) \subseteq \mathbf{Z}$).

DEF. 2.2. A meromorphic function θ on E is called a *theta function of type* (H, ψ, Φ, L) iff it satisfies the functional equation (2.1).

REMARK. Igusa's book considers three invariants (Q, ψ, L), where ψ and L are the same as in Weil and $Q = H + \Phi$ is a « quasi-hermitian form » (cf., Ch. II, § 3, p. 64); ($\Leftrightarrow Q$ is $\mathbb{C}$-linear ($\mathbb{R}$-linear) in the first (second) variable). Since every Q can be decomposed uniquely in the form $H + \Phi$, $H = \operatorname{Her} Q$, $\Phi = \operatorname{Sym} Q$; there is nothing essentially different. Although we found Igusa's exposition more didactical, we prefer the original Weil's terminology because we want to emphasize both components separately.

PROP. 2.2. *The theta function defining a positive $\mathfrak{D}$ is holomorphic.*

PROP. 2.3. If θ_1, θ_2 are theta functions of type $(H_j, \psi_j, \Phi_j, L_j)$, $(j = 1, 2)$ the product $\theta_1\theta_2$ is a theta of type $(H_1 + H_2, \psi_1\psi_2, \Phi_1 + \Phi_2, L_1 + L_2)$.

DEF 2.3. *A trivial theta* $\Leftrightarrow$ one of type $(0, 1, \Phi, L)$.

PROP. 2.4. *A theta function is trivial iff it has the form* $u \mapsto \exp \pi(\Phi(u) + L(u) + c)$ $(c \in \mathbb{C})$.

PROP. 2.5. *A trivial θ represents the zero divisor and conversely any theta representing the zero divisor is trivial.*

DEF. 2.4. A theta function of type $(H, \psi, 0, 0)$ will be called *reduced* (Weil) or normalized (Igusa). The type will be denoted by (H, ψ) for short.

The functional equation (2.1) becomes simplified as follows:

$$\theta(u + g) = \theta(u)\psi(g) \exp \pi[H(g, u) + \tfrac{1}{2}H(g, g)] \tag{2.3}$$

for a *reduced theta* of type (H, ψ).

PROP. 2.5. If θ has the type (H, ψ, Φ, L) relative to G, $\theta = \theta_0\theta'$ where θ_0 is *reduced* and θ' is trivial with $\theta'(0) = 1$.

PROP. 2.6. *Any divisor $\mathfrak{D}$ of E/G is the divisor (θ) over E/G of a theta function reduced with respect to G. θ is determined by $\mathfrak{D}$ upto a constant non zero factor. If $\mathfrak{D}$ is positive and (H, ψ) is the type of θ, H is positive definite.*

In other words: *the complete linear system $|\mathfrak{D}|$ can be identified canonically with the quotient projective space of a t-dimensional vector space $\Theta(\mathfrak{D})$ of reduced theta functions.* $(t = \operatorname{Pfaff} A)$.

DEF. 2.5. Let (H, ψ) be the symbol of the reduced theta functions attached to $|\mathfrak{D}|$ by Prop. 2.6. Since (H, ψ) are uniquely determined by $|\mathfrak{D}|$ we shall say that $|\mathfrak{D}|$ *has type* (H, ψ) or equivalently $|\mathfrak{D}|$ has type (A, ψ) (cf. § 1).

REMARKS. 1) If $t_1 \geqslant 3$, *H can be identified with the positive definite Hermitean form of the Kähler structure of $\mathcal{A}$ in $\mathbb{P}^{t-1}$ and A with its imaginary part.* It is not fair to use the condition that H is positive definite just to provide a characteristic convergence condition, as the pioneers did!

2) To make the link with the classical language, used today by Rauch, Farkas, etc. easier, it is convenient to point out that ψ plays the role of the characteristics $\begin{bmatrix} g \\ h \end{bmatrix}$ of the standard Krazer's books, in an intrinsic way the precise relationship is given in § 8, with special regard to the « half-integers » characteristics necessary for the Wirtinger case. In particular the « main case of characteristic zero is equivalent to the case that ψ is defined in terms of a natural satellite from B of A defined below (cf. Def. 2.7) in terms of a modular basis.

DEF. 2.6. An integral valued form $B\colon G\times G \to \mathbb{Z}$ is called a *satellite form of A* (Weil [2]) iff

$$A(a, b) = B(a, b) - B(b, a)\,, \qquad \big(\forall (a, b) \in G\times G\big)\,. \tag{2.4}$$

Such forms always exist, for instance if $A\colon (x, y) \mapsto \sum t_j(x_j y_{j+n} - y_j x_{j+n})$ have:

$$B(x, y) = \sum_{j=1}^{n} t_j x_j y_{j+n} \tag{2.5}$$

is a satellite form of $A\colon (x, y) \to \sum t_j(x_j y_{j+n} - x_{j+n} y_n)$. If B is a satellite form any other has the form $B + S$ where $S\colon G\times G \to \mathbb{Z}$ is $\mathbb{Z}$-bilinear symmetric.

DEF. 2.7. The satellite form defined in (2.5) will be called *natural* or of *characteristics zero* with respect to the modular basis $\mathfrak{g}$ of G.

PROP 2.7. *Let $\mathfrak{g}$ be a satellite form of A. Then the sign formula*

$$\psi_B\colon g \to \psi_B(g) = \exp \pi i B(g, g)\,, \qquad g \in G \tag{2.6}$$

defines a semicharacter of G satisfying the property $(\psi_B(g))^2 = 1$, $\forall g \in G$.

From the functional equation (1) follows

PROP. 2.8. *A semicharacter $\psi\colon G \to \mathbb{Z}$ with respect to $A|G\times G$ is a character iff the first elementary divisor t_1 is even.* However:

PROP. 2.9. *The restriction of a semicharacter ψ of G* (with respect to A) *to a cyclic submodule $\mathbb{Z}g$ $(g \in G)$ is a character.* Furthermore ψ^2 is always a *character* of the additive group G. The important « half integer » case arise, when $\psi^2(g) = 1$, $\forall g \in G$, cf. next § 8.

Because of (2.2) we have:

PROP. 2.9. *The quotient of two semicharacters of G attached to A is a character. The product of a character by a semicharacter is a semicharacter. Any semicharacter ψ has the form $\psi_B \chi$ where ψ_B is* defined in (2.6) in terms of any satellite form B of A (cf. Def. 2.6) and χ is a suitable character of G, uniquely defined by the choice of B.

DEF. 2.8. *If ψ is a semicharacter of G with respect to $A|G \times G$ the map $\psi^{-1}: G \to \mathbb{Z}$ defined by*

$$\psi^{-1}(g) = \psi(-g)\,, \qquad \forall g \in G \tag{2.7}$$

is a new semicharacter of G with respect to $A|G \times G$, called naturally the inverse of ψ.

REMARK 3. In the case $n = 1$ the Weierstrass theory of the σ functions can be regarded as the first attempt to introduce the theta functions in terms of divisors. The simplest positive divisor on an elliptic curve contains just one point. If this is the image of 0, it can be represented by $u \to \sigma(u)$. In general the divisor $\sum n_j a_j$ corresponds bijectively to $\prod (\sigma(u - a_j))^{n_j}$. H. Rauch raised the question of its relationship with the reduced thetas. Comparing (2.3) with the well-known functional equation of the σ we check that the σ are not *necessarily reduced.* Thus the reduced thetas offer a simplification of the Hurwitz-Klein theory even in the elliptic case.

3. – The reduced symplectic structure of the collineation group $\bar{G}/G$.

The Remark at the end of § 1 suggests:

DEF. 3.1. The $\mathbb{Z}$-submodule $\bar{G}$ of E defined by

$$\bar{G} = \{\bar{g} \in E | A(\bar{g}, g) \in \mathbb{Z},\ \forall g \in G\} \tag{3.1}$$

is a lattice containing G as a sublattice, called the *integral completion* or *completion* of G with respect to A, (cf. *Introduction*) $\bar{G}$ appears naturally in any intrinsic study of $|\mathfrak{D}|$ (cf. for instance, Igusa's book, : 4, page 80).

The reader can check the following immediate property

PROPOSITION 3.1. $\bar{G}$ is generated by any Siegel basis $\mathfrak{z}$ (attached to some modular basis $\mathfrak{g}$ of G) (cf. Def. 1.3 and 1.7).

Thus $\bar{G}$ *is a lattice of maximal rank* $2n$ *containing* G. We have $\bar{G} = G$ iff $T = I_n$ i.e. when G is *complete* with respect to A.

REMARK. The restriction of A to $\bar{G}\times\bar{G}$ *is not integral valued*; we have instead $A(\bar{G}\times\bar{G})\subset Q$ *since* $\mathfrak{L}^{-1}$ represents A in a Siegel basis $\mathfrak{z}$ i.e. A is represented by

$$\sum_{j=1}^{n} t_j^{-1}(x_j y_{j+n} - x_{j+n} y_j) \tag{3.2}$$

G *is characterized by the* following property, which «reverses» Def. 3.1 somehow

$$G = \{g \in \bar{G} | A(g, \bar{g}) \in \mathbb{Z},\ \forall \bar{g} \in \bar{G}\}\,. \tag{3.3}$$

PROPOSITION 3.2. *Any modular transformation* μ *has also the property* $\mu(\bar{G}) = \bar{G}$ (i.e. $\mathfrak{M}(G)$ *is a subgroup of* $\mathfrak{M}(\bar{G})$).

There is an action $\mathfrak{M}(G)\times(\bar{G}/G)\to\bar{G}/G$ defined by

$$s + G \mapsto \mu s + G = T_\mu(s + G)\,, \qquad \forall \mu \in \mathfrak{M}(G)\,. \tag{3.4}$$

To avoid the change of sign required by the Siegel's bases (cf. Def. 1.7) we shall introduce

DEF. 3.2. A basis $\mathfrak{a}$ of $\bar{G}$ is called *canonical* (with respect to A) iff

$$A(a_j, a_{j+n}) = -A(a_{j+n}, a_j) = t_j^{-1}\,, \qquad j = 1, 2, \ldots, n \tag{3.5}$$

$A(a_p, a_q) = 0$ otherwise, $a_j \in \mathfrak{a}$ $(j = 1, 2, \ldots, 2n)$.

Any canonical basis $\mathfrak{a} = \mathfrak{a}_1\mathfrak{a}_2$ of $\bar{G}$ is related to a modular basis of G (s. Def. 1.3) by $\mathfrak{a}_j = \mathfrak{g}_j T$ $(j = 1, 2)$ and conversely $\mathfrak{g} \Rightarrow \mathfrak{a}$ canonical basis of $\bar{G}$.

PROP. 3.3. *The group* $\mathfrak{C} = \bar{G}/G$ *is a finite subgroup of* E/G. Any canonical basis $\mathfrak{a}$ (cf. Def. 3.2) (or Siegel basis $\mathfrak{z} = (\mathfrak{z}_1; \mathfrak{z}_2)$) (cf. Def. 1.7) induces a direct sum decomposition $\mathfrak{C} = \mathfrak{C}_1 \oplus \mathfrak{C}_2$ where

$$\mathfrak{C}_1 = \{s_j + G | s_j \in \mathfrak{z}_1\}\,, \quad \mathfrak{C}_2 = \{s_{j+n} + G | s_{j+n} \in \mathfrak{z}_2\}\,, \quad j = 1, 2, \ldots, n\,. \tag{3.6}$$

Both are of type $(t_1, t_2, \ldots, t_n)$, i.e. they are decomposable as direct sums of cyclic groups of order t_j $(j = 1, 2, \ldots, n)$. As a consequence, $\mathfrak{C}_1 \approx \mathfrak{C}_2$. *Most of the computations used here will use basis of type* $\mathfrak{a}$.

