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Properties of normal boundary problems for elliptic even-order systems

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Properties of Normal Boundary Problems for Elliptic Even-Order Systems.

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In this paper we present a « reduction to the boundary » for normal boundary value problems for *elliptic systems* A, that is used to reduce the study of coerciveness inequalities

$$\text{(1)} \qquad \operatorname{Re}(Au, u) \geqslant c\|u\|_K^2 - \lambda\|u\|_0^2, \qquad u \in D(A_B),$$

for realizations A_B of A, to related coerciveness inequalites for the pseudo-differential operators acting in certain vector bundles over the boundary. The reduction is also used to establish a perturbation formula, from which we deduce a new asymptotic estimate for the negative eigenvalues of selfadjoint elliptic realizations of strongly elliptic systems. (The study of (1) requires a more delicate reduction than those given in [18], [25].)

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The paper has two chapters. In Chapter I (Sections 1-4), we study the coerciveness problem from an abstract viewpoint and work out a theory, that would be applicable also to other boundary problems than those for elliptic operators (e.g. the parabolic and hyperbolic cases treated in Lions-Magenes [21, vol. 2], and certain degenerate elliptic operators). (The difficulty in such applications of course resides in the *interpretation* of the more or less abstract statements.) Let A_0, A_γ and A_1 be linear, closed, densely defined operators in a Hilbert space H, such that $A_0 \subseteq A_\gamma \subseteq A_1$; A_γ being bijective. Then the closed linear operators $\tilde{A}$ lying between A_0 and A_1 are in a 1-1 correspondence with the closed, densely defined operators $T\colon V \to W$, where V resp. W run through all closed subspaces of the null-spaces $Z(A_1)$ resp. $Z(A_0^*)$, cf. [11]. Under further hypotheses on A_0, A_γ and A_1 (in particular, that A_γ is regularly accretive and has a compact inverse), we now show that when K is any Hilbert space between H and H_γ ($H_\gamma =$ the closure of $D(A_\gamma)$ under the norm $\operatorname{Re}(A_\gamma u, u) + \text{const.}\, \|u\|_H^2$), and U is any linear set between $D(A_\gamma)$ and K, then $\tilde{A}$ satisfies

$$(2) \qquad \operatorname{Re}(\tilde{A}u, u) \geqslant c\|u\|_K^2 - \lambda\|u\|_H^2\,, \quad \forall u \in D(\tilde{A}) \cap U\,,$$

for some $c > 0$, $\lambda \in \boldsymbol{R}$, if and only if T has the properties

$$(3) \qquad V \subseteq W\,, \quad \text{and} \quad \operatorname{Re}(Tz, z) \geqslant c'\|z\|_K^2 - \lambda'\|z\|_H^2\,, \quad \forall z \in D(T) \cap U\,,$$

for some $c' > 0$, $\lambda' \in \boldsymbol{R}$. This is the statement for the case where $A_1 = A_0^*$ and $A_\gamma = A_\gamma^*$ (cf. Theorem 2.13 below); in the case where $A_\gamma \neq A_\gamma^*$, the condition (3) must be replaced by a more complicated version (cf. Theorem 3.6). The result was proved earlier for quite special choices of K in [12] and [13]; the difficulty for the general case lies in concluding from (3) to (2) when $-\lambda'$ is large negative. This is overcome in the present paper by a technique that uses compactness of A_γ^{-1}, and involves a study of how the set-up changes when A_0, A_1 (etc.) are replaced by $A_0 - \mu$, $A_1 - \mu$ (etc.) for real μ outside the spectrum of A_γ.

Chapter I ends with Section 4, where the negative spectra of $\tilde{A}$ and T are set in relation, in preparation for Section 8.

The methods of Chapter I are elementary Hilbert space techniques, whereas the application to specific boundary problems in Chapter II (Sections 5-8) involves the use of more extensive theories. We consider a $2m$ order elliptic differential operator A in a q-dimensional vector bundle E over an n-dimensional compact C^∞ manifold $\bar{\Omega}$ with boundary Γ. Section 5 consists of background material. In Section 6 the *normal* boundary conditions $B\varrho u = 0$ are introduced; here ϱu denotes the Cauchy data of u, and B is a triangular matrix of differential operators, with *surjective* zero order operators in the diagonal (and possibly pseudo-differential operators below it); such conditions were studied in detail in [17]. Let A_B denote the realization of A with domain $D(A_B) = \{u \in L^2(E) | Au \in L^2(E), B\varrho u = 0\}$; then $A_0 \subseteq \subseteq A_B \subseteq A_1$, where A_0 and A_1 are the minimal, resp. the maximal, operators for A. Denoting by A_γ the realization of the Dirichlet condition $\gamma u = 0$, and assuming A_γ bijective, we have that A_B corresponds by Chapter I to an operator $T: V \to W$. It is now shown, by use of [17], that

$$(4) \qquad \gamma V = \Phi \prod_{k=0}^{m-1} H^{-k-\frac{1}{2}}(Z_k)\,, \quad \gamma W = \Psi \prod_{j=m}^{2m-1} H^{-2m+j+\frac{1}{2}}(F_j)\,,$$

where Φ and Ψ are injective (pseudo-) differential operators, and the Z_k and F_j are certain vector bundles over Γ. Moreover, T is by use of γ, Φ and Ψ carried into a pseudo-differential operator $\mathfrak{L}$ (a *representation* of T) going from $\bigoplus_{k=0}^{m-1} Z_k$ to $\bigoplus_{j=m}^{2m-1} F_j$; and the dimension of $Z(A_B)$, the codimension of $R(A_B)$ and the regularity of A_B correspond to analogous features of $\mathfrak{L}$.

Section 7 takes up the coerciveness problem (for which the assumption of normality is justified, as observed by Seeley [27]). In [17], there was given a necessary and sufficient condition (on B and A near Γ) for the weak semiboundedness estimate

$$(5) \qquad \operatorname{Re}(Au, u) \geq -\lambda \|u\|_m^2\,, \quad \forall u \in D(A_B) \cap H^{2m}(E)\,.$$

With the present notations, that condition is also equivalent with the property $\gamma V \subseteq \gamma W$ (and with $\gamma V = \gamma W$ if the range space for B has total dimension mq); in such cases, $\mathfrak{L}$ is replaced by more convenient representations. Assuming that A is strongly elliptic, we construct (from $\mathfrak{L}$ and A) a pseudo-differential operator $\mathfrak{I}: \prod_{k=0}^{m-1} H^{-k-\frac{1}{2}}(Z_k) \to \prod_{k=0}^{m-1} H^{-2m+k+\frac{1}{2}}(Z_k)$, with which we have:

THEOREM. *Let K be a Hilbert space satisfying*

$$(6) \qquad H_0^m(E) \subseteq K \subseteq L^2(E) \quad (\textit{continuous injections})$$

and containing $D(A_B) \cap H^{2m}(E)$, and let $U = K \cap \{u \in L^2(E) | Au + A'u \in \in L^2(E)\}$. K_Φ and U_Φ denote certain subspaces of $\prod_{k=0}^{m-1} H^{-k-\frac{1}{2}}(Z_k)$, derived from K and U by use of γ and Φ. There exist $c > 0$, $\lambda \in \mathbf{R}$ such that

$$(7) \qquad \operatorname{Re}(Au, u) \geq c\|u\|_K^2 - \lambda \|u\|_0^2\,, \quad \forall u \in D(A_B) \cap U\,,$$

if and only if (i) *and* (ii) *hold*:

(i) A_B *satisfies the condition for* (5).

(ii) *There exist* $c' > 0$, $\lambda' \in \mathbf{R}$ *such that*

$$\operatorname{Re}\langle \mathcal{T}\varphi, \varphi\rangle \geqslant c'\|\varphi\|^2_{K_\Phi} - \lambda'\|\varphi\|^2_{\Pi H^{-k-\frac{1}{2}}(Z_k)} \tag{8}$$

for all $\varphi \in U_\Phi$ *with* $\mathcal{T}\varphi \in \prod_{k=0}^{m-1} H^{k+\frac{1}{2}}(Z_k)$.

This is Corollary 7.8; cases of more general K and U are included in the more complicated Theorem 7.7. When $K = H^s(E)$ for some $s \in [0, m]$, (8) takes the more familiar form

$$\operatorname{Re}\langle \mathcal{T}\varphi, \varphi\rangle \geqslant c'\|\varphi\|^2_{\Pi H^{s-k-\frac{1}{2}}(Z_k)} - \lambda'\|\varphi\|^2_{\Pi H^{-k-\frac{1}{2}}(Z_k)}\,. \tag{9}$$

For $s = m$, (9) holds if and only if $\operatorname{Re}\sigma^0(\mathcal{T}) > 0$ on $T^*(\Gamma)\setminus 0$; this characterizes the realizations satisfying Garding's inequality, completing the sufficient conditions of Agmon [1] and de Figueiredo [8]. For $s = 0$, the theorem characterizes lower boundedness of A_B by the analogous property of $\mathcal{T}$. (See Theorem 7.10 and Corollary 7.11). The results generalize (and improve) those given for scalar A in [13].

Finally, we consider in Section 8 the selfadjoint realizations A_B of a formally selfadjoint elliptic operator A, and derive an *isometric* representation $\mathcal{C}$ of T, that allows for sharper correlations of properties; in particular we set up a *perturbation formula* (8.19). This is applied to a study of the negative spectrum of A_B in the case where A is strongly elliptic. We show that when A_B is elliptic and unbounded below, then the number of eigenvalues in $]-t, 0[$ satisfies the asymptotic estimate for $t \to +\infty$

$$\sum_{\lambda_j \in]-t,0[} 1 \leqslant c^-(A_B)\, t^{(n-1)/2m} + o\,(t^{(n-1)/2m})\,, \tag{10}$$

improving previously known estimates (Theorem 8.11.1°). This is derived from a more general theorem (Theorem 8.7), that also describes the number of negative eigenvalues in the lower bounded (finite) case, and allows for a discussion of the sharpness of (10), and of non-elliptic cases.

Some of the above results were announced in [14]. Moreover, we presented a sketch of part of the theory in Séminaire C. Goulaouic- L. Schwartz (and in Séminaire J. L. Lions-H. Brézis) [15], and a further developed version at « Colloque sur les équations aux dérivées partielles, Orsay 1972 » [16], which also includes a direct proof of the characterization of Garding's inequality. The author would like to thank the organizers of these meetings for the inspiring occasions.

CHAPTER I

ABSTRACT THEORY

1. – The general set-up.

The study of extensions of linear operators in Hilbert space is a well known tool in the theory of boundary value problems. The basic notions for the framework used *here* were developed in [11], additional studies were made in [12] and [13]. In the present chapter we obtain a complete discussion of abstract coerciveness inequalities (under the assumption that a certain fixed operator is compact); moreover, we derive a result on comparison of eigenvalues.

For an operator P from a topological vector space X to a topological vector space Y, we denote the domain, range and kernel by $D(P)$, $R(P)$ and $Z(P)$, respectively (*). I denotes the identity opertor in various contexts.

ASSUMPTION 1.1. There is given a Hilbert space H with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$, and a closed, densely defined, unbounded operator A_γ in H, bijective from $D(A_\gamma)$ onto H. There are given two closed, densely defined operators A_0 and A_1 in H, satisfying $A_0 \subseteq A_\gamma \subseteq A_1$. Denote $A_1^* = A_0'$, $A_0^* = A_1'$.

Clearly, $A_0' \subseteq A_\gamma^* \subseteq A_1'$ and A_γ^* maps $D(A_\gamma^*)$ bjiectively onto H. A_0 and A_0' are injective with closed ranges, A_1 and A_1' are surjective. Thus

$$H = R(A_0) \oplus Z(A_1') = R(A_0') \oplus Z(A_1) , \tag{1.1}$$

orthogonal direct sums. Define

$$\mathrm{pr}_\gamma = A_\gamma^{-1} A_1 , \quad \mathrm{pr}_\zeta = I - \mathrm{pr}_\gamma , \tag{1.2}$$

they are continuous operators in $D(A_1)$, and they project $D(A_1)$ onto its two components $D(A_\gamma)$ resp. $Z(A_1)$

$$D(A_1) = D(A_\gamma) \dotplus Z(A_1) \quad \text{(direct topological sum).}$$

We also denote $\mathrm{pr}_\gamma u = u_\gamma$, $\mathrm{pr}_\zeta u = u_\zeta$. Similarly,

$$\mathrm{pr}_\gamma' = (A_\gamma^*)^{-1} A_1' , \quad \mathrm{pr}_\zeta' = I - \mathrm{pr}_\gamma'$$

(*) $D(P)$ is provided with the graph-topology.

decompose $D(A_1') = D(A_\gamma^*) \dotplus Z(A_1')$; we write $\mathrm{pr}_\gamma' u = u_\gamma'$, $\mathrm{pr}_\zeta' u = u_\zeta'$. The *orthogonal* projection of u onto a closed subspace X of H is denoted by $\mathrm{pr}_X u$ or u_X.

$\mathcal{M}$ (resp. $\mathcal{M}'$) denotes the class of linear operators $\tilde{A}$ (resp. $\tilde{A}'$) satisfying $A_0 \subseteq \tilde{A} \subseteq A_1$ (resp. $A_0' \subseteq \tilde{A}' \subseteq A_1'$). When $u \in D(\tilde{A})$ for some $\tilde{A} \in \mathcal{M}$, we usually simplify $\tilde{A}u$ to Au; similarly $\tilde{A}'u$ is usually written $A'u$.

The closed $\tilde{A} \in \mathcal{M}$ were characterized in [11, Section II.1] as follows:

PROPOSITION 1.2. *Let $\tilde{A}$ closed $\in \mathcal{M}$. Let*

$$V = \mathrm{cl}\, \mathrm{pr}_\zeta D(\tilde{A}) , \qquad W = \mathrm{cl}\, \mathrm{pr}_\zeta' D(\tilde{A}^*) ,$$

closures in H. Then the set $G \subseteq V \times W$ defined by

$$G = \{[u_\zeta, (Au)_W] | u \in D(\tilde{A})\}$$

is the graph of a closed, densely defined operator T from V into W, with $D(T) = \mathrm{pr}_\zeta D(\tilde{A})$. Conversely, let V and W be any closed subspaces of $Z(A_1)$ resp. $Z(A_1')$, and let T be any closed, densely defined operator from V into W. Then

$$D(\tilde{A}) = \{u \in D(A_1) | u_\zeta \in D(T), (Au)_W = Tu_\zeta\}$$

is the domain of a closed operator $\tilde{A} \in \mathcal{M}$. Hereby is established a 1-1 correspondence between all closed $\tilde{A} \in \mathcal{M}$ and all such triples V, W, T.

When $\tilde{A}$ corresponds to $T\colon V \to W$ in this way, $\tilde{A}^*$ corresponds to $T^*\colon W \to V$ in the analogous way (relative to $\mathcal{M}'$).

The proof is based on the identity, valid for all $u \in D(\tilde{A})$, $v \in D(\tilde{A}^*)$,

$$0 = (Au, v) - (u, A'v) = (Au, v_\zeta') - (u_\zeta, A'v) = \big((Au)_W, v_\zeta'\big) - \big(u_\zeta, (A'v)_V\big) ,$$

which shows that $u_\zeta = 0$ implies $(Au)_W = 0$, so that G is a graph. Similarly, there is a mapping $T_1\colon v_\zeta' \mapsto (A'v)_V$. One then shows that T and T_1 are adjoints, that all triples are attained, and that $\tilde{A}$ corresponds 1-1 to T, V, W. We recall from [11. II.1] the properties:

PROPOSITION 1.3. *Let $\tilde{A}$ correspond to $T\colon V \to W$ as in Proposition 1.2. Then*

$$(1.3) \qquad D(\tilde{A}) = \{u = x + A_\gamma^{-1}(Tz + f) + z | [x, z, f] \in D(A_0) \times D(T) \times \big(Z(A_1') \ominus W\big)\} ,$$

in this decomposition $u_\gamma = x + A_\gamma^{-1}(Tz + f)$ and $u_\zeta = z$.

PROPOSITION 1.4. *Let $\tilde{A}$ correspond to $T: V \to W$ as in Proposition* 1.2. *Then*

$$Z(\tilde{A}) = Z(T)\,, \tag{1.4}$$

and

$$R(\tilde{A}) = R(T) \dotplus (H \ominus W) \quad (\textit{orthogonal direct sum}). \tag{1.5}$$

When $Z(\tilde{A}) = 0$,

$$\tilde{A}^{-1} = A_\gamma^{-1} + T^{(-1)} \quad (\textit{on } R(\tilde{A}))\,, \tag{1.6}$$

where $T^{(-1)}f = T^{-1}\,\mathrm{pr}_W f$ *for* $f \in R(\tilde{A})$.

(1.5) corrects a wrongly presented formula in [11, Theorem II 1.3] (the proof is an immediate consequence of (1.3)).

We shall now use Proposition 1.3 to discuss lower bounds. When P is an operator in H, its lower bound $m(P)$ is defined by

$$m(P) = \inf\{\mathrm{Re}\,(Pu, u)\,|\,u \in D(P),\ \|u\| = 1\}\,,$$

and P is called positive, nonnegative, lower bounded or unbounded below, according to whether $m(P)$ is > 0, $\geqslant 0$, $> -\infty$ or $= -\infty$.

2. – Inequalities in the selfadjoint set-up.

In addition to Assumption 1.1 we assume in this section

ASSUMPTION 2.1. A_γ is selfadjoint with $m(A_\gamma) > 0$. $A_1 = A_0^*$, and $D(A_0)$ is dense in $D(A_\gamma^{\frac{1}{2}})$.

Note that A_γ is the Friedrichs extension of A_0, we do not here consider a more general extension A_β as in [11].

In the following, let $\tilde{A}$ correspond to $T: V \to W$ by Proposition 1.2. ($\tilde{A}$ will of course in general not be self-adjoint.) When $V \subseteq W$, we have a simple identity:

$$\begin{aligned}(Au, v) &= (Au, v_\gamma) + ((Au)_W, v_\zeta) \\ &= (Au_\gamma, v_\gamma) + (Tu_\zeta, v_\zeta) \quad \text{for} \quad u, v \in D(\tilde{A}).\end{aligned} \tag{2.1}$$

((2.1) holds whenever $v_\zeta \in W$.) We shall also need the inequality

$$\frac{a\|x\|^2 + b\|y\|^2}{\|x+y\|^2} \geqslant \frac{ab}{a+b}\,, \qquad \text{when } a + b > 0,\ x + y \neq 0\,, \tag{2.2}$$

x and y belonging to H; it is proved as follows:

$$a\|x\|^2 + b\|y\|^2 - ab(a+b)^{-1}\|x+y\|^2 =$$
$$= (a+b)^{-1}[(a+b)a\|x\|^2 + (a+b)b\|y\|^2 - ab\|x\|^2 - ab\|y\|^2 - 2ab\,\mathrm{Re}\,(x, y)]$$
$$= (a+b)^{-1}\|ax - by\|^2 \geqslant 0\ .$$

The following results were proved in [11, Section II.2].

PROPOSITION 2.2.

(i) *If* $m(\tilde{A}) > -\infty$, *then* $V \subseteq W$ *and* $m(T) \geqslant m(\tilde{A})$.

(ii) *If* $V \subseteq W$ *and* $m(T) > -m(A_\gamma)$, *then*

$$m(\tilde{A}) \geqslant \frac{m(A_\gamma)\, m(T)}{m(A_\gamma) + m(T)}\ .$$

(iii) $\tilde{A}$ *is symmetric if and only if* $V \subseteq W$ *and* T *is symmetric as an operator in* W.

(iv) $\tilde{A}$ *is selfadjoint if and only if* $V = W$ *and* T *is selfadjoint.*

The proofs are based on (2.1); (i) furthermore uses the denseness of $D(A_0)$ in $D(A_\gamma^{\frac{1}{2}})$ (like in the proof of Theorem 2.13 below), and (ii) uses (2.2).

In order to include the cases where $m(T) \leqslant -m(A_\gamma)$, we shall study how the set-up changes when A_0, A_γ and A_1 are replaced by $A_0 - \mu$, $A_\gamma - \mu$ and $A_1 - \mu$.

DEFINITION 2.3. For $\mu \in \varrho(A_\gamma)$ (the resolvent set for A_γ), define the operators

$$E^\mu = A_1(A_\gamma - \mu)^{-1} = I + \mu(A_\gamma - \mu)^{-1}\ ,$$
$$F^\mu = (A_1 - \mu)\, A_\gamma^{-1} = I - \mu A_\gamma^{-1}\ .$$

Clearly, E^μ and F^μ are bounded operators in H, selfadjoint when μ is real; moreover

$$F^\mu E^\mu = E^\mu F^\mu = I\ , \tag{2.3}$$

since $A_\gamma^{-1} A_1 v = v = (A_\gamma - \mu)^{-1}(A_1 - \mu)\, v$ for v belonging to

$$A_\gamma^{-1} H = D(A_\gamma) = D(A_\gamma - \mu) = (A_\gamma - \mu)^{-1} H\ . \tag{2.4}$$

LEMMA 2.4. *Let K be a linear space satisfying*

$$D(A_\gamma) \subseteq K \subseteq H \, . \tag{2.5}$$

Then

$$E^\mu K = K = F^\mu K \, , \tag{2.6}$$

and

$$E^\mu[Z(A_1) \cap K] = Z(A_1 - \mu) \cap K \, , \tag{2.7}$$

$$F^\mu[Z(A_1 - \mu) \cap K] = Z(A_1) \cap K \, . \tag{2.8}$$

If K is provided with a norm $\|\cdot\|_K$ for which the injections in (2.5) *are continuous, one has for all $u \in K$*

$$\|u\|_K \leqslant c\|E^\mu u\|_K \leqslant c'\|u\|_K \tag{2.9}$$

with positive constants c and c'.

PROOF. When $u \in K$, $F^\mu u = u - \mu A_\gamma^{-1} u \in K$ since $D(A_\gamma) \subseteq K$. So $F^\mu K \subseteq \subseteq K$, and similarly (cf. (2.4)) $E^\mu K \subseteq K$. Applying E^μ resp. F^μ to these inclusions, we find $K \subseteq E^\mu K$, $K \subseteq F^\mu K$, which completes the proof of (2.6).

When the inclusions in (2.5) are continuous, we furthermore have for $u \subseteq K$

$$\begin{aligned} \|u\|_K = \|F^\mu E^\mu u\|_K &\leqslant \|E^\mu u\|_K + \|\mu A_\gamma^{-1} E^\mu u\|_K \\ &\leqslant \|E^\mu u\|_K + c_1\|A_\gamma^{-1} E^\mu u\|_{D(A_\gamma)} \leqslant \|E^\mu u\|_K + c_2\|E^\mu u\|_H \\ &\leqslant c_3\|E^\mu u\|_K \, , \end{aligned}$$

and similarly $\|E^\mu u\|_K \leqslant c_4\|F^\mu E^\mu u\|_K = c_4\|u\|_K$, proving (2.9).

Finally, let $z \in Z(A_1) \cap K$. Then $E^\mu z \in K$, and

$$(A_1 - \mu) E^\mu z = (A_1 - \mu) z + \mu (A_1 - \mu)(A_\gamma - \mu)^{-1} z = A_1 z = 0 \, ,$$

so $E^\mu z \in Z(A_1 - \mu)$. This proves $E^\mu(Z(A_1) \cap K) \subseteq Z(A_1 - \mu) \cap K$. One shows in a similar way that $F^\mu(Z(A_1 - \mu) \cap K) \subseteq Z(A_1) \cap K$, and applies (2.3) to conclude (2.7) and (2.8).

