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ELLIPTIC AND DEGENERATE-ELLIPTIC OPERATORS IN UNBOUNDED DOMAINS

D. E. EDMUNDS and W. D. EVANS

1. Introduction.

It is well known that elliptic boundary value problems in unbounded domains present difficulties which are, on the whole, more severe than those encountered in the study of similar problems in bounded domains. Sometimes these difficulties can be overcome by relatively direct means, as in the elegant work of Meyers and Serrin [17] on the exterior Dirichlet and Neumann problems for second order equations with continuous coefficients, and in the use of inversion techniques developed by Serrin and Weinberger [24] for the same kind of problem. However, there remain considerable obstacles in the way of a treatment of unbounded domain problems involving, say, uniformly elliptic equations with possibly discontinuous coefficients and an important component in the difficulties that present themselves is the lack of compact embedding theorems such as that of Rellich for bounded domains. These theorems are of vital importance in the case of problems on bounded domains: in suitable linear elliptic equations, for example, they enable the whole theory of compact linear operators in a Banach space to be brought into play, while various nonlinear problems may be handled by an application of the Leray-Schauder degree theory.

In two recent papers [5], [6], Berger and Schechter have extended the Sobolev-Kondrachov compactness and embedding theorems to the case of unbounded domains, and have applied these results to, *inter alia*, the Dirichlet problem for quasilinear elliptic equations in an unbounded domain. What they do is to identify a class M of functions such that multiplication by one such function is a continuous, or even compact, map from a certain Sobolev space to an appropriate L^p space: the application of this idea to the Dirichlet

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problem for quasilinear elliptic equations in an unbounded domain involves the imposition of conditions related to the class M on lower order terms in the differential operators which mean that these terms induce a compact map.

In the present paper we consider weighted Sobolev spaces on unbounded domains, and prove embedding and compactness theorems analogous to those derived by Berger and Schechter for the corresponding unweighted spaces. These results are applied to the study of the existence of solutions of the Dirichlet problem in an unbounded domain for quasilinear and linear equations of order $2k$ which are degenerate-elliptic in the sense of Murthy and Stampacchia [18]: in particular we are able to deal with linear equations of the form

$$-\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)+b_i\frac{\partial u}{\partial x_i}+cu=f,$$

where

$$a_{ij}(x)\,\xi_i\,\xi_j\geq m(x)\,\xi_i\,\xi_i$$

for all x in the domain and all $\xi=(\xi_i)$ in R^n. Here m is a non-negative function which is locally integrable, and such that $1/m$ satisfies certain integral growth conditions. Indeed for linear equations we are able to provide a fairly detailed discussion of the situation, giving results on existence which are close to those derived by Murthy and Stampacchia for second order equations in bounded domains, and also providing information about the essential spectrum of the maximal operator induced by the equation. We include results which are believed to be new even for the particular case of uniformly elliptic linear equations in unbounded domains such as an infinite cylinder or strip.

The plan of the paper is as follows. In § 2 there is a brief discussion of the function spaces that are needed, together with an outline of the theory of k-set contractions. The next section, § 3, is the heart of the paper, and contains the various embedding theorems for weighted Sobolev spaces that are required in the applications. These results extend those of Berger and Schechter for unweighted spaces, but in addition by singling out a particular class of domains we are able to gain new and rather precise information about various embedding maps. Thus let Ω be an unbounded domain in R^n, and denote by $B(x,d)$ the closed ball in R^n with centre x and radius d. Suppose that $p>1$ and that for some d, $0<d\leq 1$,

$$(*)\qquad \inf_{x\in\Omega}\left[\text{meas}\,(B(x,d)\setminus\Omega)/\text{meas}\,B(x,d)\right]=\delta.$$

Then we show that the norm of the natural embedding of $H_0^{1,p}(\Omega)$ in $L^p(\Omega)$ is no greater than $(1-\delta)^{1/n}$.

If instead we suppose that

$$\delta = \lim_{|x|\to\infty,}\ \inf_{x\in\Omega}\ [\text{meas}\,(B(x,d)\setminus\Omega)/\text{meas}\ B(x,d)]$$

it turns out that the natural embedding of $H_0^{1,p}(\Omega)$ in $L^p(\Omega)$ is a $(1-\delta)^{1/n}$-set contraction; and that the Poincaré inequality holds in $H_0^{k,p}(\Omega)$, where k is any positive integer, provided $\delta > 0$ for some $d > 0$. It follows that in particular the natural embedding of $H_0^{1,p}(\Omega)$ in $L^p(\Omega)$ is compact if $\lim_{|x|\to\infty,\,x\in\Omega} \text{meas}\,(B(x,1)\cap\Omega) = 0$, a result established in [6].

Section 4 deals with the Dirichlet problem for a quasilinear elliptic (or degenerate elliptic) equation of order $2k$, and proceeds by reducing the problem to an abstract one invoving a special kind of pseudo-monotone map. In § 5 the corresponding linear problem is investigated, and various results on existence are given. A number of spectral results are provided: in particular we show that provided (*) holds for some $\delta > 0$, the spectrum of the maximal operator in $L^2(\Omega)$ induced by $-\Delta$ (Δ is the Laplace operator) and $H_0^{1,2}(\Omega)$ is contained in the interval $[(1-\eta^{1/n})^2\,\eta^{-2/n},\infty)$, where $\eta = 1-\delta$. This result would apply in particular to suitable cylindrical domains.

The paper concludes with a brief discussion of future possible developments.

2. Prerequisites.

2.1. Let Ω be an unbounded domain in n dimensional Euclidean space R^n, denote by $\partial\Omega$ and $\overline{\Omega}$ the boundary and closure respectively of Ω, and represent points of R^n by $x = (x_1, x_2, \dots, x_n)$. Let m be a non negative function on Ω which is locally Lebesgue integrable on Ω. For $1 < p < \infty$ we shall let $L^p(\Omega, m)$ stand for the linear space of (equivalence classes of) complex-valued functions u on Ω which are measurable with respect to the measure $m(x)\,dx$ and which satisfy

$$\|u\|_{0,p,m} \equiv \left\{\int_\Omega |u(x)|^p\, m(x)\, dx\right\}^{1/p} < \infty.$$

When $m(x) \equiv 1$ on Ω the subscript m on $\|\cdot\|_{0,p,m}$ will be omitted.

Evidently $L^p(\Omega, m)$ becomes a Banach space when furnished with the norm $\|\cdot\|_{0,p,m}$, and the space $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω is dense in $L^p(\Omega, m)$. Moreover, $L^2(\Omega, m)$ is a Hilbert space with inner product

$$(u, v)_{0,2,m} \equiv \int_\Omega u(x)\, \overline{v(x)}\, m(x)\, dx.$$

Given any positive integer k we shall denote by $C_0^k(\Omega)$ the space of all complex-valued functions which are k times continuously differentiable on Ω and have compact support in Ω; by $H_0^{k,p}(\Omega, m)$ we shall mean the completion of $C_0^k(\Omega)$ endowed with the norm

$$\| u \|_{k,p,m} = \sum_{i=0}^{k} \| D^i u \|_{0,p,m}.$$

Here $| D^i u(x) |^2 = (\sum_{|\alpha|=i} | D^\alpha u(x) |^2)^{1/2}$, where the summation extends over all n-tuples $\alpha = (\alpha_1; \alpha_2, \dots, \alpha_n)$ of non negative integers with $|\alpha| \equiv \alpha_1 + \dots + \alpha_n = i$, and $D^\alpha u = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} u$. The case $p = 2$ is again a special one, for $H_0^{k,2}(\Omega, m)$ becomes a Hilbert space if it is given the inner product

$$(u, v)_{k,2,m} \equiv \sum_{i=0}^{k} \sum_{|\alpha|=i} (D^\alpha u, D^\alpha v)_{0,2,m},$$

and the norm

$$(u, u)_{k,2,m}^{1/2} = \left\{ \sum_{i=0}^{k} \| D^i u \|_{0,2,m}^2 \right\}^{1/2}$$

induced by this inner product is clearly equivalent to the norm $\| u \|_{k,2,m}$, since

$$(u, u)_{k,2,m}^{1/2} \leq \| u \|_{k,2,m} \leq (k+1)^{1/2} (u, u)_{k,2,m}^{1/2}.$$

If $m^{-1} \in L_{\text{loc}}^t(\Omega)$ for some $t > 1$ and $p > 1 + \dfrac{1}{t}$, then $H_0^{k,p}(\Omega, m)$ is a subspace of the space $\mathcal{D}'(\Omega)$ of distributions on Ω, and its topology is finer than the one induced by $\mathcal{D}'(\Omega)$. In other words, $H_0^{k,p}(\Omega, m) \subset \mathcal{D}'(\Omega)$, and the inclusion map is continuous. For, if $u \in H_0^{k,p}(\Omega, m)$ and $\emptyset \in C_0^\infty(\Omega)$ we have, by Hölder's inequality,

$$\left| \int_\Omega u(x)\, \Phi(x)\, dx \right| \leq \| u \|_{0,p,m} \| \emptyset \|_{0,q,1} \left\{ \int_{\text{supp}\, \emptyset} m^{-t}(y)\, dy \right\}^{1/t},$$

where $\frac{1}{q} = 1 - \frac{1}{p} - \frac{1}{pt}$. From this the continuity of the inclusion map follows.

Also if $(\emptyset_n)$ is a sequence in $C_0^\infty(\Omega)$ which converges to u in $H_0^{k,p}(\Omega, m)$, then for each α with $|\alpha| \leq k$ the sequence $(D^\alpha \emptyset_n)$ converges to the distributional derivative $D^\alpha u$ of u in $L^p(\Omega, m)$. To see this we remark that $\emptyset_n \to u$ in $L^p(\Omega, m)$, and for $0 < |\alpha| \leq k$ the sequences $(D^\alpha \emptyset_n)$ are Cauchy sequences in $L^p(\Omega, m)$ and hence converge to limits u_α, say. However, if B is a bounded subset of $C_0^\infty(\Omega)$ the members of which all have supports in a compact set K then, as above, for all $\emptyset$ in B,

$$|\langle D^\alpha \emptyset_n - u_\alpha, \emptyset \rangle| \leq \| D^\alpha \emptyset_n - u_\alpha \|_{0,p,m} \| \emptyset \|_{0,2,1} \left\{ \int_K m^{-t}(y)\, dy \right\}^{1/t}.$$

Hence $D^\alpha \Phi_n \to u_\alpha$ in $\mathcal{D}'(\Omega)$, so that u_α is the distributional derivative $D^\alpha u$ of u.

The dual of $L^p(\Omega, m)$ is isomorphic to $L^{p'}(\Omega, m)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, the duality being defined by

$$\langle f, g \rangle = \int_\Omega f(x)\, \overline{g(x)}\, m(x)\, dx$$

for $f \in L^p(\Omega, m)$ and $g \in L^{p'}(\Omega, m)$. As is customary we shall denote the dual of $H_0^{k,p}(\Omega, m)$ by $H^{-k,p'}(\Omega, m)$. Since $C_0^\infty(\Omega)$ is dense in $H_0^{k,p}(\Omega, m)$, $H^{-k,p'}(\Omega, m)$ can be identified with a space of distributions, and

$$C_0^\infty(\Omega) \subset H^{-k,p'}(\Omega, m) \subset \mathcal{D}'(\Omega),$$

where the inclusion maps are continuous. For a fuller discussion of the above spaces and their duals, when Ω is bounded, we refer to [18].

Our subsequent discussion will also involve the 'potential spaces' $H^{s,p}(R^n)$, where s is any positive real number. These are defined as follows (see [6], [16], [25]). Let G_s be the function on R^n whose Fourier transform $\widehat{G}_s$ is given by

$$\widehat{G}_s(\xi) = (1 + |\xi|^2)^{-s/2}.$$

Then for $s > 0$, $G_s \in L^1(R^n)$ and we can define a linear map $\mathcal{G}_s : L^p(R^n) \to L^p(R^n)$ by $\mathcal{G}_s(f) = G_s * f$ (the convolution of G_s and f) for f in $L^p(R^n)$. The maps $\mathcal{G}_s$, which are in fact bounded from $L^p(R^n)$ to itself, are the so-called Bessel potentials and represent the fractional operators $(-\Delta + I)^{-s/2}$, where Δ is the Laplacian operator in R^n and I is the identity map. The space $H^{s,p}(R^n)$ is then defined to be $\mathcal{G}_s(L^p(R^n))$: in other words, $f \in H^{s,p}(R^n)$ if and only if $f = \mathcal{G}_s(g) = G_s * g$ for some $g \in L^p(R^n)$. The topology on $H^{s,p}(R^n)$ is defined by the norm

$$\|f\|_{s,p} = \|g\|_{0,p} = \left(\int_{R^n} |g(x)|^p \, dx\right)^{1/p}$$

where g is the (unique) element of $L^p(R^n)$ such that $f = \mathcal{G}_s(g)$. Since we have, for $f = \mathcal{G}_s(g) \in H^{s,p}(R^n)$,

$$\|f\|_{0,p} = \|\mathcal{G}_s(g)\|_{0,p} \leq K \|g\|_{0,p} = K \|f\|_{s,p},$$

where K is constant independent of f, it follows that $H^{s,p}(R^n) \subset L^p(R^n)$, the inclusion map being continuous. It can be shown that $\mathcal{G}_s$ maps the space $\mathcal{S}$ of rapidly decreasing functions onto itself and hence, since $\mathcal{S}$ is dense in $L^p(R^n)$, the space $H^{s,p}(R^n)$ can be regarded as the completion of $\mathcal{S}$ endowed with the norm $\|\cdot\|_{s,p}$. In fact, $\mathcal{G}_s$ maps $C_0^\infty(R^n)$ onto itself ([23], p. 48), and so $H^{s,p}(R^n)$ is the completion of $C_0^\infty(R^n)$ under the norm $\|\cdot\|_{s,p}$. We also have, from $f = G_s * g$,

$$\widehat{f}(\xi) = \widehat{G}_s(\xi)\,\widehat{g}(\xi) = (1 + |\xi|^2)^{-s/2}\,\widehat{g}(\xi),$$

and so, letting $\mathcal{F}^{-1}$ stand for the inverse Fourier transformation,

$$g(x) = \mathcal{F}^{-1}\{(1 + |\xi|^2)^{s/2}\widehat{f}(\xi)\}.$$

Hence

$$\|f\|_{s,p} = \left\{\int_{R^n} |\mathcal{F}^{-1}\{(1 + |\xi|^2)^{s/2}\widehat{f}(\xi)\}(x)|^p \, dx\right\}^{1/p}$$

(see [6]).

From the definition of $\mathcal{G}_s$ it also follows that for $s \geq 0$, $t \geq 0$, and with the convention that $\mathcal{G}_0 f = f$,

$$\mathcal{G}_s \, \mathcal{G}_t = \mathcal{G}_{s+t},$$

from which we see that

$$H^{s,p}(R^n) \subset H^{t,p}(R^n) \qquad \text{if } s > t,$$

the inclusion map being continuous.

When s is a positive integer, $H^{s,p}(R^n)$ is identical with $H_0^{s,p}(R^n, 1)$ (cf. [25], chapter V, § 3.3).

Given any domain Ω in R^n, $H^{s,p}(\Omega)$ is defined to be the set of functions u which are restrictions to Ω of functions in $H^{s,p}(R^n)$. Endowed with the norm

$$\| u \|_{s,p}^{\Omega} = \inf \| v \|_{s,p},$$

where the infimum is taken over all v in $H^{s,p}(R^n)$ such that $v = u$ on Ω, $H^{s,p}(\Omega)$ is a Banach space. We shall denote by $H_0^{s,p}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in this norm. The superscript Ω in $\|\cdot\|_{s,p}^{\Omega}$ will usually be omitted, as the domain will be obvious from the context. For a discussion of these and other similar spaces we refer to [6].

2.2. Some of our results will involve the notion of a k-set contraction, which in turn rests on a measure of non-compactness of a set. Let Ω be a bounded subset of a real or complex Banach space X: the *measure of non-compactness of* Ω, $\gamma(\Omega)$, is defined by

$$\gamma(\Omega) = \inf\{\mathcal{E} > 0 : \Omega \text{ may be covered by finitely many sets of diameter} \leq \mathcal{E}\}.$$

The measure $\gamma(\Omega)$ was first introduced by Kuratowski [13] in a metric space setting: it is referred to as a measure of non-compactness because evidently Ω is relatively compact if and only if $\gamma(\Omega) = 0$.

