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LINEAR ELLIPTIC OPERATORS WITH MEASURABLE COEFFICIENTS

by NEIL S. TRUDINGER (*)

Introduction.

The study of linear elliptic operators with measurable coefficients has become increasingly prominent during the last decade. This paper treats second order, linear operators in divergence form, that is operators $\mathcal{L}$ of the form

$$\mathcal{L}u = -\frac{\partial}{\partial x_i}(a^{ij}(x)\, u_{x_j} + a^i(x)\, u) + b^i(x)\, u_{x_i} + a(x)\, u$$

whose coefficients are measurable functions on some domain Ω in Euclidean n space, E^n. Ellipticity means that the principal coefficient matrix $[a^{ij}]$ is positive definite in Ω. Principally, we shall be concerned with the existence and uniqueness of solutions to the generalized Dirichlet problem for $\mathcal{L}$, the local and global regularity of such solutions and some qualitative properties of solutions such as the weak and strong maximum principles and the Harnack inequality.

There are two features which serve to limit the scope of this work. First, it will always be possible for our coefficients to be either unbounded, discontinuous or both. The maintenance of this generality means that we cannot expect to prove regularity results beyond integral or pointwise estimates for the solutions under considerations. Another guiding general assumption is that no conditions are to be imposed on the domains Ω beyond boundedness, for global results. This generally has the effect of

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limiting the consideration of boundary value problems to only the Dirichlet problem. For domains with some smoothness, however, other boundary value problems may readily be handled by our methods.

Letting λ and Λ denote respectively the minimum and maximum eigenvalues of the symmetric part of $[a^{ij}]$, the operator, $\mathcal{L}$, is called *strictly elliptic* in Ω if λ is bounded away from zero and *uniformly elliptic* in Ω if the ratio of Λ to λ is bounded from above. Operators that are both strictly and uniformly elliptic have been treated extensively in the literature and apart from methods this paper has very few new results to offer for these operators. For equations in arbitrary n variables, treatments generally stem, directly or indirectly, from the pioneering Hölder estimates of De Giorgi [5] for generalized solutions of the equation $\mathcal{L}u = 0$, when $a^i = b^i = a = 0$. We mention, in particular, the independently derived Hölder estimate of Nash [21] and the works of Morrey [15,16], Stampacchia [24-27], Ladyzhenskaya and Uralt'seva [11,12], Moser [17,18], Serrin [22] and Trudinger [29].

Among the above mentioned works the book by Ladyzhenskaya and Uralt'seva [12] and the paper by Stampacchia [27] cover most of the aspect treated in this paper and consequently they may serve as a basis for comparison with our results. In the first place, the treatments of existence and uniqueness in these works suffer from the imposition of rather unnatural coercivity or smallness conditions on the lower order terms. We are able to avoid such restrictions while simultaneously permitting arbitrarily unbounded principal coefficients a^{ij}. The key to our approach, which generally follows that of Stampacchia, is the weak maximum principle, Theorem 3.1, which implies much of the succeeding global theory. Our treatment has the effect of bringing the global theory for the operaters $\mathcal{L}$ more firmly in line with that for operators with smooth coefficients. With regard to the weak maximum principle, it should be mentioned that in the *strictly*, *uniformly* elliptic, case, it is also an immediate consequence of the weak Harnack inequality derived by the author in [29]. A unnecessarily involved proof was also given by Chicco [2] (see also [4]) and later the result was again established, but in a more roundabout fashion, by Hervé and Hervé [7].

On the whole, our results extend those in the above papers in that they hold for operators assumed neither strictly nor uniformly (except for Corollaries 5.5 and 6.1) elliptic. The achievement of this generality has required several new test function techniques, which are delineated throughout the paper. There are two other papers dealing with non-strictly (but uniformly) elliptic operators $\mathcal{L}$ that deserve mention here. First, the paper of Kruzkov [9] treats some local estimates through the direct use of Moser's methods [17,18] while that of Stampacchia and Murthy [20] extends the work of Stampacchia [27] to a degenerate situation. Our methods have also enabled the hypotheses which these authors employ to be considerably relaxed.