Introduce the projections, for $u \in D(A_1)$,

$$\mathrm{pr}_\gamma^\mu u = (A_\gamma - \mu)^{-1} (A_1 - \mu) u = u_\gamma^\mu \, , \qquad \mathrm{pr}_\zeta^\mu u = (I - \mathrm{pr}_\gamma^\mu) u = u_\zeta^\mu \, , \tag{2.10}$$

they decompose $D(A_1) = D(A_1 - \mu)$ into the topological direct sum

$$D(A_1 - \mu) = D(A_\gamma - \mu) \dotplus Z(A_1 - \mu) .$$

The relation to the usual decomposition of $D(A_1)$ is found by observing that

$$u = u_\gamma + u_\zeta = [u_\gamma + (I - E^\mu)\, u_\zeta] + E^\mu u_\zeta ,$$

where $E^\mu u_\zeta \in Z(A_1 - \mu)$ by (2.7), and

$$(I - E^\mu)\, u_\zeta = -\mu(A_\gamma - \mu)^{-1} u_\zeta \in D(A_\gamma - \mu) ,$$

so that

$$u_\gamma^\mu = u_\gamma - \mu(A_\gamma - \mu)^{-1} u_\zeta \; ; \qquad u_\zeta^\mu = E^\mu u_\zeta . \tag{2.11}$$

We now introduce an operator in $Z(A_1)$.

DEFINITION 2.5. Let $\mu \in \varrho(A_\gamma)$. The operator G^μ in $Z(A_1)$ is defined by

$$G^\mu z = -\operatorname{pr}_{Z(A_1)} \mu E^\mu z , \quad \text{for} \quad z \in Z(A_1) \; ; \tag{2.12}$$

in other words,

$$(G^\mu z_1, z_2) = -(\mu E^\mu z_1, z_2) , \quad \text{for} \quad z_1, z_2 \in Z(A_1) . \tag{2.13}$$

Denote by $\mathscr{M}^\mu$ the class of linear operators between $A_0 - \mu$ and $A_1 - \mu$; clearly $\tilde{A} \in \mathscr{M} \Leftrightarrow \tilde{A} - \mu \in \mathscr{M}^\mu$.

PROPOSITION 2.6. *Let $\tilde{A}$ be a closed operator $\in \mathscr{M}$, corresponding to $T\colon V \to$ $\to W$ by Proposition 1.2. Let $\mu \in \varrho(A_\gamma) \cap \mathbf{R}$. Then $\tilde{A} - \mu$ corresponds to $T^\mu\colon V^\mu \to W^\mu$ (by Proposition 1.2 applied to $\mathscr{M}^\mu$), determined by*

$$D(T^\mu) = E^\mu D(T) , \quad V^\mu = E^\mu V , \quad W^\mu = E^\mu W \; ; \tag{2.14}$$

$$(T^\mu E^\mu v, E^\mu w) = (Tv, w) + (G^\mu v, w) , \quad \textit{for} \quad v \in D(T) , \quad w \in W . \tag{2.15}$$

PROOF. By (2.11), $D(T^\mu) = \operatorname{pr}_\zeta^\mu D(\tilde{A} - \mu) = E^\mu \operatorname{pr}_\zeta D(\tilde{A}) = E^\mu D(T)$. Moreover,

$$V^\mu = \operatorname{cl} \operatorname{pr}_\zeta^\mu D(\tilde{A} - \mu) = \operatorname{cl} E^\mu \operatorname{pr}_\zeta D(\tilde{A}) = E^\mu \operatorname{cl} \operatorname{pr}_\zeta D(\tilde{A}) = E^\mu V ,$$

since E^μ is an isomorphism in H. Similarly, $W^\mu = E^\mu W$. Now we have for all $w \in W$, all $u \in D(\tilde{A})$ (cf. Proposition 1.2):

$$\begin{aligned}(Tu_\zeta, w) &= (Au, w) = (Au, F^\mu E^\mu w) = (F^\mu Au, E^\mu w)\\ &= \big((A-\mu)A_\gamma^{-1}Au, E^\mu w\big) = \big((A-\mu)u_\gamma, E^\mu w\big)\\ &= \big((A-\mu)u, E^\mu w\big) - \big((A-\mu)u_\zeta, E^\mu w\big)\\ &= (T^\mu u_\zeta^\mu, E^\mu w) + (\mu u_\zeta, E^\mu w)\\ &= (T^\mu E^\mu u_\zeta, E^\mu w) + (\mu E^\mu u_\zeta, w)\end{aligned}$$

which shows (2.15), where we set $v = u_\zeta$.

Now, as long as $m(A_\gamma - \mu) > 0$, Proposition 2.2 applies to the correspondence between $\tilde{A} - \mu$ and T^μ. The treatment of large negative bounds for T then hinges on whether T^μ can be brought into the range of applicability of Proposition 2.2 (ii) for $-\mu$ large. Indeed, we shall show that $m(G^\mu) \to +\infty$ for $\mu \to -\infty$. This is obtained by use of a lemma of Rellich [24] (see Dunford-Schwartz [7, XIII. 7.22]):

LEMMA 2.7. *Let S_1 and S_2 be symmetric operators, and assume that $S_1 \subseteq S_2$, and $D(S_2) = D(\bar{S}_1) \dotplus N$, where N is finite dimensional. If $m(S_1) > -\infty$ then $m(S_2) > -\infty$.*

This will applied to a very special case:

PROPOSITION 2.8. *Assume that A_γ^{-1} is compact. Let $\lambda \in \mathbf{R}$, and denote by $T(\lambda)$ the operator λI with domain $Z(A_1)$. Let $\tilde{A}(\lambda)$ be the operator corresponding to $T(\lambda): Z(A_1) \to Z(A_1)$ by Proposition 1.2. Then $m(\tilde{A}(\lambda)) > -\infty$.*

PROOF. In view of Proposition 2.2 (ii), we may assume $\lambda \leqslant -m(A_\gamma) < 0$. By Proposition 2.2 (iv), $\tilde{A}(\lambda)$ is self-adjoint, and by Proposition 1.3,

$$D(\tilde{A}(\lambda)) = \{u = x + A_\gamma^{-1}\lambda z + z \mid [x, z] \in D(A_0) \times Z(A_1)\} ;$$

here $u_\gamma = x + A_\gamma^{-1}\lambda z$, $u_\zeta = z$. In particular,

$$Au_\gamma = Au = Ax + \lambda z \in R(A_0) \oplus Z(A_1) ,$$

so $u_\zeta = z = \lambda^{-1}\,\mathrm{pr}_{Z(A_1)} Au_\gamma$. Introduce the operator

$$\Theta = \lambda^{-1}\,\mathrm{pr}_{Z(A_1)} A_1 ,$$

it maps $D(A_\gamma)$ continuously onto $Z(A_1)$, and clearly

$$D(\tilde{A}(\lambda)) \subseteq (I+\Theta)\, D(A_\gamma)\,.$$

Conversely, if $v \in D(A_\gamma)$, we put $z = \Theta v$ and $x = v - \lambda A_\gamma^{-1} z$, then

$$Ax = Av - A\lambda A_\gamma^{-1}\lambda^{-1}\,\mathrm{pr}_{Z(A_1)} Av = (I - \mathrm{pr}_{Z(A_1)})\, Av = \mathrm{pr}_{R(A_0)} Av \in R(A_0)\,,$$

so $x \in D(A_0)$. Thus

$$(I+\Theta)\, v = x + \lambda A_\gamma^{-1} z + z \in D(\tilde{A}(\lambda))\,,$$

and we have shown

$$D(\tilde{A}(\lambda)) = (I+\Theta)\, D(A_\gamma)\,.$$

Since $R(\Theta) = Z(A_1)$,

$$\mathrm{pr}_\gamma\,(I+\Theta)\, v = v \quad \text{for} \quad v \in D(A_\gamma)\,,$$

so pr_γ serves as a left inverse to $I+\Theta$.

Now let $\tau > |\lambda|$ and let X_τ and N_τ denote the eigenspaces belonging to the eigenvalues of A_γ that are $\geqslant \tau$, resp. $< \tau$; N_τ is finite dimensional. Let $\tilde{A}_\tau(\lambda)$ be the restriction of $\tilde{A}(\lambda)$ with domain

$$D(\tilde{A}_\tau(\lambda)) = (I+\Theta)(D(A_\gamma) \cap X_\tau)\ ;$$

it is closed, and

$$D(\tilde{A}(\lambda)) = D(\tilde{A}_\tau(\lambda)) \dotplus (I+\Theta)\, N_\tau\,.$$

For $u \in D(\tilde{A}_\tau(\lambda))\setminus\{0\}$, we have $u_\gamma \in X_\tau$, so $(Au_\gamma, u_\gamma) \geqslant \tau \|u_\gamma\|^2$, and, by (2.1) and (2.2),

$$\frac{(Au, u)}{\|u\|^2} = \frac{(Au_\gamma, u_\gamma) + \lambda \|u_\zeta\|^2}{\|u_\gamma + u_\zeta\|^2} \geqslant \frac{\tau \|u_\gamma\|^2 + \lambda \|u_\zeta\|^2}{\|u_\gamma + u_\zeta\|^2} \geqslant \frac{\tau\lambda}{\tau+\lambda}\,,$$

since $\tau + \lambda > 0$. Thus $m(\tilde{A}_\tau(\lambda)) > -\infty$, and Lemma 2.7 applies to show that $m(\tilde{A}(\lambda)) > -\infty$.

DEFINITION 2.9. When A_γ^{-1} is compact, we define the function $\varphi\colon \boldsymbol{R} \to \boldsymbol{R}$ according to Proposition 2.8 by

$$\varphi(\lambda) = m(\tilde{A}(\lambda))\,. \tag{2.16}$$

It follows from Proposition 2.2 (i) that $\varphi(\lambda) \leqslant \lambda$ for all $\lambda \in \boldsymbol{R}$, so $\varphi(\lambda) \to -\infty$ for $\lambda \to -\infty$. Since $\tilde{A}(\lambda) \supseteq A_0$, we also have

$$\varphi(\lambda) \leqslant m(A_0) = m(A_\gamma) \quad \text{for all } \lambda \in \boldsymbol{R}; \tag{2.17}$$

moreover, Proposition 2.2 (ii) shows that $\varphi(\lambda) \geqslant m(A_\gamma)\big(1 + m(A_\gamma)/\lambda\big)^{-1}$ for $\lambda > -m(A_\gamma)$, so that in fact $\varphi(\lambda) \to m(A_\gamma)$ for $\lambda \to +\infty$. Define $a \in \boldsymbol{R} \cup \cup \{+\infty\}$ by

$$a = \sup \{\lambda | \varphi(\lambda) < m(A_\gamma)\} . \tag{2.18}$$

THEOREM 2.10. *Assume that A_γ^{-1} is compact, and consider G^μ for $\mu \in]-\infty, m(A_\gamma)[$. On this interval,*

$$m(G^\mu) \nearrow +\infty \quad \text{for } \mu \searrow -\infty . \tag{2.19}$$

In fact, the function $\psi: \mu \mapsto -m(G^\mu)$, defined for $\mu \in]-\infty, m(A_\gamma)[$, is the inverse of the function φ defined on $]-\infty, a[$ (cf. (2.16), (2.18)); both functions are strictly increasing and continuous. In particular, $\psi(]-\infty, 0]) =]-\infty, 0]$.

PROOF. For $\mu < \mu' < m(A_\gamma)$, $-\mu E^\mu + \mu' E^{\mu'}$ is a positive operator on H, since it equals $f(A_\gamma)$, where the function f satisfies

$$f(\tau) = -\mu\left(1 + \frac{\mu}{\tau - \mu}\right) + \mu'\left(1 + \frac{\mu'}{\tau - \mu'}\right) = \frac{(\mu' - \mu)\,\tau^2}{(\tau - \mu)(\tau - \mu')} \geqslant c > 0$$

for all $\tau \geqslant m(A_\gamma)$. Thus, by restriction to $Z(A_1)$,

$$m(G^\mu) > m(G^{\mu'}) \quad \text{for } \mu < \mu' < m(A_\gamma) ,$$

so the function $\psi: \mu \mapsto -m(G^\mu)$ is strictly increasing on $]-\infty, m(A_\gamma)[$.

Now let $\lambda \in]-\infty, a[$ and consider $\tilde{A}(\lambda)$ defined from $T(\lambda) = \lambda I$ on $Z(A_1)$, as in Proposition 2.8. For $\mu \in]-\infty, m(A_\gamma)[$, let $T^\mu(\lambda)$ denote the operator in $Z(A_1 - \mu)$ corresponding to $\tilde{A}(\lambda) - \mu \in \mathscr{M}^\mu$. Then by Proposition 2.6

$$\big(T^\mu(\lambda) E^\mu z, E^\mu z\big) = \lambda \|z\|^2 + (G^\mu z, z) , \quad \text{for all } z \in Z(A_1) .$$

Since E^μ maps $Z(A_1)$ isomorphically onto $Z(A_1 - \mu)$ for each μ, it follows that

$$m(G^\mu + \lambda) = 0 \tag{2.20}$$

holds if and only if

$$m\big(T^\mu(\lambda)\big) = 0 . \tag{2.21}$$

Since we have assumed $\mu < m(A_\gamma)$, Proposition 2.2 applies to the correspondence between $T^\mu(\lambda)$ and $\tilde{A}(\lambda) - \mu$, showing that (2.21) holds if and only if

$$m(\tilde{A}(\lambda) - \mu) = 0. \tag{2.22}$$

Now the equivalence of (2.20) and (2.22) gives: When $\lambda \in]-\infty, a[$ and we put $\mu = m(\tilde{A}(\lambda)) = \varphi(\lambda)$ ($< m(A_\gamma)$ by the definition of a), then $\lambda = -m(G^\mu) = \psi(\mu)$. Conversely, when $\mu \in]-\infty, m(A_\gamma)[$ and we put $\lambda = -m(G^\mu) = \psi(\mu)$, then $\mu = m(\tilde{A}(\lambda)) = \varphi(\lambda)$; here by (2.17), $\psi(\mu) \leqslant a$. By the monotonicity of ψ, $\mu < \mu' < m(A_\gamma)$ implies $\psi(\mu) < \psi(\mu') \leqslant a$, which shows that in fact $\psi(]-\infty, m(A_\gamma)[) \subseteq]-\infty, a[$. Altogether, we have found that $\varphi\colon]-\infty, a[\to]-\infty, m(A_\gamma)[$ and $\psi\colon]-\infty, m(A_\gamma)[\to]-\infty, a[$ are inverses of each other. Since ψ is strictly increasing, both functions are continuous and strictly increasing. Finally, $\psi(]-\infty, 0]) =]-\infty, 0]$, since

$$\psi(0) = -m(G^0) = -m(0) = 0\,.$$

Remark 2.11. It should be noted that $-\mu E^\mu$, considered as an operator on *all of H*, does *not* have a property like (2.19). In fact, if v is a normalized eigenvector for A_γ belonging to the eigenvalue τ, then

$$(-\mu E^\mu v, v) = -\mu\left(1 + \frac{\mu}{\tau - \mu}\right) = \frac{\tau}{1 - \tau\mu^{-1}} \to \tau\,, \qquad \text{for } \mu \to -\infty\,.$$

However, the eigenvectors for A_γ *do not lie in* $Z(A_1)$. In earlier, futile attempts to prove (2.19), we tried to measure and utilize the positive angle between $Z(A_1)$ and the finite dimensional eigenspaces for A_γ. In our applications to realizations of an elliptic differential operator of order $2m$, G^μ takes the form of a certain elliptic pseudo-differential operator over the boundary (cf. Remark 8.2 below). In special cases (of constant coefficient operators on $\boldsymbol{R}^n_+$) one finds here that $m(G^\mu) \geqslant c|\mu|^{1/2m}$; this is also our conjecture for the general $2m$-order elliptic operators.

We can now complete Proposition 2.2.

Theorem 2.12. (*Assumptions* 1.1 *and* 2.1). *Let A_γ^{-1} be compact, and let $\tilde{A} \in \mathscr{M}$ correspond to $T\colon V \to W$ by Proposition* 1.2. *Then $m(\tilde{A}) > -\infty$ if and only if $V \subseteq W$ and $m(T) > -\infty$. (In particular, $m(\tilde{A}) > 0 \Leftrightarrow V \subseteq W$ and $m(T) > 0$; and $m(\tilde{A}) \geqslant 0 \Leftrightarrow V \subseteq W$ and $m(T) \geqslant 0$.)*

Proof. The implications from $\tilde{A}$ to T, and from T to $\tilde{A}$ for $m(T) > -m(A_\gamma)$, are contained in Proposition 2.2. So let $V \subseteq W$ and $m(T) =$

$=\lambda\leqslant -m(A_\gamma)$. Let $\mu=\varphi(\lambda)$, then $m(G^\mu)=-\lambda$. Now, with the notations of Proposition 2.6, $V^\mu\subseteq W^\mu$, and T^μ satisfies

$$(T^\mu E^\mu z, E^\mu z)=(Tz,z)+(G^\mu z,z)\geqslant\lambda\|z\|^2-\lambda\|z\|^2=0$$

for all $z\in D(T)$, whence $m(T^\mu)\geqslant 0$. Thus $m(\tilde{A}-\mu)\geqslant 0$, i.e., $m(\tilde{A})\geqslant\mu$. Q.e.d.

We shall finally apply Theorem 2.10 to treat some more general inequalities.

Define the Hilbert space

$$H_\gamma=D(A_\gamma^{\frac{1}{2}}),\quad \text{with norm } \|v\|_{H_\gamma}=\left(\|v\|^2+\|A_\gamma^{\frac{1}{2}}v\|^2\right)^{\frac{1}{2}}, \tag{2.23}$$

it is dense in H. As is common, we identify

$$H_\gamma\subseteq H=H'\subseteq H'_\gamma, \tag{2.24}$$

the duality between H_γ and its dual space H'_γ denoted $\langle\,,\rangle$, extending the inner product $(\,,)$ in H. Considered as an operator from H_γ to H, A_0 has an adjoint from H to H'_γ that we denote $A_{1.e}$; it clearly extends A_1, so we abbreviate $A_{1.e}u$ to Au as usual. We denote $D(A_{1.e})$ by $\mathcal{K}_A$, so altogether

$$\begin{aligned} D(A_{1.e})=\mathcal{K}_A=\{u\in H\,|\,|(u,Av)|\leqslant\text{const.}\,\|v\|_{H_\gamma},\quad \text{all } v\in D(A_0)\}\\ \langle Au,v\rangle=(u,Av)\quad \text{for}\quad u\in\mathcal{K}_A,\ v\in D(A_0).\end{aligned} \tag{2.25}$$

Let $A_{\gamma.e}$ be the restriction of $A_{1.e}$ to H_γ; it is an isomorphism of H_γ onto H'_γ, extending A_γ. It is easily seen that

$$\mathcal{K}_A=H_\gamma\dotplus Z(A_1)\quad \text{(direct topological sum)}, \tag{2.26}$$

where the decomposition is defined by the projections

$$\text{pr}_\gamma=A_{\gamma.e}^{-1}A_{1.e},\quad \text{pr}_\zeta=I-\text{pr}_\gamma, \tag{2.27}$$

extending the original projections (1.2).

THEOREM 2.13. (*Assumptions* 1.1 *and* 2.1.) *Assume that* A_γ^{-1} *is compact, and let* $\tilde{A}$ *correspond to* $T\colon V\to W$ *by Proposition* 1.2. *Let* K *be a Hilbert space with norm* $\|\cdot\|_K$ *and satisfying*

$$H_\gamma\subseteq K\subseteq H\quad (\textit{continuous injections}). \tag{2.28}$$

Let U be a linear subspace of K containing $D(A_\gamma)$. There exist $c > 0$, $\lambda \in \mathbf{R}$ such that

$$\text{(2.29)} \qquad \operatorname{Re}(Au, u) \geqslant c\|u\|_K^2 - \lambda\|u\|^2, \quad \textit{for all } u \in D(\tilde{A}) \cap U,$$

if and only if (i) *and* (ii) *hold*:

(i) $D(T) \cap U \subseteq W$.

(ii) *There exist $c' > 0$, $\lambda' \in \mathbf{R}$ such that*

$$\operatorname{Re}(Tz, z) \geqslant c'\|z\|_K^2 - \lambda'\|z\|^2, \quad \textit{for all } z \in D(T) \cap U.$$

PROOF. Assume first that (2.29) holds. Let $z \in D(T) \cap U$, $f \in Z(A_1) \ominus W$, $x^n \in D(A_0)$, and set

$$u^n = x^n + A_\gamma^{-1}(Tz + f) + z,$$

it lies in $D(\tilde{A})$ by Proposition 1.3, and in U, since $D(A_\gamma) \subseteq U$. Now, using that $R(A_0) \perp Z(A_1)$,

$$\begin{aligned} \operatorname{Re}(Au^n, u^n) &= \operatorname{Re}(Au^n, u_\gamma^n) + \operatorname{Re}(Au^n, z) \\ &= (Au_\gamma^n, u_\gamma^n) + \operatorname{Re}(f, z) + \operatorname{Re}(Tz, z) \\ &\geqslant c\|u^n\|_K^2 - \lambda\|u^n\|^2 \quad \text{by assumption.} \end{aligned}$$

Let $x^n \to -A_\gamma^{-1}(Tz + f)$ in H_γ; then $u_\gamma^n \to 0$ in H_γ and thus in K and in H, so that the inequality implies

$$\operatorname{Re}(f, z) + \operatorname{Re}(Tz, z) \geqslant c\|z\|_K^2 - \lambda\|z\|^2.$$

This holds for f multiplied by any complex number; thus $(f, z) = 0$, which shows (i). When this is inserted, we find the inequality (ii) with $c' = c$, $\lambda' = \lambda$.

Conversely assume that (i) and (ii) hold. If $-\lambda' \geqslant 0$, we find for $u \in D(\tilde{A}) \cap U$, using (2.1),

$$\begin{aligned} \operatorname{Re}(Au, u) &= (Au_\gamma, u_\gamma) + (Tu_\zeta, u_\zeta) \\ &\geqslant \tfrac{1}{2}\min\bigl(1, m(A_\gamma)\bigr)\|u_\gamma\|_{H_\gamma}^2 + c'\|u_\zeta\|_K^2 \\ &\geqslant c''\|u\|_K^2, \end{aligned}$$

with $c'' > 0$, which shows (2.29). Now assume $-\lambda' < 0$. Let $\mu = \varphi(-\lambda')$. By Lemma 2.4, E^μ maps $Z(A_1) \cap K$ isomorphically onto $Z(A_1 - \mu) \cap K$.