Now let Y be another (real or complex) Banach space, and let k be a non-negative real number. A continuous map $T : X \to Y$ is said to be a *k-set contraction* if $\gamma(T(\Omega)) \leq k\,\gamma(\Omega)$ for every bounded subset Ω of X. To be absolutely precise we should, of course, write $\gamma_X(\Omega)$ and $\gamma_Y(T(\Omega))$ in order to distinguish between the two measures of non-compactness, but no ambiguity will arise from our abuse of terminology. It is clear that T is a *compact* map, that is to say a continuous map which takes bounded sets into relatively compact ones, if and only if it is a 0-set contraction. It is moreover not difficult to show that the sum of a k_1-set contraction T_1 and a k_2-set contraction T_2 is a $(k_1 + k_2)$-set contraction, and that in circumstances when the composition $T_1 \circ T_2$ is defined, it is a $k_1 k_2$-set contraction: these properties facilitate the construction of a wide variety of k-set contractions. Should T be linear, we define $\gamma(T)$ to be $\inf\{k : T \text{ is a } k\text{-set contraction}\}$: note that $\gamma(T) \leq \| T \|$, and that in many cases the inequality is strict, as for compact maps, for example. Also it can be shown that $\gamma(T_1 \circ T_2) \leq \gamma(T_1)\,\gamma(T_2)$. There are various relationships between $\gamma(T)$ and $\gamma(T^*)$, where T^* is the adjoint of T. The one that we shall need here is due to Webb [27], and states that $\gamma(T) = \gamma(T^*)$ if X and Y are both Hilbert spaces.

Lastly, we shall need to discuss the *essential spectrum* $\operatorname{ess}(T)$ of a bounded linear map T from a complex Banach space X to itself. The definition of $\operatorname{ess}(T)$ which we shall use is that adopted by Kato ([12], p. 243): according to him, $\operatorname{ess}(T)$ is the complement of the set Δ of all complex numbers λ such that $T-\lambda I$ is semi-Fredholm, so that $\operatorname{ess}(T)$ is the subset of the spectrum $\sigma(T)$ of T which consists of all those complex numbers λ such that either the range $R(T-\lambda I)$ of $T-\lambda I$ is not closed or $R(T-\lambda I)$ is closed but $\dim\ker(T-\lambda I)=\dim\{X/R(T-\lambda I)\}=\infty$ ($\ker S$ is the kernel of S). The set Δ is the union of components (connected open sets) in each of which the index $i(T-\lambda I)\equiv\dim\ker(T-\lambda I)-\dim\{X/R(T-\lambda I)\}$ is constant (see, for example, [12], Chap. IV, Theorem 5.17). The complement of the union of all those components containing points of the resolvent of T is the set taken by Browder [7] to be the essential spectrum of T. Evidently Browder's definition gives a set which contains that arising from Kato's definition, but Lebow and Schechter ([14], Theorem 6.5) have shown that whichever definition is used, the radius of the essential spectrum, $r_e(T)\equiv\sup\{|\lambda|:\lambda\in\operatorname{ess}(T)\}$, is the same.

The essential spectrum was linked up in a rather striking way with the theory of k-set contractions by Nussbaum [19], who showed that

$$r_e(T)=\lim_{n\to\infty}(\gamma(T^n))^{1/n}.$$

It follows that $r_e(T)\leq\gamma(T)$: in fact Stuart [26] has shown that $r_e(T)=\gamma(T)$ if X is a Hilbert space and T is normal, while Webb [27] has proved that the same is true for semi-normal operators. It can be shown (see Nussbaum [19]) that if $|\lambda|>r_e(T)$, $\lambda I-T$ is a Fredholm operator of index zero, so that in particular $I-T$ is a Fredholm map of index zero if T is a k-set contraction for some $k<1$.

For a comprehensive treatment of k-set contractions, including proofs of various of the assertions made above, we refer to Nussbaum [20].

3. Embedding theorems.

3.1. Given any positive real number α we define a function w_α on R^n by the rule that

$$w_\alpha(x)=\begin{cases}|x|^{\alpha-n} & \text{if } \alpha<n,\\ 1 & \text{if } \alpha>n,\\ 1-\log|x|\ (|x|\leq 1) \text{ and } 1\ (|x|>1), & \text{if } \alpha=n.\end{cases}$$

Let Q be a measurable function on a domain Ω in R^n, and set, for $1 < p < \infty$ and $d > 0$,

$$M_{\alpha, d}(|Q|^p, x) = \int_{\Omega \cap B(x, d)} |Q(y)|^p w_\alpha(x - y)\, dy$$

and

$$M_{\alpha, d}(|Q|^p) = \sup_{x \in \Omega} M_{\alpha, d}(|Q|^p, x),$$

where $B(x, d)$ is the closed ball in R^n with centre x and radius d. We shall also write

$$N_{t, d}(m^{-1}, x) = \left\{ \frac{1}{\operatorname{meas} B(x, d)} \int_{\Omega \cap B(x, d)} m^{-t}(y)\, dy \right\}^{1/t}$$

and

$$N_{t, d}(m^{-1}) = \sup_{x \in \Omega} N_{t, d}(m^{-1}, x)$$

for $1 < t < \infty$, the obvious modification being made for the case $t = \infty$. Note that if $M_{\alpha, d}(|Q|^p)$ is finite for some $d > 0$ it is finite for all $d > 0$: similarly for $N_{t, d}(m^{-1})$. When $d = 1$ we suppress the subscript d and write simply $M_\alpha(|Q|^p)$ and $N_t(m^{-1})$. We shall also write, for $r, s \in R$,

$$M_{\alpha, d}(|Q|^p N_{t, d}^s(m^{-1}, \cdot)) = \sup_{x \in \Omega} \left\{ \int_{\Omega \cap B(x, d)} |Q(y)|^p N_{t, d}^s(m^{-1}, y)\, w_\alpha(x - y)\, dy \right\}$$

and

$$N_{s, d}(|Q|^p N_{t, d}^r(m^{-1}, \cdot)) =$$
$$= \sup_{x \in \Omega} \left\{ \frac{1}{\operatorname{meas} B(x, d)} \int_{\Omega \cap B(x, d)} \{|Q(y)|^p N_{t, d}^r(m^{-1}, y)\}^s dy \right\}^{1/s}.$$

In certain instances we shall need to consider the restriction of functions Q to subdomains Ω' of Ω, and in such cases we shall write

$$Q_{\Omega'}(x) = \begin{cases} Q(x), & x \in \Omega' \\ 0, & x \notin \Omega'. \end{cases}$$

3.2. We now obtain conditions for multiplication maps Q operating on the weighted Sobolev spaces $H_0^{k,p}(\Omega, m)$ of § 2 to be bounded. The following lemma is crucial in this discussion.

LEMMA 3.1. *Let $u \in C_0^1(\Omega)$ and $d > 0$. Then for all x in Ω,*

$$|u(x)| \leq \omega_n^{-1} \int_{\Omega \cap B(x,d)} (d^{-1}|u(y)| + |Du(y)|)\, |x-y|^{1-n}\, dy, \tag{3.1}$$

where ω_n is the $(n-1)$-dimensional measure of the unit sphere S^{n-1} in R^n.

PROOF. Let $\theta \in C^1([0,\infty))$ be such that $0 \leq \theta \leq 1$ and, for some ϱ, $0 < \varrho < 1/3$,

$$\theta(r) = \begin{cases} 1, & 0 \leq r \leq \varrho d, \\ 0, & d(1-\varrho) \leq r, \end{cases}$$

with $|\theta'(r)| \leq d^{-1}(1-3\varrho)^{-1}$ for all $r \geq 0$.

Any point y in $B(x,d)$ can be written as $y = x + r\xi$, where $0 \leq r \leq d$ and $|\xi| = 1$. We extend u to the whole of R^n by putting $u(y) = 0$ for $y \notin \Omega$, and set $u(y) = u(x + r\xi) = \Phi(r, \xi)$ for y in $B(x,d)$. Then for $0 < \sigma < \varrho d$,

$$\Phi(\sigma, \xi) = -\int_\sigma^d \frac{\partial}{\partial r}[\theta(r)\,\Phi(r,\xi)]\, dr.$$

Now denote by ω the measure on S^{n-1} induced by Lebesgue measure on R^{n-1}. Then

$$\left|\int_{|\xi|=1} \Phi(\sigma,\xi)\, d\omega(\xi)\right| \leq \int_{|\xi|=1} \int_\sigma^d \left|\frac{\partial}{\partial r}[\theta(r)\,\Phi(r,\xi)]\right| dr\, d\omega(\xi)$$

$$\leq \int_{|\xi|=1} \int_0^d \left\{\left|\frac{\partial \Phi}{\partial r}(r,\xi)\right| + d^{-1}(1-3\varrho)^{-1}|\Phi(r,\xi)|\right\} r^{1-n}\, dy$$

$$= \int_{|\xi|=1} \int_0^d \{|Du(y)\cdot\xi| + d^{-1}(1-3\varrho)^{-1}|u(y)|\}\, |x-y|^{1-n}\, dy$$

$$\leq \int_{\Omega \cap B(x,d)} \{d^{-1}(1-3\varrho)^{-1}|u(y)| + |Du(y)|\}\, |x-y|^{1-n}\, dy.$$

Multiplication by $\omega_n^{-1}\sigma^{n-1}$ and subsequent integration with respect to σ over the interval $0 \le \sigma \le h < \varrho d$ gives

$$\omega_n^{-1}\left|\int_0^h \int_{|\xi|=1} \Phi(\sigma,\xi)\,\sigma^{n-1}\,d\sigma\,d\omega(\xi)\right|$$

$$\le \omega_n^{-1}\frac{h^n}{n}\int_{\Omega\cap B(x,d)} \{d^{-1}(1-3\varrho)^{-1}|u(y)| + |Du(y)|\}\,|x-y|^{1-n}\,dy.$$

Since meas $B(x,h) = \frac{h^n}{n}\omega_n$, we obtain from Lebesgue's differentiation theorem, on letting $h \to 0$,

$$|u(x)| \le (\text{meas } B(x,h))^{-1}\left|\int_{B(x,h)} u(y)\,dy\right|$$

$$\le \omega_n^{-1}\int_{\Omega\cap B(x,d)} \{d^{-1}(1-3\varrho)^{-1}|u(y)| + |Du(y)|\}\;|x-y|^{1-n}\,dy.$$

The lemma now follows since ϱ may be made arbitrarily small.

In view of the result that

$$\int_{B(x,d)\cap B(y,jd)} |x-z|^{1-n}\,|y-z|^{j-n}\,dz \le K\,w_{j+1}(x-y),$$

repeated application of lemma 3.1 gives

LEMMA 3.2. *Let k be any positive integer. Then there is a constant K, depending only on n, k and d, such that for all u in $C_0^k(\Omega)$ and all x in Ω,*

$$|u(x)| \le K\int_{\Omega\cap B(x,kd)} \sum_{i=0}^{k} |D^i u(y)|\,w_k(y-x)\,dy. \tag{3.2}$$

We can now give the embedding theorems.

THEOREM 3.3. *Let k be a positive integer, let $t > 0$, $q \ge p > 1 + \frac{1}{t}$, $\frac{1}{q} > \frac{1}{p}\left(1+\frac{1}{t}\right) - \frac{k}{n}$, and let α be a positive number satisfying*

$$\frac{\alpha - n}{q} < k - \left(1 + \frac{1}{t}\right). \tag{3.3}$$

Then we have the following:

(i) *If* $M_{\alpha, d}(|Q|^q N_{t,d}^{q/p}(m^{-1}, \cdot)) < \infty$ *then* $u \mapsto Qu$ *is a bounded map of* $H_0^{k,p}(\Omega, m)$ *into* $L^q(\Omega)$, *and there exists a constant* K, *depending only on* p, q, k, n *and* t, *such that for all* u *in* $H_0^{k,p}(\Omega, m)$,

$$\| Qu \|_{0,q} \leq K\{M_{\alpha,d}(|Q|^q N_{t,d}^{q/p}(m^{-1}, \cdot))\}^{1/q} \| u \|_{k,p,m}.$$

(ii) *If* $M_{\alpha, d}(|Q|^q N_{t,d}^{q/p}(m^{-1}, \cdot)\, m) < \infty$ *then* $u \mapsto Qu$ *is a bounded map of* $H_0^{k,p}(\Omega, m)$ *into* $L^q(\Omega, m)$, *and there exists a constant* K, *depending only on* p, q, k, n *and* t, *such that for all* u *in* $H_0^{k,p}(\Omega, m)$,

$$\| Qu \|_{0,q,m} \leq K\{M_{\alpha,d}(|Q|^q N_{t,d}^{q/p}(m^{-1}, \cdot)\, m)\}^{1/q} \| u \|_{k,p,m}.$$

PROOF. (i) It is clearly enough to establish the displayed inequality when u belongs to the dense subset $C_0^k(\Omega)$ of $H_0^{k,p}(\Omega, m)$. For convenience we put $v(y) = \sum_{i=0}^{k} |D^i u(y)|$, so that from Lemma 3.2 we have

$$|u(x)| \leq K_1 \int_{\Omega \cap B(x, kd)} v(y)\, w_k(y - x)\, dy$$

for u in $C_0^k(\Omega)$ and x in Ω.

Let λ, μ, ν be numbers ≥ 1 and satisfying

$$\frac{1}{\lambda} + \frac{1}{\nu} = \frac{1}{p}, \qquad \frac{1}{\lambda} + \frac{1}{\mu} = \frac{1}{r}, \qquad \lambda = q,$$

where $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}$. Then $\frac{1}{\lambda} + \frac{1}{\mu} + \frac{1}{\nu} = 1$. For some β to be chosen later we write

$$v(y)\, w_k(y - x) = (w_k^{\beta r/\lambda}(y-x)\, v^{p/\lambda}(y)\, m^{1/\lambda}(y))\, (w_k^{(\beta r/\mu)+1-\beta}(y - x)\, m^{-(1/\lambda)-(1/\nu)}(y))$$
$$\times (v^{p/\nu}(y)\, m^{1/\nu}(y)).$$

It then follows from an extension of Hölder's inequality ([11], p. 140) that

$$|u(x)| \leq K_1 \int_{\Omega \cap B(x,kd)} v(y)\, w_k(y-x)\, dy$$

$$\leq K_1 \left\{ \int_{\Omega \cap B(x,kd)} w_k^{\beta r}(y-x)\, v^p(y)\, m(y)\, dy \right\}^{1/\lambda}$$

$$\times \left\{ \int_{\Omega \cap B(x,kd)} w_k^{\beta r + (1-\beta)\mu}(z-x)\, m^{-\mu/p}(z)\, dz \right\}^{1/\mu}$$

$$\times \left\{ \int_{\Omega \cap B(x,kd)} v^p(y)\, m(y)\, dy \right\}^{1/\nu},$$

and so, since $\lambda = q$,

$$\int_{\Omega} |Q(x)\, u(x)|^q\, dx \tag{3.4}$$

$$\leq K_1 \int_{\Omega} \int_{\Omega \cap B(x,kd)} \left\{ |Q(x)|^q\, w_k^{\beta r}(y-x) \left(\int_{\Omega \cap B(x,kd)} w^{\beta r + (1-\beta)\mu}(z-x)\, m^{-\mu/p}(z)\, dz \right)^{q/\mu} \right.$$

$$\left. \times\, v^p(y)\, m(y)\, \|v\|_{0,p,m}^{pq/\nu} \right\} dy\, dx\,.$$

We shall show that for a suitably chosen β,

$$\sup_{y \in \Omega} \int_{\Omega \cap B(y,kd)} |Q(x)|^q\, w_k^{\beta r}(y-x) \left(\int_{\Omega \cap B(x,kd)} w_k^{\beta r + (1-\beta)\mu}(z-x)\, m^{-\mu/p}(z)\, dz \right)^{q/\mu} dx \tag{3.5}$$

$$\leq K_2\, M_{a,d}(|Q|^q\, N_t^{q/p}(m^{-1}, \cdot)).$$

It will then follow from (3.4) that

$$\|Qu\|_{0,q} \leq K\{M_{a,d}(|Q|^q\, N_t^{q/p}(m^{-1}, \cdot))\}^{1/q}\, \|v\|_{0,p,m}^{(p/q)+(p/\nu)},$$

which gives the result since $\dfrac{1}{q} + \dfrac{1}{v} = \dfrac{1}{p}$ and $\|v\|_{0,p,m} \leq \|u\|_{k,p,m}$.