The introduction of $\bar{G}$ and $\bar{G}/G = \mathfrak{C}$ is geometrically justified by

THEOREM I. *A torus translation* $u \mapsto u + a$ *leaves invariant the complete linear sistem of divisors* $|\mathfrak{D}|$ iff $a + G \in \bar{G}/G$. As a consequence we have:

1) *The finite group* $\mathfrak{C} = \bar{G}/G$ *is isomorphic with the full subgroup of the torus group* E/G *leaving invariant the complete linear system of divisors* $|\mathfrak{D}|$. *If* $t_1 \geqslant 3$, $\mathfrak{C}$ *is isomorphic to a collineation group of* $\mathbf{P}^{t-1}$ *leaving invariant the non singular, normal, Abelian variety* $\mathfrak{a} = \theta(E/G) \subset P^{t-1}$ (cf. footnote of page 46).

2) *The canonical projection* $\bar{G} \xrightarrow{\gamma} \bar{G}/G$ *induces a « reduced form »* $\mathfrak{K} = \mathfrak{C} \times \mathfrak{C} \to \mathbf{Q}/\mathbf{Z}$ *biadditive and antisymmetric with values in the additive group of rationals mod* 1.

3) *Let* $\mathfrak{m}(G)$ *be the « reduced modular group » of* $\mathfrak{C}$ *with respect to* $\mathfrak{K}$ *i.e. the subgroup of the automorphism group of* $\mathfrak{C}$ (*as a pure Abelian group*) *leaving invariant* $\mathfrak{K}$; *in symbols*:

$$\tilde{\mu} \in \mathfrak{m}(G) \Leftrightarrow \mathfrak{K}(\tilde{\mu}\alpha, \tilde{\mu}\beta) = \mathfrak{K}(\alpha, \beta), \quad \forall (\alpha, \beta) \in \mathfrak{C} \times \mathfrak{C}. \tag{3.7}$$

Then $\mathfrak{m}(G)$ is functorially related to $\mathfrak{M}(G)$ by γ as follows:

Let $\mathfrak{a}$ *be a basis of* $\bar{G}$ *canonical with respect to* A (s. Def. 3.2) *then* $\gamma(\alpha)$ *is a basis of* $\mathfrak{C}$ *canonical with respect to* $\mathfrak{K}$, i.e.:

$$\mathfrak{K}(\alpha_j, \alpha_{j+n}) = -\mathfrak{K}(\alpha_{j+n}, \alpha_j) = t_j^{-1} + \mathbf{Z}, \quad j = 1, 2, \ldots, n \tag{3.8}$$

and $\mathfrak{K}(\alpha_p, \alpha_q) = 0$ *otherwise, for* $\alpha_j \in \gamma(\alpha)$ $j = 1, 2, \ldots, 2n$.

The induced map $\mathfrak{M}(g) \xrightarrow{h} \mathfrak{m}(G)$ *is a surjective group homomorphism. The kernel of* h *is the congruende subgroup* $\Delta(G)$ *of* $\mathfrak{M}(G)$

$$\mathfrak{m}(G) \sim \mathfrak{M}(G)_a \Delta(G). \tag{3.9}$$

PROOF. Let θ be a reduced theta function (cf. Def. 2.4) representing $\mathfrak{D}$. Then if θ has type (H, ψ), the map $u \mapsto \theta(u + a)$ is again a theta function of type (H, ψ_1, Φ, L_1) relative to G where

$$\psi_1(g) = \psi(g) \exp 2\pi i A(g, a), \quad L_1(u) = L(u) + \frac{1}{2i} H(a, u) + \frac{1}{2i} \Phi(a, u). \tag{3.10}$$

(cf. § 2 and Weil [2], prop. 3, page 111).

If θ is reduced then $u \mapsto (\theta(u + a)) \exp(-\pi H(a, u))$ is also reduced of type (H, ψ_1). Thus, this function represents a divisor of $|\mathfrak{D}|$ iff $\psi_1 = \psi$. This implies the condition $A(g, a) \in \mathbf{Z}$ for any $g \in G \Leftrightarrow a \in \bar{G}$ (s. Def. 3.1).

This proves that $\mathfrak{K}$ is well-defined by

$$\mathfrak{K}(\alpha, \beta) = A(a, b) + Z \tag{3.11}$$

where $\alpha = a + G$, $\beta = b + G$, $a, b \in \bar{G}$, i.e. the definition of $\mathfrak{K}$ by (3.11) is independent of the representatives.

The biadditive property of $\mathfrak{K}$ and $\mathfrak{K}(\alpha, \beta) = -\mathfrak{K}(\beta, \alpha)$ are evident. Any $\mu \in \mathfrak{M}(G)$ induces a $\tilde{\mu}: \mathfrak{C} \to \mathfrak{C}$ preserving $\mathfrak{K}$ since $\mathfrak{K}$ is defined in (3.11) in terms of representatives. The map $\mu \to \tilde{\mu}$ is surjective since any canonical basis $\mathfrak{a}$ of $\mathfrak{C}$ can be lefted to a $\mathfrak{g}$-parallelotop for a canonical $\mathfrak{g}$ of G and, as a consequence $\mathfrak{a}$ is liftable to a canonical basis $\mathfrak{a}$ of $\bar{G}$. Applying his remark to a couple of bases of $\mathfrak{C}$ we lift any $\tilde{\mu}$ to $\mathfrak{M}(G)$.

REMARKS. 1) Th. I gives an algebraic interpretation of T since the elementary divisors t_j are the common full set of invariants of $\mathfrak{C}_1$ or $\mathfrak{C}_2$, and a fortiori Th. I, *interprets algebraically the type* $(e_1, e_2, \ldots, e_{n-1})$ *of the polarization of* E/G defined by $|\mathfrak{D}|$ (s. Def. 1.1):

2) $\Delta(G)$ is an invariant subgroup of $\mathfrak{M}(G)$. The isomorphism class of $\Delta(G)$ depends only on T. Siegel calls it the « *Kongruenzuntergruppe der Stufe* T » $\Delta(T)$ because of the following

PROP. 3.4. Let $\mathfrak{z} \overset{\mu}{\mapsto} \mathfrak{z}'$ be a modular transformation. Then μ belongs to $\Delta(G)$ iff

$$s_j \equiv s_j' \bmod G\,, \qquad j = 1, 2, \ldots, 2n \tag{3.12}$$

where s_j, s_j' are the elements of the Siegel bases $\mathfrak{z}$, $\mathfrak{z}'$ (cf. Def. 1.7).

It is an easy exercise to check that Siegel's matrix definition of $\bar{G}(T)$ is equivalent to ours when we choose Siegel's bases in E. Then a Siegel modular matrix belongs to $\Delta(T)$ iff it congruent to 1 mod T. The elements of $\mathfrak{m}(G)$ may be represented by « matrices mod T » defined in a natural way that we don't need to precise here. (cf. Siegel, 2, § 7, page 399).

We shall use in § 0 the map $\chi: \mathfrak{C} \times \mathfrak{C} \to \exp 2\pi i \mathbf{R}$ naturally defined by composition in the diagram

$$\begin{array}{ccc} \mathfrak{C} \times \mathfrak{C} & \longrightarrow & \mathbf{Q}/\mathbf{Z} \\ & \overset{\chi}{\searrow} & \downarrow \exp \\ & & \exp 2\pi i \mathbf{R} \end{array} \tag{3.13}$$

Using representatives we have:

$$\chi(\alpha, \beta) = \exp 2\pi i A(a, b) \tag{3.14}$$

again for $\alpha = a + G$, $\beta = b + G$. χ has the following property (immediately verified), needed in § 10 to construct the Hessian coordinate systems of $\mathcal{A}$.

PROP. 3.5. *For α fixed the map*

$$\beta \to \chi(\alpha, \beta) \tag{3.15}$$

is a character of $\mathfrak{C}$.

4. – Sharpening Andreotti-Igusa's duality between E/G and $E/\bar{G}$. Division of $|\mathfrak{D}|$.

PROP. 4.1. Let θ be a reduced theta of type (H, ψ) (cf. Def. 2.4); let $B\colon G \times G \to \mathbb{Z}$ be a satellite form of A (s. Def. 2.7). Let $\bar{G}$ be the completion of G (s. Def. 3.1). Then *there is a unique translation $a + \bar{G}$ on $E/\bar{G}$ such that the theta function $u \mapsto (\theta(u + a)) \exp(-\pi H(a, u))$* (relative to G) *is also reduced of type (H, ψ_B) where ψ_B is the semicharacter of type $\sqrt{1}$ defined in* (2.6).

PROOF. Since $\psi(g) = \psi_B(g)\,\chi(g)$ for some character $G \xrightarrow{\chi} \exp 2\pi i\mathbb{R}$ it would be sufficient to check that $\chi(g)$ can be expressed uniquely in the form

$$\chi(g) = \exp 2\pi i A(a, g) \tag{4.1}$$

where $a \in E$ is uniquely defined mod G. In fact $(2\pi i)^{-1} \log \chi(g)$ can be identified with the additive group $\mathbb{R}$ mod 1. Since G has maximal rank any real valued linear form $G \to \mathbb{R}$ is extendable uniquely to a real-valued $\mathbb{R}$-linear form $u \mapsto L(u)$ and since A is not degenerate it can be expressed uniquely in the form $L(u) = A(a, u)$ for a unique a. So that L is defined mod 1. Thus $A(a - a', u)$ is integral valued on G iff $L' - L$ is integral valued on G and $L'(u) = A(a', u)$. One sees as in the proof of Th. I that $a \equiv a'$ mod $\bar{G}$ (cf. Def. 3.1). Conversely, any two a's congruent mod $\bar{G}$ define the same χ.

Prop. 4.1 can be restated proving that *$E/\bar{G}$ is the «Picard variety» of E/G* (cf. Andreotti [1], [2]; Igusa, Lang.) But in order to give a more precise statement we need to use some properties of the division of complete linear systems $|\mathfrak{D}|$ on E/G. Let us recall first that in an abstract algebraic variety M the divisors L linearly equivalent to zero form a $\mathbb{Z}$-submodule $\mathfrak{Z}_l$ of the module $\mathfrak{Z}_a$ of divisors algebraically equivalent to zero. The quotient module $\mathfrak{Z}_a/\mathfrak{Z}_l$ has a structure of Abelian variety, called the Picard variety of M. A well-known duality Andreotti [1], [2], Igusa [1] claims that if $\mathcal{A}^*$ is the Picard variety of $\mathcal{A}$ then $\mathcal{A}^{**} = \mathcal{A}$. The rôle of the polarization was not made explicit in the original statements (but it occurs in the original proofs).

Since the relationship between G and $\bar{G}$ does not appear to be symmetric we try to explain here the phenomenon proving a complement of the duality in terms of the polarization type (cf. Def. 1.1).

Let $\mathcal{E}$ be a divisor on E/G such that $\mathfrak{D} \sim d\mathcal{E}$ ($\sim$ denotes linear equivalence, d a certain positive integer). As in the elliptic case we are interested in determining all the d's and $\mathcal{E}$'s satisfying this property, recalled as *the division of* $|\mathfrak{D}|$ *by* d:

PROP. 4.2. *The division of* $|\mathfrak{D}|$ *by* d *is possible iff* d *divides the degree* ($= t_1$) of A (s. Def. 1.1). If $d|t_1$ *the division is possible in exactly* d^{2n} *linearly inequivalent ways.*

PROOF. Because of Prop. 2.3 if $\mathcal{E}$ has type (A', ψ) (s. Def. 2.2) we have $A = dA'$, $\psi = \psi'^{d}$. If $\tau_1, \tau_2, \ldots, \tau_n$ are the elementary divisors of $A' t_j = d\tau_j$, $j = 1, 2, \ldots, d$ and this is possible iff $d|t_1$, (s. (1.10)).

Conversely, these conditions are sufficient. Since ψ is uniquely determined by $\psi(\mathfrak{g})$ for any modular basis $\mathfrak{g}$ of G (s. Def. 1.3) it is sufficient to select anyone of the possible d^{2n} choices of $\psi(g_j)$ such that $(\psi'(g_j))^d = \psi(g_j)$, $(j = 1, 2, \ldots, 2n)$ to obtain a « possible ψ ». Conversely an easy checking shows that any one of these d^{2g} choices really defines a semicharacter of G attached to A such that $\psi'^{d} = \psi$.