Let T^{μ} be the operator corresponding to $\tilde{A}-\mu$ (cf. Proposition 2.6), then we have for $z \in D(T) \cap U$

$$\begin{aligned}\operatorname{Re}(T^{\mu}E^{\mu}z, E^{\mu}z) &= \operatorname{Re}(Tz, z) + (G^{\mu}z, z)\\ &\geq c'\|z\|_K^2 - \lambda'\|z\|^2 + \lambda'\|z\|^2\\ &= c'\|z\|_K^2 \geq c''\|E^{\mu}z\|_K^2,\end{aligned}$$

where the constant $c''>0$. Moreover, by Lemma 2.4, $E^{\mu}(D(T)\cap U)= = E^{\mu}D(T)\cap U = D(T^{\mu})\cap U$. Then the previous argument may be applied to the correspondence between $\tilde{A}-\mu$ and T^{μ}, showing that there exists $c>0$ so that

$$\operatorname{Re}\big((\tilde{A}-\mu)u, u\big) \geq c\|u\|_K^2 \quad \text{for all } u \in D(\tilde{A})\cap U,$$

i.e.,

$$\operatorname{Re}(\tilde{A}u, u) \geq c\|u\|_K^2 - |\mu|\,\|u\|^2 \quad \text{for all } u \in D(\tilde{A})\cap U.$$

REMARK 2.14. Previously (cf. [12, Proposition 2.7]), we only had a complete result for $H_\gamma = H_0^m(\Omega)$, $K = H^s(\Omega)$ with $s \in]m-\frac{1}{2}, m]$, A_γ being the Dirichlet realization in $H = L^2(\Omega)$ of a $2m$ order elliptic operator A in a bounded open set $\Omega \subset \mathbf{R}^n$; the proof was based on trace theorems and compact injections $H^s(\Omega) \subset L^2(\Omega)$. Note that above we do not assume compactness of $K \subseteq H$.

REMARK 2.15. When $H_\gamma \subseteq K \subseteq H$, $K_1 = K \cap \mathcal{H}_A$ satisfies

$$K_1 = H_\gamma \dotplus \big(K \cap Z(A_1)\big), \tag{2.30}$$

and for $u \in K_1$,

$$\|u\|_{K_1}^2 \equiv \|u\|_K^2 + \|u\|_{\mathcal{H}_A}^2 \leq c_1(\|u_\gamma\|_{H_\gamma}^2 + \|u_\zeta\|_K^2) \leq c_2\|u\|_{K_1}^2,$$

with positive constants. The proof of Theorem 2.13 shows that (2.29) is also equivalent with

$$\operatorname{Re}(Au, u) \geq c''\|u\|_{K_1}^2 - \lambda''\|u\|^2, \quad \text{all } u \in D(\tilde{A})\cap U. \tag{2.31}$$

In particular, $\tilde{A}$ is *lower bounded* if and only if (2.31) holds with K_1 equal to $\mathcal{H}_A$ (or any space between $\mathcal{H}_A$ and H).

REMARK 2.16. It follows from Theorem 2.12 that we can also complete the results of [12] (Théorèmes 1.1, 1.2) on *variational* $\tilde{A}$ (i.e., those that are associated with coercive sesquilinear forms by the Lax-Milgram lemma): $\tilde{A}$ is variational if and only if $V = W$ and T is variational; and then the associated sesquilinear forms $\tilde{a}$ and t are connected by

$$D(\tilde{a}) = H_\gamma \dotplus D(t) \quad \text{(direct topological sum)}, \tag{2.32}$$

$$\tilde{a}(u, v) = a_\gamma(u_\gamma, v_\gamma) + t(u_\zeta, v_\zeta) \quad \text{for } u, v \in D(\tilde{a}); \tag{2.33}$$

here $a_\gamma(u_\gamma, v_\gamma) = (A_\gamma^{\frac{1}{2}} u_\gamma, A_\gamma^{\frac{1}{2}} v_\gamma)$.

3. – Inequalities in the nonselfadjoint set-up.

In this section, we assume, in addition to Assumption 1.1,

ASSUMPTION 3.1. The operator A_γ is *positive and variational* with $D(A_\gamma) = D(A_\gamma^*)$. Moreover, $D(A_0)$ equals $D(A_0')$ and is dense in H_γ, where H_γ denotes the domain $D(a_\gamma)$ of the sesquilinear form a_γ associated with A_γ:

For details on variational operators, cf. e.g. [12, Section 1.2]; some authors call such operators regularly accretive. Let us just mention that H_γ is a Hilbert space, continuously and densely injected in H, and $a_\gamma(u, v)$ is a continuous sesquilinear form on $H_\gamma \times H_\gamma$ satisfying for all $v \in H_\gamma$

$$c_\gamma \|v\|_{H_\gamma}^2 \leqslant \operatorname{Re} a_\gamma(v, v) \leqslant |a_\gamma(v, v)| \leqslant C_\gamma \|v\|_{H_\gamma}^2 \tag{3.1}$$

with positive constants c_γ and C_γ; A_γ is associated with a_γ by the Lax-Milgram lemma. We use the identification (2.24) and the there mentioned notations for the duality.

Define also the « real parts »:

$$A_\gamma^r = \tfrac{1}{2}(A_\gamma + A_\gamma^*), \qquad D(A_\gamma^r) = D(A_\gamma),$$

it is the selfadjoint positive operator associated with the sesquilinear form $\frac{1}{2}[a_\gamma(u, v) + \overline{a_\gamma(v, u)}]$ defined on H_γ; and

$$A_0^r = \tfrac{1}{2}(A_0 + A_0'), \qquad D(A_0^r) = D(A_0), \qquad \text{and} \qquad A_1^r = (A_0^r)^*.$$

The class of linear operators between A_0^r and A_1^r will be denoted $\mathcal{M}^r$.

The three operators A_1, A_1' and A_1^r are extended to operators from H to H_γ' by

$$\begin{aligned} A_{1,e}&: H \to H_\gamma' && \text{is the adjoint of} && A_0': H_\gamma \to H\,; \\ A_{1,e}'&: H \to H_\gamma' && \text{is the adjoint of} && A_0: H_\gamma \to H\,; \\ A_{1,e}^r&: H \to H_\gamma' && \text{is the adjoint of} && A_0^r: H_\gamma \to H\,; \end{aligned}$$

their domains are denoted

$$D(A_{1.e}) = \mathcal{H}_A\,,\qquad D(A'_{1.e}) = \mathcal{H}_{A'}\,,\qquad D(A^r_{1.e}) = \mathcal{H}_{A^r}\,,$$

and we as usual abreviate notations by writing $A_{1.e}u$ as Au, $A'_{1.e}u$ as $A'u$, and $A^r_{1.e}u$ as $A^r u$. (See the analogous construction (2.25).) The restrictions of these three operators with domains H_γ are *isomorphisms* $A_{\gamma.e}$, $A'_{\gamma.e}$ resp. $A^r_{\gamma.e}$ of H_γ onto H'_γ, and they clearly satisfy (cf. (3.1))

$$(3.2)\qquad \operatorname{Re}\langle A_{\gamma.e}v, v\rangle = \operatorname{Re}\langle A'_{\gamma.e}v, v\rangle = \langle A^r_{\gamma.e}v, v\rangle \geqslant c_\gamma \|v\|^2_{H_\gamma} \quad \text{for all } v\in H_\gamma .$$

Defining the projections

$$\begin{aligned} &\mathrm{pr}_\gamma = A^{-1}_{\gamma.e}A_{1.e}\,, &&\mathrm{pr}'_\gamma = (A'_{\gamma.e})^{-1}A'_{1.e}\,, &&\mathrm{pr}^r_\gamma = (A^r_{\gamma.e})^{-1}A^r_{1.e}\,,\\ &\mathrm{pr}_\zeta = I - \mathrm{pr}_\gamma\,, &&\mathrm{pr}'_\zeta = I - \mathrm{pr}'_\gamma\,, &&\mathrm{pr}^r_\zeta = I - \mathrm{pr}^r_\gamma \end{aligned}$$

(extending definitions of Section 1), we have the decompositions into topological direct sums

$$(3.3)\quad \mathcal{H}_A = H_\gamma \dotplus Z(A_1)\,,\qquad \mathcal{H}_{A'} = H_\gamma \dotplus Z(A'_1)\,,\qquad \mathcal{H}_{A^r} = H_\gamma \dotplus Z(A^r_1)\,,$$

just like in (2.26)-(2.27). We shall often write $\mathrm{pr}^r_\gamma u$ as u^r_γ, and $\mathrm{pr}^r_\zeta u$ as u^r_ζ. Clearly, the results of Sections 2 apply to the operators in $\mathcal{M}^r$.

The fact that $D(A_0) = D(A'_0) = D(A^r_0)$ with $A^r_0 = \frac{1}{2}A_0 + \frac{1}{2}A'_0$ implies easily, by use of the definitions:

LEMMA 3.2. *One has*

$$(3.4)\qquad D(A_1)\cap D(A'_1) = D(A_1)\cap D(A^r_1) = D(A'_1)\cap D(A^r_1)\ ;$$

$$(3.5)\qquad \mathcal{H}_A\cap\mathcal{H}_{A'} = \mathcal{H}_A\cap\mathcal{H}_{A^r} = \mathcal{H}_{A'}\cap\mathcal{H}_{A^r}\,,$$

and

$$(3.6)\qquad D(A_1)\cap\mathcal{H}_{A'} = D(A_1)\cap\mathcal{H}_{A^r}\,.$$

For u in any of these sets,

$$(3.7)\qquad A^r u = \tfrac{1}{2}(Au + A'u)\,.$$

Moreover, the projections fit together as follows:

LEMMA 3.3. *For* $u \in \mathcal{H}_A \cap \mathcal{H}_{A'}$, *one has*

$$\operatorname{pr}_\zeta u = \operatorname{pr}_\zeta \operatorname{pr}'_\zeta u = \operatorname{pr}_\zeta \operatorname{pr}^r_\zeta u \,, \tag{3.8}$$

$$\operatorname{pr}'_\zeta u = \operatorname{pr}'_\zeta \operatorname{pr}_\zeta u = \operatorname{pr}'_\zeta \operatorname{pr}^r_\zeta u \,, \tag{3.9}$$

$$\operatorname{pr}^r_\zeta u = \operatorname{pr}^r_\zeta \operatorname{pr}_\zeta u = \operatorname{pr}^r_\zeta \operatorname{pr}'_\zeta u \,. \tag{3.10}$$

PROOF. For $u \in \mathcal{H}_A \cap \mathcal{H}_{A'}$, we have three unique decompositions according to (3.3) (cf. (3.5))

$$u = \operatorname{pr}_\gamma u + \operatorname{pr}_\zeta u = \operatorname{pr}'_\gamma u + \operatorname{pr}'_\zeta u = \operatorname{pr}^r_\gamma u + \operatorname{pr}^r_\zeta u \,. \tag{3.11}$$

Rewriting the third member of (3.11) we find

$$u = \operatorname{pr}'_\gamma u + \operatorname{pr}_\gamma \operatorname{pr}'_\zeta u + \operatorname{pr}_\zeta \operatorname{pr}'_\zeta u \,,$$

where $\operatorname{pr}'_\gamma u + \operatorname{pr}_\gamma \operatorname{pr}'_\zeta u \in H_\gamma$ and $\operatorname{pr}_\zeta \operatorname{pr}'_\zeta u \in Z(A_1)$, so that by comparison with the second member,

$$\operatorname{pr}_\gamma u = \operatorname{pr}'_\gamma u + \operatorname{pr}_\gamma \operatorname{pr}'_\zeta u \,, \qquad \operatorname{pr}_\zeta u = \operatorname{pr}_\zeta \operatorname{pr}'_\zeta u \,. \tag{3.12}$$

This shows the first identity in (3.8); the remaining identities follow similarly.

LEMMA 3.4. *Let* $u \in D(A_1) \cap \mathcal{H}_{A'}$: *Then*

$$\operatorname{Re}\,(Au, u'_\gamma) = \langle A^r u^r_\gamma, u^r_\gamma \rangle + \operatorname{Re}\,\langle A u^r_\zeta, \operatorname{pr}'_\gamma u^r_\zeta \rangle \,.$$

PROOF. By (3.6), we can define $x = \operatorname{pr}^r_\gamma u$, $y = \operatorname{pr}^r_\zeta u$, and then $u = x + y$, where $x \in H_\gamma$, and $y \in Z(A^r_1) \cap \mathcal{H}_{A'}$. Thus

$$u_\gamma = \operatorname{pr}_\gamma x + \operatorname{pr}_\gamma y = x + \operatorname{pr}_\gamma y \,,$$
$$u'_\gamma = \operatorname{pr}'_\gamma x + \operatorname{pr}'_\gamma y = x + \operatorname{pr}'_\gamma y \,;$$

and we find by insertion

$$\begin{aligned}
\operatorname{Re}\,(Au, u'_\gamma) &= \operatorname{Re}\,(Au_\gamma, u'_\gamma) = \operatorname{Re}\big(A(x + \operatorname{pr}_\gamma y), x + \operatorname{pr}'_\gamma y\big) \\
&= \operatorname{Re}\,[\langle Ax, x\rangle + \langle Ax, \operatorname{pr}'_\gamma y\rangle + \langle A \operatorname{pr}_\gamma y, x\rangle + \langle A \operatorname{pr}_\gamma y, \operatorname{pr}'_\gamma y\rangle] \\
&= \langle A^r x, x\rangle + \operatorname{Re}\,[\langle x, A' y\rangle + \langle Ay, x\rangle] + \operatorname{Re}\,\langle A \operatorname{pr}_\gamma y, \operatorname{pr}'_\gamma y\rangle \\
&= \langle A^r x, x\rangle + \operatorname{Re}\,\langle A \operatorname{pr}_\gamma y, \operatorname{pr}'_\gamma y\rangle = \langle A^r x, x\rangle + \operatorname{Re}\,\langle Ay, \operatorname{pr}'_\gamma y\rangle \,,
\end{aligned}$$

where $\mathrm{Re}\,[\langle x, A'y\rangle + \langle Ay, x\rangle] = \mathrm{Re}\,\langle (A + A')y, x\rangle = 0$ because $y \in Z(A_1^r)$. This shows the lemma.

We shall apply the techniques of Section 2 to the operators in $\mathscr{M}^r$, so we define, for $\mu \in \varrho(A_\gamma^r)$,

$$E^\mu = I + \mu(A_\gamma^r - \mu)^{-1}\,, \qquad F^\mu = I - \mu(A_\gamma^r)^{-1}\,,$$
$$G^\mu = -\mu\, \mathrm{pr}_{Z(A_1^r)}\, E^\mu \colon Z(A_1^r) \to Z(A_1^r)\,.$$

The class of operators between $A_0^r - \mu$ and $A_1^r - \mu$ is denoted $\mathscr{M}^{r,\mu}$; and we have the projections denoted $\mathrm{pr}_\gamma^{r,\mu}\, u = u_\gamma^{r,\mu}$ and $\mathrm{pr}_\zeta^{r,\mu}\, u = u_\zeta^{r,\mu}$, decomposing $\mathcal{H}_{A^r}$ into the direct topological sum

$$\mathcal{H}_{A^r} = H_\gamma \dotplus Z(A_1^r - \mu)\,.$$

Lemma 3.5. *Let $u \in \mathcal{H}_{A^r}$. Then for $\mu \in \varrho(A_\gamma^r) \cap \boldsymbol{R}$,*

$$\langle A^r u_\gamma^r, u_\gamma^r\rangle - \mu\|u\|^2 = \langle (A^r - \mu)\, u_\gamma^{r,\mu}, u_\gamma^{r,\mu}\rangle + (G^\mu u_\zeta^r, u_\zeta^r)\,.$$

Proof. The identities (2.11) extend immediately to $u \in \mathcal{H}_{A^r}$:

$$\mathrm{pr}_\gamma^{r,\mu}\, u = u_\gamma^r - \mu(A_\gamma^r - \mu)^{-1} u_\zeta^r\,; \qquad \mathrm{pr}_\zeta^{r,\mu}\, u = E^\mu u_\zeta^r\,.$$

Denote $x = u_\gamma^r \in H_\gamma$, $y = u_\zeta^r \in Z(A_1^r)$. Then

$$\begin{aligned}
\langle (A^r - \mu)\, u_\gamma^{r,\mu}, u_\gamma^{r,\mu}\rangle &= \langle (A^r - \mu)x - \mu y, x - \mu(A_\gamma^r - \mu)^{-1} y\rangle \\
&= \langle A^r x, x\rangle - \mu(x, x) - \mu(y, x) - \mu(x, y) + \mu^2\big(y, (A_\gamma^r - \mu)^{-1} y\big) \\
&= \langle A^r x, x) - \mu(u, u) + \mu(y, y) + \mu^2\big((A_\gamma^r - \mu)^{-1} y, y\big) \\
&= \langle A^r x, x\rangle - \mu\|u\|^2 - (G^\mu y, y)\,,
\end{aligned}$$

since $\mu + \mu^2(A_\gamma^r - \mu)^{-1} = \mu E^\mu$.

Theorem 3.6. (*Assumptions* 1.1 *and* 3.1.) *Assume that A_γ^{-1} is compact, and let $\tilde{A}$ correspond to $T\colon V \to W$ by Proposition* 1.2. *Let K be a Hilbert space with norm $\|\cdot\|_K$ and satisfying*

$$H_\gamma \subseteq K \subseteq H \quad (\textit{continuous injections}), \tag{3.14}$$

and let U be a linear subspace of $\mathcal{H}_{A^r} \cap K$ containing $D(A_\gamma)$. There exist $c > 0$,

$\lambda \in \mathbf{R}$ such that

$$\text{(3.15)} \qquad \operatorname{Re}(Au, u) \geqslant c\|u\|_K^2 - \lambda\|u\|^2, \quad \textit{for all } u \in D(\tilde{A}) \cap U,$$

if and only if (i) *and* (ii) *hold:*

(i) $\mathrm{pr}'_\gamma(D(T) \cap U) \subseteq W$.

(ii) *There exist $c' > 0$, $\lambda' \in \mathbf{R}$ such that*

$$\text{(3.16)} \qquad \operatorname{Re}[(Tz, z'_\zeta) + \langle Az^r_\zeta, \mathrm{pr}'_\gamma z^r_\zeta\rangle] \geqslant c'\|z^r_\zeta\|_K^2 - \lambda'\|z^r_\zeta\|^2,$$

for all $z \in D(T) \cap U$. Here, if $-\lambda > 0 (\geqslant 0)$, $-\lambda'$ may be taken > 0 $(\geqslant 0)$, and vice versa.

PROOF. 1° Assume (3.15). When $u \in D(\tilde{A}) \cap U$, we have the three decompositions (recall (3.6) and Proposition 1.3)

$$u = x + A_\gamma^{-1}(Tz + f) + z, \qquad [x, z, f] \in D(A_0) \times D(T) \times (Z(A'_1) \ominus W),$$

$$u = u'_\gamma + u'_\zeta, \qquad \text{where } u'_\zeta = \mathrm{pr}'_\zeta u_\zeta = z'_\zeta,\ u'_\gamma = u_\gamma + \mathrm{pr}'_\gamma z,$$

$$u = u^r_\gamma + u^r_\zeta, \qquad \text{where } u^r_\zeta = z^r_\zeta,\ u^r_\gamma = u_\gamma + \mathrm{pr}^r_\gamma z.$$

Then we find by use of Lemma 3.4

$$\begin{aligned}\text{(3.17)} \qquad \operatorname{Re}(Au, u) &= \operatorname{Re}(Au, u'_\gamma) + \operatorname{Re}(Ax + Tz + f, u'_\zeta)\\ &= \langle A^r u^r_\gamma, u^r_\gamma\rangle + \operatorname{Re}\langle Au^r_\zeta, \mathrm{pr}'_\gamma u^r_\zeta\rangle + \operatorname{Re}(Tz + f, u'_\zeta)\\ &= \langle A^r u^r_\gamma, u^r_\gamma\rangle + \operatorname{Re}\langle Az^r_\zeta, \mathrm{pr}'_\gamma z^r_\zeta\rangle + \operatorname{Re}(Tz, z'_\zeta) + \operatorname{Re}(f, z'_\zeta)\\ &\geqslant c\|u\|_K^2 - \lambda\|u\|^2 \quad \text{by assumption.}\end{aligned}$$

For given $z \in D(T) \cap U$ and $f \in Z(A'_1) \ominus W$, we can choose a sequence of elements $x^n \in D(A_0)$ converging to $-A_\gamma^{-1}(Tz + f) - \mathrm{pr}^r_\gamma z$ in H_γ. Then $u^n = x^n + A_\gamma^{-1}(Tz + f) + z$ belongs to $D(\tilde{A}) \cap U$, and $\mathrm{pr}^r_\gamma u^n = x^n + A_\gamma^{-1}(Tz + f) + \mathrm{pr}^r_\gamma z \to 0$ in H_γ, so that $u^n = \mathrm{pr}^r_\gamma u^n + \mathrm{pr}^r_\zeta z \to \mathrm{pr}^r_\zeta z$ in K, in view of (3.14). Inserting u^n in (3.17) and passing to the limit, we find that

$$\text{(3.18)} \qquad \operatorname{Re}\langle Az^r_\zeta, \mathrm{pr}'_\gamma z^r_\zeta\rangle + \operatorname{Re}(f, z'_\zeta) + \operatorname{Re}(Tz, z'_\zeta) \geqslant c\|z^r_\zeta\|_K^2 - \lambda\|z^r_\zeta\|^2$$

for all $z \in D(T) \cap U$, all $f \in Z(A'_1) \ominus W$. Since f may here be multiplied by any complex number, this implies that $(f, z'_\zeta) = 0$ for all f, z, which shows (i). When this is inserted in (3.18), we find (3.16).

2° Assume conversely that (i) and (ii) hold. We first take the case where $-\lambda' \geqslant 0$. Here, by the above decomposition, we have for $u \in D(\tilde{A}) \cap U$

$$\begin{aligned}\operatorname{Re}(Au, u) &\geqslant \langle A^r u^r_\gamma, u^r_\gamma\rangle + c'\|u^r_\zeta\|^2_K \\ &\geqslant c_\gamma \|u^r_\gamma\|^2_{H_\gamma} + c'\|u^r_\zeta\|^2_K \qquad (\text{cf. } (3.2)) \\ &\geqslant c''\|u\|^2_K\end{aligned}$$

with $c'' > 0$, in view of (3.14); this shows (3.15). Next, let $-\lambda' < 0$. We then have for $u \in D(\tilde{A}) \cap U$

$$\operatorname{Re}(Au, u) \geqslant \langle A^r u^r_\gamma, u^r_\gamma\rangle + c'\|u^r_\zeta\|^2_K - \lambda'\|u^r_\zeta\|^2 .$$

Now let $\mu = \varphi(-\lambda')$ so that $m(G^\mu) = \lambda'$ (cf. Theorem 2.10); note that $\mu < 0$. Then by use of Lemma 3.5,

$$\begin{aligned}&\operatorname{Re}(Au, u) - \mu\|u\|^2 \geqslant \\ &\quad\geqslant \langle A^r u^r_\gamma, u^r_\gamma\rangle - \mu\|u\|^2 + c'\|u^r_\zeta\|^2_K - \lambda'\|u^r_\zeta\|^2 \\ &\quad= \langle (A^r - \mu) u^{r,\mu}_\gamma, u^{r,\mu}_\gamma\rangle + (G^\mu u^r_\zeta, u^r_\zeta) + c'\|u^r_\zeta\|^2_K - \lambda'\|u^r_\zeta\|^2 \\ &\quad\geqslant c_\gamma\|u^{r,\mu}_\gamma\|^2_{H_\gamma} + c'\|u^r_\zeta\|^2_K \qquad (\text{since } \mu < 0) \\ &\quad\geqslant c_\gamma\|u^{r,\mu}_\gamma\|^2_{H_\gamma} + c_1\|u^{r,\mu}_\zeta\|^2_K \qquad (\text{cf. Lemma 2.4}) \\ &\quad\geqslant c_2\|u\|^2_K ,\end{aligned}$$

with $c_2 > 0$, since $H_\gamma \subseteq K$. This proves (3.15). The last statement in the theorem is evident.