It remains to prove (3.5), and to do this we first apply Hölder's inequality to the integral in (3.5), and see that it is majorised by

$$(3.6)\quad \int\limits_{\Omega\cap B(y,kd)} |Q(x)|^q w_k^{\beta r}(y-x)\left(\int\limits_{\Omega\cap B(x,kd)} m^{-t}(z)\,dz\right)^{q/(\gamma\mu)}\left(\int\limits_{\Omega\cap B(x,kd)} w_k^{\gamma'(\beta r+(1-\beta)\mu)}(z-x)\,dx\right)^{q/(\gamma'\mu)}dx$$

$$\leq K\int\limits_{\Omega\cap B(y,kd)} |Q(x)|^q (N_{t,kd}(m^{-1},x))^{q/p} w_k^{\beta r}(y-x)$$

$$\times\left(\int\limits_{\Omega\cap B(x,kd)} w_k^{\gamma'(\beta r+(1-\beta)\mu)}(z-x)\,dz\right)^{q/(\gamma'\mu)}dx,$$

where $\gamma\mu/p = t$, so that $\gamma = (p-1)t > 1$, and $\dfrac{1}{\gamma}+\dfrac{1}{\gamma'} = 1$.

We now distinguish two cases. First suppose that

$$k-\frac{n}{p}\left(1+\frac{1}{t}\right)\leq 0.$$

Then since $p > 1+\dfrac{1}{t}$ we have from (3.3) that $k < n$ and $\alpha < n$, and so $w_k(z-x) = |z-x|^{k-n}$, $w_\alpha(z-x) = |z-x|^{\alpha-n}$. If we now put $\beta r(k-n) = \alpha - n$ in (3.6), (3.5) follows since

$$(k-n)\gamma'(\beta r+(1-\beta)\mu)+n = \gamma'\left\{(\alpha-n)\left(1-\frac{\mu}{r}\right)+\mu(k-n)\right\}+n$$

$$= \gamma'\mu\left\{(\alpha-n)\left(\frac{1}{\mu}-\frac{1}{r}\right)+(k-n)+n\left(1-\frac{1}{p}\right)\left(1-\frac{1}{\gamma}\right)\right\}$$

$$= \gamma'\mu\left\{-\frac{(\alpha-n)}{q}+k-\frac{n}{p}\left(1+\frac{1}{t}\right)\right\} > 0,$$

by (3.3).

On the other hand, if

$$k-\frac{n}{p}\left(1+\frac{1}{t}\right) > 0$$

then, since $M_{\alpha_1,d} < \infty$ if $M_{\alpha,d} < \infty$ and $\alpha < \alpha_1$, there is no loss of generality if in (3.3) we choose $\alpha > n$. Then $w_\alpha(y-x) = 1$. In (3.6) we now choose $\beta = 0$, and the result follows since

$$\gamma' \mu (k - n) + n = \gamma' \mu \left\{ k - n + n \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{\gamma}\right)\right\}$$

$$= \gamma' \mu \left\{ k - \frac{n}{p}\left(1 + \frac{1}{t}\right)\right\} > 0.$$

The proof of (i) is therefore complete. For (ii) we need only replace Q by $Qm^{1/q}$ in the preceding argument.

COROLLARY 3.4. *Let k be a positive integer, let $q \geq p > 1 + \frac{1}{t}$, $t > 0$, $s > 1$, and suppose that*

$$\frac{1}{qs'} > \frac{1}{p}\left(1 + \frac{1}{t}\right) - \frac{k}{n}, \qquad \frac{1}{s} + \frac{1}{s'} = 1.$$

Then we have

(i) *If $N_{s,d}(|Q|^q N_{t,d}^{q/p}(m^{-1}, \cdot)) < \infty$, multiplication by Q is a bounded map of $H_0^{k,p}(\Omega, m)$ into $L^q(\Omega)$, and there exists a constant K such that for all u in $H_0^{k,p}(\Omega, m)$,*

$$\| Qu \|_{0,q} \leq K \{N_{s,d}(|Q|^q N_{t,d}^{q/p}(m^{-1}, \cdot))\}^{1/q} \| u \|_{k,p,m}.$$

(ii) *If $N_{s,d}(|Q|^q N_{t,d}^{q/p}(m^{-1}, \cdot) m) < \infty$, multiplication by Q is a bounded map of $H_0^{k,p}(\Omega, m)$ into $L^q(\Omega, m)$, and there exists a constant K such that for all u in $H_0^{k,p}(\Omega, m)$,*

$$\| Qu \|_{0,q,m} \leq K \{N_{s,d}(|Q|^q N_{t,d}^{q/p}(m^{-1}, \cdot)\, m)\}^{1/q} \| u \|_{k,p,m}.$$

PROOF. From the hypothesis we have

$$\frac{n}{qs} < k - \frac{n}{p}\left(1 + \frac{1}{t}\right) + \frac{n}{q},$$

so that there clearly exists a number $\alpha > n/s$ which satisfies (3.3). For such an α it follows by the Hölder inequality that

$$M_{\alpha,d}(|Q|^q N_{t,d}^{q/p}(m^{-1}, \cdot), x) \leq K N_{s,d}(|Q|^q N_{t,d}^{q/p}(m^{-1}, \cdot)\, x)\left(\int_{\Omega \cap B(x,d)} w_\alpha^{s'}(x-y)\, dy\right)^{1/s'}$$

$$\leq K N_{s,d}(|Q|^q N_{t,d}^{q/p}(m^{-1}, \cdot), x),$$

since $s'(\alpha - n) + n = s'\left(\alpha - \frac{n}{s}\right) > 0$. Part (i) of the Corollary follows immediately from this and the Theorem: (ii) is similar.

In the sequel we shall also require the following result from [6], § 2, relating to the spaces $H_0^{s,p}(\Omega)$ when s is not necessarily an integer.

THEOREM 3.5. *Let* $s > 0$, $q \geq p > 1$, $\frac{1}{q} > \frac{1}{p} - \frac{s}{n}$, *and let* $\alpha > 0$ *satisfy*

$$\frac{\alpha - n}{q} < s - \frac{n}{p}.$$

Then if $M_{\alpha, d}(|Q|^q) < \infty$, *multiplication by* Q *is a bounded map of* $H_0^{s,p}(\Omega)$ *into* $L^q(\Omega)$, *and there exists a constant* K, *depending only on* p, q, s *and* n, *such that for all* u *in* $H_0^{s,p}(\Omega)$,

$$\| Qu \|_{0,q} \leq K \{M_{\alpha, d}(|Q|^q)\}^{1/q} \| u \|_{s,p}.$$

REMARKS 1) When $m(x) \equiv 1$ and $t = \infty$, Theorem 3.3 reduces to the special case of Theorem 3.5 in which s is an integer.

2) If in Theorem 3.3 (i) we put $Q(x) \equiv 1$ in Ω we obtain sufficient conditions for $H_0^{k,p}(\Omega, m)$ to be embedded in $L^q(\Omega)$. Of particular interest later is the case when $Q = 1$ in Ω and $p = q$. In this situation, provided $N_{t,d}(m^{-1}) < \infty$, $p > 1 + \frac{1}{t}$ and $p > \frac{n}{kt}$, then

$$H_0^{k,p}(\Omega, m) \subset L^p(\Omega), \tag{3.7}$$

the embedding being continuous.

3.3. In this subsection we investigate further the properties of the multiplication maps Q of § 3.2 and also the embeddings of $H_0^{k,p}(\Omega, m)$ in $L^p(\Omega)$ and $L^p(\Omega, m)$. Our main concern is with the maps Q as k-set contractions. As a consequence of our investigation we shall obtain a sufficient condition for the Poincaré inequality to hold in the space $H_0^{k,p}(\Omega, m)$. For these results more precise estimates are required for the norm of the map Q in Theorem 3.3 in the special case $p = q$. We also need to introduce some new notation. If T is a bounded map of a Banach space X into a Banach space Y we shall write $\| T : X, Y \|$ for the norm of T, that is the infimum of those positive numbers K such that $\| Tu \|_Y \leq K \| u \|_X$ for all u in X.

THEOREM 3.6. *Let* $t > 0$, $p > 1 + \frac{1}{t}$, $p > \frac{n}{t}$, *let* $\alpha > 0$ *satisfy* $n(1-p) + p < \alpha < p - \frac{n}{t}$, *and define*

$$C(\alpha,p,t,d,n) = \omega_n^{-1}\, n^{-1/t}\, d^{-n} \left(\frac{p - 1 - \frac{1}{t}}{p - \alpha - \frac{n}{t}} \right)^{p-1-(1/t)} (w_\alpha(d))^{-1}.$$

For x *in* Ω, *let* meas $(B(x,d) \setminus \Omega) = \delta(x)$ meas $B(x,d)$, *where* $0 \leq \delta(x) < 1$, *and set* $\delta = \inf_{x \in \Omega} \delta(x)$.

Then we have the following:

(i) *If* $M_{\alpha,d}(|Q|^p N_{t,d}(m^{-1},\cdot)) < \infty$, *multiplication by* Q *is a bounded map of* $H_0^{1,p}(\Omega, m)$ *into* $L^p(\Omega)$ *and for all* u *in* $H_0^{1,p}(\Omega, m)$,

$$(3.8) \qquad \| Qu \|_{0,p} \leq \{C(\alpha,p,t,d,n)(1-\delta)^{\left(p-\frac{n}{t}-\alpha\right)/n} M_{\alpha,d}(|Q|^p N_{t,d}(m^{-1},\cdot))\}^{1/p} \times (\| u \|_{0,p,m} + d \| Du \|_{0,p,m}),$$

so that if $d \leq 1$,

$$(3.9) \qquad \| Q : H_0^{1,p}(\Omega,m), L^p(\Omega) \| \leq \{C(\alpha,p,t,d,n)(1-\delta)^{\left(p-\frac{n}{t}-\alpha\right)/n} M_{\alpha,d}(|Q|^p N_{t,d}(m^{-1},\cdot))\}^{1/p}.$$

(ii) *If* $M_{\alpha,d}(|Q|^p N_{t,d}(m^{-1},\cdot)\, m) < \infty$, *multiplication by* Q *is a bounded map of* $H_0^{1,p}(\Omega, m)$ *into* $L^p(\Omega, m)$, *and for all* u *in* $H_0^{1,p}(\Omega, m)$,

$$(3.10) \qquad \| Qu \|_{0,p,m} \leq \{C(\alpha,p,t,d,n)(1-\delta)^{\left(p-\frac{n}{t}-\alpha\right)/n} M_{\alpha,d}(|Q|^p N_{t,d}(m^{-1},\cdot)\, m\}^{1/p} \times (\| u \|_{0,p,m} + d \| Du \|_{0,p,m}),$$

so that if $d \leq 1$,

$$(3.11) \qquad \| Q : H_0^{1,p}(\Omega,m), L^p(\Omega,m) \| \leq \{C(\alpha,p,t,d,n)(1-\delta)^{\left(p-\frac{n}{t}-\alpha\right)/n} M_{\alpha,d}(|Q|^p N_t(m^{-1},\cdot)\, m)\}^{1/p}.$$

PROOF. Let $u \in C_0^1(\Omega)$ and set $v(y) = |u(y)| + d\,|Du(y)|$, $y \in \Omega$, so that from Lemma 3.1 we have

$$|u(x)| \leq d^{-1}\,\omega_n^{-1} \int_{\Omega \cap B(x,d)} v(y)\,|x-y|^{1-n}\,dy, \quad x \in \Omega.$$

Application of Hölder's inequality to this gives

$$(3.12) \qquad |u(x)| \leq d^{-1}\,\omega_n^{-1} \left\{ \int_{\Omega \cap B(x,d)} v^p(y)\,w_\alpha(x-y)\,m(y)\,dy \right\}^{1/p}$$

$$\times \left\{ \int_{\Omega \cap B(x,d)} w_\alpha^{-p'/p}(x-y)\,|x-y|^{p'(1-n)}\,m^{-p'/p}(y)\,dy \right\}^{1/p'}$$

$$\leq d^{-1}\,\omega_n^{-1}\,(\text{meas}\,B(x,d))^{1/(tp)} \left\{ \int_{\Omega \cap B(x,d)} v^p(y)\,m(y)\,w_\alpha(x-y)\,N_{t,d}(m^{-1},x)\,dy \right\}^{1/p}$$

$$\times \left\{ \int_{\Omega \cap B(x,d)} w_\alpha^{-p'\gamma'/p}(x-y)\,|x-y|^{p'\gamma'(1-n)}\,dy \right\}^{1/(\gamma' p')},$$

where $\gamma = pt/p' = (p-1)\,t > 1$.

If $\alpha < n$ the last integral in (3.12) is

$$\int_{\Omega \cap B(x,d)} |x-y|^{p'\gamma'(1-n-(\alpha-n)/p)}\,dy = \int_{\Omega \cap B(x,d)} |x-y|^{\left(p-\alpha-\frac{n}{t}\right)/\left(p-1-\frac{1}{t}\right)-n}\,dy,$$

since

$$\frac{1}{\gamma' p'} = \left(1-\frac{1}{\gamma}\right)\left(1-\frac{1}{p}\right) = \left(p-1-\frac{1}{t}\right)/p.$$

Let us put $\beta = \left(p-\alpha-\dfrac{n}{t}\right)\Big/\left(p-1-\dfrac{1}{t}\right)$: then

$$\beta - n = \{n(1-p) + p - \alpha\}/(p-1-t^{-1}) < 0.$$

We claim that for $0 < \beta < n$,

$$(3.13) \qquad \int_{\Omega \cap B(x,d)} |x-y|^{\beta-n}\,dy \leq \omega_n\,d^\beta\,\beta^{-1}\,(1-\delta(x))^{\beta/n}.$$

For since $| x - y |^{\beta-n}$ increases towards the centre of the ball $B(x, d)$,

$$\int_{\Omega \cap B(x,d)} | x - y |^{\beta-n} \, dy \leq \int_S | x - y |^{\beta-n} \, dy,$$

where $S = \{y : |x-y| \leq \eta\}$ and meas $S =$ meas $(\Omega \cap B(x, d)) = (1-\delta(x))$ meas $B(x,d)$. Since meas $S = (1 - \delta(x)) \, \omega_n \, d^n/n$ we must have $\eta^n \, \omega_n/n = (1 - \delta(x)) \, \omega_n \, d^n/n$, giving $\eta/d = (1 - \delta(x))^{1/n}$. Therefore

$$\int_S | x - y |^{\beta-n} \, dy = \omega_n \, \eta^\beta/\beta = \omega_n \, d^\beta (1 - \delta(x))^{\beta/n}/\beta,$$

and the assertion (3.13) is proved. In (3.12) we thus have when $\alpha < n$, since $\dfrac{p}{\gamma' \, p'} = p - 1 - \dfrac{1}{t}$,

$$| u(x) |^p \leq$$

$$\leq d^{-p} \omega_n^{-p} \omega_n^{1/t} \, d^{n/t} \, n^{-1/t} \omega_n^{p-1-\frac{1}{t}} \, d^{p-\alpha-\frac{n}{t}} (1-\delta(x))^{\left(p-\alpha-\frac{n}{t}\right)/n} \left(\frac{p-1-\frac{1}{t}}{p-\alpha-\frac{n}{t}} \right)^{p-1-\frac{1}{t}}$$

$$\times \int_{\Omega \cap B(x,d)} v^p(y) \, m(y) \, w_\alpha(x-y) \, N_{t,d}(m^{-1}, x) \, dy$$

$$= C(\alpha, p, t, d, n) \, (1 - \delta(x))^{\left(p-\alpha-\frac{n}{t}\right)/n} \int_{\Omega \cap B(x,d)} v^p(y) \, m(y) \, w_\alpha(x-y) \, N_{t,d}(m^{-1}, x) \, dy.$$

From this (3.8) clearly follows. The proof is similar when $\alpha > n$, while if $\alpha = n$ we use in (3.12) the inequality

$$w_n(x - y) \geq w_n(d) \text{ for } | x - y | \leq d,$$

and the result follows as before. Part (ii) is handled just as for (i), but with Q replaced by $Q m^{1/p}$.