Let us briefly survey the contents of this paper. Sction 1 contains preparatory material, predominantly covering the $H^0(G, \Omega)$ spaces which constitute a natural framework for non-uniformly elliptic operators. The theory of these spaces, as required by us, extends the treatment of Murphy and Stampacchia [20] of certain weighted Sobolev spaces. After the development of the $H^0(G, \Omega)$ spaces, Fredholm alternatives, Theorems 2.1, 2.2, follow naturally in Section 2. The weak maximum principle for subsolutions, Theorem 3.1, is derived in Section 3, along with its numerous Corollaries culminating in the existence and uniqueness theorem for the Dirichlet problem, Theorem 3.2. In Section 4, we take up the problem of global regularity, the main result here being Theorem 4.1. We have taken the opportunity in the proof of Theorem 4.1 to introduce a test function technique for the derivation of L^∞ estimates, which proceeds through a characterization of L^∞ as an extended Orlicz space (see [6]). By dualization, a further existence and uniqueness result is derived at the end of Section 4. The local theory of Section 5, notably the strong maximum principle, continuity results and the Harnack inequality have all been approached via the weak Harnack inequality, Theorem 5.2, the proof of which is modeled on the author's paper [31] which treated the case $a^i = b^i = a = 0$. An important difference between the local and global theory is that for the local results one needs some control on the maximum eigenvalue Λ with respect to the minimum eigenvalue λ whereas our hypotheses in Sections 2, 3 and 4 relate only the lower order coefficients of $\mathcal{L}$ to λ. In the final segment of the paper, Section 6, we consider the extension of certain local estimates to the boundary of Ω.

For the simpler strictly, uniformly elliptic case, some of our methods have been illustrated in the author's lecture notes [28]. Also some of our results will appear as special cases of a treatment of quasilinear, differential inequalities [33] (see also [32]). But since the linear case presents features not shared by the quasilinear generalizations of [33], it has seemed worthwhile to present it separately. The reader will find a fuller account of some apriori estimates in [33].

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§ 1. Preliminaries.

This paper is concerned with second order, linear divergence structure differential operators of the form

$$(1.1)\qquad \mathcal{L}u = -\frac{\partial}{\partial x_i}(a^{ij}(x)\,u_{x_j} + a^i(x)\,u) + b^i(x)\,u_{x_i} + a(x)\,u$$

where the coefficients a^{ij}, a^i, b^i, a, $1 \leq i, j \leq n$ are measurable functions on a bounded domain Ω in Euclidean n space, E^n. Repeated indices indicate summation from 1 to n and $Du = \nabla u = (u_{x_1}, \dots, u_{x_n})$ is the gradient of u. Let us always assume that $n \geq 2$ so that $\mathcal{L}$ is a genuine *partial* differential operator.

As it stands the representation (1.1) is only formal. Let $\mathcal{D}(\Omega)$ $(= C_0^\infty(\Omega))$ denote the space of infinitely differentiable functions with compact support in Ω, $\mathcal{D}'(\Omega)$ the space of Schwartzian distributions on Ω, and $\mathcal{D}_1(\Omega)$ the space of strongly differentiable functions. Let us put, for u in $\mathcal{D}_1(\Omega)$,

$$(1.2)\qquad A^i(x) = a^{ij}(x)\,u_{x_j} + a^i(x)\,u,\quad A(x) = b^i(x)\,u_{x_i} + a(x)\,u$$

and let

$$(1.3)\qquad \mathcal{D}_1(\mathcal{L}, \Omega) = \{u \in \mathcal{D}_1(\Omega)\,;\, A^j,\ A \text{ are locally integrable in } \Omega\}.$$