REMARK 3.7. Note that the proof of Theorem 3.6 hinges on the decomposition, valid for $u \in D(A_B) \cap \mathcal{K}_{A^r}$ with $u'_\zeta \in W$,

$$\operatorname{Re}(Au, u) = \langle A^r u^r_\gamma, u^r_\gamma\rangle + t^r(u^r_\zeta, u^r_\zeta) ,$$

where $t^r(y, y) = \operatorname{Re}[(Ty_\zeta, y'_\zeta) + \langle Ay, \operatorname{pr}'_\gamma y\rangle]$, cf. (3.17). Here, $\operatorname{Re}(Au, u)$ might be regarded as the sum of the two « quadratic » forms $\langle A^r u^r_\gamma, u^r_\gamma\rangle = a^r_\gamma(u^r_\gamma, u^r_\gamma)$ and $t^r(u^r_\zeta, u^r_\zeta)$, like in (2.33); and the study of coerciveness inequalities (3.15) becomes part of a study of such sums of forms on subsets of $\mathcal{K}_{A^r}$.

In view of Remark 2.15, Theorem 2.13 in a special case of the above theorem. Observe also the following consequence of the proof of Theorem 3.6, which extends Theorem 2.12:

COROLLARY 3.8. (*Assumptions of Theorem* 3.6, *with* $K=H$.) *There exists* $\mu \in \boldsymbol{R}$ *so that*

$$\operatorname{Re}(Au, u) \geqslant \mu \|u\|^2 \quad \text{for all } u \in D(A) \cap U, \tag{3.19}$$

if and only if $\operatorname{pr}'_\zeta(D(T) \cap U) \subseteq W$ *and there exists* μ' *so that*

$$\operatorname{Re}[(Tz, z'_\zeta) + \langle Az^r_\zeta, \operatorname{pr}'_\gamma z^r_\zeta\rangle] \geqslant \mu' \|z^r_\zeta\|^2 \tag{3.20}$$

for all $z \in D(T) \cap U$. *If* $\mu > 0 (\mu \geqslant 0)$ *then* μ' *may be taken* > 0 $(\geqslant 0)$ *and vice versa.*

4. – The negative spectrum.

As noted in [11], Proposition 1.4 leads to methods for estimating the spectrum of $\tilde{A}$ by applying perturbation theorems to the spectrum of A_γ (or of T). Similar ideas have been used previously by Krein and by Birman (loc. cit.) and others in the study of lower semibounded operators. We shall here give a few estimates concerning the negative spectrum of selfadjoint, not necessarily lower bounded operators. The methods are quite elementary, but do however provide the basis for a new result for elliptic boundary value problems in Section 8. The application of the same techniques to the positive spectrum does not improve the very delicate estimates already known (cf. (8.22)) so we shall not here discuss the positive spectrum.

Some notations: When P is a selfadjoint operator in a Hilbert space X with discrete spectrum with finite multiplicities, the nonzero eigenvalues are arranged in the two sequences (counting multiplicities)

$$0 < \lambda_1^+(P) \leqslant \lambda_2^+(P) \leqslant \ldots \leqslant \lambda_j^+(P) \leqslant \ldots, \tag{4.1}$$

$$0 > \lambda_1^-(P) \geqslant \lambda_2^-(P) \geqslant \ldots \geqslant \lambda_j^-(P) \geqslant \ldots. \tag{4.2}$$

For $t \in]0, +\infty]$, we denote

$$N^\pm(P; t) = \sum_{|\lambda_j^\pm(P)| < t} 1 = \text{no. of eigenvalues} \gtrless 0 \text{ in }]-t, t[. \tag{4.3}$$

When $N^-(P; \infty) < \infty$, we also arrange the eigenvalues in one sequence

$$\lambda_1(P) \leqslant \lambda_2(P) \leqslant \ldots \leqslant \lambda_j(P) \leqslant \ldots \tag{4.4}$$

so that, when $0 \in \varrho(P)$, $\lambda_j(P) = \lambda^-_{N-j+1}(P)$ for $j \leqslant N = N^-(P; \infty)$, and $\lambda_j(P) = \lambda^+_{j+N}(P)$ for $j > N$.

When Q is a compact selfadjoint operator in X, the nonzero eigenvalues are arranged in the two sequences

$$\mu_1^+(Q) \geqslant \mu_2^+(Q) \geqslant \ldots \geqslant \mu_j^+(Q) \geqslant \ldots > 0 , \tag{4.5}$$

$$\mu_1^-(Q) \leqslant \mu_2^-(Q) \leqslant \ldots \leqslant \mu_j^-(Q) \leqslant \ldots < 0 , \tag{4.6}$$

and we denote by $N^\pm(Q)$ the total number of positive, resp. negative, eigenvalues. When Q is injective, $\mu_j^\pm(Q) = \lambda_j^\pm(Q^{-1})^{-1}$.

We assume in the following, that Assumptions 1.1 and 2.1 hold, and that A_γ^{-1} is compact. Moreover, $\tilde{A}$ will denote a selfadjoint operator in $\mathcal{M}$, with $0 \in \varrho(\tilde{A})$ and $\tilde{A}^{-1}$ compact, and corresponding to $T\colon V \to V$ by Proposition 1.2; so $0 \in \varrho(T)$ and T^{-1} is a compact selfadjoint operator in V, by Proposition 1.4.

Lemma 4.1. $N^-(\tilde{A}; \infty) = \infty \Leftrightarrow N^-(T; \infty) = \infty$.

Proof. Follows from Theorem 2.12, since the statements mean that $\tilde{A}$ resp. T is unbounded below.

Lemma 4.2. *For any* $t \in]0, \infty]$,

$$N^-(\tilde{A}; t) \leqslant N^-(T, t); \tag{4.7}$$

and

$$\lambda_j^-(\tilde{A}) \leqslant \lambda_j^-(T) \quad \textit{for all } j \leqslant N^-(\tilde{A}; \infty) . \tag{4.8}$$

Proof. We apply the maximum-minimum principle to the identity (cf. Proposition 1.4)

$$\tilde{A}^{-1} = A_\gamma^{-1} + T^{(-1)}, \tag{4.9}$$

where A_γ^{-1} is nonnegative, so that $(\tilde{A}^{-1}x, x) \geqslant (T^{(-1)}x, x)$. For $j \leqslant N^-(\tilde{A}; \infty) = N^-(\tilde{A}^{-1})$ we have

$$\begin{aligned} 0 > \mu_j^-(\tilde{A}^{-1}) &= \max_{X_{j-1} \subseteq H} \min \{(\tilde{A}^{-1}x, x) \,|\, \|x\| = 1,\ x \perp X_{j-1}\} \\ &\geqslant \max_{X_{j-1} \subseteq H} \min \{(T^{(-1)}x, x) \,|\, \|x\| = 1,\ x \perp X_{j-1}\} \\ &= \mu_j^-(T^{(-1)}) = \mu_j^-(T^{-1}) , \end{aligned}$$

here X_{j-1} runs through all subspaces of H of dimension $\leqslant j-1$. This shows

$$\lambda_j^-(\tilde{A}) \leqslant \lambda_j^-(T).$$

LEMMA 4.3. $N^-(\tilde{A};\infty) = N^-(T;\infty)$.

PROOF. This was shown in Lemma 4.1 for the infinite case, so we may assume $N^-(\tilde{A};\infty) < \infty$. Then $\tilde{A}$ is lower bounded, and the associated sesquilinear forms constitute a direct sum

$$\tilde{a}(u,v) = a_\gamma(u_\gamma, v_\gamma) + t(u_\zeta, v_\zeta) \tag{4.10}$$

for all $u, v \in D(\tilde{a}) = D(a_\gamma) \dotplus D(t)$, cf. Remark 2.16 (or [12]). Let $\alpha > -m(\tilde{A})$, then in particular $\alpha > -m(T)$, cf. Proposition 2.2 (i). Then $\tilde{A}+\alpha > 0$, $D(\tilde{a}) = D(\tilde{a}+\alpha) = D\big((\tilde{A}+\alpha)^{\frac{1}{2}}\big)$. Now another wellknown version of the maximum-minimum principle gives (in the notation (4.4))

$$\begin{aligned}\lambda_j(\tilde{A}+\alpha) &= \lambda_j\big((\tilde{A}+\alpha)^{\frac{1}{2}}\big)^2 = \\ &= \max_{X_{j-1}\subseteq H}\min\left\{\|(\tilde{A}+\alpha)^{\frac{1}{2}}u\|^2 \,\middle|\, u\in D\big((\tilde{A}+\alpha)^{\frac{1}{2}}\big),\ \|u\|=1,\ u\perp X_{j-1}\right\} \\ &= \max_{X_{j-1}\subseteq H}\min\left\{\tilde{a}(u,u)+\alpha\|u\|^2 \,\middle|\, u\in D(\tilde{a}),\ \|u\|=1,\ u\perp X_{j-1}\right\},\end{aligned}$$

where X_{j-1} runs through all subspaces of H of dimension $\leqslant j-1$. Now $D(t)\subseteq D(\tilde{a})$ and for $z\in D(t)$ we have $\tilde{a}(z,z) = t(z,z)$ by (4.10). Thus, for each X_{j-1},

$$\begin{aligned}&\min\{\tilde{a}(u,u)+\alpha\|u\|^2 | u\in D(\tilde{a}),\ \|u\|=1,\ u\perp X_{j-1}\} \\ &\leqslant \min\{t(z,z)+\alpha\|z\|^2 | z\in D(t),\ \|z\|=1,\ z\perp X_{j-1}\} \\ &= \min\{\|(T+\alpha)^{\frac{1}{2}}z\|^2 | z\in D\big((T+\alpha)^{\frac{1}{2}}\big),\ \|z\|=1,\ z\perp X_{j-1}\}.\end{aligned} \tag{4.11}$$

When X_{j-1} runs through the subspaces of H of dimension $\leqslant j-1$, then $\mathrm{pr}_V X_{j-1}$ runs through the subspaces of V of dimension $\leqslant j-1$. Taking the maximum in (4.11) over all X_{j-1}, we then get

$$\lambda_j(\tilde{A}+\alpha) \leqslant \lambda_j(T+\alpha)$$

for $j \leqslant \dim V$, and thus

$$\lambda_j(\tilde{A}) \leqslant \lambda_j(T) \quad \text{for } j\leqslant \dim V. \tag{4.12}$$

In particular we see that

$$N^-(\tilde{A};\infty) \geqslant N^-(T;\infty)$$

which, combined with (4.7), proves the lemma. Then for the negative eigenvalues, (4.12) is just a restatement of (4.8).

Finally, we shall determine an inequality going in the opposite direction of (4.8).

DEFINITION 4.4. For any closed subspace V of $Z(A_1)$, we define the operator S_V in V by

$$S_V v = \mathrm{pr}_V A_\gamma^{-1} v\,, \qquad v \in V.$$

The compactness of A_γ^{-1} implies the compactness of each operator S_V, moreover they are injective and nonnegative.

LEMMA 4.5. *One has for all* $j \leqslant N^-(S_V + T^{-1})$

$$\lambda_j^-(\tilde{A}) \geqslant \mu_j^-(S_V + T^{-1})^{-1}\,. \tag{4.13}$$

PROOF. For $v \in V$, we have

$$(\tilde{A}^{-1}v, v) = (A_\gamma^{-1}v, v) + (T^{-1}v, v) = \big((S_V + T^{-1})v, v\big)\,.$$

Thus

$$\begin{aligned}\mu_j^-(\tilde{A}^{-1}) &= \max_{X_{j-1}\subseteq H} \min\{(\tilde{A}^{-1}x, x)\,|\,x \in H,\ \|x\| = 1,\ x \perp X_{j-1}\}\\ &\leqslant \max_{X_{j-1}\subseteq H} \min\{(\tilde{A}^{-1}v, v)\,|\,v \in V,\ \|v\| = 1,\ v \perp X_{j-1}\}\\ &= \mu_j^-(S_V + T^{-1})\,,\end{aligned}$$

when $j \leqslant N^-(S_V + T^{-1})$ (like in the proof of Lemma 4.3).

Let us collect the results in a theorem.

THEOREM 4.6. (*Assumptions* 1.1 *and* 2.1). *Assume that* A_γ^{-1} *is compact. Let* $\tilde{A} \in \mathcal{M}$ *be selfadjoint with* $0 \in \varrho(\tilde{A})$ *and* $\tilde{A}^{-1}$ *compact;* $\tilde{A}$ *corresponds to* $T\colon V \to V$ *by Proposition* 1.2, *where* T *is selfadjoint with* $0 \in \varrho(T)$ *and* T^{-1} *compact. Then*

1° $N^-(\tilde{A}; \infty) = N^-(T; \infty)$.

2° *For any* $j \in \mathbb{N}$ *with* $j \leqslant N^-(\tilde{A}; \infty)$, $\lambda_j^-(\tilde{A}) \leqslant \lambda_j^-(T)$.

3° $N^-(\tilde{A}; t) \leqslant N^-(T; t)$ *for all* $t > 0$.

4° *For any* $j \in \mathbb{N}$ *with* $j \leqslant N^-(S_V + T^{-1})$,

$$\lambda_j^-(\tilde{A}) \geqslant \mu_j^-(S_V + T^{-1})^{-1}\,,$$

where S_V *is defined in Definition* 4.4.

CHAPTER II.

APPLICATIONS TO ELLIPTIC SYSTEMS

5. – Preliminaries.

The abstract results of Chapter I will now be applied to normal boundary value problems for elliptic systems. A treatment of scalar elliptic operators was given in [13], and much of the background material presented there carries over to the case of systems (or rather operators between vector bundles) without any trouble; such results will just be stated without details here.

For scalar operators, a normal boundary condition consists of a finite set of boundary conditions of distinct orders, where in each condition the order equals the normal order and the coefficient of the highest normal derivative is an *invertible* function (Aronszajn-Milgram [5]). For operators on vector-valued functions, one groups together the boundary conditions of the same normal order, and normality means (following Seeley [27]) that in each of those sets, the coefficient matrix of the highest normal derivatives is *surjective* (a precise statement is given below in Section 6). This more general concept requires the introduction of new techniques; the resulting theorems contain and in a sense simplify the statements in [13].

Normal boundary conditions for systems of differential operators were studied extensively in [17], which we shall build on here. We recall that the requirement of normality, which is unnecessary for existence and regularity studies (cf. [4], [18]) is justified in the study of semiboundedness inequalities (cf. e.g. [17, Remark 2.2]).

Let A be a $2m$ order properly elliptic differential operator in a hermitian C^∞ vector bundle E of fiber dimension q over a compact n-dimensional riemannian manifold $\overline{\Omega}$ with boundary Γ (and interior $\overline{\Omega}\setminus\Gamma$ denoted Ω); m, q and n are positive integers (²). On E and $E|_\Gamma$ one defines the usual Sobolev spaces, the norm on $H^s(E)$ or $H^s(E|_\Gamma)$ being denoted $\|\cdot\|_s$. In particular, $H^0(E) = L^2(E)$, $H^0(E|_\Gamma) = L^2(E|_\Gamma)$, and L^2-inner products will be denoted $(\cdot, \cdot)$; their extensions to (sesquilinear) dualities will be denoted $\langle\cdot, \cdot\rangle$ (these will mostly occur over Γ). $C_0^\infty(E)$ shall denote the space of C^∞ sections in E with support in Ω, and $H_0^s(E)$ its closure in $H^s(E)$ for $s \geq 0$. All differential operators will be assumed to have C^∞ coefficients (when expressed in local coordinates).

(²) Some results in the following are only interesting (or meaningful) for $n > 1$; this should be clear from the context.

Define for s and $t \in \mathbf{R}$

$$(5.1) \qquad \mathcal{H}_A^{s,t}(E) = \{u \in H^s(E) \mid Au \in H^t(E)\}\,,$$

it is a Hilbert space with the graph norm $\|u\|_{\mathcal{H}_A^{s,t}(E)} = (\|u\|_s^2 + \|Au\|_t^2)^{\frac{1}{2}}$. Define also

$$(5.2) \qquad Z_A^s(E) = \{u \in H^s(E) \mid Au = 0\}\,,$$

it is a closed subspace of $H^s(E)$ and of any $\mathcal{H}_A^{s,t}(E)$. Analogous spaces are defined for the formal adjoint A' of A. Introduce the index sets

$$M = \{0, \ldots, 2m-1\}\,, \qquad M_0 = \{0, \ldots, m-1\}\,, \qquad M_1 = \{m, \ldots, 2m-1\};$$

and, with γ_k denoting the k-th normal derivative $\gamma_k : u \mapsto (D_n^k u)|_\Gamma$ for $u \in C^\infty(E)$, define

$$\varrho u = \{\gamma_k u\}_{k \in M}\,, \qquad \gamma u = \{\gamma_k u\}_{k \in M_0}\,, \qquad \nu u = \{\gamma_k u\}_{k \in M_1}\,,$$

the Cauchy data, Dirichlet data, resp. Neumann data of u. By an easy generalization of [21], γ and ν extend to continuous mappings

$$(5.3) \qquad \gamma : \mathcal{H}_A^{s,-m}(E) \to \prod_{k \in M_0} H^{s-k-\frac{1}{2}}(E|_\Gamma)$$

$$(5.4) \qquad \nu : \mathcal{H}_A^{s,0}(E) \to \prod_{k \in M_1} H^{s-k-\frac{1}{2}}(E|_\Gamma)$$

for all $s \in \mathbf{R}$. For the norms in the latter spaces we shall use the notation

$$(5.5) \qquad \|\varphi\|_{\prod_{k \in N} H^{s_k}(E_k)} = \|\varphi\|_{\{s_k\}, k \in N}\,.$$

One has the following *Green's formula* (cf. [17], or [25], [13]): For $s \in [0, 2m]$, $u \in \mathcal{H}_A^{s,0}(E)$ and $v \in \mathcal{H}_{A'}^{2m-s,0}(E)$,

$$\begin{aligned}(5.6) \qquad (Au, v) - (u, A'v) &= \langle \mathcal{A}\varrho u, \varrho v\rangle \\ &= \langle \mathcal{A}^{00}\gamma u, \gamma v\rangle + \langle \mathcal{A}^{01}\nu u, \gamma v\rangle + \langle \mathcal{A}^{10}\gamma u, \nu v\rangle \\ &= \langle \chi u, \gamma v\rangle - \langle \gamma u, \chi' v\rangle\,,\end{aligned}$$

where $\mathcal{A} = (\mathcal{A}_{jk})_{j,k \in M}$ is a certain invertible skew-triangular matrix of differential operators $\mathcal{A}_{jk}$ in $E|_\Gamma$ of orders $2m-1-j-k$;

$$\mathcal{A}^{\delta\varepsilon} = (\mathcal{A}_{jk})_{j\in M_\delta, k\in M_\varepsilon} \quad \text{for } \delta, \varepsilon = 0, 1 \ ; \tag{5.7}$$

and

$$\chi u = \mathcal{A}^{01}\nu u + \tfrac{1}{2}\mathcal{A}^{00}\gamma u \ , \qquad \chi' v = -\mathcal{A}^{10*}\nu v - \tfrac{1}{2}\mathcal{A}^{00*}\gamma v \ . \tag{5.8}$$

Define now A_0, A_γ and A_1 as the operators in $L^2(E)$ sending u into Au and with domains

$$D(A_0) = H_0^{2m}(E) \ , \qquad D(A_\gamma) = H_0^m(E) \cap H^{2m}(E) \ , \qquad D(A_1) = \mathcal{H}_A^{0,0}(E) \ . \tag{5.9}$$

We shall assume

Assumption 5.1. A maps $D(A_\gamma)$ bijectively onto $L^2(E)$.

Define the analogous operators for A', then we have from well known theorems on elliptic operators

$$A_0' = A_1^* \ , \qquad A_\gamma' = A_\gamma^* \ , \qquad A_1' = A_0^* \ ,$$

so that the introduced operators altogether satisfy the hypotheses of Section 1. The graph-norms on $D(A_0)$ and $D(A_\gamma)$ are equivalent with the H^{2m}-norms, and since $\overline{\Omega}$ is compact, A_γ^{-1} is a *compact operator*. The operators $\tilde{A}$ in $\mathcal{M}$ (i.e. satisfying $A_0 \subseteq \tilde{A} \subseteq A_1$) are now called the *realizations* of A. Clearly, we have

$$Z(A_1) = Z_A^0(E) \ , \qquad Z(A_1') = Z_{A'}^0(E) \ . \tag{5.10}$$

In particular, when A is *strongly elliptic*, we may assume that a constant has added to A so that, with $c_m > 0$,

$$\operatorname{Re}(Au, u) \geq c_m \|u\|_m^2 \ , \qquad \text{all } u \in C_0^\infty(E) \ ; \tag{5.11}$$

then Assumption 3.1 is satisfied, and

$$H_\gamma = H_0^m(E) \ , \qquad \mathcal{H}_A = \mathcal{H}_A^{0,-m}(E) \ , \qquad \mathcal{H}_{A'} = \mathcal{H}_{A'}^{0,-m}(E) \ ; \tag{5.12}$$

moreover, the operators A_0^r, A_γ^r, A_1^r etc. are the analogous realizations of the formally selfadjoint strongly elliptic operator

$$A^r = \tfrac{1}{2}(A + A') \ . \tag{5.13}$$

Proposition 1.2 is turned into a correspondence between realizations and boundary conditions by means of the following theorems:

PROPOSITION 5.2. *For* $t \geqslant -m$, $t \neq -\frac{1}{2}, -\frac{3}{2}, \ldots, -m+\frac{1}{2}$, *and for* $-\infty < s \leqslant t+2m$, *the mapping* $\{A, \gamma\}$ *defines an isomorphism*

$$\{A, \gamma\}: \mathcal{K}_A^{s,t}(E) \to H^t(E) \times \prod_{k \in M_0} H^{s-k-\frac{1}{2}}(E|_\Gamma) .$$

Here, $\{A, \gamma\}^{-1}(H^t(E) \times \{0\}) = H_0^m(E) \cap H^{t+2m}(E)$, *and* $\{A, \gamma\}^{-1}(\{0\} \times \times \prod_{k \in M_0} H^{s-k-\frac{1}{2}}(E|_\Gamma)) = Z_A^s(E)$. *With* γ_Z *defined as the isomorphism*

$$\gamma: Z_A^s(E) \xrightarrow{\sim} \prod_{k \in M_0} H^{s-k-\frac{1}{2}}(E|_\Gamma) , \tag{5.14}$$

the operators

$$\mathrm{pr}_\gamma = I - \mathrm{pr}_\zeta , \quad \mathrm{pr}_\zeta = \gamma_Z^{-1} \circ \gamma \tag{5.15}$$

coincide with the projections pr_γ *and* pr_ζ *defined in Sections* 1 *and* 3, *and they decompose* $\mathcal{K}_A^{s,t}(E)$ *into the direct topological sum*

$$\mathcal{K}_A^{s,t}(E) = \big(H_0^m(E) \cap H^{t+2m}(E)\big) \dotplus Z_A^s(E) . \tag{5.16}$$

PROPOSITION 5.3. *The composed operator*

$$P_{\gamma,\nu} = \nu \circ \gamma_Z^{-1}: \prod_{k \in M_0} H^{s-k-\frac{1}{2}}(E|_\Gamma) \to \prod_{j \in M_1} H^{s-j-\frac{1}{2}}(E|_\Gamma) \tag{5.17}$$

is an $M_1 \times M_0$*-matrix of pseudo-differential operators in* $E|_\Gamma$, *it is of type* $(-k, -j)_{j \in M_1, k \in M_0}$, *and its principal symbol is at each point in* $T^*(\Gamma) \setminus 0$ *obtained by the analogous construction for a related ordinary differential operator.*

Proposition 5.2 follows from well known theorems on the well-posedness of the Dirichlet problem, extended to general spaces by Lions and Magenes [21], see [13, Theorem 2.1] for a detailed account. Proposition 5.3 follows from Boutet de Monvel [6]; the related ordinary differential operator is obtained from the principal part of A by freezing the coefficients at a point of Γ and Fourier transforming in the tangential variables. A matrix $(P_{jk})_{j \in N_1, k \in N_2}$ of pseudo-differential operators P_{jk} from E_k to F_j (vector bundles over a manifold X) is said to be *of type* $(t_k, s_j)_{j \in N_1, k \in N_2}$ if it is continuous from $\prod_{k \in N_2} H^{t_k}(E_k)$ to $\prod_{j \in N_1} H^{s_j}(F_j)$. (For the case where A is scalar, we showed in [13] how the result follows from [18], [25], and calculated the principal symbol of $P_{\gamma,\nu}$.)