Remarks. Note that if we replace Q by $Q_{\Omega \setminus B(0,R)}$, then in (3.8) and (3.9) we can put

$$\delta = \inf_{x \in \Omega \setminus B(0,R)} \{\text{meas}\,(B(x, d) \setminus \Omega)/\text{meas}\, B(x, d)\}.$$

Note also that the general character of the results of the Theorem is unaltered if the hypothesis that $\alpha > n(1-p)+p$ is dropped. For we could work with an $\alpha_1 > n(1-p)+p$, obtain the results given from this α_1, and then use an inequality of the form $M_{\alpha_1, d} \leq \text{const.}\, M_{\alpha, d}$, where the constant depends on d, α_1, and α.

COROLLARY 3.7. *Let* $t>0$, $s>1$, $p>1+\frac{1}{t}$, $d\leq 1$, *and* $p>n\left(\frac{1}{t}+\frac{1}{s}\right)$, *and set* $\frac{1}{r}=\frac{1}{s}+\frac{1}{t}$. *Suppose that* $\text{meas}\,(B(x,d)\setminus \Omega)=\delta(x)\,\text{meas}\,B(x,d)$, $x\in\Omega$, *and write* $\delta=\inf_{x\in\Omega}\delta(x)$. *Then:*

(i) *If* $N_{s,d}(|Q|^p N_{t,d}(m^{-1},\cdot))<\infty$,

$$\| Q : H_0^{1,p}(\Omega, m), L^p(\Omega)\| \tag{3.14}$$

$$\leq \left\{ n^{-1/r}\left(\frac{p-\frac{1}{r}}{p-\frac{n}{r}}\right)^{p-\frac{1}{r}} (1-\delta)^{\frac{p}{n}-\frac{1}{r}} N_{s,d}(|Q|^p N_{t,d}(m^{-1},\cdot))\right\}^{1/p}.$$

(ii) *If* $N_{s,d}(|Q|^p N_{t,d}(m^{-1},\cdot)\,m)<\infty$,

$$\| Q : H_0^{1,p}(\Omega, m), L^p(\Omega, m)\| \tag{3.15}$$

$$\leq \left\{ n^{-1/r}\left(\frac{p-\frac{1}{r}}{p-\frac{n}{r}}\right)^{p-\frac{1}{r}} (1-\delta)^{\frac{p}{n}-\frac{1}{r}} N_{s,d}(|Q|^p N_{t,d}(m^{-1},\cdot)\,m)\right\}^{1/p}.$$

PROOF Since $p-\frac{n}{t}>\frac{n}{s}$, $s>1$, and $p>1+\frac{1}{t}$, there exists a number α which satisfies, provided $n>1$,

$$0<\alpha<n,\quad n/s<\alpha,\quad n-p(n-1)<\alpha<p-(n/t).$$

For any such α, we have by Hölder's inequality,

$$M_{\alpha,d}(|Q|^p N_{t,d}(m^{-1},\cdot),x)=\int_{\Omega\cap B(x,d)} |Q(y)|^p N_{t,d}(m^{-1},y)\,|x-y|^{\alpha-n}\,dy$$

$$\leq \left(\int_{\Omega \cap B(x,d)} (|Q(y)|^p N_{t,d}(m^{-1}, y))^s \, dy \right)^{1/s} \left(\int_{\Omega \cap B(x,d)} |x - y|^{s'(\alpha - n)} \, dy \right)^{1/s'}$$

$$\leq \{ (\text{meas } B(x,d))^{1/s} N_{s,d}(|Q|^p N_{t,d}(m^{-1}, \cdot)) \} \left\{ \frac{\omega_n d^{s'(\alpha - (n/s))}}{s'\left(\alpha - \frac{n}{s}\right)} (1 - \delta(x))^{\frac{s'}{n}\left(\alpha - \frac{n}{s}\right)} \right\}^{1/s'}$$

as in (3.13). If we now substitute in (3.8) we have for all u in $H_0^{1,p}(\Omega, m)$,

(3.16) $\quad \| Qu \|_{0,p}$

$$\leq \left\{ \frac{n^{-1/r}}{\left[s'\left(\alpha - \frac{n}{s}\right)\right]^{1/s'}} \left(\frac{p - 1 - \frac{1}{t}}{p - \alpha - \frac{n}{t}} \right)^{p-1-\frac{1}{t}} (1-\delta)^{\frac{p}{n} - \frac{1}{r}} N_{s,d}(|Q|^p N_{t,d}(m^{-1}, \cdot)) \right\}^{1/p}$$

$$\times (\| u \|_{0,p,m} + d \| Du \|_{0,p,m}).$$

It is readily shown that

$$\left[s'\left(\alpha - \frac{n}{s}\right)\right]^{-1/s'} \left(\frac{p - 1 - \frac{1}{t}}{p - \alpha - \frac{n}{t}} \right)^{p-1-\frac{1}{t}},$$

regarded as a function of α, attains a minimum value of

$$\left(\frac{p - \frac{1}{r}}{p - \frac{n}{r}} \right)^{p - \frac{1}{r}}$$

when

$$\alpha = \left\{ p\left(\frac{1}{s'} + \frac{n}{s}\right) - n\left(\frac{1}{s} + \frac{1}{t}\right) \right\} \Big/ \left(p - \frac{1}{s} - \frac{1}{t} \right).$$

Note also that for $n > 1$, this value of α satisfies

$$\alpha - \frac{n}{s} = \left(p - \frac{n}{s} - \frac{n}{t} \right) \Big/ \left\{ s'\left(p - \frac{1}{s} - \frac{1}{t} \right) \right\} > 0,$$

$$\alpha - \left(v - \frac{n}{t}\right) = \left(p - \frac{n}{s} - \frac{n}{t}\right)\left(1 + \frac{1}{t} - p\right) \Big/ \left(p - \frac{1}{s} - \frac{1}{t}\right) < 0,$$

$$\alpha - n = p\,(1 - n) \Big/ \left\{ s' \left(p - \frac{1}{s} - \frac{1}{t}\right)\right\} < 0,$$

and

$$\alpha = (n - p\,(n-1)) = p\,(1-n)\left(1 + \frac{1}{t} - p\right) \Big/ \left(p - \frac{1}{s} - \frac{1}{t}\right) > 0.$$

Substitution of this value of α in (3.16) now gives (3.14) for $n > 1$. When $n = 1$, this result is easily obtained directly from (3.1) by the repeated use of Hölders's inequality as above. We omit the details. For (3.15) put $Qm^{1/p}$ in (3.14).

An especially interesting particular case of Corollary 3.7 (ii) is that which arises when $Q = E$, the identity map. We state this below.

Corollary 3.8. *Let δ, p, r, s and t be as in Corollary 3.7 and suppose that $N_{s,d}(N_{t,d}(m^{-1}, \cdot)\, m) < \infty$. Then for all u in $H_0^{1,p}(\Omega, m)$,*

$$(3.17) \qquad \| u \|_{0,p,m} \leq \left\{ n^{-1/r} \left(\frac{p - \frac{1}{r}}{p - \frac{n}{r}} \right)^{p - \frac{1}{r}} (1-\delta)^{\frac{p}{n} - \frac{1}{r}} N_{s,d}(N_{t,d}(m^{-1}, \cdot)\, m) \right\}^{1/p} \times (\| u \|_{0,p,m} + d\, \| Du \|_{0,p,m}) .$$

When $d \leq 1$,

$$(3.18) \qquad \| E : H_0^{1,p}(\Omega, m), L^p(\Omega, m) \| \leq \left\{ n^{-1/r} \left(\frac{p - \frac{1}{r}}{p - \frac{n}{r}} \right)^{p - \frac{1}{r}} (1-\delta)^{\frac{p}{n} - \frac{1}{r}} N_{s,d}(N_{t,d}(m^{-1}, \cdot)\, m) \right\}^{1/p} .$$

In particular, when $d \leq 1$, $m \equiv 1$, $t = \infty$ and $s = \infty$, we have for all $p > 1$,

$$(3.19) \qquad \| E : H_0^{1,p}(\Omega), L^p(\Omega) \| \leq (1 - \delta)^{1/n} .$$

Before discussing various consequences of Theorem 3.6 and its Corollaries we shall show that when restricted to bounded subdomains of Ω, the maps Q and E behave in a way which does not affect the properties of these maps that we wish to develop. In the following we shall denote $B(0, R)$ by $B(R)$.

LEMMA 3.9. *Let* $t > n/p$, $p > 1 + \frac{1}{t}$, $0 < \alpha < p - \frac{n}{t}$, *and suppose* $\overline{\Omega} \neq R^n$. *Then we have that:*

(i) *If* $M_\alpha(|Q|^p N_t(m^{-1}, \cdot), x)$ *is locally bounded, then for all* u *in* $H_0^{1,p}(\Omega, m)$,

$$\| Q_{\Omega \cap B(R)} u \|_{0,p} \leq K \| Du \|_{0,p,m},$$

where K *is a positive constant depending only on* R, p, t *and* n.

(ii) *If* $M_\alpha(|Q|^p N_t(m^{-1}, .)\, m, x)$ *is locally bounded, there is a positive constant* K, *depending only on* R, p, t *and* n, *such that for all* u *in* $H_0^{1,p}(\Omega, m)$,

$$\| Q_{\Omega \cap B(R)} u \|_{0,p,m} \leq K \| Du \|_{0,p,m}.$$

PROOF. Let $u \in C_0^1(\Omega)$. Since $\overline{\Omega} \neq R^n$, there exist $x_0 \in R^n$ and $\delta > 0$ such that $B(x_0, 2\delta) \cap \Omega = \emptyset$. Given any x in $\Omega \cap B(R)$ let $C(x)$ denote the closed cone, with vertex x, which is tangential to $B(x_0, \delta)$ and has base determined by the appropriate diametral plane through $B(x_0, \delta)$. Let $\Gamma(x)$ be the intersection of $C(x)$ and the sphere with centre x and radius 1. Then we may express any point $y \in C(x)$ in the form $y = x + t\xi$, where $0 \leq t \leq R(x, \xi)$ say, and $\xi \in \Gamma(x)$: note that $|\xi| = 1$ and $t = |y - x|$. Since $u = 0$ on $B(x_0, \delta)$ we have, for any $\xi \in \Gamma(x)$,

$$u(x) = -\int_0^{R(x,\xi)} \frac{\partial}{\partial t} u(x + t\xi)\, dt,$$

so that

$$|u(x)| \leq \int_0^{R(x,\xi)} \left| \frac{\partial}{\partial t} u(x + t\xi) \right| dt.$$

Integration over $\Gamma(x)$ with respect to the angular coordinates now gives

$$(\text{meas}\, \Gamma(x))\, |u(x)| \leq \int_{\xi \in \Gamma(x)} \int_0^{R(x,\xi)} \left| \frac{\partial}{\partial t} u(x + t\xi) \right| dt\, d\omega(\xi)$$

$$\leq \int_{C(x)} |Du(y)|\, |y - x|^{1-n}\, dy.$$

It is clear that there exists $T > 0$ such that $R(x, \xi) \leq T$ for all ξ in $\Gamma(x)$ and all x in $\Omega \cap B(R)$: moreover, there is a number $\gamma > 0$ such that meas $\Gamma(x) \geq \gamma$ for all x in $\Omega \cap B(R)$. Hence

$$|u(x)| \leq \gamma^{-1} \int_{\Omega \cap B(x, T)} |Du(y)| \ |x - y|^{1-n} \, dy.$$

The result now follows exactly as in the proof of Theorem 3.6.

We are now in a position to give a sufficient condition for the Poincaré inequality to hold for elements of $H_0^{k,p}(\Omega, m)$.

THEOREM 3.10. *Let* $s > 1$, $t > 0$, $p > 1 + \frac{1}{t}$, *and* $p > n\left(\frac{1}{t} + \frac{1}{s}\right)$. *Suppose that there exist positive numbers* d *and* δ *such that*

$$\liminf_{|x| \to \infty, \, x \in \Omega} \text{meas}\,(B(x, d) \setminus \Omega) \geq \delta \, \text{meas}\, B(d) \tag{3.20}$$

and

$$\liminf_{R \to \infty} N_{s,d}(N_{t,d}(m_{\Omega(R)}^{-1}, \cdot)\, m_{\Omega(R)})\, (1 - \delta)^{\frac{p}{n} - \frac{1}{r}} < n^{1/r} \left(\frac{p - \frac{n}{r}}{p - \frac{1}{r}} \right)^{p - \frac{1}{r}}, \tag{3.21}$$

where $\Omega(R) = \Omega \setminus B(R)$ *and* $\frac{1}{r} = \frac{1}{s} + \frac{1}{t}$. *Suppose also that* $N_{s,d}(N_{t,d}(m^{-1}, \cdot)\, m) < \infty$. *Then there exists a constant* K, *depending only on* p, n, s *and* t, *such that for all* u *in* $H_0^{1,p}(\Omega, m)$,

$$\| u \|_{0,p,m} \leq K \| Du \|_{0,p,m}.$$

Moreover, given any positive integer k *there is a constant* K_1, *independent of* u, *such that for all* u *in* $H_0^{k,p}(\Omega, m)$ *and all* i, $0 \leq i \leq k - 1$,

$$\| D^i u \|_{0,p,m} \leq K_1 \| D^k u \|_{0,p,m},$$

so that the norm on $H_0^{k,p}(\Omega, m)$ *is equivalent to the norm*

$$| u |_{k,p,m} \equiv \| D^k u \|_{0,p,m}.$$

PROOF. We write the embedding $E: H_0^{1,p}(\Omega, m) \rightarrow L^p(\Omega, m)$ in the form $E = E_{\Omega \cap B(R)} + E_{\Omega(R)}$. From (3.20), (3.21) and (3.17), and taking account of the remark at the end of the proof of Theorem 3.6, it is clear that there exist R and C, $0 < C < 1$, such that for all u in $H_0^{1,p}(\Omega, m)$,

$$\| E_{\Omega(R)} u \|_{0,p,m} \leq C(\| u \|_{0,p,m} + d \| Du \|_{0,p,m}). \tag{3.22}$$

Choosing α such that $\frac{n}{s} < \alpha < p - \frac{n}{t}$, we also have from Lemma 3.9 (ii), since (3.20) implies that $\overline{\Omega} \neq R^n$, that for all u in $H_0^{1,p}(\Omega, m)$,

$$\| E_{\Omega \cap B(R)} u \|_{0,p,m} \leq K \| Du \|_{0,p,m} .$$

Thus

$$\| u \|_{0,p,m} = \| Eu \|_{0,p,m} \leq \| E_{\Omega \cap B(R)} u \|_{0,p,m} + \| E_{\Omega(R)} u \|_{0,p,m}$$

$$\leq C \| u \|_{0,p,m} + (Cd + K) \| Du \|_{0,p,m} .$$

Since $C < 1$, the Poincaré inequality

$$\| u \|_{0,p,m} \leq \text{const.} \| Du \|_{0,p,m}$$

follows immediately. The rest of the theorem is now clear.

COROLLARY 3.11. *Let $p > 1$ and let k be a positive integer. Suppose there exist positive numbers d and δ such that*

$$\liminf_{|x| \to \infty,\ x \in \Omega} \operatorname{meas}(B(x, d) \setminus \Omega) \geq \delta \operatorname{meas} B(d). \tag{3.23}$$

Then there exists a constant K, depending only on p and n, such that for all u in $H_0^{k,p}(\Omega)$ and all i, $0 \leq i \leq k-1$,

$$\| D^i u \|_{0,p} \leq K \| D^k u \|_{0,p} .$$

PROOF. Set $m \equiv 1$, $r = s = t = \infty$ in Theorem 3.10.

We remark that condition (3.23) is not necessary for the Poincaré inequality to hold in $H_0^{k,p}(\Omega)$. For if Ω is the « spiny urchin » (see [10]) then (3.23) is not satisfied although the Poincaré inequality does hold. Excepting such pathological regions, the negation of (3.23) intuitively suggests that Ω contains balls of arbitrarily large radius. However, for such a domain Ω the Poincaré inequality is known to be false (see [9], Corollary to Theorem 1). We also note that if Ω does contain arbitrarily large balls then in (3.19)

$\delta = 0$, and so we merely have the obvious result that

$$\| E : H_0^{1,p}(\Omega), L^p(\Omega) \| \leq 1.$$

To conclude this section we investigate the properties of the multiplication maps Q regarded as k-set contractions, and obtain as a consequence conditions for these maps to be compact.