COROLLARIES. 1) *Any two solutions* $|\mathcal{E}_1|$, $|\mathcal{E}_2|$ *of the division of* $|\mathfrak{D}|$ *by* d (assuming $d|t_1$) *are algebraically equivalent.* This implies that *the algebraic division by* d *is always possible in a unique way iff* $d|t_1$.

2) $|\mathfrak{D}|$ *is indivisible* (both mod $\mathfrak{Z}_a$, $\mathfrak{Z}_l$ (s. § 8.1) *iff* $t_1 = 1$).

3) *There are always indivisible linear and algebraic systems of polar divisors on* $E/\bar{G}$.

THEOREM II. *Let* $E/G = \mathcal{A}$ *be algebraically polarized of type* e *by* $|\mathfrak{D}|$ (s. Def. 1.1) ($\Leftrightarrow$ polarized by A). *Let* G *be the completion of* G *with respect to* A (s. Def. 3.1). *Then* E/G *is the Picard variety of* $E/\bar{G}$, *algebraically polarized by* $t_n A|\bar{G}\times\bar{G}$ *of type* e^* (s. Def. 1.1).

Iterating, the construction, let $\bar{\bar{G}}$ *be the completion of* $\bar{G}$ *with respect to* $t_n A$. *Then* $E/\bar{\bar{G}} = \mathcal{A}^{**}$ *is canonically isomorphic with* $\mathcal{A}$.

We know that two divisors $\mathfrak{D}_1$, $\mathfrak{D}_2$ of E/G are algebraically equivalent iff there is a torus translattion t_a in E/G such that $T_a(\mathfrak{D}_1)$ is linearly equivalent to $\mathfrak{D}_2$ (cf. [4.1]). As a consequence for a variable $T_a(\mathfrak{D})$ one obtains a representative for any linear equivalence class contained in the algebraic class of $\mathfrak{D}$ and since $T_a(\mathfrak{D})$ is linearly equivalent to $\mathfrak{D}$ *iff* $a \in \bar{G}$ (cf. Prop. 4.1) (we see immediately that $E/\bar{G} = (E/G)/\mathfrak{C}$ represents bijectively the set of linear equivalent classes contained in the algebraic class $\{\mathfrak{D}\}$. But $t_n A$

satisfies « Riemann existence conditions » for $\bar{G}$ (it is non degenerate, integral valued on $\bar{G}$ and $u \to t_n A(iu, u) = t_n H(u)$ is also positive definite). Thus $E/\bar{G}$ is a polarized Abelian variety.

Furthermore if $\mathfrak{g} = (\mathfrak{g}_1; \mathfrak{g}_2)$ is a basis of G modular with respect to $A|G \times G$ the basis

$$\mathfrak{a}^* = (a_n, a_{n-1}, \ldots, a_1; a_{2n}, a_{2n-1}, \ldots, a_{n+1}) = (\mathfrak{a}_1^*; \mathfrak{a}_2^*)$$

obtained reversing the ordering of both halves $\mathfrak{a}_1$, $\mathfrak{a}_2$ of the modular basis $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2)$ (s. Def. 1.3) associated with $\mathfrak{g}$, is again modular of type e^* for $t_n A$. Let us apply the same construction to $E/\bar{G}$: Then, we obtain a torus $E/\bar{\bar{G}} \approx E/G$ as a polarized Abelian variety, because the canonical basis b of $\bar{\bar{G}}$ associated to $\mathcal{A}$ (regarded as a modular basis of $\bar{\bar{G}}$ with respect to $t_n A$) is a modular basis of G with respect to $t_n^2 A$. The isomorphism induced by the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\ t_n^{-1}\ } & E \\ \uparrow & & \uparrow \\ G & \xrightarrow{\ t_n^{-1}|G\ } & \bar{\bar{G}} \end{array}$$

where the vertical bars are inclusions, t_n^{-1} symbolizes $u \mapsto t_n^{-1} u (\forall u \in E)$ and $t_n^{-1}|G$ is its restriction to G (inducing a bijection $G \leftrightarrow \bar{\bar{G}}$). Finally the trivial verification $A(x, y) = (t_n^2 A)(t_n^{-1} x, t_n^{-1} y)$, $(\forall (x, y) \in E \times E)$ completes the proof.

Remark. Since the polarization type e has an algebraic interpretation in terms of the collineation group $\bar{G}/G$ (which is algebraically defined) and since there is a non-singular projective embedding for any $t_1 \geqslant 3$ we can reformulate Theorem II algebraically in terms of non singular normal projective algebraic varieties. Mumford does it effectively in [2] as an easy consequence of the fact that $\mathfrak{C}$ is a two-step nilpotent group and this allows us to introduce the reduced symplectic structure of $\mathfrak{C}$ in a pure algebraic way. The next property (and similar results) can be established easily to strenghten the extension of Klein and Hurwitz research to the general $\mathcal{A}$:

Prop. 4.3. *The d^{2n} linear equivalence classes $\mathcal{E}$ satisfying $d\mathcal{E} \sim \mathfrak{D}$ $(d|t_1)$ contain just one effective divisor $d = t_j$ $(j = 1, 2, \ldots, n)$. In particular this happens in the Kummer-Wirtinger case.*

The division theory of $|\mathfrak{D}|$ is well-known, also with the pure algebraic approach (cf. Lang). We shall complete the study of the collineation group $H(\mathcal{A})$ by introducing the symmetries (cf. § 7).

5. – Distinguished sublattices and overlattices of G.

The intrinsic construction of suitable bases of reduced theta functions representing $|\mathfrak{D}|$ (cf. § 10) requires the introduction of certain overlattices of G; since they are defined in terms of some sublattices, we start with them.

DEF. 5.1. Let $\mathfrak{g} = (\mathfrak{g}_1; \mathfrak{g}_2)$ be a modular basis of G (cf. Def. 1.3). We shall need the two sublattices G_1, G_2 defined by

$$G_j = \mathbb{Z}\mathfrak{g}_j\,, \quad j = 1, 2\,. \tag{5.1}$$

They have the properties

1)
$$G = G_1 \oplus G_2 \tag{5.2}$$

2)
$$A|G_j \times G_j = 0\,, \quad j = 1, 2 \tag{5.3}$$

3) A induces a non singular bilinear pairing between G_1 and G_2.

We shall need some types of lattices L satisfying the properties:

$$G \subset L \subset \bar{G}\,, \quad A(L\times L) \subset \mathbb{Z}\,. \tag{5.4}$$

DEF. 5.2. An overlattice L of G (i.e. satisfying (5.2) is called of the τ-type iff the matrix $\tau = \operatorname{diag}\{\tau_1\tau_2 \dots \tau_n\}$ $(1|\tau_1|\tau_2| \dots |\tau_n)$ indicates the elementary divisors of $A|L\times L(\tau_i \in \mathbb{Z},\ i = 1, 2, \dots, n)$.

We are particularly interested in the A-overlattices of G of the unit type: $\tau = 1$ (« *1-overlattices* » for short). We have obviously:

PROP. 5.1. *Any modular transformation $\mu \in \mathfrak{M}(G)$ transforms an overlattice $L \subset G$ in another overlattice of the same type. In particular 1-overlattices are permuted by the action of $\mathfrak{M}(G)$.*

REMARKS. 1) There exists always overlattices of the 1-type. Precisely we are mainly interested in those of the form

$$\Gamma_1 = \mathbb{Z}\mathfrak{a}_1 \oplus G_2\,, \quad \Gamma_2 = G_1 \oplus \mathbb{Z}\mathfrak{a}_z \tag{5.5}$$

where $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2)$ is a canonical basis of $\bar{G}$ (cf. Def. 3.2) and G_1, G_2 are those of Def. 5.1 where $(\mathfrak{a}_j = \mathfrak{g}_j T \ (j = 1, 2))$, and $\mathfrak{g} = (\mathfrak{g}_1, \mathfrak{g}_2))$ is a modular basis of G associated with $\mathfrak{a}$.

2) Γ_1 and Γ_2 satisfy the following conditions

a) Both Γ_j are of the unit type

b) $\Gamma_1 \cap \Gamma_2 = G$, $\Gamma_1 + \Gamma_2 = \bar{G}$

c) $A|\Gamma_j \times \Gamma_j = 0$, $j = 1, 2$

d) A induces a non singular pairing between Γ_1 and Γ_2:

The reduction mod G of these properties according to Th. I is

a') The subgroups $\mathfrak{C}_j = \Gamma_j/G \ (j = 1, 2)$ are finite and isomorphic of type T $\mathfrak{C}_1 \cap \mathfrak{C}_2 = 0$, $\mathfrak{C} = \mathfrak{C}_1 \oplus \mathfrak{C}_2$;

b') $\mathfrak{K}(\sigma_j, \sigma_{j+n}) = t_j^{-1} + \mathbb{Z} \Leftrightarrow \chi(\sigma_j, \sigma_{j+n}) = \exp 2\pi i t_j^{-1}$ where $\sigma_j = s_j + G$, $s_j \in \mathfrak{z}$, $j = 1, 2, \dots, 2n$.

DEF. 5.3. A subset $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 = (T_1, T_2, \dots, T_n; T_{n+1}, \dots, T_{2n})$ of $2n$ self-collineations T_j of P^{t-1} belonging to $\mathfrak{C}$ is called *a canonical basis* of $\mathfrak{C}$ iff:

1) T_j and T_{j+n} are cyclic of degree t_j $(j = 1, 2, \dots, n)$.

2) $\mathcal{T}_1 = \{T_1, T_2, \dots, T_n\}$ and $\mathcal{T}_2 = \{T_{n+1}, \dots, T_{2n}\}$ generate two subgroups $\mathfrak{C}_1$ (or $\mathfrak{C}_2$) isomorphic to the direct sum of the n cyclic subgroups $\{T_j^{m_j}\}\{T_{j+n}^{m_j}\}$ respectively $(m_j = 0, 1, \dots, t_j - 1; j = 1, 2, \dots, n)$.

3) The T_j's satisfy the conditions:

$$\chi(T_j, T_{j+n}) = \big(\chi(T_{j+n}, T_j)\big)^{-1} = \mathfrak{z}_j \tag{5.6}$$

$$\chi(T_r, T_s) = 1 \qquad \text{otherwise} \tag{5.7}$$

where ε_j will denote in the sequel $\exp 2\pi i t_j^{-1}$; $j = 1, 2, \dots, n$. As a consequence we have:

$$\chi\left(\prod_{j=1}^{n} T_j^{m_j}, \prod_{j=1}^{n} T_{j+n}^{u_j}\right) = \exp 2\pi i \left(\sum_{j=1}^{n} m_j \mu_j t_j^{-1}\right) = \prod_{j=1} \varepsilon_j^{m_j \mu_j}. \tag{5.6}$$

REMARK. Notice that (5.4), (5.5), (5.6) are nothing else that the « exponential restatement » of the fact that $\mathfrak{C}$ has a reduced symplectic structure and the basis defined in (5.3) is *canonical* in the symplectic sense.

6. – Extensions of a semicharacter.

Let us keep the notations of the § 5. Since $A|\Gamma_j \times \Gamma_j \subset \mathbb{Z}$ $(j = 1, 2)$ it makes sense to consider semi-characters on Γ_j, relative to A. To construct the Hessian coordinate systems in § 11 we shall need to know all the extensions of a given semi-character ψ of G to Γ_1 (or to Γ_2). We shall fix an over-lattice $\Gamma = \mathbb{Z}\mathfrak{a}_1 + G$ where $(\mathfrak{g}_1, \mathfrak{g}_2)$ is a symplectic basis of the 1-over-lattice L of G $(\Leftrightarrow A|L \times L \subset \mathbb{Z}$ has type 1). Let $(\mathfrak{g}_1, \mathfrak{g}_2)$ be a modular basis of G $(A|G \times G$ of type T, $\mathfrak{a}_1 = \mathfrak{g}_1 T)$.