The analogous concepts are introduced for A' (and A^r in the strongly elliptic case), with notations γ'_z, $P'_{\gamma,\nu}$ (and γ^r_z, $P^r_{\gamma,\nu}$). Define furthermore

$$P_{\gamma,\chi} = \chi \circ \gamma_z^{-1} = \mathcal{A}^{01} P_{\gamma,\nu} + \tfrac{1}{2}\mathcal{A}^{00} \tag{5.18}$$

and the analogous operators relative to A' and A^r; we note that for the operators $\mathcal{A}'$ and $\mathcal{A}^r$ entering in the Green's formula,

$$\mathcal{A}' = -\mathcal{A}^*, \qquad \mathcal{A}^r = \tfrac{1}{2}(\mathcal{A} + \mathcal{A}') = \tfrac{1}{2}(\mathcal{A} - \mathcal{A}^*). \tag{5.19}$$

Finally, define the pseudo-differential boundary operator μ by

$$\mu u = \chi u - P_{\gamma,\chi}\gamma u \qquad \text{for} \qquad u \in \mathcal{H}_A^{0,0}(E) = D(A_1);$$

μ' and μ^r are defined analogously relative to A' and A^r. It has the properties (cf. [11], [13]):

PROPOSITION 5.4. *For* $u \in D(A_1)$,

$$\mu u = \chi u - P_{\gamma,\chi}\gamma u = \mathcal{A}^{01}(\nu u - P_{\gamma,\nu}\gamma u) = \mathcal{A}^{01}\nu A_\gamma^{-1} A u, \tag{5.20}$$

and μ *maps* $D(A_1)$ *continuously onto* $\prod_{k \in M_0} H^{k+\frac{1}{2}}(E|_\Gamma)$; *moreover,*

$$(Au, z) = \langle \mu u, \gamma z\rangle \qquad \textit{for all } z \in Z(A'_1). \tag{5.21}$$

The mapping $\{\gamma, \mu\}$ *is surjective from* $D(A_1)$ *onto* $\prod_{k\in M_0} H^{-k-\frac{1}{2}}(E|_\Gamma) \times \prod_{j \in M_0} H^{j+\frac{1}{2}}(E|_\Gamma)$, *with kernel* $H_0^{2m}(E)$.

Now, when $\tilde{A}$ is a closed realization of A and corresponds to $T\colon V \to W$ by Proposition 1.2, we can use the isomorphisms in Proposition 5.2 to carry T into an operator $L\colon X \to Y'$, where

$$X = \gamma V, \qquad Y = \gamma W, \qquad \text{and} \qquad L = (\gamma_W^*)^{-1} T \gamma_V^{-1}, \tag{5.22}$$

γ_V and γ_W denoting the restrictions of γ to isomorphisms from V to X resp. from W to Y. Here

$$\begin{aligned} (Tu_\zeta, w) &= \langle L\gamma_V u_\zeta, \gamma_W w\rangle = \langle L\gamma u, \gamma w\rangle, \\ (Tu_\zeta, w) &= (Au, w) = \langle \mu u, \gamma w\rangle \end{aligned} \tag{5.23}$$

for all $u \in D(\tilde{A})$, $w \in W$; so we find that $\tilde{A}$ corresponds to $L\colon X \to Y'$ by the formula

$$D(\tilde{A}) = \{u \in D(A_1) | \gamma u \in D(L),\, L\gamma u = i_Y^* \mu u\} ,$$

where i_Y^* is the adjoint of the injection $i_Y \colon Y \to \prod_{k \in M_0} H^{-k-\frac{1}{2}}(E|_\Gamma)$.

We refrain from further details (see [11], [13]), since we shall now restrict the attention to normal boundary conditions, for which one may construct other representations of T, where X and Y are replaced by whole Sobolev-spaces over Γ and L is replaced by a pseudo-differential operator.

6. – Realizations of normal boundary conditions.

Recall the set-up of [17]: There are given $2m$ hermitian C^∞ vector bundles F_j over Γ, of fiber dimensions $p_j \geqslant 0$; $j = 0, 1, \ldots, 2m-1$. There is given a matrix $B = (B_{jk})_{j,k \in M}$ of differential operators B_{jk} from $E|_\Gamma$ to F_j of orders $j-k$, respectively (differential operators of negative order being zero), i.e., B is of type $(-k, -j)_{j,k\in M}$. In the present paper, we also permit the B_{jk} with $j-k>0$ to be pseudo-differental operators (cf. Remarks 1.14 and 2.7 of [17]); pseudo-differential operators occurring below will be called (pseudo-) differential operators if they are differential operators when the B_{jk} are so.

B is lower triangular, and is split into four blocks (compare (5.7))

$$B = \begin{pmatrix} B^{00} & 0 \\ B^{10} & B^{11} \end{pmatrix} , \qquad B^{\delta\varepsilon} = (B_{jk})_{j\in M_\delta,\, k\in M_\varepsilon} . \tag{6.1}$$

B defines the boundary condition $B\varrho u = 0$, i.e.

$$B^{00}\gamma u = 0 , \quad B^{10}\gamma u + B^{11}\nu u = 0 . \tag{6.2}$$

The *diagonal part* of B

$$B_d = (\delta_{jk} B_{jk})_{j,k\in M} , \tag{6.3}$$

consisting of zero order differential operators, may be viewed as a *vector bundle morphism* from $\bigoplus_{k\in M} E|_\Gamma$ to $\bigoplus_{j\in M} F_j$, an identification we shall use throughout. We denote $B - B_d = B_s$, the *subtriangular part* of B.

We assume from now on that *the following definition holds*:

DEFINITION 6.1. B (or the boundary condition $B\varrho u = 0$) is said to be *normal*, when B_d is a *surjective* vector bundle morphism (i.e., B_{jj} is a surjective morphism from $E|_\Gamma$ to F_j for each $j \in M$; in particular $p_j \leqslant q$ for all j).

Under this assumption, B itself is surjective from $\prod_{k\in M} H^{s-k}(E|_\Gamma)$ to $\prod_{j\in M} H^{s-j}(F_j)$ for all $s \in \mathbb{R}$, and it has a right inverse $C = (C_{jk})_{j,k\in M}$ consisting of (pseudo-) differential operators C_{jk} from F_k to $E|_\Gamma$ of orders $j-k$; C is lower triangular and injective; cf. [17, Section 1.3]. Let $C^{\delta\varepsilon} = (C_{jk})_{j\in M_\delta, k\in M_\varepsilon}$ ($\delta, \varepsilon = 0, 1$), then C^{00} is the analogous right inverse of B^{00}, and C^{11} is the right inverse of B^{11}.

With the notation (for $s \in \mathbb{R}$)

$$Z^s(B) = \left\{\varphi \in \prod_{k\in M} H^{s-k-\frac{1}{2}}(E|_\Gamma) | B\varphi = 0\right\} \tag{6.4}$$

and analogous notations for $Z^s(B^{00})$ and $Z^s(B^{11})$, we showed in [17, Lemma 1.11]:

$$Z^s(B) = (I - CB) \prod_{k\in M} H^{s-k-\frac{1}{2}}(E|_\Gamma) \; ; \tag{6.5}$$

$$Z^s(B^{\varepsilon\varepsilon}) = (I - C^{\varepsilon\varepsilon}B^{\varepsilon\varepsilon}) \prod_{k\in M_\varepsilon} H^{s-k-\frac{1}{2}}(E|_\Gamma) , \quad \varepsilon = 0, 1 . \tag{6.6}$$

Note the easy consequence

LEMMA 6.2. *For $t < s$, $Z^s(B)$ (resp. $Z^s(B^{\varepsilon\varepsilon})$, $\varepsilon = 0, 1$) is dense in $Z^t(B)$ (resp. $Z^t(B^{\varepsilon\varepsilon})$, $\varepsilon = 0, 1$) in the norm $\|\varphi\|_{\{t-k-\frac{1}{2}\}, k\in M}$ (resp. $\|\varphi\|_{\{t-k-\frac{1}{2}\}, k\in M_\varepsilon}$).*

We shall now study the realization A_B of A defined by

$$D(A_B) = \{u \in D(A_1) | B^{00}\gamma u = 0, \quad B^{10}\gamma u + B^{11}\nu u = 0\} . \tag{6.7}$$

Clearly, A_B is a closed operator in $L^2(E)$. (Because of the extended definitions (5.3), (5.4), we do not need to restrict the domain to $H^{2m}(E)$ as in [17].) Let

$$V = \operatorname{cl} \operatorname{pr}_\zeta D(A_B) , \quad W = \operatorname{cl} \operatorname{pr}'_\zeta D(A_B^*) , \tag{6.8}$$

closures in $L^2(E)$ (as in Proposition 1.2), and let

$$X = \gamma V = \operatorname{cl} \gamma D(A_B) , \quad Y = \gamma W = \operatorname{cl} \gamma D(A_B^*) , \tag{6.9}$$

closures in $\prod_{k\in M_0} H^{-k-\frac{1}{2}}(E|_\Gamma)$ (as in (5.22)); the restrictions of γ to isomorphisms from V to X, resp. W to Y, are denoted γ_V resp. γ_W: X and Y are analyzed as follows:

PROPOSITION 6.3.

(i) *For each* $k \in M$, *let* Z_k *denote the* $(q - p_k)$*-dimensional subbundle of* $E|_\Gamma$ *defined as the kernel of the morphism* B_{kk}; *let* i_{Z^0} *denote the injection of* $\bigoplus_{k\in M_0} Z_k$ *into* $\bigoplus_{k\in M_0} E|_\Gamma$; *and denote by* Φ *the injective (pseudo-) differential operator from* $\bigoplus_{k\in M_0} Z_k$ *to* $\bigoplus_{k\in M_0} E|_\Gamma$

$$\Phi = (I - C^{00} B_s^{00})\, i_{Z^0}\,, \tag{6.10}$$

it is of type $(-k, -j)_{j,k\in M_0}$. *Then one has*

$$X = Z^0(B^{00}) = \Phi \prod_{k\in M_0} H^{-k-\frac{1}{2}}(Z_k)\,. \tag{6.11}$$

(ii) *Let* Ψ *denote the injective (pseudo-) differential operator from* $\bigoplus_{j\in M_1} F_j$ *to* $\bigoplus_{j\in M_0} E|_\Gamma$

$$\Psi = (\mathcal{A}^{01*})^{-1} B^{11*}\,, \tag{6.12}$$

it is of type $(-2m + k + 1, -j)_{j\in M_0, k\in M_1}$. *Then one has*

$$Y = \Psi \prod_{j\in M_1} H^{-2m+j+\frac{1}{2}}(F_j)\,. \tag{6.13}$$

PROOF. (i) By (6.2), $\gamma D(A_B) \subseteq Z^0(B^{00})$. On the other hand, $Z^{2m}(B^{00}) \subseteq \subseteq \gamma D(A_B)$, since, for given $\varphi \in Z^{2m}(B^{00})$, we can always find $u \in H^{2m}(E)$ with

$$\gamma u = \varphi \qquad \nu u = - C^{11} B^{10} \varphi\ ;$$

such functions u satisfy (6.2). Since $Z^{2m}(B^{00})$ is dense in $Z^0(B^{00})$, it follows that $X = Z^0(B_{00})$. The second identity in (6.11) was proved in [17, (1.34)].

(ii) For the determination of Y, we have by Green's formula (5.6)

$$D(A_B^*) \supseteq \{v \in H^{2m}(E) \,|\, \langle \mathcal{A}\varrho u, \varrho v\rangle = 0 \text{ for } u \in D(A_B)\}\ ; \tag{6.14}$$

$$D(A_B^*) \subseteq \{v \in D(A_1') \,|\, \langle \mathcal{A}\varrho u, \varrho v\rangle = 0 \text{ for } u \in D(A_B) \cap H^{2m}(E)\}\,. \tag{6.15}$$

This implies, by use of (6.5), that $(I - CB)^* \mathcal{A}^* \varrho v = 0$ for $v \in D(A_B^*)$ or more precisely

$$Z^{2m}\big((I - CB)^* \mathcal{A}^*\big) \subseteq \varrho D(A_B^*) \subseteq Z^0\big((I - CB)^* \mathcal{A}^*\big)\,.$$

The range space $\bigoplus_{k\in M} E|_\Gamma$ for the (pseudo-) differential operator $(I - CB)^* \mathcal{A}^*$ has so large fiber dimension that this operator will usually not be surjective. However, we showed in [17, Section 2.4] how the operator can be replaced by a surjective (pseudo-) differential operator B' with smaller range space (B' defining a *normal* boundary condition adjoint to the given one); and this we can treat as in (i). The calculations of [17] imply in particular (cf. [17, (2.48)])

$$(\mathcal{A}^{01*})^{-1}B^{11*}\prod_{j\in M_1} H^{j+\frac{1}{2}}(F_j) \subseteq \gamma D(A_B^*) \subseteq (\mathcal{A}^{01*})^{-1}B^{11*}\prod_{j\in M_1} H^{-2m+j+\frac{1}{2}}(F_j)\,,$$

from which (6.13) follows, since $\prod_{j\in M_1} H^{j+\frac{1}{2}}(F_j)$ is dense in $\prod_{j\in M_1} H^{-2m+j+\frac{1}{2}}(F_j)$. Q.e.d.

By [17, Lemma 1.12], Φ has the left inverse

$$\Phi^{(-1)} = \mathrm{pr}_{Z^0}\,(I + C_d^{00}B_s^{00})\,, \tag{6.16}$$

where pr_{Z^0} is the orthogonal projection of $\bigoplus_{k\in M_0} E|_\Gamma$ onto $\bigoplus_{k\in M_0} Z_k$; so we have

$$\Phi^{(-1)}\Phi = I\,, \quad \text{and} \quad \Phi\Phi^{(-1)}\varphi = \varphi \quad \text{for } \varphi \in X\,. \tag{6.17}$$

Ψ has the left inverse

$$\Psi^{(-1)} = C^{11*}\mathcal{A}^{01*} \tag{6.18}$$

so that

$$\Psi^{(-1)}\Psi = I\,, \quad \text{and} \quad \Psi\Psi^{(-1)}\psi = \psi \quad \text{for } \psi \in Y\,. \tag{6.19}$$

In particular, we have now found *isomorphisms*

$$\gamma_V^{-1}\Phi\colon \prod_{k\in M_0} H^{-k-\frac{1}{2}}(Z_k) \simeq V\,, \quad \text{with inverse } \Phi^{(-1)}\gamma\,; \tag{6.20}$$

$$\gamma_W^{-1}\Psi\colon \prod_{j\in M_1} H^{-2m+j+\frac{1}{2}}(F_j) \simeq W\,, \quad \text{with inverse } \Psi^{(-1)}\gamma\,; \tag{6.21}$$

the operator in (6.21) has the adjoint

$$\Psi^*(\gamma_W^*)^{-1}\colon W \simeq \prod_{j\in M_1} H^{2m-j-\frac{1}{2}}(F_j)\,. \tag{6.22}$$

We use these to represent T by an operator from $\prod_{k\in M_0} H^{-k-\frac{1}{2}}(Z_k)$ to $\prod_{j\in M_1} H^{2m-j-\frac{1}{2}}(F_j)$.

THEOREM 6.4. (*Assumption* 5.1.) *Let* A_B, *defined by* (6.7), *correspond to* $T\colon V \to W$ *by Proposition* 1.2. *Denote by* $\mathfrak{L}_1$ *the operator from* $\prod_{k\in M_0} H^{-k-\frac12}(Z_k)$ *to* $\prod_{j\in M_1} H^{2m-j-\frac12}(F_j)$ *induced from* $T\colon V\to W$ *by the isomorphisms* (6.20) *and* (6.22), *i.e.*,

$$D(\mathfrak{L}_1) = \Phi^{(-1)}\gamma D(T)\,,$$

$$\mathfrak{L}_1 = \Psi^*(\gamma_W^*)^{-1} T \gamma_V^{-1}\Phi\,,$$

or in other words

$$(Tv, w) = \langle \mathfrak{L}_1 \Phi^{(-1)}\gamma v, \Psi^{(-1)}\gamma w\rangle \quad \text{for } v\in D(T),\ w\in W\,. \tag{6.23}$$

Denote by $\mathfrak{L}$ *the pseudo-differential operator* (*continuous from* $\prod_{k\in M_0} H^{-k-\frac12}(Z_k)$ *to* $\prod_{j\in M_1} H^{-j-\frac12}(F_j)$) *defined by*

$$\mathfrak{L} = -(B^{10} + B^{11}P_{\gamma,\nu})\,\Phi\,. \tag{6.24}$$

Then $\mathfrak{L}_1$ *is exactly the restriction of* $\mathfrak{L}$ *with domain*

$$D(\mathfrak{L}_1) = \Big\{\varphi \in \prod_{k\in M_0} H^{-k-\frac12}(Z_k) \,|\, \mathfrak{L}\varphi \in \prod_{j\in M_1} H^{2m-j-\frac12}(F_j)\Big\}\,. \tag{6.25}$$

PROOF. Let $u\in D(A_B)$ and $w\in W$. Let $\varphi = \Phi^{(-1)}\gamma u$ and $\psi = \Psi^{(-1)}\gamma w$ Then (cf. Proposition 5.4)

$$\begin{aligned}(Tu_\zeta, w) &= (Au, w) = \langle \mu u, \gamma w\rangle\\ &= \langle \mathcal{A}^{01}(\nu u - P_{\gamma,\nu}\gamma u), (\mathcal{A}^{01*})^{-1}B^{11*}\psi\rangle\end{aligned}$$

(duality between $\prod_{k\in M_0} H^{k+\frac12}(E|_\Gamma)$ and $\prod_{k\in M_0} H^{-k-\frac12}(E|_\Gamma)$)

$$= \langle B^{11}\nu u - B^{11}P_{\gamma,\nu}\gamma u, \psi\rangle$$

(duality between $\prod_{j\in M_1} H^{2m-j-\frac12}(F_j)$ and $\prod_{j\in M_1} H^{-2m+j+\frac12}(F_j)$)

$$\begin{aligned}&= \langle -B^{10}\gamma u - B^{11}P_{\gamma,\nu}\gamma u, \psi\rangle \quad \text{(using (6.2))}\\ &= \langle -(B^{10}+B^{11}P_{\gamma,\nu})\Phi\varphi, \psi\rangle\\ &= \langle \mathfrak{L}\varphi, \psi\rangle\,.\end{aligned}$$

This shows that $\mathfrak{L}_1$ acts like $\mathfrak{L}$, and that, when $\varphi\in D(\mathfrak{L}_1)$, $\mathfrak{L}\varphi \in \prod_{j\in M_1} H^{2m-j-\frac12}(F_j)$, so $\subseteq$ holds in (6.25).

Conversely, let $\varphi \in \prod_{k\in M_0} H^{-k-\frac{1}{2}}(Z_k)$ with $\mathfrak{L}\varphi \in \prod_{j\in M_1} H^{2m-j-\frac{1}{2}}(F_j)$. Then $\eta = \mathcal{A}^{01} C^{11} \mathfrak{L}\varphi$ belongs to $\prod_{k\in M_0} H^{k+\frac{1}{2}}(E|_\Gamma)$, in view of the types of the operators involved; and $\Phi\varphi$ belongs to $\prod_{k\in M_0} H^{-k-\frac{1}{2}}(E|_\Gamma)$. By Proposition 5.4, there exists $u \in D(A_1)$ satisfying

$$\gamma u = \Phi\varphi\,, \qquad \mu u = \eta\,.$$

Here $\gamma u \in X$, so $B^{00}\gamma u = 0$. Moreover, since $\mu u = \mathcal{A}^{01}(\nu u - P_{\gamma,\nu}\gamma u)$, we have that

$$\begin{aligned} \nu u - P_{\gamma,\nu}\gamma u &= (\mathcal{A}^{01})^{-1}\eta = C^{11}\mathfrak{L}\varphi \\ &= C^{11}(-B^{10} - B^{11}P_{\gamma,\nu})\gamma u \\ &= -C^{11}B^{10}\gamma u - C^{11}B^{11}P_{\gamma,\nu}\gamma u\,, \end{aligned}$$

and thus

$$B^{11}\nu u = -B^{11}C^{11}B^{10}\gamma u - B^{11}C^{11}B^{11}P_{\gamma,\nu}\gamma u + B^{11}P_{\gamma,\nu}\gamma u = -B^{10}\gamma u\,.$$

Then $u \in D(A_B)$, which shows the inclusion $\supseteq$ in (6.25). Q.e.d.

Any operator obtained from $T: V \to W$ by replacing V and W by spaces isomorphic to them, will be called a *representation of* T. In special cases, e.g. when $X = Y$, it will be convenient to use other representations of T than $\mathfrak{L}_1$.

COROLLARY 6.5. *When $X = Y$, define the representation $\mathcal{M}_1$ of T by*

$$\mathcal{M}_1 = \mathfrak{L}_1 \Phi^{(-1)} \Psi \colon \prod_{j\in M_1} H^{-2m+j+\frac{1}{2}}(F_j) \to \prod_{j\in -1} H^{2m-j-\frac{1}{2}}(F_j)\,. \tag{6.26}$$

It is the restriction of the pseudo-differential operator in $\bigoplus_{j\in M_1} F_j$ of type $(-2m + k + \frac{1}{2}, -j - \frac{1}{2})_{j,k\in M_1}$

$$\mathcal{M} = -(B^{10} + B^{11}P_{\gamma,\nu})(\mathcal{A}^{01*})^{-1}B^{11*} \tag{6.27}$$

with domain

$$D(\mathcal{M}_1) = \{\varphi \in \prod_{j\in M_1} H^{-2m+j+\frac{1}{2}}(F_j) \mid \mathcal{M}\varphi \in \prod_{j\in M_1} H^{2m-j-\frac{1}{2}}(F_j)\} \tag{6.28}$$

and it satisfies

$$(Tv, w) = \langle \mathcal{M}_1 \Psi^{(-1)}\gamma v, \Psi^{(-1)}\gamma w\rangle\,, \quad \text{all } v \in D(T),\ w \in W. \tag{6.29}$$

PROOF. Compose $\mathfrak{L}_1$ to the right with the isomorphism $\Phi^{(-1)}\Psi$.