Then clearly $\mathcal{L}$ maps $\mathcal{D}_1(\mathcal{L}, \Omega)$ into $\mathcal{D}'(\Omega)$. Accordingly, defining for φ in $\mathcal{D}(\Omega)$, u in $\mathcal{D}_1(\mathcal{L}, \Omega)$

$$(1.4)\qquad \mathcal{L}(u, \varphi) = \int_\Omega (A^i(x)\,\varphi_{x_i} + \varphi A)\,dx$$

we have the following definitions of solution, subsolution, and supersolution. Let T be a distribution in $\mathcal{D}'(\Omega)$. Then u is a *solution* (*subsolution, supersolution*) of the equation $\mathcal{L}u = T$ if

$$(1.5)\qquad \mathcal{L}(u, \varphi) - T(\varphi) = 0 \qquad (\leq 0, \geq 0)$$

for all nonnegative φ in $\mathcal{D}(\Omega)$.

The operator $\mathcal{L}$ is *elliptic* in Ω if the coefficient matrix $\mathcal{A} = [a^{ij}]$ is positive almost everywhere in Ω. Since the redefinition of $\mathcal{A}$ on a set of measure zero does not effect the value of $\mathcal{L}(u, \varphi)$, our ellipticity definition is equivalent, in the sense of generalized solutions as introduced above, to the classical one. Let $\lambda(x)$, $\Lambda(x)$ denote respectively the minimum, maximum eigenvalues of $\mathcal{A}^s = [a_s^{ij}]$, the symmetric part of $\mathcal{A}$ so that

$$(1.6)\qquad \lambda(x)\,|\,\xi\,|^2 \leq a^{ij}(x)\,\xi_i\,\xi_j \leq \Lambda(x)\,|\,\xi\,|^2$$

for all ξ in E^n, x in Ω. We will refer to $\mathcal{L}$ as *strictly elliptic* in Ω if λ^{-1} is essentially bounded in Ω, and *uniformly elliptic* if $\gamma = \Lambda\lambda^{-1}$ is essentially bounded in Ω. Note that uniformly elliptic operators that are not strictly elliptic have sometimes been referred to as *degenerate elliptic* [21]. In our context it is the strict ellipticity rather than the ellipticity which degenerates.

The function spaces $H^0(\mathcal{A}, \mu, \Omega)$, $H(\mathcal{A}, \mu, \Omega)$.

There are various function spaces that are associated naturally with an elliptic operator of the form (1.1). Let Ω be a bounded domain in E^n, $p \geq 1$ and λ a nonnegative measurable function on Ω such that λ^{-1} lies in $L_t(\Omega)$ for some $t \geq 1$. We write $L_p(\lambda, \Omega)$ for the Banach space of (equivalence classes of) functions satisfying

$$\| u \|_{L_p(\lambda, \Omega)} = \left(\int_\Omega \lambda \, | u |^p \, dx \right)^{1/p} < \infty. \tag{1.7}$$

and observe that by Hölder's inequality, a function u belonging to $L_p(\lambda, \Omega)$ satisfies

$$\| u \|^p_{L_q(\Omega)} \leq \| \lambda^{-1} \|_{L_t(\Omega)} \int_\Omega \lambda \, | u |^p \, dx \tag{1.8}$$

provided $\dfrac{1}{q} = \left(1 + \dfrac{1}{t}\right) \dfrac{1}{p} \leq 1$.

Now, let $\mathcal{A} = [a^{ij}]$ be a positive definite, measurable, $n \times n$ matrix valued function on Ω and μ a nonnegative measurable scalar function on Ω with λ and Λ defined as in (1.6). Under the assumption that Λ and μ belong to $L_1(\Omega)$ the form

$$(u, v) = \int_\Omega (a^{ij}_s u_{x_i} v_{x_j} + \mu u v) \, dx \tag{1.9}$$

defined a real, scalar product on $\mathcal{D}(\Omega)$. The Hilbert space $H^0(\mathcal{A}, \mu, \Omega)$ is subsequently defined as the completion of $\mathcal{D}(\Omega)$ under (1.9). With the further assumption that μ is positive on a subset of Ω of positive measure, the Hilbert space, $H^1(\mathcal{A}, \mu, \Omega)$, is obtained by completing $C^\infty(\Omega)$ under (1.9). Also, let $W^1_2(\mathcal{A}, \mu, \Omega)$ denote the class of strongly differentiable functions, u, satisfying