Def. 6.1. A semicharacter ψ_Γ of Γ attached to $A|\Gamma$ is called *an extension of* ψ *iff*

$$\psi_\Gamma | G = \psi \,. \tag{6.1}$$

Prop. 6.1. *There is one and only one extension* ψ_Γ *of any given semicharacter* $\psi\colon G \to \mathbb{C}_1^\times$ *attached to* $A|G \times G$ *to a semicharacter* ψ_Γ *of* Γ *attached to* $A|\Gamma \times \Gamma$ *assuming prescribed values integral* $\psi_\Gamma(a_j)$ *for every* $a_j \in \mathfrak{a}_1$, $j = 1, 2, \ldots, n$ *satisfying the necessary conditions*

$$\big(\psi_\Gamma(a_j)\big)^{t_j} = \psi_\Gamma(t_j a_j) = \psi(t_j a_j)\,, \qquad j = 1, 2, \ldots, n\,. \tag{6.2}$$

Corollary. *There are exactly* $t = t_1 t_2 \ldots t_n =$ Pfaff A *prolongations of* ψ *from* G *to* Γ.

Proof. The restriction of the functional equation (2.6) for a, b belonging to a cyclic subgroup of G reduces to $\psi(ab) = \psi(a)\psi(b)$. As a consequence the value $\psi_\Gamma(a_j)$ should satisfy (6.2). Conversely, (6.2) determines uniquely the value of ψ_Γ at any element of G_1 and we can check that actually ψ_Γ is a semicharacter of Γ prolonging ψ.

Another direct proof can be given using a satellite form of A:

Let B be a satellite form of A in Γ (cf. Def. 2.3) and B_G its restriction to G. Obviously B_G is a satellite of $A|G$. We know that every semicharacter of $L(G)$ attached to A has the form $\psi_B \chi\; g \to \big(\exp \pi i B(g, g)\big)\chi(g)$ (for every $g \in L(G)$) where χ is a character of $L(G)$. The multiplicative group $\exp 2\pi i \mathbb{R}$ is injective, thus *every character of* G *can be extended to a character of* L. We can use Prop. 2.4 and show as before that a character χ of G has t extensions $\chi_1, \chi_2, \ldots, \chi_t$ to L and $\psi_B \chi_j$ $(j = 1, 2, \ldots, t)$ give all the extensions of ψ.

We are going to use in § 11 a standard ordering of the extensions of ψ to L.

DEF. 6.2. Let $(m_1, m_2, \ldots, m_n)$ be a reduced representative mod T where $m_j \in \mathbb{Z}$ and takes the values

$$m_j = 0, 1, \ldots, t_j - 1 \quad \text{for } j = 1, 2, \ldots, n \,. \tag{6.3}$$

We define the extension $\chi[m_1, m_2, \ldots, m_n]$ of the character $\chi\colon G \to \mathbf{C}_1^\times$ to L by the conditions

$$\chi[m_1 m_2, \ldots, m_n](a_j) = \exp 2\pi i m_j t_j^{-1} \,, \quad j = 1, 2, \ldots, m \tag{6.4}$$

and

$$\psi[m_1 m_2, \ldots, m_n] = \psi[0, 0, \ldots, 0]\chi[m_1 m_2, \ldots, m_n] \tag{6.5}$$

where $\psi[0, 0, \ldots, 0](a_j)$ is equal to the t_j^{th} root of $\psi(g_j)$ with minimal argument φ $(0 \leqslant \varphi < 2\pi)$.

The standard ordering of the prolongations $\psi_1, \ldots, \psi_t$ *is the lexicographical ordering of the* $[m_1 m_2, \ldots, m_n]$ *satisfying* (6.3).

REMARK. *A reduced theta function of type* (H, ψ_Γ) *can be regarded also as a reduced theta function of type* (H, ψ_G), $(\psi_G = \psi_\Gamma | G)$.

Since $A|\Gamma \times \Gamma$ has unit type (cf. Def. 5.2) the vector space of theta functions of type (H, ψ_Γ) is one dimensional. Any non-zero θ of this space represents a unique divisor $\mathcal{E}$ on the torus E/Γ. Lifting $\mathcal{E}$ to E/G by means of the canonical map $E/G \xrightarrow{\pi} E/\Gamma$ one obtains a divisor $\mathfrak{D} = \mathcal{E} \circ \pi$ of E/G invariant by an Abelian finite subgroup $\mathfrak{C}_1$ of type T of the torus group E/G (cf. next § 9).

7. – Symmetries. The fix-point set and the full collineation group $\mathcal{K}(\mathcal{A})$.

We need to study the geometrical meaning of ψ (cf. Int., § 2 and next § 8). To do that we need to recall in this § 7 some well-known facts, essentially that *if E/G has general moduli* there are no other birational transformations of the abstract Abelian variety E/G in itself besides the torus translations and the symmetries, (defined below (cf. Conforto's book). (cf. next Prop. 7.1).

Let us start recalling the definition of a *symmetry* (1).

DEF. 7.1. The mapping $S_\omega\colon E/G \to E/G$ characterized by

$$u + S_\omega(u) = \omega \,, \quad \forall u \in E/G \tag{7.1}$$

(1) We avoid the classical terminology of maps of 1-st or 2-nd kind because of the reasons pointed out by Conforto, Ch. II, § 10, page 175, footnote. The distinction between translations and symmetries is much clearer.

is called a *symmetry of type* ω ($\omega \in E/G$). If S_ω leaves invariant $|\mathfrak{D}|$ *we say that* S_ω *is a symmetric collineation.*

Obviously $S_\omega \neq S_\tau$ if $\omega \neq \tau$. Besides, S_ω is *involutive*, i.e. $S_\omega^2 = id$ for any $\omega \in E/G$.

Usually (7.1) is defined in terms of a lifting to E:

$$u + \textstyle\sum_\omega u \equiv a, \ \mathrm{mod}\, G, \qquad \omega = a + G. \tag{7.1'}$$

DEF. 7.2. We call *fix-point* of $\mathcal{A} = \Theta(E/G)$ any point γ fixed by some symmetric collineation S_ω (cf. Def. 7.1).

A coincidence point γ of any symmetry S_ω (not necessarily a collineation) is characterized by $2\gamma = \omega$ ($\gamma \in E/G$). As a consequence we have:

Any symmetry S_ω *has* 2^{2n} *coincidence points.*

PROP. 7.1. Let S_ω be a symmetry, let T_α be a torus translation ($\omega, \alpha \in E/G$). We have:

$$T_\alpha S_\omega = S_{\alpha+\omega}, \qquad S_\omega T_\alpha = S_{\omega-\alpha}, \qquad T_\alpha S_\omega T_\alpha^{-1} = S_{\omega+2\alpha} \tag{7.2}$$

in particular,

$$T_\alpha S_\omega T_\alpha^{-1} = S_\omega \quad \text{iff } 2\alpha = 0 \text{ in } E/G, \tag{7.3}$$

which means in words:

PROP. 7.2. *The group of torus translations in* E/G *leaving invariant any prescribed symmetry* S_ω *is identical with the group of periodic elements of period two* (cf. (7.2)). (for any ω) Furthermore, $S_{\omega_2} S_{\omega_1} = T_{\omega_2 - \omega_1}$. (7.2) tell us the following well-known result.

PROP. 7.3. *The set* $\mathfrak{B}(E/G)$ *set of symmetries and torus translations of* E/G *form a Lie group with two connected components, the one containing the identity, coincides with the group of torus translations and the other component contains all the symmetries.* $\mathfrak{B}(E/G)$ *will be called the automorphism group of* E/G *although this is the full automorphism group just when* E/G *has general moduli*:

An immediate consequence is the following:

COROLLARY. *The complete linear system of divisors* $|\mathfrak{D}|$ *is invariant by* $t^2 = \# \mathfrak{C}$ *symmetries. If* S_ω *leaves invariant* $|\mathfrak{D}|$ *any other symmetry with the same property is given by* $S_{\omega+\alpha}$ *for every* $\alpha \in \mathfrak{G}$.

In other words, we have the following property:

PROP. 7.4. *The subgroup* $\mathfrak{K}(\mathcal{A}) \subset \mathfrak{B}(E(G))$ *leaving invariant* $|\mathfrak{D}|$ *is a finite group of order* $2t^2$ ($t = \mathrm{Pfaff}\, A$) *containing* $\mathfrak{C}$ *as a subgroup of index two* (cf. Prop. 7.2).

Prop. 7.4 is also well-known. $\mathcal{H}(\mathcal{A})$ will be called the *full collineation group* of $\mathcal{A}$ with abuse of language, although this is true just if E/G has general moduli.

Our justification to recall these well-known facts is the following Prop. 7.5, which will help us in next § 8 to make clear the rôle of the origin, (cf. next Remark) which has been neglected so far. Let us assume that the origin $0 \in \mathcal{A}$, where $0 = \Theta(0)$, $(0 \in E)$ is one of the centers of a possible symmetry S_0 leaving invariant $|\mathfrak{D}|$. Then we have:

PROP. 7.5. *The transformed function* $u \mapsto \eta(u) = \theta(-u)$ *of a reduced theta of type* (H, ψ) (cf. Def. 2.2) *is a reduced theta of type* (H, ψ^{-1}) (cf. 2.2) and Def. 2.8. As a consequence. *The complete linear system* $|\mathfrak{D}|$ *of type* (H, ψ) (cf. Def. 2.4) *is invariant by* S_0 *iff* $\psi^2 = 1$, i.e. iff $\psi(g) = \psi(-g)$ for every $g \in G$.

The proof is a straightforward verification. Of course, the invariancy of $|\mathfrak{D}|$ in Prop. 7.5 means that any positive divisor of $|\mathfrak{D}|$ is mapped in some other divisor belonging also to $|\mathfrak{D}|$. Let us study now the case when a divisor is fixed by S_ω.

PROP. 7.6. *Let* (H, ψ) *be a vector space of reduced theta functions with* $\psi^2 = 1$. *The divisor* D *represented by a theta function* $\vartheta \in (H, \psi)$ *is invariant by* S_0 *iff* ϑ *is an even or odd function.*

PROOF. Since $\vartheta(-u)$ represents also D and it is also reduced $\vartheta(-u) = \lambda\vartheta(u)$, $\lambda \in \mathbf{C}$ and by iteration we see that $\lambda^2 = 1$. Conversely any even (odd) theta function of (H, ψ) is invariant by the symmetry with respect to the origin.

The unique decomposition $\vartheta(u) = \frac{1}{2}(\vartheta(u) + \vartheta(-u)) + \frac{1}{2}(\vartheta(u) - \vartheta(-u))$ as a sum of an even and odd theta function, which is the same as the direct sum decomposition in eigenspaces corresponding to ± 1 by the mapping induced by S_0 in (H, ψ) describes completely the behavior of S_0 in (H, ψ).

In the recent proof by Farkas-Rauch of the Schottky-Jung relations the parity of the « first order » theta functions play an important rôle. We shall study in the next two sections what is the intrinsic meaning of the parities in terms of H, ψ alone.

8. – Parity properties of the reduced theta functions.

DEF. 8.1. Let (H, ψ) be a vector space of reduced theta functions with $\psi^2 = 1$. A period $g \in G$ is called *even* or *odd* according to the value $\psi(g) = \pm 1$.

WARNING. Def. 8.1 is consistent with the classical criterion for parity when $\psi = \psi_0$ is the natural semicharacter attached to a canonical basis of G when $t_1 = t_2 = \dots = t_n = 1$ (cf. next § 9). However the names even, odd are really bad! *It is false that the sum* $g_1 + g_2$ *of two periods of the same* (*opposite*) *parity is even* (*odd*) *as we can check looking at the perturbing sign factor* $\exp \pi i A(g_1, g_2)$ in the functional equation (2.2) of ψ.

Let D be a divisor invariant by the symmetry S_0. Since the involutive torus translation $u \mapsto u + \gamma/2$ $(\gamma \in G)$ leaves invariant S_0 the translated divisor $T_{\gamma/2}(D)$ should be also invariant by S_0. In the language of the reduced thetas: A reduced theta θ representing D defines by translation $T_{\gamma/2}$ a reduced theta (up to an exp factor). Since θ is either even or odd the same happens to the translated function. The following elementary computation makes precise the behavior in terms of the parity of γ.