THEOREM 3.12. *Let* $t > 0$, $p > 1 + \frac{1}{t}$, $p > \frac{n}{t}$, $0 < d \leq 1$, *let* $\alpha > 0$ *satisfy* $p + n(1-p) < \alpha < p - \frac{n}{t}$, *and define*

$$C(\alpha, p, t, d, n) = \omega_n^{-1} n^{-1/t} d^{-n} \left(\frac{p - 1 - \frac{1}{t}}{p - \alpha - \frac{n}{t}} \right)^{p-1-(1/t)} (w_\alpha(d))^{-1},$$

$$\delta = \liminf_{|x| \to \infty,\, x \in \Omega} \operatorname{meas}(B(x,d) \setminus \Omega)/\operatorname{meas} B(d).$$

Suppose also that $N_t(m^{-1}, x)$ *is locally bounded.*
Then:

(i) *If* $M_{\alpha,d}(|Q_{\Omega \cap B(R)}|^p) < \infty$ *for all large enough* R, *and*

$$\liminf_{R \to \infty} \{ C(\alpha,p,t,d,n)(1-\delta)^{\left(p - \frac{n}{t} - \alpha\right)/n} M_{\alpha,d}(|Q_{\Omega(R)}|^p N_{t,d}(m^{-1}_{\Omega(R)}, \cdot))\}^{1/p} = k_0, \tag{3.24}$$

then $Q : H_0^{1,p}(\Omega, m) \to L^p(\Omega)$ *is a* k_0*-set contraction.*

(ii) *If* $M_{\alpha,d}(|Q_{\Omega \cap B(R)}|^p m) < \infty$ *for all large enough* R, *and*

$$\liminf_{R \to \infty} \{ C(\alpha,p,t,d,n)(1-\delta)^{\left(p - \frac{n}{t} - \alpha\right)/n} M_{\alpha,d}(|Q_{\Omega(R)}|^p N_{t,d}(m^{-1}_{\Omega(R)}, \cdot) m)\}^{1/p} = k_0, \tag{3.25}$$

then $Q : H_0^{1,p}(\Omega, m) \to L^p(\Omega, m)$ *is a* k_0*-set contraction.*

PROOF. (i) From (3.9) and (3.24), and taking account of the remark at the end of the proof of Theorem 3.6, we have that

$$\liminf_{R \to \infty} \| Q_{\Omega(R)} : H_0^{1,p}(\Omega, m), L^p(\Omega) \| \leq k_0 . \tag{3.26}$$

For each $R > 0$ let $\theta_R \in C_0^\infty(B(2R))$ be such that $0 \leq \theta_R \leq 1$, with $\theta_R(x) = 1$ for all x in $B(R)$, and write $Q = Q\theta_R + Q(1-\theta_R)$. Since $|Q(1-\theta_R)| \leq |Q_{\Omega(R)}|$, it follows from (3.26) that given any $\mathcal{E} > 0$ there exists $R > 0$ such that

$$\| Q(1-\theta_R) : H_0^{1,p}(\Omega, m), L^p(\Omega) \| \leq k_0 + \mathcal{E}. \tag{3.27}$$

We shall prove that $Q\theta_R : H_0^{1,p}(\Omega, m) \to L^p(\Omega)$ is compact: once this has been done it will follow that Q is a $(k_0 + \mathcal{E})$-set contraction, and part (i) will be immediate, since $\mathcal{E}$ may be chosen arbitrarily small.

Given any u in $H_0^{1,p}(\Omega, m)$ it is clear that $\theta_R u \in H_0^{1,p}(\Omega \cap B(2R), m)$ and that

$$\| \theta_R u \|_{1,p,m} \leq K \| u \|_{1,p,m},$$

where K depends only on θ_R. Moreover, if $v \in H_0^{1,p}(\Omega \cap B(2R), m)$ then with $\frac{1}{\tau} = 1 + \frac{1}{t}$ and $i = 0,1$,

$$\|D^i v\|_{0,p\tau}^{p\tau} = \int_{\Omega \cap B(2R)} |D^i v(x)|^{p\tau} m^\tau(x) m^{-\tau}(x)\, dx$$

$$\leq \left\{ \int_{\Omega \cap B(2R)} |D^i v(x)|^p m(x)\, dx \right\}^\tau \left\{ \int_{\Omega \cap B(2R)} m^{-t}(x)\, dx \right\}^{1/(1+t)}$$

since $(1/\tau)' = 1 + t$.

Hence

$$\| D^i v \|_{0,p\tau} \leq \| D^i v \|_{0,p,m} \left\{ \int_{\Omega \cap B(2R)} m^{-t}(x)\, dx \right\}^{1/(pt)}.$$

It follows that multiplication by θ_R is a bounded map from $H_0^{1,p}(\Omega, m)$ to $H_0^{1,p\tau}(\Omega \cap B(2R))$. Also, for any $s < 1$ the natural embedding of $H_0^{1,p\tau}(\Omega \cap B(2R))$ in $H_0^{s,p\tau}(\Omega \cap B(2R))$ is compact (see [23], Chap. 2, Theorem 4.4). From Theorem 3.5 we see that multiplication by Q is a bounded map of $H_0^{s,p\tau}(\Omega \cap B(2R))$ into $L^p(\Omega)$ provided that

$$\frac{1}{p} > \frac{1}{p\tau} - \frac{s}{n} \text{ and } \frac{\alpha - n}{p} < s - \frac{n}{p\tau},$$

that is to say provided that $s > \frac{n}{pt}$ and $\alpha < sp - \frac{n}{t}$. But since $1 > \frac{n}{pt}$

and $\alpha < p - \frac{n}{t}$ there is evidently an $s < 1$ which satisfies these requirements, and so $Q\theta_R : H_0^{1,p}(\Omega, m) \to L^p(\Omega)$ is compact. This concludes the proof of (i): that of (ii) is similar.

Similar results hold under the conditions of Corollary 3.7, namely

COROLLARY 3.13. *Let* $t > 0$, $s > 1$, $p > 1 + \frac{1}{t}$, $p > n\left(\frac{1}{t} + \frac{1}{s}\right)$, $0 < d \leq 1$, $\frac{1}{r} = \frac{1}{s} + \frac{1}{t}$, *and* $\delta = \liminf_{|x| \to \infty,\, x \in \Omega} \operatorname{meas}(B(x, d) \setminus \Omega)/\operatorname{meas} B(d)$. *Then if* $N_{s,d}(|Q|^p_{\Omega \cap B(R)}) < \infty$ *for all large enough* R, *and*

$$\liminf_{R \to \infty} \left\{ n^{-1/r} \left(\frac{p - \frac{1}{r}}{p - \frac{n}{r}} \right)^{p - \frac{1}{r}} (1-\delta)^{\frac{p}{n} - \frac{1}{r}} N_{s,d}(|Q_{\Omega(R)}|^p N_{t,d}(m^{-1}_{\Omega(R)}, \cdot)) \right\}^{1/p} = k_0, \tag{3.28}$$

the map $Q : H_0^{1,p}(\Omega, m) \to L^p(\Omega)$ *is a* k_0*-set contraction. A similar result holds for* $Q : H_0^{1,p}(\Omega, m) \to L^p(\Omega, m)$.

The particular case of Corollary 3.13 obtained when $Q = E$, $m = 1$, and $r = s = t = \infty$ is important enough to state as a separate Corollary, as follows:

COROLLARY 3.14. *Let* $p > 1$, $0 < d \leq 1$, *and*

$$\delta = \liminf_{|x| \to \infty,\, x \in \Omega} \operatorname{meas}(B(x, d) \setminus \Omega)/\operatorname{meas} B(d).$$

Then the natural embedding of $H_0^{1,p}(\Omega)$ *in* $L^p(\Omega)$ *is a* $(1-\delta)^{1/n}$*-set contraction.*

If in Theorem 3.12 or Corollary 3.13 we have $k_0 = 0$, then the maps Q are compact. The next Corollary deals with this situation.

COROLLARY 3.15. *Let the conditions of Theorem 3.12 be satisfied. Then the map* $Q : H_0^{1,p}(\Omega, m) \to L^p(\Omega)$ *is compact if any one of the following conditions is satisfied:*

a) $\lim_{|x| \to \infty,\, x \in \Omega} \operatorname{meas}(B(x, d) \cap \Omega) = 0$;

b) $\liminf_{R \to \infty} N_{t,d}(m^{-1}_{\Omega(R)}) = 0$ *and* $M_{\alpha,d}(|Q|^p) < \infty$;

c) $\liminf_{R \to \infty} N_{p,d}|(Q_{\Omega(R)})| = 0$ *and* $N_{t,d}(m^{-1}) < \infty$.

Similar results hold for Q regarded as a map from $H_0^{1,p}(\Omega, m)$ to $L^p(\Omega, m)$, to $L^p(\Omega, m)$, with Q replaced by $Q m^{1/p}$ above.

PROOF. Parts (*a*) and (*b*) follow immediately from (3.24). For (*c*), suppose first that $\alpha < n$. Then by Hölder's inequality, with $\gamma > 1$,

$$(3.29) \qquad \int_{\Omega \cap B(x,1)} |Q(y)|^p |x-y|^{\alpha-n} dy \leq \left\{ \int_{\Omega \cap B(x,1)} |Q(y)|^p dy \right\}^{1/\gamma} \times \left\{ \int_{\Omega \cap B(x,1)} |Q(y)|^p |x-y|^{\gamma'(\alpha-n)} dy \right\}^{1/\gamma'}$$

Now set $\varrho - n = \gamma'(\alpha - n)$. Then $\varrho < n$, and since $\varrho = \gamma' \alpha - \dfrac{n}{\gamma - 1}$ we may and shall choose γ so large that $\varrho > \alpha$. Hence $M_\varrho(|Q|^p) < \infty$, and (*c*) follows from (3.29) and (3.24). The proof for $\alpha \geq n$ is similar.

The special case of Corollary 3.15 (*c*) when $m \equiv 1$ was obtained by Schechter in [22], Corollary 4.12.

Taking $Q = E$ in Corollary 3.15 we obtain

COROLLARY 3.16. *Let $t > 0$, $p > 1 + \frac{1}{t}$, $p > n/t$, and suppose $N_t(m^{-1}) < \infty$. Then the canonical embedding of $H_0^{1,p}(\Omega, m)$ in $L^p(\Omega)$ is compact if either*

(*a*) $\lim_{|x| \to \infty,\, x \in \Omega} \operatorname{meas}(B(x,d) \cap \Omega) = 0$ *or* (*b*) $\liminf_{R \to \infty} N_t(m^{-1}_{\Omega(R)}) = 0.$

Berger and Schechter ([6], Theorem 2.8) obtained the particular case of Corollary 3.16(*a*) where $m \equiv 1$. Note that when $m \equiv 1$ condition (*a*) is not a necessary condition for the compactness of the embedding of $H_0^{1,p}(\Omega)$ in $L^p(\Omega)$, as the case in which Ω is the spiny urchin shows (see [10]). In fact the compactness of this embedding for unbounded domains Ω has received considerable attention in recent years: see in particular [1], where a necessary and sufficient condition is obtained. For some results concerning certain weighted spaces we refer to [2]. By way of examples of weight functions m which satisfy condition (*b*) of Corollary 3.16 we mention $m(x) = (1 + g(x))^\theta$ ($\theta > 0$), where g is any positive measurable function which tends to infinity as $|x| \to \infty$ in Ω, and also the function $m(x) = |x|^\theta$ for suitably small positive θ.

In conclusion, it should be noted that it follows from Corollary 3.13 that $H_0^{1,p}(\Omega, m)$ is compactly embedded in $L^p(\Omega, m)$ if

$$\lim_{|x| \to \infty,\, x \in \Omega} \operatorname{meas}(B(x,d) \cap \Omega) = 0.$$

4. The quasilinear Dirichlet problem.

The usefulness of the proceding theory can now be illustrated by applying the embedding theorems to various situations involving partial differential equations in unbounded domains. In the present section we provide sufficient conditions for there to be a solution of a Dirichlet problem for a quasilinear equation of elliptic type: the idea of the proof is to reduce the problem to that of solving an abstract operator equation involving a coercive pseudo-monotone map from a Banach space to its dual, and then to invoke the theory of pseudo-monotone maps. All the functions and spaces occurring in this section will be real, so that this theory of pseudo-monotone maps may be used.

Let k be a positive integer, and let $s(k)$ denote the number of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers α_i such that $|\alpha| \leq k$. For each α with $|\alpha| \leq k$ a continuous function $A_\alpha : \Omega \times R^{s(k)} \to R$ is defined, and the differential operator we shall consider to begin with is given by

$$Au(x) = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi_k(u)(x)),$$

where $\xi_k(u)(x) = \{D^\alpha u(x) : |\alpha| \leq k\}$. Let p satisfy $1 < p < \infty$: we shall give conditions on the A_α which are sufficient to ensure that A induces a bounded and continuous map from $H_0^{k,p}(\Omega, m)$ to its dual.

Let $t > \max(1, n/p)$, $p > 1 + \dfrac{1}{t}$, and suppose that $N_t(m^{-1}) < \infty$. We shall assume that there are non-negative functions $a_{\alpha\beta}$ such that for all α with $|\alpha| \leq k$, all x in Ω, and all $z = (z_\beta)$ and $z' = (z'_\beta)$ in $R^{s(k)}$,

$$(4.1) \qquad |A_\alpha(x, z) - A_\alpha(x, z')| \leq \sum_{|\beta| \leq k} a_{\alpha\beta}(x)(|z_\beta| + |z'_\beta|)^{p-2} |z_\beta - z'_\beta|,$$

and that for each α, $|\alpha| \leq k$,

$$(4.2) \qquad m^{-1} A_\alpha(\cdot, 0) \in L^{p'}(\Omega, m).$$

The functions $a_{\alpha\beta}$ are restricted by the following assumptions:

(i) if $|\alpha| = |\beta| = k$, then $m^{-1} a_{\alpha\beta} \in L^\infty(\Omega)$;

(ii) if $|\alpha| = k$ and $|\beta| < k$, then $M_{\mu(\alpha,\beta)}(a_{\alpha\beta}^{p'} m^{1-p'}) < \infty$ for some $\mu(\alpha, \beta)$ such that $0 < \mu(\alpha, \beta) < p(k - |\beta|) - \dfrac{n}{t}$;

(iii) if $|\alpha| < k$ and $|\beta| = k$, then $M_{\mu(\alpha,\beta)}(a^p_{\alpha\beta} m^{1-p}) < \infty$ for some $\mu(\alpha, \beta)$ such that $0 < \mu(\alpha, \beta) < p(k - |\alpha|) - \frac{n}{t}$;

(iv) if $|\alpha| \leq k-1$ and $|\beta| \leq k-1$, then $M_{\mu(\alpha,\beta)}(a_{\alpha\beta}) < \infty$ for some $\mu(\alpha, \beta)$ such that $0 < \mu(\alpha, \beta) < p\{k - \max(|\alpha|, |\beta|)\} - \frac{n}{t}$.

Now define

$$a(u, \Phi) = \int_\Omega \sum_{|\alpha| \leq k} A_\alpha(x, \xi_k(u)(x)) D^\alpha \Phi(x)\, dx$$

for $u, \Phi \in H_0^{k,p}(\Omega, m)$. The following lemma is important in our discussion of the Dirichlet problem.