$$\| u \|_{\mathcal{A}, \mu, \Omega} = (u, u)^{1/2} < \infty. \tag{1.10}$$

From (1.6) and (1.8) (with $t = 1$, $p = 2$, $u = | Du |$), it follows that $H^0(\mathcal{A}, \mu, \Omega)$ is a subspace of $W^1_2(\mathcal{A}, \mu, \Omega)$ if λ^{-1} belongs to $L_1(\Omega)$. Also, if in addition, μ^{-1} belongs to $L_1(\Omega)$, it follows that $W^1_2(\mathcal{A}, \mu, \Omega)$ is a Banach space under

(1.10) and $H^0(\mathcal{A}, \mu, \Omega)$ and $H^1(\mathcal{A}, \mu, \Omega)$ are then respectively the closures of $\mathcal{D}(\Omega)$ and $C^\infty(\Omega)$ in $W_2^1(\mathcal{A}, \mu, \Omega)$

Let us write, for simplicity, $H^0(\mathcal{A}, \Omega) = H^0(\mathcal{A}, 0, \Omega)$; $H^1(\mathcal{A}, \Omega) = H^1(\mathcal{A}, \Lambda, \Omega)$ and introduce the local spaces

$$(1.11) \qquad H^{\mathrm{loc}}(\mathcal{A}, \Omega) = \{u \in \mathcal{D}_1(\Omega)\,;\ \eta u \in H^0(\mathcal{A}, \Omega)\ \ \forall\ \eta \in \mathcal{D}(\Omega)\}.$$

The following proposition is then pertinent to our treatment of local estimates.

PROPOSITION 1.1. $H(\mathcal{A}, \Omega) \subset H^{\mathrm{loc}}(\mathcal{A}, \Omega)$.

PROOF. Let $u \in H(\mathcal{A}, \Omega)$ and suppose the sequence $\{u^{(m)}\}_{m=1}^\infty \in C^\infty(\Omega) \cap W_2^1(\mathcal{A}, \Omega)$ converges to u. For a fixed m, set $v = u^{(m)} - u$ so that

$$\begin{aligned}(\eta v, \eta v) = \int_\Omega & (a^{ij}\eta_{x_i}\eta_{x_j}v^2 + a^{ij}v_{x_i}v_{x_j}\eta^2 + a^{ij}v_{x_i}\eta_{x_j}v\eta \\ & + a^{ij}\eta_{x_i}v_{x_j}v\eta + \Lambda\eta^2v^2)\,dx \\ & \leq (2\sup|D\eta|^2 + \sup|\eta|^2)\int_\Omega \Lambda v^2\,dx \\ & + 2\sup|\eta|^2\int_\Omega a^{ij}v_{x_i}v_{x_j}\,dx \\ & \leq C(v, v)\end{aligned}$$

where C depends on $\sup|\eta|\ \sup|D\eta|$. Hence $\eta u^{(m)}$ converges to ηu, whence $\eta u \in H^0(\mathcal{A}, \Omega)$. ||

Since $\mathcal{D}(\Omega)$ is dense in $H^0(\mathcal{A}, \mu, \Omega)$, the dual space of $H^0(\mathcal{A}, \mu, \Omega)$, which we designate $[H^0(\mathcal{A}, \mu, \Omega)]^*$, will be a subspace of $\mathcal{D}'(\Omega)$ and by virtue of the Riesz representation theorem it will be isomorphic to $H^0(\mathcal{A}, \mu, \Omega)$. We give now a more concrete representation of $[H^0(\mathcal{A}, \mu, \Omega)]^*$. Let $\mathcal{B} = [b_{ij}]$ and ν satisfy the same hypotheses as $\mathcal{A}$ and μ with the exception that ν can be infinite on a set of positive measure. Then $H^{-1}(\mathcal{B}, \nu, \Omega)$ denotes the space of distributions, T, that can be represented in the form