PROP. 8.1. *Let* θ *be a reduced theta function of type* (H, ψ) *and* $\psi^2 = 1$. *Let* $\frac{1}{2}\gamma$ *be a half period* $(\gamma \in G)$.

If θ *is even* (odd) *and* $\psi(g) = +1(-1)$ *then the reduced theta function*

$$u \mapsto \theta \begin{bmatrix} \frac{\gamma}{2} \end{bmatrix} (u) = \exp \left(-\pi H \left(\frac{\gamma}{2}, u \right) \right) \theta \left(\frac{\gamma}{2} + u \right) \tag{8.1}$$

(cf. Weil [2], page 111) *is even. In the other two cases* $\theta[\gamma/2]$ *is odd.*

PROOF. Let us prove it in the case that θ is even and $\psi(g) = \pm 1$. (The other cases are totally similar). We have:

$$\begin{aligned} \theta \begin{bmatrix} \frac{\gamma}{2} \end{bmatrix} (-u) &= \exp \left(\pi H \left(\frac{\gamma}{2}, u \right) \right) \theta \left(\frac{\gamma}{2} - u \right) = \theta \left(u - \frac{\gamma}{2} \right) \exp \left(\pi H \left(\frac{\gamma}{2}, u \right) \right) = \\ &= \theta \left(u + \frac{\gamma}{2} \right) \exp \left(\pi H \left(\frac{\gamma}{2}, u \right) \right) \psi(-\gamma) \exp \left(\frac{\pi}{2} \left(H(\gamma, \gamma) + 2H \left(-\gamma, u + \frac{\gamma}{2} \right) \right) \right) = \\ &= \theta \left(u + \frac{\gamma}{2} \right) \exp \left(\pi H \left(\frac{\gamma}{2}, u \right) \right) = -\theta \begin{bmatrix} \frac{\gamma}{2} \end{bmatrix} (u) . \end{aligned} \tag{8.2}$$

Immediate consequences of Prop. 8.1 are the following:

1) If $\psi = \psi_0$ is the natural semicharacter (cf. Def. 2.7) and $t_1 = t_2 = \dots = t_n = 0$, a direct verification shows that the classical first order theta function is even. Accordingly: $u \mapsto \theta[\gamma/2](u) = \exp(-\pi H(\gamma/2, \mathrm{J}))\theta(u)$ is a reduced theta function, such that $\theta[\gamma/2]$ is even or odd according to the property $\psi_0(\gamma) = \pm 1$.

2) If $t_1 = t_2 = \dots = t_n = 2$ and $\psi(g) = 1$ $\forall g \in G$, any « second order theta function » of type $(2H, 1)$ is *even*.

3) PROP 8.2. *Let θ be an even (odd) reduced theta function of type (H, ψ) with $\psi^2 = 1$. Let $\frac{1}{2}\gamma$ be a half period ($\gamma \in G$. Then if $\psi(\gamma) = -1(+1)$ we have*

$$\theta(\tfrac{1}{2}\gamma) = 0 .$$

An immediate direct verification of Prop. 8.2 can be obtained showing that $\theta(-\frac{1}{2}\gamma) = \pm \psi(\gamma)\theta(\frac{1}{2}\gamma) = 0$ since the exponent of the exponential factor $\pi[H(\gamma_1 - \gamma/2) + \frac{1}{2}H(\gamma_1\gamma)]$ vanishes (cf. (2.3)).

9. – The semicharacter and the characteristics.

It is clear for any knowledgeable reader that ψ « plays the same role as the *characteristics* $\begin{bmatrix} g \\ h \end{bmatrix}$ » in the classical Krazer's approach. As a matter of fact the introduction of ψ is a progress, since it is intrinsic and the old $\begin{bmatrix} g \\ h \end{bmatrix}$ are not. To make precise the relationship let us choose an origin $O = \Theta(0)$ in $\mathcal{A}$ and a modular basis $\mathfrak{g}$ of G (cf. Def. 1.4) and the Def. 8.1. Let (H, ψ) be a vector space of reduced theta functions with $\psi^2 = 1$. Let ψ_B: $G \to C_1^*$ be the semicharacter defined in terms of the natural satellite form B (2.11) (cf. Def. 2.7) relative to $\mathfrak{g}$. According to Prop. 2.7 and Prop. 7.1 we have:

$$\psi(g) = \psi_B(g)\,\chi(g) = \psi_B(g) \exp 2\pi i A(c, g)\,, \qquad \forall g \in G \tag{9.1}$$

where χ is a well defined character of G and c is a vector of E *defined modulo* $\bar{G}$. Def. 9.1 is our « linking » formal definition; it is convenient to explain first, informally what the role of both intrinsic invariants (H, ψ) of $|\mathfrak{D}|$ is. *H does not change by any torus translation*; in other words *H depends only on the algebraic equivalence class of $\mathfrak{D}$.* Thus ψ fixes the position of $|\mathfrak{D}|$ in the Picard variety $E/\bar{G}$ of E/G (cf. § 4). Precisely:

DEF. 9.1. Let $|\mathfrak{D}|$ be a complete linear system of divisors on E/G of type (H, ψ) (cf. Def. 2.4); we say that $|\mathfrak{D}|$ ($\Leftrightarrow$ any vector space (H, ψ, Φ, L) (cf. § 2) representing $|\mathfrak{D}|$) has *characteristic $\gamma \in E/\bar{G}$ with respect to the origin $O = \Theta(0)$* iff ψ satisfies (9.1). In particular: *ψ has characteristic zero with respect to O iff $\psi = \psi_B$ for the natural satellite form B,* (cf. Def. 2.7). In other words: *Any $|\mathfrak{D}|$ can be represented in the form (H, ψ_B) by a suitable choice of the origin* (cf. Introduction).

The transition to « coordinates » is done easily, first by lifting of γ to E, if $\gamma = c + \bar{G}$ we may say as well that $|\mathfrak{D}|$ *has characteristics c with respect to O*; then c is determined mod $\bar{G}$. A second step is to represent c in the

$\mathbb{R}$-basis $\mathfrak{g}$. In the classical Krazer's approach instead of c (or $c+\bar{G}$) was considered the components $g_1, g_2, \ldots, g_n$, $h_1 h_2, \ldots, h_n$ ($\in \mathbb{R}$) with respect to c of the n vectors of $\mathfrak{g}_1$ or $\mathfrak{g}_2$ respectively. It did not appear so clear to me that, classical geometers realized that in the general case the vector c is defined mod $\bar{G}$ (not mod G).

The special emphasis on the half-integers characteristics appears to be geometrically justified by the following property which is just a rephrasing, using coordinates of Prop. 7.4.

The property $\psi^2 = 1$ holds *iff* $|\mathfrak{D}|$ *has half integers characteristics with respect to* $\mathfrak{g}$.

10. – Intrinsic series expansions of the reduced thetas.

We shall show that the *reduced theta functions of type* (H, ψ) (associated to the basic t Fourier thetas, for a fixed arbitrary choice of the origin O in $\mathcal{A}$ and a canonical *basis* $\mathfrak{a} = (\mathfrak{a}_1 : \mathfrak{a}_2)$, cf. Def. (3.2), in G) *can be expressed intrinsically*, without any use of coordinates, by series expansions (10.1) involving just: 1) *The variable point* $P \in E$ (or better *the vector* $u = OP$); 2) The intrinsic invariants: H, ψ (cf. § 2); 3) The lattice of periods G, or rather an overlattice $\Gamma_1 \supset G$ of the type considered in § 5 (cf. (5.3))

$$\Gamma_1 = Z\mathfrak{a}_1 \oplus Z\mathfrak{a}_2 P \,. \tag{10.1}$$

The canonical basis $\mathfrak{a}$ does not play any essential role, (although it is handy to define Γ_1; (cf. (10.1)) in fact one can see that *the original* basis $\mathfrak{a}$ might be replaced by any other congruent mod $\Delta(G)$, since *the expressions* (10.2) *shows clearly that they are invariant by* $\Delta(G)$. In other words, we can express Γ_1 in the form

$$\Gamma_1 = p^{-1}\mathfrak{C}_1 \qquad \text{(cf. Introduction)}$$

where $\mathfrak{C} = \mathfrak{C}_1 \oplus \mathfrak{C}_2$ is a canonical decomposition of the collineation group $\mathfrak{C} = \bar{G}/G$ with respect to its reduced symplectic structure (cf. § 3):

In order to make more clear the independence of (10.1) from the bases we write down the intrinsic expansions and we shall prove a posteriori the relationship with the ordinary Fourier thetas.

There is, still a curious fact (cf. § 6) about the choice of anyone $\psi^{(j)}$ of the t prolongations of ψ from G to Γ_1 (cf. Th. III below). They play a very useful role in next § 14 in the definition of the Hessian basis. In spite of the choice of $\psi^{(j)}$, the Fourier thetas are defined as usual choosing a quadratic exponential factor in such a way that the product becomes a Fourier theta strictly periodic with respect to the last n basic periods, which remain *well determined.*

THEOREM III. *Let* ψ_j *be a fixed extension to* Γ_1 (cf. (10.1)) *of the semi-character* ψ *of* G *attached to* A ((cf. Prop. 6.1): $j = 1, 2, \ldots, t$). *Let* H *be the positive definite Hermitean form defined by* $\operatorname{Im} H = A$ (cf. § 2, formula (2.2)). *Let* λ *be any of the* t ($= \operatorname{Pfaff} A|G\times G$) *classes of* Γ_1 mod G. Then, we have:

a) *The series*

$$\theta_\lambda^{(j)}(u) = \sum_{l\in\lambda} \psi_j^{-1}(l) \exp\left(-\frac{\pi}{2}\left(H(l,l) + 2H(l,u)\right)\right) \qquad \forall u \in E \tag{10.2}$$

converges uniformly in every compact set of E.

b) *The function* $u \mapsto \theta_\lambda^{(j)}(u)$ *is a reduced theta function of type* (H, ψ).

c) *For* $\lambda \in \Gamma_1/G$ *the* t *series* $\theta_\lambda^{(j)}$ *are* $\mathbf{C}$*-linearly independent*; *thus they form a basis of the vector space* $\Theta(\mathfrak{D})$ *of reduced theta functions attached to the complete linear system* $|\mathfrak{D}|$ *of divisors on the torus* E/G.

d) *Every series* (10.1) *is invariant under the congruence subgroup* $\Delta(G)$ *of* $\mathfrak{M}(G)$.

e) If $\mu \in \Gamma_1$ we have:

$$\theta_\lambda^{(j)}(u+\mu) = \psi_j(\mu)\left(\exp\frac{\pi}{2} H(\mu,\mu)\right)\left(\exp \pi H(\mu, u)\right)\theta_{\mu+\lambda}^{(j)}(u) \tag{10.3}$$

f) *c*) implies: If $t_1 \geqslant 3$ *the projective Abelian variety* $\mathcal{A} = \theta(E/G)$ *represented by the* t *series* (10.2) *is invariant under a collineation group* $\mathfrak{C} \approx \bar{G}/G$ *permuting the hyperplane sections* $\theta_\lambda^{(j)}(u) = 0$ *according to* (10.2).

REMARKS. 1) The main statement of Th. III is that *the series are intrinsic*; *no basis are needed and they make clear the functional dependence from* H, ψ *and the overlattice* $\Gamma_1 = p^{-1}\mathfrak{C}_1$ (appearing in the chosen prolongation).

2) Since the summation is taken for the lattice vector of $\bar{G}$ belonging to a given class of $\Gamma_1/G \subset \bar{G}/G$, *d*) becomes *evident* (in contrast with the three page proof of the corresponding Satz 9 (400-403) in Siegel [2]).

3) Before proving the theorem let us check that (10.2) is still a « theta series » in the traditional sense of the standard references specially [ ] § 4.2 as « n-dimensional » generalizations of $\sum_{-\infty}^{+\infty} \exp(am^2 + 2bm + c)$ $a, b, c \in \mathbf{C}$; $m \in \mathbf{Z}$. In fact, according to § 2 we can write ψ in exponential form as $\psi^{-1}(v) = \exp\left(-\pi i\left(B(g,g) + 2A(a,g)\right)\right)$. Then, the general term of (10.1) contains as exponent a quadratic (non necessarily homogeneous) polynomial ni $2n$ indices, with negative definite imaginary part $H(g, g)$.