LEMMA 4.1. *Let $t > \max(1, n/p)$, $p > 1 + \frac{1}{t}$, $N_t(m^{-1}) < \infty$, and suppose that* (4.1), (4.2) *and conditions* (i)-(iv) *hold. Then there is a constant C such that for all u and Φ in $H_0^{k,p}(\Omega, m)$,*

$$|a(u, \Phi)| \leq C \|\Phi\|_{k,p,m} (1 + \|u\|^{p-1}_{k,p,m}).$$

PROOF. It is clear that

$$|a(u, \Phi)| \leq \sum_{|\alpha| \leq k} \int_\Omega \{|A_\alpha(x, 0)| + \sum_{|\beta| \leq k} a_{\alpha\beta}(x) |D^\beta u(x)|^{p-1}\} |D^\alpha \Phi(x)|\, dx,$$

and to obtain the required result we estimate the various kinds of terms separately. First we have by (4.2) that for all α, by Hölder's inequality,

$$\int_\Omega |A_\alpha(x, 0)|\ |D^\alpha \Phi(x)|\, dx \leq \|A_\alpha(\cdot, 0) m^{-1}\|_{0,p',m} \|D^\alpha \Phi\|_{0,p,m}.$$

If $|\alpha| = |\beta| = k$, then by condition (i) we see that

$$\int_\Omega a_{\alpha\beta}(x) |D^\beta u(x)|^{p-1} |D^\alpha \Phi(x)|\, dx \leq \text{const.} \|D^\alpha \Phi\|_{0,p,m} \|D^\beta u\|^{p-1}_{0,p,m}.$$

To handle the situation in which either $|\alpha|$ or $|\beta|$ is less than k we use the embedding result contained in Theorem 3.3. Suppose $|\alpha| = k$ and $|\beta| < k$.

Then clearly

$$(4.3)\qquad \int_\Omega a_{\alpha\beta}(x)\,|D^\beta u(x)|^{p-1}\,|D^\alpha \Phi(x)|\,dx \leq \|D^\alpha \Phi\|_{0,p,m}\,\|m^{-1}a_{\alpha\beta}\,|D^\beta u|^{p-1}\|_{0,p',m}$$

$$= \|D^\alpha \Phi\|_{0,p,m}\,\|(m^{-1}a_{\alpha\beta})^{1/(p-1)}D^\beta u\|^{p-1}_{0,p,m}\,.$$

It is now a simple matter to verify that Theorem 3.3 (ii) may be applied to show that multiplication by $(m^{-1}a_{\alpha\beta})^{1/(p-1)}$ is a bounded map of $H_0^{k-|\beta|,p}(\Omega,m)$ into $L^p(\Omega,m)$, so that the right hand side of (4.3) is majorised by

$$\text{const.}\,\|D^\alpha \Phi\|_{0,p,m}\,\|D^\beta u\|^{p-1}_{k-|\beta|,p,m}\,.$$

Similarly, when $|\alpha| < k$ and $|\beta| = k$, we have that

$$(4.4)\qquad \int_\Omega a_{\alpha\beta}(x)\,|D^\beta u(x)|^{p-1}\,|D^\alpha \Phi(x)|\,dx \leq \|D^\beta u\|^{p-1}_{0,p,m}\,\|m^{-1}a_{\alpha\beta}D^\alpha \Phi\|_{0,p,m}\,,$$

and in view of the conditions imposed in the lemma we may invoke Theorem 3.3(ii) again to show that multiplication by $(m^{-1}a_{\alpha\beta})$ is a bounded map of $H_0^{k-|\alpha|,p}(\Omega,m)$ into $L^p(\Omega,m)$. Hence the integral in (4.4) is dominated by

$$\text{const.}\,\|D^\alpha \Phi\|_{k-|\alpha|,p,m}\,\|D^\beta u\|^{p-1}_{0,p,m}\,.$$

It remains to deal with the terms with $|\alpha| < k$ and $|\beta| < k$. In this case we apply Hölder's inequality to obtain

$$(4.5)\qquad \int_\Omega a_{\alpha\beta}(x)\,|D^\beta u(x)|^{p-1}\,|D^\alpha \Phi(x)|\,dx \leq$$

$$\leq \|(m^{-1}a_{\alpha\beta})^{1/p}D^\alpha \Phi\|_{0,p,m}\,\|(m^{-1}a_{\alpha\beta})^{1/p'}\,|D^\beta u|^{p-1}\|_{0,p',m}$$

$$= \|(m^{-1}a_{\alpha\beta})^{1/p}D^\alpha \Phi\|_{0,p,m}\,\|(m^{-1}a_{\alpha\beta})^{1/p}D^\beta u\|^{p-1}_{0,p,m}\,.$$

Once more we appeal to Theorem 3.3(ii) to show that multiplication by $(m^{-1}a_{\alpha\beta})^{1/p}$ is a bounded map of $H_0^{k-|\alpha|,p}(\Omega,m)$ (and $H_0^{k-|\beta|,p}(\Omega,m)$) into $L^p(\Omega,m)$. The integral in (4.5) is therefore dominated by

$$\text{const.}\,\|D^\alpha \Phi\|_{k-|\alpha|,p,m}\,\|D^\beta u\|^{p-1}_{k-|\beta|,p,m}\,.$$

The proof of the lemma is now complete.

In view of Lemma 4.1 it is clear that under the conditions of that Lemma, the map $\Phi \mapsto a(u, \Phi)$ is, for each fixed u in $H_0^{k,p}(\Omega, m)$, a bounded linear functional $T(u)$ on $H_0^{k,p}(\Omega, m)$. Thus if we write $X = H_0^{k,p}(\Omega, m)$ and write X^* for the dual of X, a map $T: X \to X^*$ is defined by

$$a(u, \Phi) = (T(u), \Phi)$$

for all u, Φ in X: here $(T(u), \Phi)$ stands for the value of the linear functional $T(u)$ at Φ.

LEMMA 4.2. *Suppose $p \geq 2$ and the conditions of Lemma 4.1 hold, and let $T: X \to X^*$ be the map induced by the form $a(u, \Phi)$. Then T is bounded and continuous.*

PROOF. The boundedness of T is clear from Lemma 4.1. As for the continuity, let (u_j) be a sequence in X which converges to $u \in X$. Then for all Φ in X and all j,

$$\begin{aligned} |(T(u) - T(u_j), \Phi)| &= |a(u, \Phi) - a(u_j, \Phi)| \\ &\leq \int_\Omega \sum_{|\alpha| \leq k} |A_\alpha(x, \xi_k|(u)(x)) - A_\alpha(x, \xi_k(u_j)(x)|\ |D^\alpha \Phi(x)|\, dx \\ &\leq \int_\Omega \sum_{|\alpha|, |\beta| \leq k} a_{\alpha\beta}(x)(|D^\beta u(x)| + |D^\beta u_j(x)|)^{p-2} |D^\beta(u - u_j)(x)| \\ &\qquad \times |D^\alpha \Phi(x)|\, dx \\ &= \sum_{|\alpha|, |\beta| \leq k} I_{\alpha\beta} \text{ say.} \end{aligned}$$

We now estimate the various terms in much the same way as in the proof of Lemma 4.1. If $|\alpha| = |\beta| = k$, then by Hölder's inequality,

$$\text{(4.6)} \qquad I_{\alpha\beta} \leq \text{const.}\, \|D^\alpha \Phi\|_{0,p,m}$$

$$\cdot \left\{ \int_\Omega (|D^\beta u| + |D^\beta u_j|)^{p(p-2)/(p-1)} |D^\beta(u - u_j)|^{p/(p-1)}\, m dx \right\}^{(p-1)/p}$$

$$\leq \text{const.}\, \|D^\alpha \Phi\|_{0,p,m} \|D^\beta(u - u_j)\|_{0,p,m} \|\,|D^\beta u| + |D^\beta u_j|\,\|_{0,p,m}^{p-2}$$

$$\leq \text{const.}\, \|D^\alpha \Phi\|_{0,p,m} \|D^\beta(u - u_j)\|_{0,p,m} (\|D^\beta u\|_{0,p,m} + \|D^\beta u_j\|_{0,p,m})^{p-2}$$

$$\leq \text{const.}\, \|\Phi\|_{k,p,m} \|u - u_j\|_{k,p,m} (\|u\|_{k,p,m} + \|u_j\|_{k,p,m})^{p-2}.$$

If $|\alpha| = k$ and $|\beta| < k$ we have that

$$I_{\alpha\beta} \leq \| D^\alpha \Phi \|_{0,p,m}$$

$$\cdot \left\{ \int_\Omega (m^{-1} a_{\alpha\beta})^{p/(p-1)} (|D^\beta u| + |D^\beta u_j|)^{p(p-2)/(p-1)} |D^\beta (u-u_j)|^{p/(p-1)} m dx \right\}^{(p-1)/p}$$

$$\leq \| D^\alpha \Phi \|_{0,p,m} \| (m^{-1} a_{\alpha\beta})^{1/(p-1)} D^\beta (u-u_j) \|_{0,p,m} \| (m^{-1} a_{\alpha\beta})^{1/(p-1)} (|D^\beta u| + |D^\beta u_j|) \|_{0,p,m}^{p-2}.$$

Since by Theorem 3.3 (ii) multiplication by $(m^{-1} a_{\alpha\beta})^{1/(p-1)}$ is a bounded map from $H_0^{k-|\beta|,p}(\Omega, m)$ to $L^p(\Omega, m)$ we again obtain an estimate of the form of (4.6). In the case when $|\alpha| < k$ and $|\beta| = k$ we see that

$$I_{\alpha\beta} \leq \| D^\beta (u - u_j) \|_{0,p,m}$$

$$\cdot \left\{ \int_\Omega (m^{-1} a_{\alpha\beta})^{p/(p-1)} (|D^\beta u| + |D^\beta u_j|)^{p(p-2)/(p-1)} |D^\alpha \Phi|^{p/(p-1)} m dx \right\}^{(p-1)/p}$$

$$\leq \| D^\beta (u - u_j) \|_{0,p,m} \| (m^{-1} a_{\alpha\beta}) D^\alpha \Phi \|_{0,p,m} \| |D^\beta u| + |D^\beta u_j| \|_{0,p,m}^{p-2},$$

and an estimate like (4.6) follows from Theorem 3.3 (ii).

Finally, when $|\alpha| < k$ and $|\beta| < k$,

$$(4.7) \quad I_{\alpha\beta} \leq \| (m^{-1} a_{\alpha\beta})^{1/p} D^\alpha \Phi \|_{0,p,m} \| (m^{-1} a_{\alpha\beta})^{1/p'} (|D^\beta u| + |D^\beta u_j|)^{p-2} D^\beta (u-u_j) \|_{0,p',m}$$

$$\leq \| (m^{-1} a_{\alpha\beta})^{1/p} D^\alpha \Phi \|_{0,p,m} \| (m^{-1} a_{\alpha\beta})^{1/p} (|D^\beta u| + |D^\beta u_j|) \|_{0,p,m}^{p-2}$$

$$\cdot \| (m^{-1} a_{\alpha\beta})^{1/p} D^\beta (u - u_j) \|_{0,p,m}.$$

The customary use of Theorem 3.3 (ii) yields an estimate like (4.6) once more.

It follows that there is a constant C, independent of u, u_j and Φ, such that

$$|(T(u) - T(u_j), \Phi)| \leq C \| \Phi \|_{k,p,m} (\| u \|_{k,p,m} + \| u_j \|_{k,p,m})^{p-2} \| u - u_j \|_{k,p,m}.$$

Hence

$$(4.8) \quad \| T(u) - T(u_j) \| \leq C (\| u \|_{k,p,m} + \| u_j \|_{k,p,m})^{p-2} \| u - u_j \|_{k,p,m}.$$

The continuity of T is now obvious.

Under sharper restrictions on the differential operator it can be proved that the induced map from X to X^* is compact. Let

$$Bu(x) = \sum_{|\alpha| \leq k-1} (-1)^{|\alpha|} D^\alpha B_\alpha (x, \xi_{k-1}(u)(x)),$$

where each $B_\alpha : \Omega \times R^{s(k-1)} \to R$ is continuous and satisfies a strengthened form of condition (iv). We let $t > \max\left(1, \frac{n}{p}\right)$, $p \geq 2$, and suppose $N_t(m^{-1}) < \infty$. Assume further that there are functions $b_{\alpha\beta}$ such that for all α with $|\alpha| \leq k-1$, all x in Ω, and all $z = (z_\beta)$ and $z' = (z'_\beta)$ in $R^{s(k-1)}$,

$$(4.9) \qquad | B_\alpha(x, z) - B_\alpha(x, z') | \leq \sum_{|\beta| \leq k-1} b_{\alpha\beta}(x) (| z_\beta | + | z'_\beta |)^{p-2} | z_\beta - z'_\beta | ,$$

where the $b_{\alpha\beta}$ satisfy:

(v) $M_{\nu(\alpha,\beta)}(b_{\alpha\beta}) < \infty$ for some $\nu(\alpha, \beta)$ such that

$$0 < \nu(\alpha, \beta) < p\,\{k - \max(|\alpha|, |\beta|)\} - \frac{n}{t},$$

and $\liminf_{R\to\infty} N_1((b_{\alpha\beta})_{\Omega(R)}) = 0$. Suppose also that $m^{-1} B_\alpha(\cdot, 0) \in L^{p'}(\Omega, m)$ for all α.

It is now clear that the form

$$b(u, \Phi) = \int_\Omega \sum_{|\alpha| \leq k-1} B_\alpha(x, \xi_{k-1}(u)(x))\, D^\alpha \Phi(x)\, dx$$

induces a bounded and continuous map $S : X \to X^*$ by the rule that

$$b(u, \Phi) = (S(u), \Phi)$$

for all u and Φ in X. However, we also have

LEMMA 4.3. *Under condition* (v), S *is completely continuous, that is,* $S u_j \to S u$ *whenever* $u_j \to u$ *weakly.*

PROOF. Let (u_j) be a sequence in X which converges weakly to $u \in X$, and let $\Phi \in X$.

Just as in the proof of Lemma 4.2 we obtain the equivalent of (4.7):

$$(4.10) \qquad \int_\Omega b_{\alpha\beta} (| D^\beta u | + | D^\beta u_j |)^{p-2} \, | D^\beta (u - u_j) | \, | D^\alpha \Phi | \, dx$$

$$\leq \| (m^{-1} b_{\alpha\beta})^{1/p} D^\alpha \Phi \|_{0,p,m} \, \| (m^{-1} b_{\alpha\beta})^{1/p} (| D^\beta u | + | D^\beta u_j |) \|_{0,p,m}^{p-2}$$

$$\cdot \| (m^{-1} b_{\alpha\beta})^{1/p} D^\beta (u - u_j) \|_{0,p,m} .$$

The first two factors on the right hand side of (4.10) are handled as in the proof of Lemma 4.2, by an application of Theorem 3.3(ii), and for the last factor we are able to appeal to condition (v) and Corollary 3.15(c) to show that multiplication by $(m^{-1} b_{\alpha\beta})^{1/p}$ is a compact map from $H_0^{1,p}(\Omega,m)$ to $L^p(\Omega,m)$, and so is compact regarded as a map from $H_0^{k-|\beta|,p}(\Omega, m)$ to $L^p(\Omega, m)$. Since we have, as in Lemma 4.2,

$$\| S(u) - S(u_j) \|$$
$$\leq C \sum_{|\alpha|,\, |\beta| \leq k-1} (\| u \|_{k,p,m} + \| u_j \|_{k,p,m})^{p-2} \; \| (m^{-1} b_{\alpha\beta})^{1/p} D^\beta (u - u_j) \|_{0,p,m},$$

the compactness of S follows immediately if we note that the weakly convergent sequence (u_j) is bounded.

We shall now apply these results about boundedness, continuity and compactness to a Dirichlet problem. Let

$$\text{(4.11)} \qquad \begin{aligned} Au(x) &= \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha A_\alpha (x, \xi_k(u)(x)), \\ Bu(x) &= \sum_{|\alpha| \leq k-1} (-1)^{|\alpha|} D^\alpha B_\alpha (x, \xi_{k-1}(u)(x)), \end{aligned}$$

suppose the conditions of Lemma 4.2 and 4.3 hold, and write

$$\text{(4.12)} \qquad Nu(x) = Au(x) + Bu(x) = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha N_\alpha (x, \xi_k(u)(x)).$$

We require further that for all u and v in $X = H_0^{k,p}(\Omega, m)$,

$$\text{(4.13)} \qquad \sum_{|\alpha| \leq k} \int_\Omega \{ A_\alpha (x, \xi_k(u)(x)) - A_\alpha (x, \xi_k(v)(x)) \} \; D^\alpha (u - v)(x)\, dx \geq 0,$$

and that there should be a function $c : R_+ \to R$ with $c(r) \to \infty$ as $r \to \infty$, such that for all u in X,

$$\text{(4.14)} \qquad \sum_{|\alpha| \leq k} \int_\Omega N \;\; (x, \xi_k(u)(x))\, D^\alpha u(x)\, dx \geq c(\| u \|_{k,p,m}) \, \| u \|_{k,p,m}.$$

The Dirichlet problem we shall consider is that of finding a function u such that in some sense

$$\text{(4.15)} \qquad \begin{cases} Nu = f \text{ (a prescribed function) in } \Omega, \\ \text{with } D^\alpha u = 0 \text{ on } \partial\Omega, \text{ for all } \alpha \text{ such that } |\alpha| \leq k-1. \end{cases}$$

To be more precise we shall look for a function $u \in X$ such that for all Φ in $C_0^\infty(\Omega)$,

$$(4.16) \qquad \sum_{|\alpha| \leq k} \int_\Omega N_\alpha(x, \xi_k(u)(x))\, D^\alpha \Phi(x)\, dx = \int_\Omega f(x)\, \Phi(x)\, dx .$$

We shall refer to such a function u as a variational solution of the Dirichlet problem (4.15).