$$(1.12) \qquad T\varphi = \int_\Omega (f_i\varphi_{x_i} + f\varphi)\,dx \quad \text{for } \varphi \text{ in } \mathcal{D}(\Omega),$$

where f_i, $i = 1, \ldots, n$ and f are measurable functions on Ω satisfying

$$\int (b^{ij} f_i f_j + \nu f^2)\, dx < \infty. \tag{1.13}$$

Note that $f^2 \nu$ is understood to vanish where ν is infinite. We now have

PROPOSITION 1.2. $[H^0(\mathcal{A}, \mu, \Omega)]^* = H^{-1}(\mathcal{A}^{-1}, \mu^{-1}, \Omega)$.

PROOF. It is easy to see, by virtue of Schwarz's inequality for $\mathcal{A}$, that any T of the form (1.12) is a bounded linear functional on $H^0(\mathcal{A}, \mu, \Omega)$. The reverse inclusion follows by considering the natural imbedding of $H^0(\mathcal{A}, \mu, \Omega)$ into the space of $n+1$ vector valued functions on Ω, $\underset{\sim}{g} = (g_0, g_1, \ldots, g_n)$, satisfying

$$\| \underset{\sim}{g} \|^2 = \int (a^{ij} g_i g_j + \mu g_0^2)\, dx < \infty, \tag{1.14}$$

which we designate $L^2(\mathcal{A}, \mu, \Omega)$. Since the dual of $L^2(\mathcal{A}, \mu, \Omega)$ is $L^2(\mathcal{A}^{-1}, \mu^{-1}, \Omega)$ (see [23]), the result follows. ||

Let us now pause briefly in our development of the properties of $H^0(\mathcal{A}, \mu, \Omega)$ and $H^1(\mathcal{A}, \mu, \Omega)$ to indicate some well-known specific examples. Suppose that $\mathcal{A}$ is the coefficient matrix of a strictly, uniformly elliptic operator, $\mathcal{L}$. Then the spaces $H^0(\mathcal{A}, \Omega)$, $H^1(\mathcal{A}, \Omega)$, $W_2^1(\mathcal{A}, \Omega)$ coincide respectively with the Sobolev spaces $\overset{\circ}{W}{}_2^1(\Omega)$, $H_2^1(\Omega) = W_2^1(\Omega)$ which have previously been used extensively by Stampacchia in his treatment of strictly, uniformly elliptic operators (see for example [27[). More generally, suppose that $\mathcal{A}$ is the coefficient matrix of a uniformly elliptic operator, $\mathcal{L}$. Then the spaces $H^0(\mathcal{A}, \lambda, \Omega)$ and $H^1(\mathcal{A}, \lambda, \Omega)$ coincide with the function epaces, introduced and studied by Murthy and Stampacchia in [20]. We note here that for these cases, the argument of Meyers and Serrin [14] for the Sobolev space case is applicable and we consequently have

PROPOSITION 1.3. $H^1(\mathcal{A}, \mu, \Omega) = W_2^1(\mathcal{A}, \mu\, \Omega)$ *provided* $\gamma = \Lambda/\lambda \in L_\infty(\Omega)$.

Some fundamental lemmas.

We formulate now some important properties of strongly differentiable functions, which we cast in the framework of the $H^0(\mathcal{A}, \mu, \Omega)$ and $H^1(\mathcal{A}, \mu, \Omega)$ spaces. The following lemma consists of simple extensions via inequality (1.8), for some variants of the well known Sobolev imbedding theorem (see [16], [27], and [30]). For a function, u, integrable on a measurable set S, we write

$$u_S = \frac{1}{|S|}\int_S u\,dx \tag{1.15}$$

for the integral mean value of u.