4) It is possible to obtain a proof of *a*) just «rephrasing» the classical one, for the Fourier thetas.

5) In the unit case $t_j = 1$, $n = 1, 2, \ldots, n$, $\Gamma_1 = G$, $t = 1$, we can drop the unnecessary indexes j, λ. Let us choose $\psi = \psi_0 = \psi_0^{-1}$ to be the *natural semicharacter* (cf. Def. 2.1). Then we have the unique theta series

$$\theta(u) = \sum_{\gamma\in G} \psi_0(\gamma) \exp\left(-\frac{\pi}{2}(H(\gamma, \gamma) + 2H(\gamma, u))\right). \qquad \forall u \in E \tag{10.4}$$

The whole theory can be developed starting from (10.4) as in the classical Fourier case. Because of the emphasis on the intrinsic invariants H, ψ we give below a direct sketchy proof of *a*) and *b*).

PROOF. $H(l, l) = S(l, l)$ where $(x, y) \mapsto S(x, y) = -A(ix, y)$ is the real part of H. S is a symmetric, real valued, bilinear form on $E\times E$, thus actually the real part of the homogeneous quadratic term of *the exponential of the generic l term is negative definite*. This is a characteristic uniform convergence condition in compact sets of E known over a century ago. (Cf. Krazer's book).

As a consequence (10.2) represents an entire function $E\to\mathbf{C}$ in the complex space E. In order to check the functional equations (2.9), we introduce a decomposition of $\theta_\lambda^{(j)}(u)$ as a product $\{\exp[(\pi/2)S(u, u)]\}\zeta_\lambda^{(j)}(u)$, where

$$\zeta_\lambda^{(j)}\zeta(u) = \sum_{l\in\lambda} \psi_j^{-1}(l)\left\{\exp\left[-\frac{\pi}{2}S(l+u, l+u)\right]\right\}\exp\left[-\frac{\pi}{2}iA(l, u)\right]. \tag{10.5}$$

The translation $u \mapsto u + g$ applied to both previous factors of $\theta_\lambda^{(j)}(u)$ induces the automorphy factor $\exp\{(\pi/2)(S(g, g) + 2S(g, u))\}$, thus it would be sufficient to check the following functional equation of $\zeta_\lambda^{(j)}(u)$:

$$\zeta_\lambda^{(j)}(u+g) = \psi_j(g)\cdot\exp\pi i A(g, u)\cdot\zeta_\lambda^{(j)}(u+g) \tag{10.6}$$

We have

$$\begin{aligned}
&\zeta_\lambda^{(\lambda)}(u+g) = \qquad\qquad (10.7)\\
&= \sum_{l\in\lambda}\psi_j^{-1}(l)\exp\left[-\frac{\pi}{2}S(l+u+g, l+u+g)\right]\exp[-\pi i A(l, u+g)] = \\
&= \psi_j(g)\sum_{l\in\lambda}\psi_j^{-1}(l+g)\exp\left[-\frac{\pi}{2}S(l+u+g, l+u+g)\right]\exp(-\pi i A(l, u)) = \\
&= \psi_j(g)\exp\pi i A(g, u)\sum_{l\in\lambda}\psi_j^{-1}(l+g)\exp\left[-\frac{\pi}{2}S(l+u+g, l+u+g)\right]\cdot \\
&\qquad\qquad\cdot\exp(-\pi i A(l+g, u)) = \\
&= \{\psi_j(g)\exp\pi i A(g, u)\}\zeta_\lambda^{(j)}(u) = \{\psi(g)\exp(\pi i A(g, u))\}\zeta_\lambda^{(j)}(u)
\end{aligned}$$

because $\psi_j(g) = \psi(g)$ (since $g \in G$).

In (10.7) we used, the functional equation of ψ_j in the form $\psi_j^{-1}(l) = \psi_j(g)\psi_j^{-1}(l+g)\exp(\pi i A(l, g))$, the decomposition $H = S + iA$ and then a trivial change of variable summation vector $l \mapsto l + g$, which does not change λ since $g \in G$.

REMARK. The interest of $\zeta_\lambda^{(j)}$ is that, since the automorphy factor appearing in (10.5) belongs to $\mathbf{C}_1^\times$ we have:

The product $\zeta_\lambda^{(j)}\bar{\zeta}_\lambda^{(j)}$ *is a* C^∞*function on* E *everywhere non negative absolutely periodic with* G *as its lattice of periods* (cf. Igusa's book, Ch. II, Lemma 4, p. 69).

Thus *a*) and *b*) are proved. *c*) is clear since the t different series contain all different terms (the summations are taken with respect to each one of the t classes $\Gamma_1 \bmod G$). Furthermore this proves also *d*) by the definition of $\Delta(G)$ (cf. Th. I, page 60). *e*) is clear by a direct computation very similar to (10.4)-(10.6). Besides it is a particular case of *b*) according to our previous Remark 5), formula (10.4).

Finally *f*) is the projective interpretation of *e*) in the more expressive case when $\mathcal{A} = \Theta(E/G)$ is a non singular normal, projective variety in $\mathbf{P}^{t-1}$, injective image of E/G (cf. Weil [2], Siegel [2] Conforto. Since the three factors preceeding $\theta^{(j)}(u)$ in (10.2) are independent of λ, they are non essential, thus the collineation group Γ_1/G permutes the coordinate hyperplanes isomorphically by $\lambda \mapsto \lambda + \mu$ where now, we can replace μ by its class $\mu + G (\bmod G)$.

REMARKS. 1) The usual proof of the property $\dim_{\mathbf{C}}|\mathfrak{D}| = \operatorname{Pfaff} A$ is obtained remarking that it is possible to choose Φ and L (cf. § 2, Prop. 2.1)) in such a way that a prescribed $\mathfrak{D}$ is represented by a theta of type (H, ψ, Φ, L) such that $\theta(u+g) = \theta(u)$ for every $g \in G_2$. Then the existence and uniqueness of the Fourier expression and the linear independence of the t thetas corresponding to the classes of $\Gamma_1 \bmod G$ complete the result. Cf. Weil [1, 2]. It is possible to eliminate this last appeal to the non reduced thetas but it is outside the scope of our presentation to do so here. It can be done using the group representation approach, as in Ch. I of Igusa's book, cf. also Satake.

11. – The Hessian coordinate systems of $\mathcal{A}$.

The property *e*) of Th. III, § 10 gives a simple faithful representation of the direct summand $\mathfrak{C}_1$ of the group $\mathfrak{C} = \mathfrak{C}_1 \oplus \mathfrak{C}_2$ (cf. § 6) as a collineation group of $\mathbf{P}^{t-1}$ leaving invariant $\mathcal{A} = \Theta(E/G)$. The representation of $\mathfrak{C}_2$ is not so simple if we keep the basis of (10.1). We shall change this theta basis *introducing a new one which gives a very simple representation of both*

$\mathfrak{C}_1$ and $\mathfrak{C}_2$. *This representation is independent of the moduli of $\mathcal{A}$.* (Cf. literature quoted in the Introduction).

Let us consider a symplectic basis $\mathfrak{T}$, $\mathfrak{T}_2 \oplus \mathfrak{T}_2$ of the group $\mathfrak{C}$, where $\mathfrak{T}_1$, $\mathfrak{T}_2$ are canonical basis of the direct summands $\mathfrak{C}_1 \mathfrak{C}_2$ (cf. Def. 6.1) and the conditions (6.2), (6.3), (6.4) are fulfilled. To fix the ideas we shall represent $T_j \in \mathfrak{T}$ as a torus translation $u \mapsto u + a_j$ where $a_j \in \mathfrak{a} = (\mathfrak{a}_1; \mathfrak{a}_2)$. $\mathfrak{a}$ is the canonical basis (cf. Def. (3.2)) of the completion $\bar{G}$ (cf. Def. (4.1) of G) associated to a modular basis of G (cf. Def. (1.3)).

PROP. 11.1. *The sum*

$$(11.1) \qquad \theta^{(j)}(u) = \sum_{\lambda \in \Gamma_1/G} \theta_\lambda^{(j)}(u) = \sum_{\gamma \in \Gamma} \psi_j(\gamma) \exp\left(-\frac{\pi}{2}\left(H(\gamma, \gamma) + 2H(\gamma, u)\right)\right)$$

of the t series (11.1) *is a reduced theta of type* $(H, \psi^{(j)})$ *relative to the lattice* $\Gamma_1 = \mathbb{Z}\mathfrak{a}_1 + G$.

We shall need now to display both indices j, λ in (10.1) replacing them by $\{m_1, m_2, \ldots, m_n\}$ and $[\lambda_1 \lambda_2, \ldots, \lambda_k]$ as representatives $\bmod T$ $(m_j, \mu_j \in \mathbb{Z}$; $m_j, \mu_j = 0, 1, \ldots, t_j - 1$; $j = 1, 2, \ldots, n)$, precisely we write

$$(11.2) \qquad \vartheta^{(j)} = \vartheta\{m_1 m_2, \ldots, m_n\}, \qquad \theta^{(1)} = \theta\{0, 0, \ldots, 0\}$$

iff j is the ordinal number of $\{m_1, \ldots, m_n\}$ in the lexicographical ordering.

Now we have all the tools to prove next Th. IV, but we want to describe first its geometric content, in terms of the canonical map, $E/G \xrightarrow{p} E/\Gamma_1$, where $\Gamma_1 = \pi^{-1}\mathfrak{C}_1$ $(\pi\colon \bar{G} \to G)$. The restriction $A|\Gamma_1 \times \Gamma_1$ has unit type. As a consequence $\dim \theta(E/\Gamma_1) = 1$ and *the corresponding* (unique!) *divisor $\mathfrak{D}$ on E/Γ_1 represented by any fixed reduced non-zero theta θ_0 on E/Γ_1 of type (H, ψ_1)* (ψ_j any prolongation of ψ from G to Γ_1) *is linearly isolated.* Moreover, since no translation in E/Γ_1 can leave fixed $\mathfrak{D}_0$ the images of the translations of the group Γ_2 acting on $\mathfrak{D}_0$ are t linearly independent divisors $\tau(\mathfrak{D}_0)$ $(\tau \in \mathfrak{C}_2)$ corresponding to the θ-functions $\tau(\theta_0)$ $(\tau \in \mathfrak{B}_2)$. Lifting these t divisors $\tau(\mathfrak{D}_0)$ to E/G we obtain *t linearly independent divisors of E/G* of type (H, ψ) $(t = \operatorname{Pfaff} A|G \times G)$ belonging to $|\mathfrak{D}|$ *represented by the corresponding theta functions* $\tau(\theta)$ $(\tau \in \mathfrak{C}_2)$, but interpreted now in relation with the original lattice $G \subset \Gamma$. These $\tau(\theta)$ form the Hessian basis (cf. Def. 11.1). The behavior of the full group $\mathfrak{C}$ is the following:

Every $\mathfrak{C}_1$ leaves invariant anyone of the t Hessian divisors $\tau(\mathfrak{D}_0)$, $\tau \in \mathfrak{C}_2$. $\mathfrak{C}_2$ induces permutation group isomorphic with $\mathfrak{C}_2$ *among the* $\{\tau(\mathfrak{C}_2), \tau \in \mathfrak{C}_2\}$.

The consideration of the reduced θ's, $\tau(\theta_0)$ representing $\tau(\mathfrak{D}_0)$ $(\forall \tau \in \mathfrak{C}_2)$ enables a more careful and totally explicit description of $\mathfrak{C}$ in terms of the isomorphic collineation group corresponding to the projective coordinate system represented by the $\tau(\theta_0)$ and $\sum_{\tau \in \mathfrak{C}_2} \tau(\theta_0)$.