For the case where $X \subseteq Y$, we shall make do with

COROLLARY 6.6. *Assume that* $X \subseteq Y$. *Then for* $z \in D(T) \cap \mathcal{H}_{A'}$, *one has*

$$(6.30) \qquad (Tz, \mathrm{pr}'_\zeta z) = \langle -\Phi^* \mathcal{A}^{01} C^{11}(B^{10} + B^{11} P_{\gamma,\nu}) \Phi\varphi, \varphi\rangle$$

(*duality between* $\prod_{k\in M_0} H^{k+\frac12}(Z_k)$ *and* $\prod_{k\in M_0} H^{-k-\frac12}(Z_k)$), *where* $\varphi = \Phi^{(-1)}\gamma z \in D(\mathfrak{L}_1)$.

PROOF. When $z \in D(T) \cap \mathcal{H}_{A'}$, $\mathrm{pr}'_\zeta z = (\gamma'_z)^{-1}\gamma z \in (\gamma'_z)^{-1} Y = W$ (cf. (5.15)). By Theorem 6.4 we have that $D(\mathfrak{L}_1) = \Phi^{(-1)}\gamma D(T)$, and that, since $X \subseteq Y$,

$$\begin{aligned}(Tz, \mathrm{pr}'_\zeta z) &= \langle \mathfrak{L}_1 \Phi^{(-1)}\gamma z, \Psi^{(-1)}\gamma\, \mathrm{pr}'_\zeta z\rangle \\ &= \langle \mathfrak{L}_1 \Phi^{(-1)}\gamma z, \Psi^{(-1)}\gamma z\rangle \\ &= \langle \Phi^* \Psi^{(-1)*} \mathfrak{L}_1 \Phi^{(-1)}\gamma z, \Phi^{(-1)}\gamma z\rangle \\ &= \langle \Phi^* \Psi^{(-1)*} \mathfrak{L}_1 \varphi, \varphi\rangle\end{aligned}$$

where $\Phi^* \Psi^{(-1)*} \mathfrak{L}_1 = -\Phi^* \mathcal{A}^{01} C^{11}(B^{10} + B^{11} P_{\gamma,\nu})\Phi$, by (6.18) and (6.24).

Concerning the generality of $\mathfrak{L}_1$ we have the following important observation

PROPOSITION 6.7. *Given a system* $\{F_j\}_{j\in M}$ *of* $2m$ *vector bundles over* Γ *of dimensions* $p_j \leqslant q$, *and given two normal (pseudo-) differential operators* B^{00} *and* B^{11} *of types* $(-k, -j)_{j,k\in M_\varepsilon}$, *from* $\bigoplus_{k\in M_\varepsilon} E|_\Gamma$ *to* $\bigoplus_{j\in M_\varepsilon} F_j$ $(\varepsilon = 0, 1)$. *When* B^{10} *runs through all pseudo-differential operators from* $\bigoplus_{k\in M_0} E|_\Gamma$ *to* $\bigoplus_{j\in M_1} F_j$ *of type* $(-k, -j)_{j\in M_1, k\in M_0}$, *then* $\mathfrak{L}$ (*derived as above from* A_B *defined by the boundary condition* (6.7)) *runs through all pseudo-differential operators from* $\bigoplus_{k\in M_0} Z_k$ *to* $\bigoplus_{j\in M_1} F_j$ *of type* $(-k, -j)_{j\in M_1, k\in M_0}$.

PROOF. $\mathfrak{L}$ is derived from B^{10} by (6.24). Conversely, when $\mathfrak{L}$ is given (of the above mentioned type), a solution B^{10} of (6.24) is

$$B^{10} = -B^{11} P_{\gamma,\nu} - \mathfrak{L}\Phi^{(-1)}\,.$$

Note however that the $\mathfrak{L}$ obtained when B^{10} runs through strictly *differential* operators, form a special subclass of the *pseudo-differential* operators of the mentioned type.

We shall now prove a general theorem concerning existence, uniqueness and regularity.

THEOREM 6.8. (*Assumption* 5.1). *Let* A_B *be the realization defined by* (6.7) *and let* $\mathfrak{L}$ *and* $\mathfrak{L}_1$ *be the operators defined in Theorem* 6.4, *cf.* (6.24), (6.25). *Then*

1° $\dim Z(A_B) = \dim Z(\mathfrak{L}_1)$.

2° *The ranges of* A_B *and* $\mathfrak{L}_1$ *in* $L^2(E)$, *resp. in* $\prod_{j\in M_1} H^{2m-j-\frac{1}{2}}(F_j)$, *are simultaneously closed, and they have the same codimension.*

3° *Let* $t \geqslant 0$, $0 \leqslant s \leqslant t+2m$. A_B *satisfies*

$$u \in D(A_B)\,, \qquad Au \in H^t(E) \Rightarrow u \in H^s(E)\,, \tag{6.31}$$

if and only if $\mathfrak{L}$ *satisfies*

$$\varphi \in \prod_{k\in M_0} H^{-k-\frac{1}{2}}(Z_k)\,, \qquad \mathfrak{L}\varphi \in \prod_{j\in M_1} H^{t+2m-j-\frac{1}{2}}(F_j) \Rightarrow \varphi \in \prod_{k\in M_0} H^{s-k-\frac{1}{2}}(Z_k)\,. \tag{6.32}$$

PROOF. 1° and 2° are immediate consequences of Proposition 1.4, since $\mathfrak{L}_1$ is a representation of T.

To prove 3°, let us first assume that (6.31) holds. Let $\varphi \in \prod_{k\in M_0} H^{-k-\frac{1}{2}}(Z_k)$ with $\mathfrak{L}\varphi \in \prod_{j\in M_1} H^{t+2m-j-\frac{1}{2}}(F_j)$. We shall construct $u \in D(A_B)$ so that (6.31) can be applied. To do this, let v be the solution of

$$A'Av = 0\,, \qquad \gamma v = 0\,, \qquad \nu v = C^{11}\mathfrak{L}\varphi\,, \tag{6.33}$$

it is a Dirichlet problem for the strongly elliptic operator $A'A$, clearly the solution is unique.
$C^{11}\mathfrak{L}\varphi \in \prod_{j\in M_1} H^{t+2m-j-\frac{1}{2}}(E|_\Gamma)$ implies $v \in H^{t+2m}(E)$, by an application of Proposition 5.2. Furthermore, let $z = \gamma_Z^{-1}\Phi\varphi$. Now $u = v + z$ satisfies

$$\gamma u = \gamma z = \Phi\varphi \in X = Z^0(B^{00})\,,$$

and

$$\begin{aligned} B^{11}\nu u &= B^{11}(\nu v + \nu z) = B^{11}C^{11}\mathfrak{L}\varphi + B^{11}P_{\gamma,\nu}\gamma z \\ &= -B^{10}\gamma u - B^{11}P_{\gamma,\nu}\gamma u + B^{11}P_{\gamma,\nu}\gamma u \\ &= -B^{10}\gamma u\,, \end{aligned}$$

so $u \in D(A_B)$. Moreover, $Au = Av \in H^t(E)$. Then by (6.31), $u \in H^s(E)$ and, since $v \in H^{t+2m}(E)$ and $s \leqslant t+2m$, $z = u - v \in Z_A^s(E)$. Thus finally $\varphi = \Phi^{(-1)}\gamma z \in \prod_{k\in M_0} H^{s-k-\frac{1}{2}}(Z_k)$. This shows (6.32).

Conversely, assume that (6.32) holds. Let $u \in D(A_B)$ with $Au \in H^t(E)$. Then $\mathrm{pr}_\gamma u = A_\gamma^{-1} Au \in H^{t+2m}(E)$ by Proposition 5.2, so furthermore $\mu u = \mathcal{A}^{01} \nu \, \mathrm{pr}_\gamma u \in \prod_{k \in M_1} H^{t+k+\frac{1}{2}}(E|_\Gamma)$ by (5.4) and (5.20). Let $\varphi = \Phi^{(-1)} \gamma u$, so $\gamma u = \Phi \varphi$. Then

$$\begin{aligned} \mathfrak{L}\varphi &= -B^{10}\gamma u - B^{11} P_{\gamma,\nu} \gamma u = B^{11}(\nu u - P_{\gamma,\nu}\gamma u) \\ &= B^{11}(\mathcal{A}^{01})^{-1} \mu u \in \prod_{j \in M_1} H^{t+2m-j-\frac{1}{2}}(F_j) , \end{aligned}$$

in view of the types of the operators involved. By (6.32) it follows that $\varphi \in \prod_{k \in M_0} H^{s-k-\frac{1}{2}}(Z_k)$, and hence $\gamma u = \Phi \varphi \in \prod_{k \in M_0} H^{s-k-\frac{1}{2}}(E|_\Gamma)$ and $\mathrm{pr}_\zeta u = \gamma_Z^{-1} \gamma u \in Z_A^s(E)$. Since $s \leqslant t + 2m$, $u = u_\gamma + u_\zeta \in H^s(E)$, which shows (6.31).

REMARK 6.9. We have not bothered to give an « abstract » version of the result in 3°. Let us just mention that the argument from (6.31) to (6.32) generalizes to all closed $\tilde{A}$, where as the other direction only holds under certain assumptions on T.

COROLLARY 6.10. *A_B is the realization of an* elliptic *boundary value problem* (³) *if and only if $\mathfrak{L}$ is* elliptic *(as a pseudo-differential operator of type $(-k, -j)_{j \in M_1, k \in M_0}$). Then* $\dim \bigoplus_{k \in M_0} Z_k = \dim \bigoplus_{j \in M_1} F_j$, *i.e.,*

$$\sum_{j \in M} p_j = mq \ ;$$

and

$$D(\mathfrak{L}_1) = \prod_{k \in M_0} H^{2m-k-\frac{1}{2}}(Z_k) \, . \tag{6.35}$$

The « reduction to the boundary » in the above theorem is different from those introduced in Hörmander [18] or Seeley [25], where the involved vector bundles over Γ have dimension $2mq$ or more. An advantage of our theorem is that it keeps track of the dimensions of the kernel and the cokernel individually (cf. 1° and 2°), not just of the index. Otherwise, our statement 3° resembles Hörmander's characterization of regularity in [18, Theorem 2.2.3], which treats more general (non-normal) boundary conditions. The strength of the present theory rather lies in its ability to treat semiboundedness and spectral problems, as will be shown in the following sections.

(³) i.e., A_B is a Fredholm operator satisfying (6.31) with $s = t + 2m$, all $t \geqslant 0$.

7. – Semiboundedness and coerciveness.

We shall now characterize various inequalities. The weakest one is defined as follows:

DEFINITION 7.1. A realization $\tilde{A}$ of A is called weakly semibounded if there exist $c > 0$, $\theta \in \mathbf{R}$ such that

$$\operatorname{Re} e^{i\theta}(Au, u) \leqslant c\|u\|_m^2 \qquad \text{for all } u \in D(\tilde{A}) \cap H^{2m}(E) .$$

Note that A_0 is always weakly semibounded, simply because A is of order $2m$. Furthermore, a realization must be weakly semibounded in order to be symmetric (i.e., $\operatorname{Re} i(Au, u) = 0$) or selfadjoint, or satisfy any of the usual coerciveness inequalities (for an $s \in [0, m]$)

$$\operatorname{Re}(Au, u) \geqslant c\|u\|_s^2 - \lambda\|u\|_0^2 , \qquad u \in D(\tilde{A}) . \tag{7.1}$$

Weakly semibounded A_B were characterized in [17], from which we quote some results:

THEOREM 7.1. *Let A_B be the realization of A defined by* (6.7). *The following statements are equivalent*:

(*a*) *A_B is weakly semibounded.*

(*b*) $(I - C^{00}B^{00})^* \mathcal{A}^{01}(I - C^{11}B^{11}) = 0$.

(*c*) $X \subseteq Y$

$$[\textit{i.e.}, \; Z^0(B^{00}) \subseteq (\mathcal{A}^{01*})^{-1}B^{11*}\prod_{j\in M_1} H^{-2m+j+\frac{1}{2}}(F_j)] .$$

(*d*) *There exists $c > 0$ such that $|(Au, v)| \leqslant c\|u\|_m\|v\|_m$ for all $u \in D(A_B) \cap H^{2m}(E)$, all $v \in H^m(E)$ with $B^{00}\gamma v = 0$.*

For the proofs, see [17], Theorem 2.4, Remarks 2.5 and 2.7, and, for (*c*), Lemma 2.8, (2.18). A further analysis of (*b*) and (*c*) leads to

THEOREM 7.2. 1° *When A_B is weakly semibounded, then $\sum_{j\in M} p_j \geqslant mq$.*

2° *When A_B is weakly semibounded, then $\sum_{j\in M} p_j = mq$ holds if and only if $X = Y$, and if and only if A_B^* is weakly semibounded.*

3° *Let $\sum_{j\in M} p_j = mq$. Then weak semiboundedness of A_B is equivalent* *with*

(*e*) $X = Y$

$$[\text{i.e., } Z^0(B^{00}) = (\mathcal{A}^{01*})^{-1} B^{11*} \prod_{j \in M_1} H^{-2m+j+\frac{1}{2}}(F_j)] \, .$$

(*f*) $X \supseteq Y$ [i.e., $B^{00}(\mathcal{A}^{01*})^{-1} B^{11*} = 0$].

4° *For any given* B^{00} (*or* B^{11}) *there exists* B^{11} (*resp.* B^{00}) *so that* (*e*) *holds; such* B^{11} (*resp.* B^{00}) *are determined up to a* (*pseudo-*)*differential isomorphism.*

For proofs and further details, see [17], Theorems 2.11, 2.13 and 2.15, and Corollaries 2.14 and 2.20.

The condition $\sum_{j \in M} p_j = mq$ is usually assumed in the study of boundary problems. It assures us that $\dim \bigoplus_{k \in M_0} Z_k = \sum_{k \in M_0} (q - p_k) = \sum_{j \in M_1} p_j = \dim \bigoplus_{j \in M_1} F_j$, which gives the best chance of having both existence and uniqueness for the boundary problem (cf. Theorem 6.8). When $X = Y$, we shall of course use the representation $\mathcal{M}_1$ of T introduced in Corollary 6.5.

Selfadjoint A_B clearly satisfy $\sum p_j = mq$, since for those, both A_B and A_B^* are weakly semibounded. The proof of [13, Corollary 4.3] easily generalizes to give

THEOREM 7.3. *Let* A_B *be the realization of* A *defined by* (6.7). *Then* A_B *is selfadjoint if and only if* (i) - (iv) *hold:*

(i) A *is formally selfadjoint.*

(ii) $\sum_{j \in M} p_j = mq$.

(iii) $B^{00}(\mathcal{A}^{01*})^{-1} B^{11*} = 0$.

(iv) $\mathcal{M}$, *defined by* (6.27), *is formally selfadjoint, and* $\prod_{j \in M_1} C^\infty(F_j)$ *is dense in* $D(\mathcal{M}_1)$ (*cf.* (6.28)) *in the graphtopology.*

In particular, the density requirement in (iv) is satisfied if $\mathcal{M}$ is elliptic or subelliptic.

We shall now restrict the attention to strongly elliptic A, to which we shall apply the results of Section 3, in the same manner as done for the scalar case in [13]. We assume

ASSUMPTION 7.4. A is strongly elliptic, and a constant has been added so that, with $c_m > 0$,

$$\operatorname{Re}(Au, u) \geqslant c_m \|u\|_m^2 \quad \text{for all } u \in D(A_0) \, . \tag{7.2}$$

Then $A^r = \frac{1}{2}(A + A')$ is selfadjoint elliptic, and the operators A_0, A_γ A_1, A_0', A_γ^*, A_1', A_0^r, A_γ^r and A_1^r satisfy the assumptions of Sections 1 and 3.

The following two statements are immediate generalizations of [13, p. 60-61]

DEFINITION 7.5. The quadratic form $q(\varphi, \varphi)$ on $\gamma[\mathcal{H}_A \cap \mathcal{H}_{A^r}]$ is defined by

$$q(\varphi, \varphi) = \mathrm{Re}\langle A(\gamma_Z^r)^{-1}\varphi \,, \mathrm{pr}_\gamma' (\gamma_Z^r)^{-1}\varphi\rangle \,.$$

Recall from [13] that $q(\varphi, \varphi) \leqslant 0$ for all φ. Recall from Lemma 3.3 that $\mathcal{H}_A \cap \mathcal{H}_{A^r} = \mathcal{H}_A \cap \mathcal{H}_{A'}$, and $D(A_1) \cap D(A_1^r) = D(A_1) \cap D(A_1')$.

PROPOSITION 7.6. *Denote by Q the pseudo-differential operator* (cf. (5.18)-(5.19))

$$Q = -P_{\gamma,\chi^r}^r + \tfrac{1}{2}[P_{\gamma,\chi} + P_{\gamma,\chi'}'] = \tfrac{1}{2}[-(\mathcal{A}^{01} - \mathcal{A}^{10*})P_{\gamma,\nu}^r + \mathcal{A}^{01}P_{\gamma,\nu} - \mathcal{A}^{10*}P_{\gamma,\nu}'] \,, \tag{7.3}$$

it is of type $(-k, -2m + 1 + j)_{j,k\in M_0}$. For $\varphi \in \gamma[D(A_1) \cap D(A_1^r)]$, $Q\varphi \in \in \prod_{j\in M_0} H^{j+\frac{1}{2}}(E|_\Gamma)$ and

$$q(\varphi, \varphi) = \langle Q\varphi, \varphi\rangle \,, \tag{7.4}$$

duality between $\prod_{j\in M_0} H^{j+\frac{1}{2}}(E|_\Gamma)$ and $\prod_{j\in M_0} H^{-j-\frac{1}{2}}(E|_\Gamma)$.

Now we find

THEOREM 7.7. (*Assumption* 7.4) *Let A_B be the realization of A defined by* (6.7). *Let K be a Hilbert space and U a linear set, satisfying*:

$$H_0^m(E) \subseteq K \subseteq L^2(E) \quad (\textit{algebraically and topologically}) \tag{7.5}$$

$$D(A_\gamma) \subseteq U \subseteq K \cap \mathcal{H}_{A^r} \quad (\textit{algebraically}) \tag{7.6a}$$

$$\gamma D(A_B) \cap \gamma U \quad \textit{is dense in } X. \tag{7.6b}$$ [4]

Let $K_\Phi = \Phi^{(-1)}\gamma(K \cap \mathcal{H}_{A^r} \cap V)$ (cf. (6.8)-(6.11) *and* (6.16)-(6.17)), *and provide K_Φ with the norm*

$$\|\varphi\|_{K_\Phi} = \|(\gamma_Z^r)^{-1}\Phi\varphi\|_K \quad \textit{for } \varphi \in K_\Phi \,. \tag{7.7}$$

[4] Equations (7.6*a*-*b*) ensure that K and U do not impose extra boundary conditions.

Then there exist $c > 0$, $\lambda \in \mathbf{R}$ *so that*

$$\mathrm{Re}(Au, u) \geqslant c\|u\|_K^2 - \lambda\|u\|_0^2 \quad \textit{for all } u \in D(A_B) \cap U\,, \tag{7.8}$$

if and only if (i) *and* (ii) *hold:*

(i) $\sum_{j\in M} p_j \geqslant mq$ *and* $(I - C^{00}B^{00})^* \mathcal{A}^{01}(I - C^{11}B^{11}) = 0$.

(ii) *There exist* $c' > 0$, $\lambda' \in \mathbf{R}$, *so that for all* $\varphi \in D(\mathfrak{L}_1) \cap \Phi^{(-1)}(\gamma U \cap X)$

$$\mathrm{Re}\langle \mathcal{K}\varphi, \varphi\rangle + q(\Phi\varphi, \Phi\varphi) \geqslant c'\|\varphi\|_{K\Phi}^2 - \lambda'\|\varphi\|_{\{-k-\frac{1}{2}\},k\in M_0}^2; \tag{7.9}$$

here $\mathcal{K}$ *is the pseudo-differential operator*

$$\mathcal{K} = -\Phi^* \mathcal{A}^{01} C^{11}(B^{10} + B^{11}P_{\gamma,\nu})\Phi\,, \tag{7.10}$$

and $D(\mathfrak{L}_1)$, *defined in Theorem* 6.4, *satisfies*

$$D(\mathfrak{L}_1) = \Big\{\varphi \in \prod_{k\in M_0} H^{-k-\frac{1}{2}}(Z_k) \,|\, \mathcal{K}\varphi \in \prod_{k\in M_0} H^{k+\frac{1}{2}}(Z_k)\Big\}\,. \tag{7.11}$$

When $-\lambda > 0$ (*resp.* $\geqslant 0$) *in* (7.8), $-\lambda'$ *may be taken* > 0 (*resp.* $\geqslant 0$) *in* (7.9), *and vice versa.*

PROOF. We shall apply Theorem 3.6. Recall that $\mathrm{pr}'_\zeta = (\gamma'_Z)^{-1}\gamma$ and $\mathrm{pr}_\zeta = \gamma_Z^{-1}\gamma$, so that $\gamma\,\mathrm{pr}_\zeta u = \gamma\,\mathrm{pr}'_\zeta u = \gamma u$ for $u \in \mathcal{H}_A \cap \mathcal{H}_{A'}$ (cf. Proposition 5.2), and note that $D(A_\gamma) \subseteq U$ implies $\gamma(U \cap Z(A_1)) = \gamma U$. So

$$\begin{aligned}\gamma\,\mathrm{pr}'_\zeta(D(T) \cap U) = \gamma(D(T) \cap U) &= \gamma D(T) \cap \gamma(U \cap Z(A_1)) \\ &= \gamma D(T) \cap \gamma U = \gamma D(A_B) \cap \gamma U\end{aligned}$$

which is a dense subset of X (cf. Proposition 6.3) by (7.6*b*). Thus

$$\mathrm{pr}'_\zeta(D(T) \cap U) \subseteq W \Leftrightarrow \gamma D(A_B) \cap \gamma U \subseteq Y \Leftrightarrow X \subseteq Y\,.$$

By use of Theorems 7.1 and 7.2, condition (i) of Theorem 3.6 then takes the form of the present condition (i).

Concerning (ii), we note that

$$\begin{aligned}\Phi^{(-1)}(\gamma D(T) \cap \gamma U) &= \Phi^{(-1)}\gamma D(T) \cap \Phi^{(-1)}(\gamma U \cap X) = \\ &= D(\mathfrak{L}_1) \cap \Phi^{(-1)}(\gamma U \cap X)\,,\end{aligned}$$

by Theorem 6.4 and (6.17). Then, using Corollary 6.6 and Definition 7.5, we have for $z \in D(T) \cap U$, $\varphi = \Phi^{(-1)}\gamma z$,

$$Re[(Tz, \mathrm{pr}'_\zeta z) + \langle A\, \mathrm{pr}^r_\zeta z, \mathrm{pr}'_\gamma \mathrm{pr}^r_\zeta z\rangle]$$
$$= \mathrm{Re}\langle -\Phi^* \mathcal{A}^{01} C^{11}(B^{10} + B^{11}P_{\gamma,\nu})\Phi\varphi, \varphi\rangle + q(\Phi\varphi, \Phi\varphi) \;;$$

moreover, by (7.7),

$$\|\varphi\|_{K\Phi} = \|(\gamma^r_Z)^{-1}\Phi\varphi\|_K = \|(\gamma^r_Z)^{-1}\gamma z\|_K = \|\mathrm{pr}^r_\zeta z\|_K \,.$$

We also have, with positive constants c_j,

$$\|\mathrm{pr}^r_\zeta z\|_0 \leqslant c_1\|\gamma z\|_{\Pi H^{-k-\frac{1}{2}}(E|_\Gamma)} = c_1\|\Phi\varphi\|_{\Pi H^{-k-\frac{1}{2}}(E|_\Gamma)}$$
$$\leqslant c_2\|\varphi\|_{\Pi H^{-k-\frac{1}{2}}(Z_k)} \leqslant c_3\|\Phi\varphi\|_{\Pi H^{-k-\frac{1}{2}}(E|\Gamma)}$$
$$\leqslant c_4\|\mathrm{pr}^r_\zeta z\|_0 \,,$$

using various isomorphisms accounted for in the preceding sections. Thus condition (ii) in Theorem 3.6 may be formulated as the present condition (ii). The statement of Theorem 3.6 then implies the statement of Theorem 7.7.