THEOREM 4.4. *Let the hypotheses of Lemmas* 4.2 *and* 4.3 *hold, and suppose that* (4.13) *and* (4.14) *are satisfied. Then given any f such that $m^{-1} f \in L^{p'}(\Omega, m)$, there is a variational solution u of the Dirichlet problem* (4.15).

PROOF. Since the map $\Phi \longmapsto \int_\Omega f(x)\, \Phi(x)\, dx$ evidently induces a continuous linear functional, g say, on X, it is clear from the previous discussion that (4.16) is equivalent to the equation

$$(4.17) \qquad T(u) + S(u) = g.$$

The operators S and T map X to X^*, S being completely continuous and T bounded and continuous, in view of Lemmas 4.2 and 4.3. Moreover, condition (4.13) implies that $(T(u) - T(v), u - v) \geq 0$ for all u, v in X, so that T is monotone, while (4.14) means that $(S(u) + T(u), u) \geq c(\| u \|_{k,p,m}) \| u \|_{k,p,m}$ for all u in X, and this says that $S + T$ is coercive. Hence $S + T$ is a coercive map which is pseudo-monotone (see Lions [15], Chap. 2, pp. 179-182), and by the fundamental theorem of pseudo-monotone maps (see, for example, Lions [15], Chap. 2, Théorème 2.7), (4.17) has a solution u. It follows that there is a variational solution of (4.15).

REMARKS. 1) The monotonicity condition (4.13) is plainly satisfied if we require that for all x in Ω and all z, $z' \in R^{s(k)}$.

$$\sum_{|\alpha| \leq k} \{ A_\alpha(x, z) - A_\alpha(x, z') \} (z_\alpha - z'_\alpha) \geq 0.$$

2) Some relaxation of the conditions imposed on the coefficients $b_{\alpha\beta}$ in order to ensure that S is completely continuous is possible if the domain Ω has good enough properties, e.g. if $\lim_{|x| \to \infty,\, x \in \Omega} \operatorname{meas}(B(x, 1) \cap \Omega) = 0$,

3) As a specific example of a nonlinear operator N to which Theorem 4.4 may be applied we cite

$$Nu = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\left(\frac{\partial u}{\partial x_i}\right)^3\right) + u^3,$$

with $n = 3$, $p = 4$, and $m(x) \equiv 1$. It is a routine matter to verify that the various hypotheses of Theorem 4.4 are fulfilled, and so we omit the details. We also leave to the reader the exercise of constructing more complicated nonlinear operators to which the theorem may be applied. Linear operators are dealt with more fully in § 5.

4) The assumptions made about the coefficients in A and B, though admittedly already somewhat intricate, may be weakened quite considerably. For example, condition (4.1), which limits the growth in z of $A_\alpha(x, z)$ to be no faster than $|z|^{p-1}$, may be weakened so as to permit different growths for different α's: this is what Berger and Schechter [6] do in their discussion of the case $m(x) \equiv 1$. Again, the operator $S+T$ which arises as a consequence of our hypotheses is the sum of a monotone and a completely continuous operator, and as such is a particularly simple example of a pseudo-monotone operator. More sophisticated examples would arise if we were to proceed in the manner of Browder [8] and to weaken the monotonicity assumption (4.13). The technical complications which these two lines of development would necessitate in the case of unbounded domains are quite considerable, and accordingly we prefer not to present them here.

5. The linear Dirichlet problem.

5.1. In this section we shall be concerned with the linear case of the differential operator discussed in § 4, and we shall confine our attention to the case $p = 2$. The differential expression is now given by

$$Au(x) = \sum_{|\alpha|, |\beta| \le k} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x)), \tag{5.1}$$

where the coefficients $a_{\alpha\beta}$ are measurable (real or complex valued) on the unbounded domain $\Omega \subset R^n$. We shall begin by obtaining conditions for there to exist variational solutions of the Dirichlet problem

$$Au = f \text{ in } \Omega, \quad D^\alpha u = 0 \text{ on } \partial\Omega, \quad \text{for all } \alpha \text{ with } |\alpha| < k,$$

for a given f in $L^2(\Omega)$ (or $L^2(\Omega, m)$). More precisely, we shall prove that given any f in $L^2(\Omega)$ (or $L^2(\Omega, m)$), there exists a function $u \in H_0^{k,2}(\Omega, m)$ such that

$$a(u, v) = (f, v)_{0,2} \qquad (\text{or } (f, v)_{0,2,m})$$

for all v in $H_0^{k,2}(\Omega, m)$, where $a(u, v)$ is the sesquilinear form

$$a(u, v) = \sum_{|\alpha|, |\beta| \leq k} \int_\Omega a_{\alpha\beta}(x) D^\beta u(x) \overline{D^\alpha v(x)} \, dx . \tag{5.2}$$

The first result is a linear counterpart of Lemma 4.1. In stating it we take $d = 1$ and write M_a, N_t as indicated in § 3.1.

LEMMA 5.1. *Let $t > \max(1, n/2)$ and suppose that the coefficients $a_{\alpha\beta}$ satisfy the following conditions on Ω:*

(i) *if $|\alpha| = |\beta| = k$, let $m^{-1} a_{\alpha\beta} \in L^\infty(\Omega)$;*

(ii) *if $|\alpha| = k$ or $|\beta| = k$, $|\alpha| + |\beta| \leq 2k - 1$, let*

$$M_{\mu(\alpha,\beta)}(|a_{\alpha\beta}|^2 N_t(m^{-1}, \cdot)\, m^{-1}) < \infty$$

for some $\mu(\alpha, \beta)$ satisfying

$$0 < \mu(\alpha, \beta) < 2(k - \min[|\alpha|, |\beta|]) - \frac{n}{t} ;$$

(iii) *if $|\alpha| \leq k - 1$ and $|\beta| \leq k - 1$, let $M_{\mu(\alpha,\beta)}(|a_{\alpha\beta}| N_t(m^{-1}, \cdot)) < \infty$ for some $\mu(\alpha, \beta)$ satisfying*

$$0 < \mu(\alpha, \beta) < 2(k - \max[|\alpha|, |\mu|]) - \frac{n}{t} .$$

Then $a(u, v)$ is a bounded sesquilinear form on $H_0^{k,2}(\Omega, m) \times H_0^{k,2}(\Omega, m)$; that is, there is a positive constant K such that for all u, v in $H_0^{k,2}(\Omega, m)$,

$$|a(u, v)| \leq K \| u \|_{k,2,m} \| v \|_{k,2,m} .$$

PROOF. Apart from the facts that $N_t(m^{-1})$ is no longer required to be finite and the $a_{\alpha\beta}$ are measurable, the Lemma is a special case of Lemma 4.1, obtainable by setting $p = 2$ and $A_\alpha(x, 0) = 0$ in that Lemma. The difference in hypothesis does not alter the proof in any essential way, and accordingly we shall not present a separate proof of Lemma 5.1.

We shall prove the existence of a variational solution of the Dirichlet problem by an appeal to the Lax-Milgram theorem. To be able to do this we have to prove that $a(u, u)$ is coercive on $H_0^{k,2}(\Omega, m)$. This amounts to showing that there is a constant $K > 0$ such that for all u in $H_0^{k,2}(\Omega, m)$, $|a(u, u)| \geq K \| u \|_{k,2,m}^2$. For this we shall need to assume that

$$a_0(u, u) \equiv \sum_{|\alpha| = |\beta| = k} \int_\Omega a_{\alpha\beta} D^\beta u \, D^\alpha \bar{u} \, dx$$

is coercive on $H_0^{k,2}(\Omega, m)$, so that there exists $c_0 > 0$ such that for all u in $H_0^{k,2}(\Omega, m)$,

$$|a_0(u, u)| \geq c_0 \| u \|_{k,2,m}^2. \tag{5.3}$$

If all the functions and function spaces discussed are assumed to be real, it is enough for this to assume that A satisfies the degenerate elliptic condition

$$\sum_{|\alpha| = |\beta| = k} a_{\alpha\beta}(x) \xi^\alpha \xi^\beta \geq c_1 m(x) |\xi|^{2k} \tag{5.4}$$

for all x in Ω and all $\xi \in R^n$ (here c_1 is a positive constant independent of x and ξ), provided that the Poincaré inequality holds for the space $H_0^{k,2}(\Omega, m)$, i.e. there is a positive constant c_2 such that for all u in $H_0^{k,2}(\Omega, m)$;

$$\| u \|_{k,2,m} \leq c_2 | u |_{k,2,m} \tag{5.5}$$

(see Theorem 3.10). Provided that the lower order terms are not too large, in a certain sense, the coercivity of $a(u, u)$ may then be established. In what follows any one of (5.3) or (5.4) (and (5.5)) can be assumed, with the proviso that the latter would mean that real function spaces are implied.

LEMMA 5.2. *Suppose* $t > \max(1, n/2)$, *that conditions* (ii) *and* (iii) *of Lemma* 5.1 *are satisfied, and that* (5.3) (*or* (5.4) *and* (5.5)) *holds. Then for all* u *in* $H_0^{k,2}(\Omega, m)$,

$$\begin{aligned} |a(u, u)| \geq \Big\{ c_0 &- \sum_{\substack{|\alpha| \text{ or } |\beta| = k \\ |\alpha| + |\beta| \leq 2k-1}} \| m^{-1} a_{\alpha\beta} : H_0^{k-\min[|\alpha|, |\beta|], 2}(\Omega, m), L^2(\Omega, m) \| \\ &- \sum_{|\alpha|, |\beta| \leq k-1} \| \, |a_{\alpha\beta}|^{\frac{1}{2}} : H_0^{k-|\alpha|, 2}(\Omega, m), L^2(\Omega) \| \; \| \, |a_{\alpha\beta}|^{\frac{1}{2}} : \\ &\qquad H_0^{k-|\beta|, 2}(\Omega, m), L^2(\Omega) \| \Big\} \| u \|_{k,2,m}^2 \\ &\equiv c_3 \| u \|_{k,2,m}^2, \end{aligned} \tag{5.6}$$

so that $a(u, u)$ *is coercive on* $H_0^{k,2}(\Omega, m)$ *if* $c_3 > 0$.

PROOF. It is clear that

$$|a(u,u)| \geq |a_0(u,u)| - \sum_{\substack{|\alpha| \text{ or } |\beta| = k \\ |\alpha|+|\beta| \leq 2k-1}} \int_\Omega |a_{\alpha\beta} D^\alpha u \, D^\beta \bar{u}| \, dx$$

$$- \sum_{|\alpha|, |\beta| \leq k-1} \int_\Omega |a_{\alpha\beta} D^\beta u \, D^\alpha \bar{u}| \, dx .$$

The Lemma now follows by the arguments used in the proof of Lemma 4.1, using (5.3) (or (5.4) and (5.5)).

As an illustration of these results, let us consider

$$\mathrm{A}u(x) \equiv -\Delta u(x) + Q(x)\, u(x)$$

on a cylindrical region Ω. We have

COROLLARY 5.3. *Let Ω be a cylindrical domain in R^n such that*

$$\inf_{x \in \Omega} \operatorname{meas}(B(x,1) \setminus \Omega) \geq (1-\eta) \operatorname{meas} B(1)$$

for some η, $0 < \eta < 1$. Suppose that $N_s|Q| < \infty$ for some $s > \max(1, n/2)$, and that

$$N_s|Q| < n^{1/s} \left(\frac{2s-n}{2s-1}\right)^{2-(1/s)} \eta^{(1/s)-(2/n)} (1-\eta^{1/n})^2.$$

Then $a(u,v) \equiv \int_\Omega \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_i} + Qu\bar{v}\right) dx$ is a bounded, coercive, sesquilinear form on $H_0^{1,2}(\Omega)$.

PROOF. From (3.19) with $\delta = 1 - \eta$ we obtain

$$\| u \|_{0,2} \leq \eta^{1/n} (\| u \|_{0,2} + \| Du \|_{0,2}),$$

so that

$$\| u \|_{0,2} \leq \left(\frac{\eta^{1/n}}{1-\eta^{1/n}}\right) | Du \|_{0,2}$$

and $\| u \|_{1,2} \leq (1-\eta^{1/n})^{-1} | u |_{1,2}$.

Moreover, since

$$a_0(u,u) = \sum_{i=1} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 dx = | u |_{1,2}^2 \geq (1-\eta^{1/n})^2 \| u \|_{1,2}^2 ,$$

we have that (5.3) holds with $c_0 = (1-\eta^{1/n})^2$.

Again, from (3.15) with $m = 1$, $t = \infty$ we have

$$\| \, | Q |^{\frac{1}{2}} : H_0^{1,2}(\Omega), L^2(\Omega) \| \leq \left\{ n^{-1/s} \left(\frac{2s-1}{2s-n} \right)^{2-(1/s)} \eta^{(2/n)-(1/s)} N_s(Q) \right\}^{\frac{1}{2}} .$$

The result now follows from Lemmas 5.1 and 5.2.

Lemmas 5.1 and 5.2 enable an existence theorem to be established.

THEOREM 5.4. *Let the conditions of Lemmas* 5.1 *and* 5.2 *be satisfied, let* $c_3 > 0$, *and suppose* $N_t(m^{-1}) < \infty$. *Then given any* f *in* $L^2(\Omega)$ (*or* $L^2(\Omega, m)$), *there is a unique* u *in* $H_0^{k,2}(\Omega, m)$ *such that for all* $v \in H_0^{k,2}(\Omega, m)$,

$$a(u,v) = (f, v)_{0,2} \quad (\text{or } (f,v)_{0,2,m}) .$$

PROOF. Let $f \in L^2(\Omega)$ and $v \in H_0^{k,2}(\Omega, m)$. Then since $H_0^{k,2}(\Omega, m) \subset L^2(\Omega)$ (see (3.7)),

$$|(f,v)_{0,2}| \leq \|f\|_{0,2} \| v \|_{0,2} \leq K \| f \|_{0,2} \| v \|_{k,2,m} .$$

Thus the map $v \longmapsto \overline{(f, v)}_{0,2}$ is a continuous linear functional on $H_0^{k,2}(\Omega, m)$, and the theorem follows immediately from the Lax-Milgram theorem. The case when $f \in L^2(\Omega, m)$ is handled in a similar fashion.

In fact, the Lax-Milgram theorem implies that given any F in the dual $H^{-k,2}(\Omega, m)$ of $H_0^{k,2}(\Omega, m)$, there exists a unique u in $H_0^{k,2}(\Omega, m)$ such that for all v in $H_0^{k,2}(\Omega, m)$,

$$a(u,v) = \langle F, v \rangle_{k,2,m} , \tag{5.7}$$

where $\langle \cdot, \cdot \rangle_{k,2,m}$ denotes the duality between $H_0^{k,2}(\Omega, m)$ and $H^{-k,2}(\Omega, m)$. Thus the above theorem merely reasserts the properties

$$H_0^{k,2}(\Omega, m) \subset L^2(\Omega) \subset H^{-k,2}(\Omega, m), \; H_0^{k,2}(\Omega, m) \subset L^2(\Omega, m) \subset H^{-k,2}(\Omega, m),$$

the duals of the L^2 spaces being identified with the spaces themselves, the duality being expressed by their inner products.