LEMMA 1.1. *Let Ω be a ball in E^n and suppose that either u is a function in $H^0(\mathcal{A}, \Omega)$ or that u is a function in $H^0(\mathcal{A}, \Omega)$ satisfying $u_\Omega = 0$. Then if λ^{-1} belongs to $L_t(\Omega)$ for $1 + 1/t > 2/n$, u lies in $L_{t^*}(\Omega)$ where*

$$\frac{1}{t^*} = \frac{1}{2}\left(1 + \frac{1}{t}\right) - \frac{1}{n} \tag{1.16}$$

and we have the estimate

$$\| u \|^2_{L_{t^*}(\Omega)} \leq C(n, t)\, \| \lambda^{-1} \|_{L_t(\Omega)} \int_\Omega \lambda\, |Du|^2\, dx \tag{1.17}$$

where C depends on n, t. Alternatively, if $1 + 1/t = 2/n$, i.e. $t = \infty$, $n = 2$, u belongs to the Orlicz space $L_\varphi(\Omega)$ associated with the function

$$\varphi = e^{t^2} - 1 \tag{1.18}$$

and in this case, we have

$$\| u \|^2_{L_\varphi(\Omega)} \leq C(|\Omega|) \int_\Omega |Du|^2\, dx \tag{1.19}$$

where C depends on $|\Omega|$.

Note, of course, that for the $H^0(\mathcal{A}, \Omega)$ case above, Ω can be an arbitrary bounded domain in E^n. The next lemma, also derivable from an imbedding theorem (see [8] or [30]) was the key step in Moser's Harnack inequality [18].

LEMMA 1.2. *Let Ω be a ball in E^n and u a strongly differentiable function in Ω, either of compact support in Ω or satisfying $u_\Omega = 0$. Suppose there exists a constant K, such that for any ball, B, in E^n,*

$$\int_{\Omega \cap B} |Du|\, dx \leq K\, |B|^{1-1/n}. \tag{1.20}$$

Then there exist positive constants p_0 and γ depending only on n such that

$$\int_\Omega e^{p_0 K^{-1} |u|}\, dx \leq \gamma\, |\Omega|. \tag{1.21}$$

We will also require the chain rule for strong differentiation [16].

LEMMA 1.3. *Let u be a strongly differentiable function on Ω and g a uniformly Lipschitz continuous function on E^1. Then the composite function $g(u)$ is strongly differentiable on Ω and the chain rule applies, i.e.*

$$Dg(u) = g'(u) \cdot Du \text{ a.e. } (\Omega). \tag{1.22}$$

Furthermore if $g(0) = 0$ and u belongs to $H^0(\mathcal{A}, \mu, \Omega)$ or $H(\mathcal{A}, \mu, \Omega)$, then $g(u)$ again belongs to $H^0(\mathcal{A}, \mu, \Omega)$ or $H(\mathcal{A}, \mu, \Omega)$ respectively.

The last statement of Lemma 1.2 follows from a simple approximation argument.

Boundedness in $H^0(\mathcal{A}, \mu, \Omega)$.

To complete our brief study of the spaces $H^0(\mathcal{A}, \mu, \Omega)$, we introduce some concepts or boundedness. Let g be a nonnegative, measurable function on Ω. Then g is said to be *bounded* on $H^0(\mathcal{A}, \mu, \Omega)$ if the space $H^0(\mathcal{A}, \mu, \Omega)$ may be continuously imbedded in the space $L^2(g, \Omega)$, i.e. if $H^0(\mathcal{A}, \mu, \Omega)$ is equivalent to $H^0(\mathcal{A}, \mu + g, \Omega)$. Thus g is bounded on $H^0(\mathcal{A}, \mu, \Omega)$ if and only if there exists a constant K such that

$$\int_\Omega g\varphi^2\,dx \leq K(\varphi, \varphi) \text{ for all } \varphi \text{ in } \mathcal{D}(\Omega). \tag{1.23}$$

Let us now call g *compactly bounded* on $H^0(\mathcal{A}, \mu, \Omega)$ if for any $\varepsilon > 0$, there exists a constant K_ε depending on ε such that

$$\int_\Omega g\varphi^2 \leq \varepsilon(\varphi, \varphi) + K_\varepsilon\Big(\int_\Omega |\varphi|\,dx\Big)^2 \text{ for all } \varphi \text{ in } \mathcal{D}(\Omega). \tag{1.24}$$