THEOREM IV. Def. 11.1. *a) The lexicographically ordered set* $\theta^{(j)} = \theta\{m_1 m_2, \ldots, m_n\}$ *of reduced theta functions of type* (H, ψ) (cf. Def. 2.3) *representing the complete linear system of divisors* $|\mathfrak{D}|$ on E/G (cf. § 1) *form a basis of the t-dimensional vector space* $\Theta(\mathfrak{D})$ ($t = \operatorname{Pfaff} A$, cf. § 1) *called the Hessian basis of* $\Theta(\mathfrak{D})$ *attached to* $\mathfrak{C} = \mathfrak{C}_1 \oplus \mathfrak{C}_2$.

b) Every mapping of $\mathfrak{C}_1$ *leaves invariant each one of the t hyperplane sections* $X_j = 0$ *represented by* $\theta^{(j)}$ $(j = 1, 2, \ldots, t)$. *As a consequence the maps of* $\mathfrak{C}_1$ *are represented in diagonal form with respect to the* $\theta^{(j)}$ *basis. The eigenvalues are* t_n^{th} *roots of one.*

Precisely, we have:

$$\varrho\theta\{m_1 m_2, \ldots, m_n\}\Big(u + \sum_{j=1}^{n} \mu_j a_j\Big) = \prod_{j=1}^{n} \varepsilon_j^{m_j \mu_j} \theta\{m_1 m_2, \ldots, m_n\}(u) \tag{11.3}$$

where ϱ is an omitted uninteresting exponential common factor (cf. § 10).

c) The group $\mathfrak{C}_2$ is represented as a permutation group of the t coordinate spaces $X\{m_1 m_2, \ldots, m_n\} = 0$. Precisely, we have

$$\varrho\theta\{m_1 m_2, \ldots, m_n\}\Big(u + \sum_{j=1}^{n} \mu_j a_{j+n}\Big) = \theta\{m_1 + \mu_1, m_2 + \mu_2, \ldots, m_n + \mu_n\}\,. \tag{11.4}$$

PROOF. Any $\vartheta^{(j)}$ (cf. (11.1) can be expressed by

$$\theta\{m_1 m_2, \ldots, m_n\} = \sum_{\lambda \in \Gamma_1/G} \Big(\prod_{j=1}^{n} \varepsilon_j^{-m_j \lambda_j}\Big) \theta_\lambda^{(1)} \tag{11.5}$$

as a linear combination of the $\theta_\lambda^{(1)}$ because of the properties proved in § 5, in particular formula (5.5). The matrix μ of coefficients in (10.5) is the tensor product of n matrices of the Vandermonde type $(\varepsilon_j^{-m_j \mu_j})$, $j = 1, 2, \ldots, n$. As a consequence $\det \mu \neq 0$, *thus the t theta functions* $\theta^{(j)}$ *are linearly independent.*

b) is a consequence of the fact that the $\theta^{(j)}$ are theta functions relative to Γ_1. Then we obtain (11.3) supposing the common non zero factor $\psi_1(\sum \mu_j a_j)$, corresponding to the first canonical prolongation of ψ to Γ_1 (cf. again (5.5)).

c) is an immediate consequence of (6.1), and § 5, in particular (5.6), (5.7).

As in Rosati's papers it suffices to consider the representation of $2n$ generators of $\mathfrak{C}$. To generate the full group $\mathcal{H}(\mathcal{A})$ according to § 8 it would be sufficient to represent one symmetry. Because of Prop. 7.1, Prop. 8.3 we can assume without loss of generality that $O = \Theta(0)$ is a center of symmetry ($\Leftrightarrow \psi^2 = 1$). Then the image of $u \mapsto -u$ ($u \in E$) is given by

$$\chi\{m_1, m_2, \ldots, m_n\} \to \chi\{-m_1, -m_2, \ldots, -m_n\} \tag{11.6}$$

when m_j are regarded in (11.6) as defined mod t_j ($j = 1, 2, \ldots, n$) in (10.6).

REMARKS. 1) The previous result gives an algebraic-geometric interpretation of the *reduced symplectic structure* of the group $\mathfrak{C}$ (defined in Th. I, § 4) and its full automorphism group $\mathfrak{m}(\mathfrak{C}) = \mathfrak{m}(G) = \mathfrak{M}(G)/\Delta(G)$ (cf. Def. 4.3).

2) The particular case when $\mathcal{A} = \eta_m$ ($m \geqslant 3$) is the normal elliptic curve of order m in $\mathbf{P}^{m-1}$ is well known after Klein-Hurwitz (cf. Introduction) to extend the properties of the « *configuration* (1) of flexes » of a non singular plane cubic η_3. In fact: The group $H(\eta_m)$ contains an Abelian subgroup $\mathfrak{C}$ of collineations, image of suitable torus translations. $\mathfrak{C}$ has order m^2 and can be decomposed (non uniquely!) as a direct sum $\mathfrak{C} = \mathfrak{C}_1 \oplus \mathfrak{C}_2$ of two cyclic subgroups of order m: $\mathfrak{C}_1 \approx \mathfrak{C}_2$, generalizing to every m the well-known *couples of conjugate triangles of flexes of* η_3. η_3 is represented in any one of such natural projective coordinate systems in *Hessian form* as indicated in the Introduction.

12. – Hermitian metrics on $\Theta(\mathfrak{D})$.

The t-dimensional complex vector space $\theta(\mathfrak{D})$ of reduced theta functions attached to the complete linear system $|\mathfrak{D}|$ of divisors over E/G has a natural Hermitean metrics, studied by several authors; Siegel [2] introduces it (implicitly) in § 3 using the ordinary Fourier thetas. In Cartier's, Satake and Igusa's approach by means of group representations these metrics play a fundamental role, (almost « by definition »). Explicitly the scalar product $\langle\theta_1, \theta_2\rangle$ of two reduced theta functions $\theta_1, \theta_2 \in \theta(\mathfrak{D})$ is given by

$$\langle\theta_1, \theta_2\rangle = \int\limits_{E/G} \theta_1 \bar{\theta}_2 \left\{\exp\left(-\frac{\pi}{2} H\right)\right\} \omega \tag{12.1}$$

(1) The word « *configuration* » has a precise meaning in XIX-th century mathematics (cf. the well-known book of HILBERT-COHN-VOSSEN, *Geometry and the imagination*. We use here the word informally. The analogies are not « perfect ».

where $\omega =$ is the $2n$-real « volume » differential form of degree $2n$ associated with the real positive quadratic form $u \mapsto H(u, u)$. Siegel proves that his ordinary Fourier basic theta functions can be normalized (by eventual multiplication by a non zero complex number) in such a way *that they form an orthonormal basis with respect to* (13.1). This property is also true for the reduced thetas (it suffices to change Siegel differential form $\varphi(u)\omega$ by the simpler one $\exp[-(\pi/2)H(u)]\omega$ (which is, by the way the probability density used by the Statisticians in Gauss « normal distribution » of errors ...). We want to show that this very convenient property is also true for the Hessian basis.

THEOREM IV. *We can normalize the t basic theta functions of a Hessian coordinate system* (cf. Def. 11.1), *by multiplication with a non zero common factor, in such a way that any Hessian coordinate system becomes also orthonormal with respect to H.*

PROOF. It is an immediate consequence of the fact that the transformation matrix from an orthonormal « ordinary » basis of $\theta(\mathfrak{D})$ to the Hessian one is a tensor product of t Vandermonde matrices M_j and $M_j^t \overline{M}_j = t_j$. As a consequence $t_j^{-\frac{1}{2}} M_j$ $(j = 1, 2, ..., n)$ is a unitary matrix and $t^{-\frac{1}{2}}\Big(\bigoplus_{j=1}^{n} M_j\Big)$ is also unitary matrix, q.e.d.

13. – Siegel's moduli program. The Normalgleichung. Expression of the Chow point of $\mathcal{A}$ by Thetanulls.

Siegel summarizes his research program in [2] page 376 in four points freely translated below. We add here and in the next § 14 our own remarks and contributions to the items 1), 2), respectively.

1) We should represent $\mathcal{A}$ by a well-defined irreducible algebraic equation valid for any $Z \in \mathfrak{S}_n$ (Siegel's upper-half plane of dimension n).

2) We want to express the ratios of the coefficients of previous equations as functions of Z by means of the so-called thetanulls (or « theta constants ») investigating by which modular substitutions of level (Stufe) T they remain invariant.

3) We should construct regular functions of Z no vanishing simultaneously, by means of the found coefficients in such a way that (after a further symmetrization with respect to the modular group $\mathfrak{M}(G)$) their ratios generate the field of all modular functions of type T.

4) We should indicate the algebraic equations satisfied by the previously found generators.

Siegel solved 1) in his paper by expressing the *Chow coordinates* of $\mathcal{A} = \Theta(E/G)$ in terms of the thetanulls. In fact the Chow homogeneous coordinates can be defined as the coefficients (ordered in a certain fixed way) of any one of the well-known *associate forms of* $\mathcal{A}$. Z depends on H and a canonical basis of G. (cf. our next § 14).

Siegel's *Normalgleichung* (used also by Andreotti-Mayer) is one of the generic projections known in Algebraic Geometry (cf. Samuel's book I, 9-4 page 44 or Weil's *Foundations*, 2nd Ed.), leading to the « Chow coordinates » and closely related to any one of the well-known standard associate forms (cf. Van der Waerden or Hodge-Pedoe, II).

An intrinsic version of the « Normalgleichung » (valid for any projective irreducible algebraic variety is the following:

Let M_d be an irreducible algebraic variety embedded in the projective space $\mathbf{P}_N(k)$. The standard associate forms of V_d are defined in terms of the irreducible divisor representing the set $\{S_{n-d-1} \cap V_d \neq \emptyset\}$ in the Grassmann manifold $\mathcal{G}(N-d-1; N)$ of $(N-d-1)$-spaces of $\mathbf{P}_N$. Let $V_{N+1}(k)$ be the vector space attached to $\mathbf{P}_n(k)$ by the definition: $\mathbf{P}_N = V_{N+1} - \{0\}/k^*$. Let V^* be its dual space; Chow's *zugeordnete Form* ψ: $V^{*d+1} \to k$ and Severi's associate form $\mathcal{S}$: $V^{n+d} \to k$ are both non-zero, determined up to a non-zero factor of k. Either one are characterized by their vanishing properties. Let $0 \neq u_j \in V^*$ $(j = 1, 2, ..., d+1)$ or $0 \neq x_h \in V$ $(h = 1, 2, ..., N-d)$. Then $\psi(u_1, u_2, ..., u_{d+1}) = 0$ $\big(\mathcal{S}(x_1, x_2, ..., x_{N+d}) = 0\big)$ if and only if the $d+1$ hyperplanes $\big((N-d)$ points$\big)$ represented by the u's (x's) intersect in (span a) subspace S_{N-d-1} meeting M_d. The $\mathcal{S}$ form was preferred by Severi because *for* $d = N-1$, $\mathcal{S}(x) = 0$ *becomes the equation of the hypersurface* M_{N-1}, i.e., $\mathcal{S}$ can be regarded really as a true extension of « the equation of M » to higher codimensions. If we fix $N-d-1$ linearly independent elements of V: $a_1, a_2, ..., a_{N-d-1}$ the equation in x: $\mathcal{S}(a_1, a_2, ..., a_{N-d-1}, x) = 0$ represents the projecting cone of M_d from the S_{N-d-2} space spanned by the points of $\mathbf{P}_N$ represented by the a's. If we « move » the a's, M_d *is represented as intersection of all these projecting cones from a variable* S_{N-d-2}!

The two forms ψ, $\mathcal{S}$ may be regarded as extreme cases of forms

$$F_{rs}(x_1, x_2, ..., x_r; u_1, u_2, ..., u_s) = 0 \tag{13.1}$$

for $0 \neq x_j \in V$ $(j = 1, 2, ..., r)$, $0 \neq u_h \in V^*$ $(h = 1, 2, ..., q)$ iff the space S_{r-1} spanned by the points of $\mathbf{P}_N$ represented by the x's and the S_{N-s} intersection of the q hyperplanes determined (in the appropriate way) an S_{N-d-1} meeting M_d.