When $U \subseteq D(A^r_1)$, (7.4) yields more explicit statements, for example:

COROLLARY 7.8. *Let A_B be as in Theorem* 7.7; *let $H^m_0(E) \subseteq K \subseteq L^2(E)$ (alg. and top.) with $D(A_B) \cap H^{2m}(E) \subseteq K$ (alg.), and let $U = K \cap D(A^r_1)$. Then* (7.8) *is valid for some $c > 0$, $\lambda \in \mathbf{R}$, if and only if* (i) *and* (ii) *hold:*

(i) $\sum_{j \in M} p_j \geqslant mq$ *and* $(I - C^{00}B^{00})^* \mathcal{A}^{01}(I - C^{11}B^{11}) = 0$.

(ii) *There exist $c' > 0$, $\lambda' \in \mathbf{R}$ such that*

$$\langle(\mathcal{K} + \Phi^* Q\Phi)\varphi, \varphi\rangle \geqslant c'\|\varphi\|^2_{K\Phi} - \lambda'\|\varphi\|^2_{\{-k-\frac{1}{2}\},k\in M_0}$$

for all $\varphi \in D(\mathfrak{L}_1) \cap \Phi^{(-1)}(\gamma U \cap X)$ (*cf.* (7.3), (7.7), (7.10-11)).

In the case where $\sum_{j\in M} p_j = mq$, we may use Ψ instead of Φ, and $\mathcal{M}$ enters in the formulae instead of $\mathfrak{L}$, which gives simpler calculations (cf. (6.12), (6.18) and Corollary 6.5).

COROLLARY 7.9. *Let A_B, K and U be as in Corollary* 7.8, *and assume in addition that $\sum_{j\in M} p_j = mq$. Then* (7.8) *holds if and only if*:

(i) $B^{00}(\mathcal{A}^{01*})^{-1}B^{11*} = 0$.

(ii) *There exist $c' > 0$, $\lambda' \in \mathbf{R}$ such that*

$$\mathrm{Re}\langle(\mathcal{M} + \Psi^* Q\Psi)\psi, \psi\rangle \geqslant c'\|\psi\|^2_{K\Psi} - \lambda'\|\psi\|^2_{\{-2m+j+\frac{1}{2}\}j\in M_1}$$

for all $\psi \in \prod_{j\in M_1} H^{-2m+j+\frac{1}{2}}(F_j) \cap \Psi^{(-1)}(\gamma U \cap X)$ *for which* $\mathcal{M}\psi \in \prod_{j\in M_1} H^{2m-j-\frac{1}{2}}(F_j)$; *here* $\|\psi\|_{K_\Psi} = \|(\gamma_Z^r)^{-1}\Psi\psi\|_K$, *and*

$$\Psi = (\mathcal{A}^{01*})^{-1}B^{11*}, \qquad \mathcal{M} = -(B^{10} + B^{11}P_{\gamma,\nu})\Psi .$$

Consider the special case where $K = H^s(E)$, $s \in [0, m]$. (We let $U = H^{2m}(E)$ for simplicity; more general choices are covered above.)

THEOREM 7.10. (*Assumption* 7.4). *Let* A_B *be defined by* (6.7) *and let* $K = H^s(E)$ *for some* $s \in [0, m]$. *Then there exist* $c > 0$, $\lambda \in \mathbf{R}$ *such that*

$$\operatorname{Re}(Au, u) \geqslant c\|u\|_s^2 - \lambda\|u\|_0^2 \quad \text{for all } u \in D(A_B) \cap H^{2m}(E)\,, \tag{7.12}$$

if and only if: *Theorem* 7.7 (i) *holds, and there exist* $c' > 0$, $\lambda' \in \mathbf{R}$ *so that for all* $\varphi \in \prod_{k\in M_0} H^{2m-k-\frac{1}{2}}(Z_k)$

$$\operatorname{Re}\langle(\mathcal{K} + \Phi^*Q\Phi)\varphi, \varphi\rangle \geqslant c'\|\varphi\|_{\{s-k-\frac{1}{2}\},k\in M_0} - \lambda'\|\varphi\|_{\{-k-\frac{1}{2}\},k\in M_0}\,. \tag{7.13}$$

In particular, (7.13) *holds with* $s = m$ *if and only if*

$$\sigma^0(\mathcal{K}) + \sigma^0(\mathcal{K})^* + 2\sigma^0(\Phi^*Q\Phi) > 0 \quad \text{on } S(\Gamma) \tag{7.14}$$

(*the cotangent sphere bundle*); *in that case,* $D(A_B) \subseteq H^{2m}(E)$. (*If* $\sum_{j\in M} p_j = mq$, A_B *is then elliptic in the sense of Corollary* 6.10.)

PROOF. $K_\Phi = \Phi^{(-1)}\gamma(H^s(E) \cap \mathcal{K}_{A^r} \cap V) = \Phi^{(-1)}\gamma(\mathcal{K}_{A^r}^{s,-m}(E) \cap V)$ with the norm $\|\varphi\|_{K_\Phi} = \|(\gamma_Z^r)^{-1}\Phi\varphi\|_s = \|\varphi\|_{\{s-k-\frac{1}{2}\},k\in M_0}$. When $s = m$, (7.13) means that $\mathcal{K} + \Phi^*Q\Phi$ is strongly elliptic (both terms are of type $(m-k-\frac{1}{2}, -m+ j+\frac{1}{2})_{j,k\in M_0}$), which is equivalent with (7.14) by a well known result of Hörmander, Lax and Nirenberg. Since Q is nonpositive, (7.14) implies ellipticity of $\mathcal{K} = \Phi^*\Psi^{(-1)*}\mathcal{L}$, so that $\mathcal{L}\varphi \in \prod_{j\in M_1} H^{t+2m-j-\frac{1}{2}}(F_j) \Rightarrow \mathcal{K}\varphi \in \prod_{k\in M_0} H^{t+k+\frac{1}{2}}(Z_k) \Rightarrow \varphi \in \prod_{k\in M_0} H^{t+2m-k-\frac{1}{2}}(Z_k)$ for all $t \in \mathbf{R}$. Theorem 6.8.3° then shows that $D(A_B) \subseteq H^{2m}(E)$; in fact (6.31) holds with $s = t + 2m$ for all $t \geqslant 0$. If $\sum_{j\in M} p_j = mq$, $\mathcal{L}$ itself is elliptic. Q.e.d.

The inequality (7.12) with $s = m$ is often called Garding's inequality. In [17], we showed how Theorem 7.7 (i) complements the sufficient conditions of Agmon [1] and de Figueiredo [8] for (7.12) (formulated by sesquilinear forms) for the case of *differential* boundary conditions. For $s = m - \frac{1}{2}$, Fujiwara treated (7.12) for a special class of boundary conditions in [9];

and Melin gave a complete discussion of (7.13) in [22] for the scalar case, i.e., $\sum_{j\in M_1} p_j = 1$. Results seem lacking for $s < m - \frac{1}{2}$ and for systems.

In each of the above results, $-\lambda$ may be taken > 0 ($\geqslant 0$) if and only if $-\lambda'$ may be taken > 0 ($\geqslant 0$), cf. Theorem 7.7. For $s = 0$, Theorem 7.10 gives a statement on lower semiboundedness; however, Corollary 3.8 leads to the sharper result:

COROLLARY 7.11. (*Notations of Theorem* 7.7.) *Let* $D(A_\gamma) \subseteq U \subseteq \mathcal{K}_{A^r}$, *with* $\gamma D(A_B) \cap \gamma U$ *dense in* X. *There exists* $\mu \in \mathbb{R}$ *so that*

$$\mathrm{Re}(Au, u) \geqslant \mu \|u\|_0^2 \quad \textit{for all } u \in D(A_B) \cup U \,, \tag{7.15}$$

if and only if: *Theorem* 7.7 (i) *holds and there exists* $\mu' \in \mathbb{R}$ *so that for all* $\varphi \in D(\mathcal{L}_1) \cap \Phi^{(-1)}(\gamma U \cap X)$

$$\mathrm{Re}\langle \mathcal{K}\varphi, \varphi\rangle + q(\Phi\varphi, \Phi\varphi) \geqslant \mu' \|\varphi\|^2_{\{-k-\frac{1}{2}\}.k\in M_0} \,. \tag{7.16}$$

Here μ *may be taken* > 0 ($\geqslant 0$) *if and only if* μ' *may be taken* > 0 ($\geqslant 0$). (*Simplifications as in Corollaries* 7.8-9.)

REMARK 7.12. When $n = 1$, $D(A_1) = H^{2m}(E)$, and $Z(A_1)$ is *finite dimensional*. Then Theorem 7.7 (i) *alone* is necessary and sufficient for lower semiboundedness (7.15), and when it holds, A_B satisfies Garding's inequality (7.12) (without requiring (7.14)).

8. – Perturbation theory; the negative spectrum.

The results in Section 7 were qualitative, in that only the *signs* of the constants c, λ, c', λ' were discussed, not their values. More precise evaluations require, among other things, that one fixes the norms in the various Sobolev spaces and keeps track of the various isomorphisms in an exact way. Similar efforts have to be made if one wants to use the formula $\tilde{A}^{-1} = A_\gamma^{-1} + T^{(-1)}$ in Proposition 1.4. For this, it is important to choose a representation $\mathcal{T}$ of T that is derived from T by *isometries*. We shall show how to do that, and thereby give a key to the application of perturbation theorems, for selfadjoint A_B (with a remark on the non-selfadjoint case). We give one application, namely to the study of the negative spectrum, by use of the results in Section 4. (In the construction of $\mathcal{T}$, the compactness of the manifold $\bar{\Omega}$ plays no essential role, and could be replaced by uniform bounds on the symbols; the calculations are local. However, our application to spectral theory is concerned with the compact case.)

Assume in the following that A is formally selfadjoint satisfying Assumption 5.1, and that A_B is a selfadjoint realization defined by a boundary condition (6.7); such realizations are characterized in Theorem 7.3. We shall need two auxiliary pseudo-differential operators $\mathfrak{R}$ and $\mathcal{S}$ in Γ ($\mathfrak{R}$ was described for general elliptic A in [13, Example 6.3]).

PROPOSITION 8.1. *For all* $z, z_1 \in Z(A_1)$,

$$(8.1) \qquad (z, z_1) = \langle \mathfrak{R}\gamma z, \gamma z_1 \rangle ,$$

$$(8.2) \qquad (A_\gamma^{-1} z, z_1) = \langle \mathcal{S}\gamma z, \gamma z_1 \rangle$$

(*dualities between* $\prod_{k\in M_0} H^{k+\frac12}(E|_\Gamma)$ *and* $\prod_{k\in M_0} H^{-k-\frac12}(E|_\Gamma)$), *where, for* $\varphi \in \prod_{k\in M_0} H^{-k-\frac12}(E|_\Gamma)$, $\mathfrak{R}\varphi$ *and* $\mathcal{S}\varphi$ *are defined as follows*:

$$(8.3) \qquad \mathfrak{R}\varphi = \mathcal{A}^{01}\nu v ,$$

where v *is the solution in* $\mathcal{K}^{2m,0}_{A^2}(E)$ *of*

$$(8.4) \qquad A^2 v = 0 , \quad \gamma v = 0 , \quad \gamma A v = \varphi ;$$

and

$$(8.5) \qquad \mathcal{S}\varphi = \mathcal{A}^{01}\nu w ,$$

where w *is the solution in* $\mathcal{K}^{3m,0}_{A^3}(E)$ *of*

$$(8.6) \qquad A^3 w = 0 , \quad \gamma w = \gamma A w = 0 , \quad \gamma A^2 w = \varphi .$$

Here, $\mathfrak{R}$ *is a strongly elliptic selfadjoint pseudo-differential operator in* $\bigoplus_{k\in M_0} E|_\Gamma$ *of type* $(-k-\frac12, j+\frac12)_{j,k\in M_0}$, *positive with respect to the norm* $\|\varphi\|_{\{-k-\frac12\},k\in M_0}$; *and* $\mathcal{S}$ *is, when* A *satisfies Assumption* 7.4, *a strongly elliptic selfadjoint pseudo-differential operator in* $\bigoplus_{k\in M_0} E|_\Gamma$ *of type* $(-k-\frac12, 2m+j+\frac12)_{j,k\in M_0}$, *positive with respect to the norm* $\|\varphi\|_{\{-m-k-\frac12\},k\in M_0}$.

PROOF. Note first, that for any $n \in \mathbf{N}$, the boundary value problem

$$(8.7) \qquad A^n u = f , \quad \gamma u = \varphi_0 , \ \gamma A u = \varphi_1, \ldots, \gamma A^{n-1} u = \varphi_{n-1} ;$$

is *elliptic* and uniquely solvable, since the solution u is determined by solving a succession of Dirichlet problems for A:

$$(8.8) \qquad A u_{n-1} = u_n , \quad \gamma u_{n-1} = \varphi_{n-1}; \ldots ; \quad A u_0 = u_1 , \quad \gamma u_0 = \varphi_0 ,$$

where $f = u_n$, $u = u_0$. Denote by γ^n the Dirichlet boundary operator for A^n, then the pseudo-differential operator over Γ defined as follows (compare Proposition 5.3)

$$P^{A^n}_{\gamma^n,\{\gamma,\gamma A,\ldots,\gamma A^{n-1}\}}\colon \gamma^n v \mapsto \{\gamma v, \ldots, \gamma A^{n-1} v\} \qquad \text{for } v \in Z^s_{A^n}(E)$$

is elliptic and invertible. In particular, we may define

$$\mathcal{R} = \mathcal{A}^{01} \circ P^{A^2}_{\gamma^2,\nu} \circ (P^{A^2}_{\gamma^2,\{\gamma,\gamma A\}})^{-1} \circ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.9}$$

and

$$\mathcal{S} = \mathcal{A}^{01} \circ P^{A^3}_{\gamma^3,\nu} \circ (P^{A^3}_{\gamma^3,\{\gamma,\gamma A,\gamma A^2\}})^{-1} \circ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{8.10}$$

which are pseudo-differential operators in $\bigoplus_{k\in M_0} E|_\Gamma$ of types $(-k-\frac{1}{2}, j+\frac{1}{2})_{j,k\in M_0}$ resp. $(-k-\frac{1}{2}, 2m+j+\frac{1}{2})_{j,k\in M_0}$. Now let $\varphi \in \prod_{k\in M_0} H^{-k-\frac{1}{2}}(E|_\Gamma)$, and let $z = \gamma_z^{-1}\varphi$. The solution of (8.4) is exactly $v = A_\gamma^{-1} z$, and we find for any $z_1 \in Z(A_1)$, by use of Green's formula

$$\begin{aligned}(z, z_1) = (Av, z_1) &= (Av, z_1) - (v, Az_1) \\ &= \langle \mathcal{A}^{01}\nu v, \gamma z_1\rangle = \langle \mathcal{R}\varphi, \gamma z_1\rangle \\ &= \langle \mathcal{R}\gamma z, \gamma z_1\rangle\,.\end{aligned}$$

Then $\mathcal{R}$ satisfies (8.1); and since $\langle \mathcal{R}\gamma z, \gamma z\rangle = \|z\|_0^2 \simeq \|\gamma z\|_{\{-k-\frac{1}{2}\},k\in M_0}$, we see that $\mathcal{R}$ is selfadjoint strongly elliptic and positive as indicated.

The solution of (8.6) is exactly $w = A_\gamma^{-1} v = A_\gamma^{-2} z$, and we find for any $z_1 \in Z(A_1)$

$$\begin{aligned}(A_\gamma^{-1} z, z_1) = (Aw, z_1) = (Aw, z_1) - (w, Az_1)& \\ = \langle \mathcal{A}^{01}\nu w, \gamma z_1\rangle = \langle \mathcal{S}\varphi, \gamma z_1\rangle = \langle \mathcal{S}\gamma z, \gamma z_1\rangle&\,.\end{aligned}$$

Thus $\mathcal{S}$ satisfies (8.2); and when A furthermore satisfies Assumption 7.4, we have $\langle \mathcal{S}\gamma z, \gamma z\rangle = (A_\gamma^{-1} z, z) = \|A_\gamma^{-\frac{1}{2}} z\|_0^2 \simeq \|z\|_{-m}^2 \simeq \|\gamma z\|_{\{-m-k-\frac{1}{2}\},k\in M_0}$, so that the selfadjoint pseudo-differential operator $\mathcal{S}$ is strongly elliptic and positive as indicated.

REMARK 8.2. By related considerations, one finds that G^μ (Definition 2.5) satisfies

$$(8.11) \qquad (G^\mu z, z) = \langle \mathcal{A}^{01}(P^A_{\gamma,\nu} - P^{A-\mu}_{\gamma,\nu})\gamma z, \gamma z\rangle$$

for all $z \in Z(A_1)$; here $\mathcal{A}^{01}(P^A_{\gamma,\nu} - P^{A-\mu}_{\gamma,\nu})$ is an elliptic positive selfadjoint pesudo-differential operator in $\bigoplus_{k\in M_0} E|_\Gamma$ of type $(-k-\frac{1}{2}, j+\frac{1}{2})_{j,k\in M_0}$. (See also Remark 2.11.)

The symbols of $\mathfrak{R}$ and $\mathfrak{S}$ are found by use of (8.9) and (8.10), cf. Proposition 5.3 and the remarks there.

Introduce the notation for the bundle $\bigoplus_{j\in M_1} F_j$

$$(8.12) \qquad \bigoplus_{j\in M_1} F_j = F^1$$

and choose a pseudo-differential isomorphism Λ of $\prod_{j\in M_1} H^0(F_j) = L^2(F^1)$ onto $\prod_{k\in M_0} H^{-2m+j+\frac{1}{2}}(F_j)$. Recall the definition of Ψ from Proposition 6.3; Ψ is continuous and injective (with a continuous left inverse) from $\prod_{j\in M_1} H^{-2m+j+\frac{1}{2}}(F_j)$ into $\prod_{k\in M_0} H^{-k-\frac{1}{2}}(E|_\Gamma)$. The composed operator $\Lambda^*\Psi^*\mathfrak{R}\Psi\Lambda$ is then a selfadjoint pseudo-differential operator of order 0 in $L^2(F^1)$. Moreover, for all $\psi \in L^2(F^1)$,

$$(\Lambda^*\Psi^*\mathfrak{R}\Psi\Lambda\psi, \psi) = \langle \mathfrak{R}\Psi\Lambda\psi, \Psi\Lambda\psi\rangle \geqslant c_1 \|\Psi\Lambda\psi\|_{\{-k-\frac{1}{2}\},k\in M_0} \geqslant c_2 \|\psi\|_{L^2(F^1)},$$

so $\Lambda^*\Psi^*\mathfrak{R}\Psi\Lambda$ is strongly elliptic and positive, and we may define

$$(8.13) \qquad \Xi = (\Lambda^*\Psi^*\mathfrak{R}\Psi\Lambda)^{-\frac{1}{2}};$$

a selfadjoint positive elliptic pseudo-differential operator in F^1 of order 0 with $\sigma^0(\Xi) = \sigma^0(\Lambda^*\Psi^*\mathfrak{R}\Psi\Lambda)^{-\frac{1}{2}}$, by the calculus of Seeley [26].

PROPOSITION 8.3. *The mapping*

$$(8.14) \qquad J = \gamma_V^{-1}\Psi\Lambda\Xi : L^2(F^1) \to V$$

is an isometry, *with inverse* $J^{-1} = \Xi^{-1}\Lambda^{-1}\Psi^{(-1)}\gamma = J^*$.

PROOF. Let $v \in V$ and let $\psi = \Xi^{-1}\Lambda^{-1}\Psi^{(-1)}\gamma v$. Then, since $V \subseteq Z(A_1)$,

$$\begin{aligned}\|v\|^2_{L^2(E)} &= (v, v) = \langle \mathfrak{R}\gamma v, \gamma v\rangle = \langle \mathfrak{R}\Psi\Lambda\Xi\psi, \Psi\Lambda\Xi\psi\rangle \\ &= \langle \Xi\Lambda^*\Psi^*\mathfrak{R}\Psi\Lambda\Xi\psi, \psi\rangle \\ &= \langle \Xi\Xi^{-2}\Xi\psi, \psi\rangle = \|\psi\|^2_{L^2(F^1)},\end{aligned}$$

which shows the proposition.

THEOREM 8.4. *Let A be formally selfadjoint satisfying Assumption* 5.1, *and let A_B be a selfadjoint realization defined by a boundary condition* (6.7). *Let $T: V \to V$ be the operator corresponding to A_B by Proposition* 1.2, *and let $\mathcal{T}_1$ be the representation of T*

$$\mathcal{T}_1 = J^* T J . \tag{8.15}$$

Then $\mathcal{T}_1$ is the restriction of the pseudo-differential operator in F^1 of order $2m$

$$\mathcal{T} = \Xi\Lambda^* \mathscr{M} \Lambda\Xi , \tag{8.16}$$

with domain

$$D(\mathcal{T}_1) = \{\varphi \in L^2(F^1) | \mathcal{T}\varphi \in L^2(F^1)\} . \tag{8.17}$$

Moreover, $\mathcal{T}_1$ has the same spectrum as T, and its eigenvectors are mapped into the corresponding eigenvectors of T by the isometry J.

PROOF. By the isomorphism $\Lambda\Xi$ from $L^2(F^1)$ to $\prod_{j\in M_1} H^{-2m+j+\frac12}(F_j)$, the representation $\mathscr{M}_1: \prod_{j\in M_1} H^{-2m+j+\frac12}(F_j) \to \prod_{j\in M_1} H^{2m-j-\frac12}(F_j)$ (cf. Corollary 6.5) is carried into the representation

$$\mathcal{T}_1 = \Xi\Lambda^* \mathscr{M}_1 \Lambda\Xi$$

acting in $L^2(F^1)$; it clearly satisfies (8.15) and is the restriction of (8.16) with domain (8.17). The involved operators have the following continuity properties

$$H^s(F^1) \xrightarrow{\Lambda\Xi} \prod_{j\in M_1} H^{s-2m+j+\frac12}(F_j) \xrightarrow{\mathscr{M}} \prod_{j\in M_1} H^{s-j-\frac12}(F_j) \xrightarrow{(\Lambda\Xi)^*} H^{s-2m}(F^1) ,$$

so $\mathcal{T}$ is of order $2m$. The last statement is evident in view of Proposition 8.3.

Note that when $\mathcal{C}$ is a given selfadjoint pseudo-differential operator in F^1 of order $\leqslant 2m$, then the « realization » $\mathcal{C}_1$ in $L^2(F^1)$ defined by (8.17) is a selfadjoint operator in $L^2(F^1)$ when $\mathcal{C}$ has a certain regularity; in particular if $\mathcal{C}$ is elliptic of order $s \in]0, 2m]$.