Given any f *in* $L^2(\Omega)$, let us write the unique solution u of

$$a(u, v) = (f, v)_{0,2} \text{ for all } v \in H_0^{k,2}(\Omega, m)$$

as $u = Gf$. Then $G: L^2(\Omega) \to H_0^{k,2}(\Omega, m)$, and $a(Gf, v) = (f, v)_{0,2}$. Substitution of $u = Gf$ in (5.6) yields

$$\begin{aligned} c_3 \| Gf \|_{k,2,m}^2 &\leq | a(Gf, Gf) | = | (f, Gf)_{0,2} | \\ &\leq \|f\|_{0,2} \; \|Gf\|_{0,2} \\ &\leq \| E: H_0^{k,2}(\Omega, m), L^2(\Omega) \| \; \|f\|_{0,2} \; \|Gf\|_{k,2,m} . \end{aligned}$$

Hence G is a bounded linear map, and

$$\| G: L^2(\Omega), H_0^{k,2}(\Omega, m) \| \leq c_3^{-1} \| E: H_0^{k,2}(\Omega, m), L^2(\Omega) \|. \tag{5.8}$$

Equation (5.7) implies the existence of a map $\mathcal{G}: H^{-k,2}(\Omega,m) \to H_0^{k,2}(\Omega,m)$ such that $a(\mathcal{G}F, v) = \langle F, v \rangle_{k,2,m}$ for all F in $H^{-k,2}(\Omega, m)$ and all v in $H_0^{k,2}(\Omega, m)$. As for G above, we have from (5.6) that $\mathcal{G}$ is bounded and

$$\| \mathcal{G}: H^{-k,2}(\Omega, m), H_0^{k,2}(\Omega, m) \| \leq c_3^{-1} . \tag{5.9}$$

Clearly $\mathcal{G}^{-1}$ exists, and also from

$$K \| \mathcal{G}F \|_{k,2,m}^2 \geq | a(\mathcal{G}F, \mathcal{G}F) | = | \langle F, \mathcal{G}F \rangle_{k,2,m} |$$

it follows that $\mathcal{G}^{-1}$ is bounded on its domain in $H_0^{k,2}(\Omega, m)$. But the domain of $\mathcal{G}^{-1}$ is dense in $H_0^{k,2}(\Omega,m)$; for if we suppose that there exists $F \in H^{-k,2}(\Omega,m)$ which satisfies $\langle F, v \rangle_{k,2,m} = 0$ for all v in the domain of $\mathcal{G}^{-1}$, we obtain

$$0 = \langle F, \mathcal{G}F \rangle_{k,2,m} = a(\mathcal{G}F, \mathcal{G}F) \geq c_3 \| \mathcal{G}F \|_{k,2,m}^2 .$$

Thus $\mathcal{G}F = 0$, and consequently $F = 0$ as required. Hence $\mathcal{G}$ is an isomorphism of $H^{-k,2}(\Omega, m)$ onto $H_0^{k,2}(\Omega, m)$.

Now let $E^*: L^2(\Omega) \to H^{-k,2}(\Omega, m)$ denote the adjoint of the embedding $E: H_0^{k,2}(\Omega, m) \to L^2(\Omega)$. Then for all f in $L^2(\Omega)$ and all v in $H_0^{k,2}(\Omega, m)$,

$$a(Gf, v) = (f, v)_{0,2} = (f, Ev)_{0,2} = \langle E^* f, v \rangle_{k,2,m} = a(\mathcal{G}E^* f, v),$$

Hence $G = \mathcal{G} E^*$, and so, considered as an operator in $L^2(\Omega)$ we have

$$(5.10) \qquad EG = E\mathcal{G}E^*.$$

Note that if E is a k_0-set contraction then EG is a k-set contraction for some $k \leq c_3^{-1} k_0^2$, in view of (5.8). Of particular interest is the case when $c_3^{-1} k_0^2 < 1$, as then EG is a Fredholm map of index zero.

A similar analysis can be carried out when $L^2(\Omega)$ is replaced by $L^2(\Omega, m)$.

5.2. We shall now use the preceding results to investigate the properties of the operator $\mathcal{A}$ defined in $L^2(\Omega)$ as follows: let the domain of $\mathcal{A}$ be

$$\mathcal{D}(\mathcal{A}) = \{ u \in H_0^{k,2}(\Omega, m) : Au \in L^2(\Omega) \},$$

and define $\mathcal{A}$ by $\mathcal{A}u = Au$ for $u \in \mathcal{D}(\mathcal{A})$. Here we understand by A the formal expression associated with the Dirichlet form $a(u, v)$, so that for all v in $H_2^{k,2}(\Omega, m)$ and all u in $\mathcal{D}(\mathcal{A})$, $(Au, v)_{0,2} = a(u, v)$: it is assumed throughout that the conditions of Lemmas 5.1 and 5.2 hold, and that $N_t(m^{-1}) < \infty$. Hence $u \in \mathcal{D}(\mathcal{A})$ if and only if there exists f in $L^2(\Omega)$ such that for all v in $H_0^{k,2}(\Omega, m)$,

$$a(u, v) = (f, v)_{0,2},$$

and we write in this case $Au = f$. In other words, $\mathcal{D}(\mathcal{A})$ is the range of the operator EG in § 5.1.

We note that $\mathcal{D}(\mathcal{A})$ is dense in $L^2(\Omega)$, for if there exists g in $L^2(\Omega)$ for which $(EGf, g)_{0,2} = 0$ for all f in $L^2(\Omega)$, then

$$0 = (EGf, g)_{0,2} = (E\mathcal{G}E^* f, g)_{0,2} = (f, E\,\mathcal{G}^*\,E^* g)_{0,2},$$

so that $E\,\mathcal{G}^*\,E^*\,g = 0$. This in turn implies that $E^*\,g = 0$, and hence $g = 0$ since E has dense range (containing $C_0^\infty(\Omega)$) in $L^2(\Omega)$.

From these remarks it is apparent that $\mathcal{A}^{-1}$ exists, is bounded on $L^2(\Omega)$, and

$$(5.11) \qquad \mathcal{A}^{-1} = EG = E\mathcal{G}E^*.$$

Also, if E is a k_0-set contraction, $\mathcal{A}^{-1}$ is a k-set contraction for some $k \leq c_3^{-1} k_0^2$. Hence the essential spectrum $\sigma_e(\mathcal{A}^{-1})$ of $\mathcal{A}^{-1}$ satisfies

$$\sigma_e(\mathcal{A}^{-1}) \subset \{ \lambda \in \mathbb{C} : |\lambda| \leq c_3^{-1} k_0^2 \}.$$

Consequently

$$(5.12) \qquad \sigma_e(\mathcal{A}) \subset \{ \lambda \in \mathbb{C} : |\lambda| \geq c_3\, k_0^{-2} \}.$$

If E is compact (see Corollary 3.15) then so is $\mathcal{A}^{-1}$, and thus $\sigma_e(\mathcal{A}^{-1})$ can consist merely of the point $\lambda = 0$, the spectrum being otherwise discrete. In this case the spectrum of $\mathcal{A}$ is also discrete.

The operator associated with the adjoint form $a^*(u, v) = \overline{a(v, u)}$ is the adjoint $\mathcal{A}^*$ of $\mathcal{A}$, and hence if $a(u, v)$ is symmetric, i.e. $a(u, v) = \overline{a(v, u)}$ for all u, v in $H_0^{k,2}(\Omega, m)$, or $a_{\alpha\beta} = \bar{a}_{\beta\alpha}$ for all α and β, then $\mathcal{A}$ is self-adjoint. It is also positive definite if $a(u, u)$ is real and positive, for then

$$(5.13) \qquad (\mathcal{A}u, u)_{0,2} = a(u, u) \geq c_3 \| u \|_{k,2,m}^2 \geq c_3 K \| u \|_{0,2}^2$$

for all u in $\mathcal{D}(\mathcal{A})$, where $K = \| E : H_0^{k,2}(\Omega, m),\ L^2(\Omega) \|^{-2}$.

As an illustration we shall apply the preceding results to the positive definite self adjoint operator $\mathcal{A}_0$ defined as above by the expression

$$A_0 u(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(m(x)\frac{\partial u}{\partial x_i}\right), \quad x \in \Omega.$$

THEOREM 5.5. *Suppose that*

$$\inf_{x \in \Omega} \operatorname{meas}(B(x, 1) \setminus \Omega) = (1 - \eta) \operatorname{meas} B(1), \qquad 0 \leq \eta \leq 1.$$

Let $t > 1$, $s > 1$, $\frac{1}{r} \equiv \frac{1}{s} + \frac{1}{t} < \frac{2}{n}$, *and suppose that* $N_t(m^{-1}) < \infty$ *and* $N_s(N_t(m^{-1}, \cdot)\, m) < \infty$. *Also, writing*

$$\gamma(r) = \left\{ n^{-1/r} \left(\frac{2r - 1}{2r - n}\right)^{2-(1/r)} \right\}^{\frac{1}{2}},$$

suppose that

$$(5.14) \qquad \gamma(r)\, N_s^{\frac{1}{2}}(N_t(m^{-1}, \cdot)\, m)\, \eta^{(1/n)-(1/2r)} < 1.$$

Then $\mathcal{A}_0$ *is a positive definite self-adjoint operator with lower bound not less than*

$$l(\mathcal{A}_0) = \frac{\{1 - \gamma(r)\, N_s^{\frac{1}{2}}(N_t(m^{-1}, \cdot)\, m)\, \eta^{(1/n)-1/(2r)}\}^2}{\gamma^2(t)\, \eta^{(2/n)-(1/t)}\, N_t(m^{-1})},$$

so that the spectrum $\sigma(\mathcal{A}_0)$ *of* $\mathcal{A}_0$ *satisfies*

$$\sigma(\mathcal{A}_0) \subset \{\lambda : \lambda \geq l(\mathcal{A}_0)\}.$$

In particular, if $m = 1$, $s = r = t = \infty$, *and* $\eta < 1$,

$$l(\mathscr{A}_0) = (1 - \eta^{1/n})^2 \eta^{-2/n}.$$

Moreover, if either

(i) $\lim_{|x| \to \infty,\, x \in \Omega} \operatorname{meas}(B(x, 1) \setminus \Omega) = \operatorname{meas} B(1)$ (*or equivalently* $\lim_{|x| \to \infty,\, x \in \Omega} \operatorname{meas}(B(x, 1) \cap \Omega) = 0$),

or

(ii) $\lim_{|x| \to \infty} N_t(m^{-1}, x) = 0$,

then the spectrum of $\mathscr{A}_0$ *is discrete.*

PROOF. From Corollary 3.8 with $p = 2$, $d = 1$ and $\delta = 1 - \eta$,

$$\| u \|_{0,2,m} \leq \gamma(r) N_s^{\frac{1}{2}}(N_t(m^{-1}, \cdot)\, m)\, \eta^{(1/n) - 1/(2r)} (\| u \|_{0,2,m} + \| Du \|_{0,2,m})$$

for all u in $H_0^{1,2}(\Omega, m)$. Hence

$$\| u \|_{0,2,m} \leq \left\{ \frac{\gamma(r) N_s^{\frac{1}{2}}(N_t(m^{-1}, \cdot)\, m)\, \eta^{(1/n) - 1/(2r)}}{1 - \gamma(r) N_s^{\frac{1}{2}}(N_t(m^{-1}, \cdot)\, m)\, \eta^{(1/n) - 1/(2r)}} \right\} \| Du \|_{0,2,m},$$

and

$$\| u \|_{1,2,m} \leq \{1 - \gamma(r) N_s^{\frac{1}{2}}(N_t(m^{-1}, \cdot)\, m)\, \eta^{(1/n) - 1/(2r)}\}^{-1} | u |_{1,2,m}. \tag{5.15}$$

We therefore obtain

$$a_0(u, u) = \sum_{i=1}^{n} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 m(x)\, dx = | u |_{1,2,m}^2 \tag{5.16}$$

$$\geq \{1 - \gamma(r) N_s^{\frac{1}{2}}(N_t(m^{-1}, \cdot)\, m)\, \eta^{(1/n) - 1/(2r)}\}^2 \| u \|_{1,2,m}^2.$$

Also, from Corollary 3.7 with $Q = E: H_0^{1,2}(\Omega, m) \subset L^2(\Omega)$ and $s = \infty$, we have

$$\| E : H_0^{1,2}(\Omega, m), L^2(\Omega) \| \leq \gamma(t) N_t^{\frac{1}{2}}(m^{-1})\, \eta^{(1/n) - 1/(2t)}. \tag{5.17}$$

The first result now follows from (5.14). Under conditions (i) or (ii), E is compact and so the spectrum of $\mathscr{A}_0$ is discrete.

THEOREM 5.6. *Suppose the conditions of Theorem 5.5 are satisfied and that for some α with $0 < \alpha < 2 - \frac{n}{t}$, $M_\alpha(|Q|^2 N_t(m^{-1}, \cdot)) < \infty$. Suppose also that one of the following conditions holds:*

(i) $\lim_{|x| \to \infty,\ x \in \Omega} \operatorname{meas}(B(x, 1) \cap \Omega) = 0$;

(ii) $\liminf_{|x| \to \infty,\ x \in \Omega} N_t(m^{-1}, x) = 0$ *and* $M_\alpha(|Q|^2) < \infty$;

(iii) $\liminf_{|x| \to \infty,\ x \in \Omega} N_2(|Q|, x) = 0$.

Then $A_0 + Q$ defines a closed operator $\mathcal{A} = \mathcal{A}_0 + Q$ with domain $\mathcal{D}(\mathcal{A}_0)$ in $L^2(\Omega)$ which satisfies $\sigma_e(\mathcal{A}) = \sigma_e(\mathcal{A}_0)$. If Q is real, $\mathcal{A}$ is self-adjoint. Under conditions (i) *or* (ii) *the spectrum of $\mathcal{A}$ is discrete, i.e. $\sigma_e(\mathcal{A}) = \emptyset$.*

PROOF. Under the conditions of the theorem it follows from Theorem 3.6 and Corollary 3.15 that Q is a compact operator from $H_0^{1,2}(\Omega, m)$ to $L^2(\Omega)$. In particular, Q is defined on $\mathcal{D}(\mathcal{A}_0)$ $(\subset H_0^{1,2}(\Omega, m))$. If $\mathcal{B}$ is a bounded set in $\mathcal{D}(\mathcal{A}_0)$ endowed with the graph norm, we have from (5.16) for $u \in \mathcal{B}$,

$$\| u \|_{1,2,m}^2 \leq K \,|\, a_0(u, u) \,| = K(\mathcal{A}_0 u, u)_{0,2}$$

$$\leq K(\| u \|_{0,2}^2 + \| \mathcal{A}_0 u \|_{0,2}^2).$$

Hence $\mathcal{B}$ is bounded in $H_0^{1,2}(\Omega, m)$, and consequently $Q(\mathcal{B})$ is relatively compact in $L^2(\Omega)$. In other words, Q is $\mathcal{A}_0$-compact, and the Theorem follows from [12], Chapter IV, Theorems 1.11 and 5.35. We have, of course, already seen that under conditions (i) or (ii) $\mathcal{A}_0$ has a discrete spectrum.

Perturbation results for more general operators, but with $m = 1$ and $\Omega = R^n$, may be found in [4] and [22]. For such operators, and with $m(x) \not\equiv 1$ or $\Omega \neq R^n$, the methods of the present paper should be useful.

6. Concluding remarks.

Although the principal use to which our embedding theorems of § 3 have been put in this paper is to discuss the existence and properties of variational solutions of a Dirichlet problem for linear and quasilinear equations of elliptic or degenerate elliptic type, it is clear that there are a number of other useful applications of the theory, and we should like to mention two of these.

First, existence theorems for weak solutions of Cauchy-Dirichlet problems for equations of parabolic type in an unbounded space domain can be obtained: in the parabolic counterpart of the quasilinear theory of § 4, for example, one can proceed in much the same way as for the elliptic case, relying eventually on the abstract work on evolution equations which may be found in Lions [15], Chap. 2, § 7. We shall not elaborate on this here in view of the general similarity of the methods needed to those already given for elliptic problems.

Second, it seems probable that by the use of the methods of § 3 some progress may be made in the study of bifurcation theory corresponding to eigenvalue problems associated with quasilinear partial differential aquations in unbounded domains. The corresponding theory for bounded domains has been discussed rather neatly by Rabinowitz [21], while Stuart [26] has been able to classify the situation for some ordinary differential equations on the half-line. We hope to return to this topic in a later paper.

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