Accordingly, for $r > s$, $r - s = N - d$, $S_{N-d-1} = S_{r-1} \cap S_{N-s}$ and in the case $s > r$, $s - r = d + 1$ and S_{N-d-1} is the join of S_{r-1}, S_{N-s} ($S_{r-1} \cap S_{N-s} = \emptyset$). In fact the construction (and uniqueness up to a non-zero constant of any F_{rs}) is insured by the well-known relations between both kinds of Grassmann coordinates p_j, p^h of an S_{N-d-1} and the well-known Grassmannian expressions

$$(13.2) \qquad \psi(u_1, u_2, \dots, J_{d+1}) = \bar{\psi}(\dots, p_j, \dots)\,, \qquad \mathrm{S}(x_1, \dots, x_{N-d}) = \tilde{\mathrm{S}}(\dots, p^h, \dots)$$

(always existing, but not necessarily unique, cf. Hodge-Pedoe, II).

The case $r = 1$, $s = d + 2$ (thus $s > r$) leads naturally to Siegel's Normalgleichung. Let us particularize (13.1); the condition:

$$(13.3) \qquad F_{d+2,1}(x;\, u_1, u_2, \dots, u_{d+2}) = 0$$

represents the cone projecting M_d from the generic space S_{N-d-2} intersection of the $d + 2$ linearly independent hyperplanes u_j ($j = 1, 2, \dots, d + 2$). *For a variable* S_{N-d-2} (with $S_{N-d-1} \cap M_d = \emptyset$) *we obtain similar representation as in the Severi form*, with the only difference that the center of projection S_{N-d-2} is now constructed as intersection of hyperplanes instead as a join of $N - d - 1$ points! Now Siegel's Normal equation can be constructed as follows: Let us consider the space $W_{N-d-1} \subset V_{N+1}$ representing S_{N-d-2} and its « orthogonal » $\tilde{N}_{d+2} \subset V^*$. $\tilde{W}_{d+2}$ represents the *star of hyperplanes* $\tilde{\mathbf{P}}_{d+1}$ passing through S_{N-d-2}. $\tilde{P}_{d+1}$ has a natural structure of projective space of dimension $d + 1$. The $d + 2$ linearly independent elements $u_1, u_2, \dots, u_{d+2}$ determine $\left(\text{together with the « unit » } \sum_{j=1}^{d+2} u_j\right)$ a projective coordinate system. Since $S_{N-d-2} \cap M_d = \emptyset$ the map $M_d \to \mathbf{P}_{d+1}$ well-defined by $\xi \to \big(u_1(\xi), u_2(\xi), \dots, u_{d+2}(\xi)\big)$ (not all the homogeneous coordinates $u_j(\xi) \in k$ can vanish) gives a birationally equivalent image of M_d, (iff the u_j are indeterminates) which is an irreducible hypersurface of $\mathbf{P}_{d+1}$. Let us call η the point of $\tilde{\mathbf{P}}_{d+1}$ with homogeneous coordinates $n_j = u_j(\xi)$ ($j = 1, 2, \dots, d + 2$). Then, the equation of this hypersurface.

$$(13.4) \qquad N(\eta;\, u_1, u_2, \dots, u_{d+2}) = 0$$

defines Siegel Normal form N. Siegel uses coordinates and N becomes well-defined in terms of a $(d + 2) \times (N + 1)$ hyperplane coordinate matrix S with algebraically independent indeterminates adjoint to k and $d + 2$ algebraically independent homogeneous joint coordinates $\eta_1, \eta_2, \dots, \eta_d$.

The relationship between N and S is clear by means of $\xi \to \eta$ or in matrix form by $\eta = S\xi$.

Summarizing: All the mentioned « associated forms » have essentially the same ordered set of coefficients (not all zero and determined up to a non-zero common factor); as a consequence they can be taken as the homogeneous coordinates of $\mathcal{A}$ (« Chow *coordinates* ») in a convenient well-known projective space $\mathbf{P}_N(k)$ (k ring the ground field; the theory is valid for abstract commutative ground fields of any characteristic.) If $X = \sum \lambda_i X_i$ (X_i irreducible of dimension d, $\lambda_i \in Z$, $\lambda_i > 0$) the associated form of any type t is $\prod F_i^{\lambda_i}$ where F_i is the a.f. of type t of the irreducible component X_i.

Siegel and Andreotti-Mayer used the Chow coordinates of $\mathcal{A}$ and also partially solved problem 2), although the explicit expressions of the Chow coordinates in terms of the the anulls are far from being explicit! Their methods can be applied also for our Hessian basis. Thus, we want to show in next § 14 how the dependence of the Chow point $\mathrm{Ch}(\mathcal{A}) \in \mathbf{P}^N$ (for a suitable N) from the Hessian basis can be obtained theoretically (explicit computations are difficult!) from a unitary representation of the reduced modular group $\mathfrak{m}(G)$.

14. – Unitary representation of the reduced modular group. Construction of the variety of moduli in terms of $\mathfrak{m}(G)$.

Our embedding of E/G in $\mathbf{P}^{t-1}$ depends on the choice of a *Hessian basis* $\mathcal{H}(\tilde{\mathfrak{c}})$ of $\Theta(\mathfrak{D})$ (cf. Def. 11.1). $\mathcal{H}(\tilde{\mathfrak{c}})$ is well defined in terms of a canonical basis of the group $\mathfrak{C}$ endowed with its reduced symplectic structure (cf. Th. I). We introduce $\tilde{\mathfrak{c}}$ in the notation to study the map $\tilde{\mathfrak{c}} \to \mathcal{H}(\tilde{\mathfrak{c}})$:

THEOREM V. Let $\mathfrak{m}(G) = \mathfrak{M}(G)/\Delta(G)$ *be the reduced modular group* (cf. § 1) *acting in a simply transitive way on the finite set of canonical bases of* $\mathfrak{C}$. *The map* $\tilde{\mathfrak{c}} \to H(\tilde{\mathfrak{c}})$ *defines a unitary faithful representation* (cf. Th. IV, § 12).

$$\mathfrak{m}(G) \to GL(\Theta(\mathfrak{D})) \tag{14.1}$$

in the vector space $\theta(\mathfrak{D})$ *of reduced theta functions attached to* $|\mathfrak{D}|$.

The corresponding projective representation

$$\mathfrak{m}(G) \to PGL(P^{t-1}) \tag{14.2}$$

in $\mathbf{P}^{t-1}$ *is also faithful. The image group of* (14.2) *permutes the* $v = |\mathfrak{m}(G)|$ *models* $\mathcal{A}(\tilde{\mathfrak{c}})$ *of the Abelian variety* E/G.

PROOF. The only non-trivial collineation of $\mathfrak{C}$ leaving invariant each one of the t coordinate hyperplanes $X[m_1 m_2, \ldots, m_n] = 0$ of a Hessian coordinate system in $\mathbf{P}^{t-1}$ are those belonging to $\mathfrak{C}_1$; in fact those belonging to $\mathfrak{C}_2$ permute them without leaving any one of them fixed. As a consequence if another canonical basis $\mathfrak{c}$ defines $\mathfrak{C} = \mathfrak{C}_2' \oplus \mathfrak{C}_2'$ and $\mathcal{H}(\tilde{\mathfrak{c}}) = \mathcal{H}(\tilde{\mathfrak{c}}')$ should be $\mathfrak{C}_1 = \mathfrak{C}_2$ and because the reduced symplectic structure of $\mathfrak{C}$ is preserved by $\mathfrak{m}(G)$ we have also $\mathfrak{C}_2 = \mathfrak{C}_2'$. Then the unique map $\tilde{\mathfrak{c}} \to \tilde{\mathfrak{c}}'$ $(\in \mathfrak{m}(G))$ should be a permutation of pairs (T_j, T_{j+n}). But since $\{m_1 m_2, \ldots, m_n\}$ depends on $\mathfrak{C}_2$ if $\mathcal{H}(\mathfrak{c}) = \mathcal{H}(\tilde{\mathfrak{c}}')$ we should have $\mathfrak{c} = \mathfrak{c}'$. The faithfulness of (14.2), is also clear since any two proportional reduced theta functions define the same divisor and we can apply again the previous argument. The fact that the representation is unitary comes from Th. IV, § 2, i.e., because the normalized Hessian basis are orthonormal.

REMARK. The restriction of any non-singular projective model of the Cartesian product $\prod_{\mu \in \mathfrak{m}(G)} \mathcal{A}(\mu \mathfrak{c})$ to those ν-tuples of points $\prod_{\mu \in \mathfrak{m}(G)} \mu(x)$ corresponding to the image collineation group in $\mathbf{P}^{t-1}$ (cf. Th. IV) gives a *canonical projective embedding* of E/G associated to $|\mathfrak{D}|$. The simplest one is obtained from the Segre variety Ω representing the product of ν projective spaces $P(V^{(j)})$ $j \equiv 1, 2, \ldots, \nu$ well-defined by

$$(14.3) \qquad P(x_1) \times P(x_2) \times \ldots \times P(x_\nu) \to P(x_1 \otimes \ldots \otimes x_n) \in P\left(\bigotimes_{j=1}^{\nu} V^{(j)}\right)$$

where $P\colon x \to P(\boldsymbol{x})$ denotes the canonical projection $V - \{0\} \to P(V)$ of a finite dimensional vector space in its quotient projective space. (cf. Hodge-Pedoe).

We are interested in the symmetric product $\prod/\mathfrak{S} = \Omega/\mathfrak{S}_\nu$ ($\mathfrak{S}_\nu$ symmetric group acting on the set of ν copies of V) replacing $\overset{n}{\otimes}$ by the symmetric product $\bigvee$ of Bourbaki's Multilinear algebra

$$(14.4) \qquad \big(P(x_1), P(x_2), \ldots, P(x_\nu)\big) \to P\left(\bigvee_{j=1}^{\nu} x_j\right).$$

Let us come back to the case $V = C^t$ and the symmetric power $S_\nu(P^{t-1})$ of $\mathbf{P}^{t-1}$ (the « Veronese variety » of the Italian geometers).

PROP. 14.1. *The restriction $S_\nu(\mathcal{A})$ of $S_\nu(\mathbf{P}^{t-1})$ to the ν-tuples $\prod_{\mu \in \mathfrak{m}(G)} \mu(x)$ represents E/G in a canonical way.*

It is well known that $S^\nu(\mathcal{A})$ is a non-singular projective algebraic variety lying in a well-known complex projective space $\mathbf{P}^d$.

If we fix any point in E/G (that we can assume to be $O + G$) the image point in P^d depends only on the moduli of E/G. Precisely, this construction gives a *projective model of the variety of moduli.*

The dependence of the series (14.1) from the « moduli » becomes clear if we recall again that $H(u) = -A(Ju, u)$ (cf. § 1). Let us keep the structure of $2n$ dimensional real vector space on E and the lattice G. To change the complex structure of E we interpret the map $u \xrightarrow{J} iu$ (of the fixed complex structure) as a

$$(14.5) \qquad J: E \to E \qquad J \in GL(E; \mathbb{R}), \quad J^2 = -1 .$$

Conversely for any choice of J satisfying (14.5) the invariance condition

$$(14.6) \qquad A(Jx, Jy) = A(x, y), \qquad (x, y) \in E \times E$$

and

$$(14.7) \qquad A(Ju, u) \geqslant 0, \qquad A(Ju, u) = 0, \quad \text{iff } u = 0$$

we have a complex structure in E/G and a canonical θ embedding of E/G.

We hope to give full details of this construction in another forthcoming paper. Let us examine again item 2) of Siegel's program. Siegel proved (cf. Satz 9) that the Chow point Ch$(\mathcal{A})$ of $\mathcal{A}$ is invariant by the congruence subgroup $\Delta(G)$. Our contributions show the geometrical meaning of such fact, since any map of $\Delta(G)$ leaves invariant element wise the vector space $\Theta(\mathfrak{D})$ of reduced theta functions, thus is only the quotient group $\mathfrak{m}(G) = \mathfrak{M}(G)/\Delta(G)$ that counts! We believe that to try to make explicit such dependence is an important task that deserves full attention.

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