Let us collect some facts about the correspondence between A_B and $\mathcal{C}$. (In 2° and 4°, cf. Remark 7.12 for $n = 1$.)

THEOREM 8.5. *Let A_B, $\mathcal{C}$ and $\mathcal{C}_1$ be defined as in Theorem 8.4.*

1° *For all u, $v \in D(A_B)$*

$$(Au, v) = (Au_\gamma, v_\gamma) + (\mathcal{C}J^{-1}u_\zeta, J^{-1}v_\zeta)\,. \tag{8.18}$$

2° *$D(A_B) \subseteq H^{2m}(E)$ if and only if $\mathcal{C}$ is elliptic of order $2m$. If $\mathcal{C}$ is elliptic of order $s \in]0, 2m]$, then $D(A_B) \subseteq H^s(E)$.*

3° $\dim Z(A_B) = \dim Z(\mathcal{C}_1)$ *and* $\operatorname{codim} R(A_B) = \operatorname{codim} R(\mathcal{C}_1)$.
In case $\dim Z(\mathcal{C}_1) = 0$,

$$A_B^{-1} = A_\gamma^{-1} + J\mathcal{C}_1^{-1}J^* \operatorname{pr}_V\,. \tag{8.19}$$

4° *When A satisfies Assumption 7.4, then $m(A_B) > 0$, $\geqslant 0$ or $> -\infty$ if and only if $m(\mathcal{C}_1) > 0$, $\geqslant 0$ or $> -\infty$, respectively. Here, when $\mathcal{C}$ is elliptic of order $s \in]0, 2m]$, $m(\mathcal{C}_1) > -\infty$ if and only if $\sigma^0(\mathcal{C}) > 0$ on $S(\Gamma)$.*

PROOF. 1° follows from (2.1). 2° follows from Theorem 6.8, carried over from $\mathcal{L}_1$ to the representation $\mathcal{C}_1$. 3° is an immediate consequence of Proposition 1.4, in view of (8.15). 4° similarly follows from Theorem 2.12, together with a wellknown result on elliptic pseudo-differential operators (described e.g. in [12, Appendix]).

THEOREM 8.6. *Let $\{F_j\}_{j\in M_1}$ be any system of vector bundles over Γ with $\dim F_j \leqslant q$, and let B^{11} be any normal (pseudo-)differential operator from $\bigoplus_{k\in M_1} E|_\Gamma$ to $\bigoplus_{j\in M_1} F_j$ (as in Section 6). Then $\{F_j\}_{j\in M_0}$ and $B^{00}\colon \bigoplus_{k\in M_0} E|_\Gamma \to \bigoplus_{j\in M_0} F_j$ may be chosen, uniquely up to isomorphisms, so that $X = Y$. (Similar statement with B^{11} and B^{00} interchanged.) Let $\mathcal{C}$ be any pseudo-differential operator in $F^1 = \bigoplus_{j\in M_1} F_j$ of order $\leqslant 2m$, for which $\mathcal{C}_1$ defined by (8.17) is selfadjoint in $L^2(F^1)$. Then one may choose B^{10} so that A_B defined by (6.7) is selfadjoint and corresponds to $\mathcal{C}$ as in Theorem 8.4.*

PROOF. Follows from Proposition 6.7 and Theorem 7.2.4°. (More details may be found in [17, Section 2.3].)

Concerning the negative spectrum, we have as a direct consequence of Theorem 4.6:

THEOREM 8.7. *Let A be formally selfadjoint satisfying Assumption* 7.4. *Let A_B be selfadjoint with $0 \in \varrho(A_B)$ and A_B^{-1} compact, and let $\mathcal{T}$ and $\mathcal{T}_1$ be as in Theorem* 8.4. *Then*

1° $N^-(A_B; \infty) = N^-(\mathcal{T}_1; \infty)$.

2° *For any $j \in \mathbb{N}$ with $j \leqslant N^-(\mathcal{T}_1; \infty)$,*

$$\lambda_j^-(A_B) \leqslant \lambda_j^-(\mathcal{T}_1) . \tag{8.20}$$

3° $N^-(A_B; t) \leqslant N^-(\mathcal{T}_1; t)$ *for all $t > 0$.*

4° *Let $\mathcal{S}_\Psi = \Xi \Lambda^* \Psi^* \mathcal{S} \Psi \Lambda \Xi$; it is a strongly elliptic selfadjoint pseudo-differential operator in F^1 of order $-2m$, positive with respect to the norm $\|\varphi\|_{-m}$: For any $j \in \mathbb{N}$ with $j \leqslant N^-(\mathcal{S}_\Psi + \mathcal{T}_1^{-1})$,*

$$\lambda_j^-(A_B) \geqslant \mu_j^-(\mathcal{S}_\Psi + \mathcal{T}_1^{-1})^{-1} . \tag{8.21}$$

PROOF. Only 4° requires comments; it is shown by observing that for $v \in V$, $\varphi = J^{-1}v$,

$$\begin{aligned}(S_V v, v) = (A_\gamma^{-1} v, v) = \langle \mathcal{S}\gamma v, \gamma v\rangle \qquad &\text{(cf. Proposition 8.1)}\\ = \langle \mathcal{S}\Psi\Lambda\Xi\varphi, \Psi\Lambda\Xi\varphi\rangle = (\mathcal{S}_\Psi\varphi, \varphi) ,&\end{aligned}$$

so $\mathcal{S}_\Psi + \mathcal{T}_1^{-1}$ is the operator in $L^2(F^1)$ derived from the operator $S_V + T^{-1}$ in V by the isometry J^{-1}.

We shall now prove a consequence of these theorems, that improves the previously known asymptotic estimates on $N^-(A_B; t)$. Let us first recall the known results.

Let A be strongly elliptic and formally selfadjoint. When A_B is a lower bounded selfadjoint elliptic realization of A, then the eigenvalues satisfy

$$N^+(A_B; t) - c(A)\, t^{n/2m} = \mathcal{O}(t^{(n-\theta)/2m}) \qquad \text{for } t \to \infty, \tag{8.22}$$

with any $\theta < \frac{1}{2}$ (Agmon [3]), and with any $\theta < 1$ if the eigenvalues of $\sigma^0(A)\cdot$ $\cdot(x, \xi)$ are simple (consequence of Hörmander [19] by standard arguments); here $c(A)$ is a constant derived from A like in (8.25) below. When A_B is not lower bounded, the negative spectrum is also infinite and, according to a statement in Agmon [3], (8.22) holds and

$$N^-(A_B; t) = O(t^{(n-\theta)/2m}) \quad \text{for } t \to \infty \tag{8.23}$$

with $\theta < \frac{1}{2}$; it seems plausible that Hörmander [19] implies the validity for $\theta < 1$ when the eigenvalues of $\sigma^0(A)$ are simple.

We shall show (independently of these assertions) that in fact

$$N^-(A_B; t) \leqslant c^-(A_B)\, t^{(n-1)/2m} + o(t^{(n-1)/2m}) \tag{8.24}$$

for $t \to \infty$, see the precise statements below. For this we shall use an estimate like (8.22) pertaining to pseudo-differential operators on the compact manifold without boundary Γ; recall that it is of dimension $n-1$ (which we assume *positive* in the following). Hörmander and Seeley proved

PROPOSITION 8.8. *Let P be a strongly elliptic, selfadjoint pseudo-differential operator of order $r > 0$ in a vector bundle F over Γ. Let*

$$c(P) = (2\pi)^{1-n} \int_\Gamma dy \int_{|\eta|=1} \Sigma \lambda_j(\sigma^0(P)(y,\eta))^{(1-n)/r}\, d\eta, \tag{8.25}$$

where the sum is over the eigenvalues of $\sigma^0(P)(y,\eta)$ at each $(y,\eta) \in S(\Gamma)$, the cotangent sphere bundle. Then

$$N^+(P; t) - c(P)\, t^{(n-1)/r} = R(t) \quad \textit{for } t \to \infty, \tag{8.26}$$

where $R(t) = O(t^{(n-2)/r})$ if the eigenvalues of $\sigma^0(P)(y,\eta)$ are simple (Hörmander [19]) and $R(t) = o(t^{(n-1)/r})$ in general (Seeley [26]).

This is extended to the indefinite case as follows:

PROPOSITION 8.9. *Let P be an elliptic selfajoint invertible pseudo-differential operator of order $r > 0$ in a vectorbundle F over Γ. Let*

$$c^\pm(P) = (2\pi)^{1-n} \int_\Gamma dy \int_{|\eta|=1} \Sigma |\lambda_j^\pm(\sigma^0(P)(y,\eta))|^{(1-n)/r}\, d\eta\,, \tag{8.27}$$

where the sum is over the positive, resp. over the negative, eigenvalues of $\sigma^0(P)(y,\eta)$. Then

$$N^\pm(P; t) - c^\pm(P)\, t^{(n-1)/r} = R(t)\,, \quad \textit{for } t \to \infty\,, \tag{8.28}$$

where $R(t) = O(t^{(n-2)/r})$ if the eigenvalues of $\sigma^0(P)(y,\eta)$ are simple and $n > 2$, and $R(t) = o(t^{(n-1)/r})$ in general ($n > 1$).

PROOF. We use the calculus established in Seeley [26]. Let $|P| = (P^2)^{\frac{1}{2}}$, it is a positive, selfadjoint, elliptic pseudo-differential operator in F of order r. Let

$$(8.29) \qquad P^+ = \tfrac{1}{2}(P + |P|), \quad P^- = \tfrac{1}{2}(P - |P|);$$

so in particular

$$(8.30) \qquad \sigma^0(P^+) = \tfrac{1}{2}(\sigma^0(P) + \sigma^0(|P|)), \quad \sigma^0(P^-) = \tfrac{1}{2}(\sigma^0(P) - \sigma^0(|P|)).$$

Then P^+ acts like P on the positive eigenspace of P, and is zero on the negative eigenspace, whereas P^- acts like P on the negative eigenspace of P and is zero on the positive eigenspace; similar statements hold for $\sigma^0(P^+)(y, \eta)$ and $\sigma^0(P^-)(y, \eta)$. Now choose $0 < a < 1$ so that

$$P_a = |P| + aP \quad \text{and} \quad P_{-a} = |P| - aP$$

(which are positive elliptic) have simple eigenvalues in the principal symbol if P has; this may be done since Γ is compact. Then in fact

$$(8.31) \quad P_a = (1 + a)P^+ - (1 - a)P^-, \quad P_{-a} = (1 - a)P^+ - (1 + a)P^-,$$

and it follows from (8.25), (8.27) and (8.30) that

$$(8.32) \qquad \begin{cases} c(P_a) = (1 + a)^{(1-n)/r} c^+(P) + (1 - a)^{(1-n)/r} c^-(P), \\ c_{-a}(P) = (1 - a)^{(1-n)/r} c^+(P) + (1 + a)^{(1-n)/r} c^-(P). \end{cases}$$

Now (8.31) implies that for all $t > 0$,

$$(8.33) \qquad \begin{cases} N^+((1 + a)P_a; t) = N^+((1 + a)^2 P^+; t) + N^-((1 + a)(1 - a)P^-; t), \\ N^+((1 - a)P_{-a}; t) = N^+((1 - a)^2 P^+; t) + N^-((1 - a)(1 + a)P^-; t), \end{cases}$$

from which we obtain for P^+:

$$\begin{aligned}
(8.34) \qquad & N^+((1 + a)^2 P^+; t) - N^+((1 - a)^2 P^+; t) \\
&= N^+((1 + a)P_a; t) - N^+((1 - a)P_{-a}; t) \\
&= c(P_a)\left(\frac{t}{1 + a}\right)^{(n-1)/r} - c(P_{-a})\left(\frac{t}{1 - a}\right)^{(n-1)/r} + R_1(t) \\
&= [c(P_a)(1 + a)^{(1-n)/r} - c(P_{-a})(1 - a)^{(1-n)/r}]\, t^{(n-1)/r} + R_1(t) \\
&= [(1 + a)^{2(1-n)/r} - (1 - a)^{2(1-n)/r}]\, c^+(P)\, t^{(n-1)/r} + R_1(t),
\end{aligned}$$

by use of (8.26) and (8.32); $R_1(t)$ is $\mathcal{O}(t^{(n-2)/r})$ or $o(t^{(n-1)/r})$ according to whether the eigenvalues of $\sigma^0(P)(y, \eta)$ are simple or not. Now write

$$N^+(P^+; t) = c^+(P)\, t^{(n-1)/r} + g(t) \tag{8.35}$$

(where $g(t)$ is to be determined), and insert this in (8.34); then we find

$$g\big(t(1+a)^{-2}\big) - g\big(t(1-a)^{-2}\big) = R_1(t) \quad \text{for } t > 0 .$$

By an application of Lemma 8.10 below (with $c = (1-a)^{-2}(1+a)^2$ and $q = (n-2)/r$ resp. $(n-1)/r$), we conclude that

$$g(t) = \mathcal{O}(t^{(n-2)/r}) \quad \text{resp. } o(t^{(n-1)/r}) \text{ for } t \to \infty ,$$

which proves the proposition for $N^+(P; t)$. The proof for $N^-(P; t)$ is analogous.

LEMMA 8.10. *Let $g(t)$ be a locally bounded function on $[0, \infty[$ satisfying, for some $c > 1$, $q > 0$,*

$$|g(ct) - g(t)| \leqslant h(t)\, t^q \quad \textit{for } t \in [0, \infty[,$$

where $h(t)$ is bounded. Then $g(t)t^{-q}$ is bounded. If $h(t) \to 0$ for $t \to \infty$, then $g(t)\,t^{-q} \to 0$ for $t \to \infty$.

PROOF. Let $k = \sup\{|g(t)|\,|\, t \in [0, 1]\}$. Then for any $s \in\,]0, 1]$, any $n \in \mathbf{N}$,

$$\begin{aligned} |g(c^n s)| &\leqslant |g(c^n s) - g(c^{n-1} s)| + |g(c^{n-1} s) - g(c^{n-2} s)| + \ldots + |g(s)| \\ &\leqslant (c^n s)^q\, [h(c^{n-1} s)\, c^{-q} + h(c^{n-2} s)\, c^{-2q} + \ldots + h(s)\, c^{-nq}] + k \\ &\leqslant (c^n s)^q\, M c^{-q} (1 - c^{-q})^{-1} + k , \end{aligned}$$

where $M = \sup\{|h(t)|\,|\, t \in [0, \infty[\}$. This gives the first part of the lemma, by setting $t = c^n s$. For the second part, let $s \in\,]0, c^{-1}[$, and define

$$M(p, s) = \{\sup |h(t)|\,|\, t \geqslant c^p s\} ,$$

then for any positive integers $p \leqslant n$,

$$|g(c^n s)| \leqslant (c^n s)^q\, [M(p, s)\, c^{-q} (1 - c^{-q})^{-1} + c^{-(n-p)q} M (1 - c^{-q})^{-1}] + k .$$

For given $\varepsilon > 0$, we may choose p so large that $M(p, s)\, c^{-q}(1-c^{-q})^{-1} < \frac{1}{2}\varepsilon$, and choose $n(\geqslant p)$ so large that $c^{-(n-p)q} M(1-c^{-q})^{-1} < \frac{1}{2}\varepsilon$, then the expression in [ ] is $< \varepsilon$ for $s' \geqslant s$. This implies the second part of the lemma, by setting $t = c^n s'$.

Applying Proposition 8.9 to $\mathfrak{C}$, we now find the special consequences of Theorem 8.7:

THEOREM 8.11. *Let A be formally selfadjoint satisfying Assumption* 7.4, *and let A_B be a selfadjoint realization defined by a boundary condition* (6.7). *Let $\mathfrak{C}$ be the pseudo-differential operator in F^1 derived from A_B by Theorem* 8.4. *Assume $n > 1$.*

1° *If A_B is elliptic* (*i.e.* $D(A_B) \subseteq H^{2m}(E)$), *then A_B has infinitely many negative eigenvalues if and only if $c^-(\mathfrak{C}) \neq 0$, and then*

(8.36) $$N^-(A_B; t) \leqslant c^-(\mathfrak{C})\, t^{(n-1)/2m} + R(t)\,,$$

where $R(t) = O(t^{(n-2)/2m})$ for $t \to \infty$ if the eigenvalues of $\sigma^0(\mathfrak{C})(y, \eta)$ are simple (*in particular if* $\sum_{j \in M_1} p_j = 1$) *and $n > 2$, and $R(t)$ is $o(t^{(n-1)/2m})$ in general.*

2° *If $D(A_B) \subseteq H^s(E)$ for some $s \in\,]0, 2m[$, then there exists a constant $c > 0$ so that*

(8.37) $$N^-(A_B; t) \leqslant c t^{(n-1)/s} \quad \text{for } t \in\,]0, \infty[\,.$$

PROOF. 1° follows from Theorem 8.5.2° and 4°, Theorem 8.7.3° and Proposition 8.9, by using that $\mathfrak{C}$ is elliptic in F^1 of order $2m$. For 2°, we use that $D(A_B) \subseteq H^s(E)$ implies $D(\mathfrak{C}_1) \subseteq H^s(F^1)$, so that, by a theorem of Paraska [23] (improving results of Agmon [2]), $N^-(\mathfrak{C}_1; t) \leqslant \text{const. } t^{(n-1)/s}$, which gives (8.37) by Theorem 8.7.3°.

Note that we have as a special case of 2°, that when $\mathfrak{C}$ is elliptic of order $s \in\,]0, 2m[$, then

(8.38) $$N^-(A_B; t) \leqslant c^-(\mathfrak{C})\, t^{(n-1)/s} + R(t)\,,$$

with $R(t) = O(t^{(n-2)/s})$ or $o(t^{(n-1)/s})$ as usual.

In the converse direction we find:

THEOREM 8.12. *Let A be formally selfadjoint satisfying Assumption* 7.4. *Let B^{11}* (*or B^{00}*) *be given arbitrarily, and choose B^{00}* (*or B^{11}*), *so that $X = Y$. Assume $n > 1$ and let $s \in\,]0, 2m]$. For any $c > 0$ we may choose B^{10}* (*pseudo-*

differential) so that the realization A_B defined by (6.7) *is selfadjoint with* $D(A_B) \subseteq \subseteq H^s(E)$, *and satisfies*

$$N^-(A_B; t) \geq ct^{(n-1)/s} + O(t^{(n-2)/s}) \quad \textit{for } t \to \infty, \tag{8.39}$$

and moreover, if $s < 2m - 2m/n$,

$$N^-(A_B; t) - ct^{(n-1)/s} = o(t^{(n-1)/s}) \quad \textit{for } t \to \infty. \tag{8.40}$$

PROOF. Given $s \in]0, 2m]$ and $c > 0$. Let P be an elliptic selfadjoint pseudo-differential operator in F^1 of order $-s$, positive w.r.t. $\|\varphi\|_{-s/2}$, with simple eigenvalues in $\sigma^0(P)(y, \eta)$, and with $c(P^{-1}) = c$. Then $\mathcal{S}_\varphi + P$ is elliptic selfadjoint of order $-s$ (since $-s \geq -2m$) and positive in the above sense, so we may define

$$\mathcal{C} = -(\mathcal{S}_\varphi + P)^{-1};$$

it is elliptic of order $+s$, so $D(\mathcal{C}_1) = H^s(F^1)$, and

$$\mathcal{S}_\varphi + \mathcal{C}_1^{-1} = -P \quad \text{on } L^2(F^1). \tag{8.41}$$

For A_B (defined according to Theorem 8.6), we then get by Theorem 8.7.4°

$$\lambda_j^-(A_B) \geq \mu_j^-(\mathcal{S}_\varphi + \mathcal{C}_1^{-1})^{-1} = \lambda_j^-(-P^{-1}),$$

for all $j \in \mathbf{N}$, whence by Proposition 8.8

$$N^-(A_B; t) \geq N^-(-P^{-1}; t) = ct^{(n-1)/s} + O(t^{(n-2)/s})$$

for $t \to \infty$. This proves (8.39).

Assume now furthermore that $s < 2m - 2m/n$; then $(n-1)/s > n/2m$. By (8.19), (8.41),

$$\begin{aligned} A_B^{-1} &= A_\gamma^{-1} + J\mathcal{C}_1^{-1}J^* \operatorname{pr}_V \\ &= A_\gamma^{-1} - J\mathcal{S}_\varphi J^* \operatorname{pr}_V - JPJ^* \operatorname{pr}_V \\ &= C - JPJ^* \operatorname{pr}_V, \end{aligned}$$

where the operator C is continuous from $L^2(E)$ into $H^{2m}(E)$, so that the eigenvalues satisfy $\mu_j(|C|) \leq \text{const.}\, j^{-2m/n}$ (Paraska [23]). The positive eigenvalues of the nonnegative operator $C_1 = JPJ^* \operatorname{pr}_V$ satisfy

$$\mu_j^+(C_1) = \mu_j^+(P) \sim c_0 j^{-s/(n-1)} \quad \text{for } j \to \infty,$$

where $c_0 = c^{-s/(n-1)}$, by Proposition 8.8. Since $-s/(n-1) > -2m/n$, a theorem of Ky Fan [20] implies that

$$\mu_j^+(|A_B^{-1}|) \sim c_0 j^{-s/(n-1)} \quad \text{for } j \to \infty ,$$

which gives

$$N^-(A_B; t) \leqslant N^+(|A_B|; t) = ct^{(n-1)/s} + o(t^{(n-1)/s}) .$$

Together with (8.39) this shows (8.40). (Similar arguments seem to lie behind M. Gehtman's estimate of positive eigenvalues for certain lower bounded realizations of the Laplacian [10]).

Without doubt it would be worthwhile to scan the literature for further useful perturbation theorems and apply them to the correspondence between A_B and $\mathcal{T}$, by use of Theorems 8.5-8.7 etc.

REMARK 8.13. When A_B is a nonselfadjoint realization of a formally selfadjoint A, and $X = Y$, we still have the key formulae (8.18) and (8.19), to which perturbation theorems may be applied. When A itself is nonselfadjoint, we get partial information from the formula, valid when $D(A_B) \subseteq \subseteq D(A_1^r)$ and $X = Y$:

$$\operatorname{Re}(Au, u) = (A^r u_\gamma^r, u_\gamma^r) + (\mathcal{T}^r J^{-1} u_\zeta^r, J^{-1} u_\zeta^r) , \tag{8.42}$$

where $\mathcal{T}^r$ is the $2m$ order pseudo-differential operator

$$\mathcal{T}^r = \Xi \Lambda^* (\mathcal{M} + \Psi^* Q \Psi) \Lambda \Xi$$

in F^1 (Λ and Ξ as in the text preceding Proposition 8.3, constructed relative to A^r), and J is the *isometry* $J = (\gamma_z^r)^{-1} \Psi \Lambda \Xi$ of $L^2(F^1)$ onto $(\gamma_z^r)^{-1} X$ ($\subset L^2(E)$). (In fact, $(\mathcal{T}^r J^{-1} u_\zeta^r, J^{-1} u_\zeta^r) = \operatorname{Re}[(Tu_\zeta, u'_\zeta) + (Au_\zeta^r, \operatorname{pr}'_\gamma u_\zeta^r)]$, cf. Remark 3.7, from which $\mathcal{T}^r$ and J are found by Corollary 6.5 and Propositions 7.6 and 8.3.)

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