Clearly compact boundedness is a stronger property than boundedness. L_p conditions guaranteeing compact boundedness are provided by

LEMMA 1.4. *Let λ^{-1} belong to $L_t(\Omega)$ and suppose that for $1 + 1/t < 2/n$, g belongs to $L_s(\Omega)$ where*

$$\frac{1}{t} + \frac{1}{s} = \frac{2}{n} \tag{1.25}$$

and that for $t = \infty$, $n = 2$, g belongs to $L \log L(\Omega)$. Then g is compactly bounded on $H^0(\mathcal{A}, \Omega)$.

PROOF. Except for the case where $s = \infty$, the estimate (1.24) is obtained by decomposing g into the sum of a bounded function and a function with arbitrarily small L_s or $L \log L$ norm and then using a Hölder inequality combined with Lemma 1.1. The argument is similar to that employed in [20]. The case $s = \infty$ seems more difficult. Here we may use the fact that λ^{-1} belongs to an Orlicz space strictly contained in $L_t(\Omega)$ and then apply the imbedding theorem in [6] to show that any φ in $H^0(\mathcal{A}, \Omega)$ belongs to an Orlicz space L_ψ strictly contained in $L_2(\Omega)$. Interpolating $\| \varphi \|_{L_2(\Omega)}$ between $\| \varphi \|_{L_\psi(\Omega)}$ and $\| \varphi \|_{L_1(\Omega)}$ then yields the estimate (1.24). ||

For certain estimates a specification of a uniform dependence of K on ε is required. The following interpolation lemma follows from Lemma 1.1.

LEMMA 1.5. *Let λ^{-1}, g lie in $L_t(\Omega)$, $L_s(\Omega)$ respectively where*

$$\frac{1}{t} + \frac{1}{s} < \frac{2}{n}. \tag{1.26}$$

Then g is compactly bounded on $H^0(\mathcal{A}, \mu, \Omega)$ with the constant K in (1.24) given by

$$K = C\varepsilon^{-\nu} \tag{1.27}$$

where the constant C depends on n, s, t and $\| \lambda^{-1} \|_{L_t(\Omega)} \cdot \| g \|_{L_p(\Omega)}$ and $\nu > 0$ depends on n, s, t.

Let us call a function g, compactly bounded on $H^0(\mathcal{A}, \mu, \Omega)$, *$\varepsilon^\nu$-compactly bounded* if the constant K in (1.24) satisfies an inequality of the form (1.27)

Lemma 1.5 has a Morrey space generalization.

LEMMA 1.5a. *Let λ^{-1}, g lie in $L_t(\Omega)$, $L_s(\Omega)$ respectively where*

$$\frac{1}{t} + \frac{1}{s} = \frac{2}{n} \tag{1.25}$$

and suppose there exist constants γ and $\delta > 0$ such that for any ball B in E^n,

$$\| \lambda^{-1} \|_{L_t(B \cap \Omega)} \, \| g \|_{L_s(B \cap \Omega)} \leq \gamma \, | B |^\delta . \tag{1.28}$$

Then g is ε^ν-compactly bounded on $H^0(\mathcal{A}, \mu, \Omega)$ for some $\nu > 0$ depending on n, s, t and δ.

PROOF. For $s < \infty$, $n > 2$, Lemma 1.5a follows from Lemma 1.1 with the aid of a partition of unity. For $n = 2$, Lemma 1.5a is proved in Morrey [16]. Eor $s = \infty$, Lemma 1.5a may be derived through an imbedding theorem of Campanato [1]. Note that the case $t = \infty$ coincides with the case $\alpha = 2$ in Lemma 5.1 of [29]. ||

In order to apply the Fredholm alternative to the operator $\mathcal{L}$, we need the following property of compactly bounded functions on $H^0(\mathcal{A}, \mu, \Omega)$.

LEMMA 1.6. *Let g be compactly bounded on $H^0(\mathcal{A}, \mu, \Omega)$. Then the natural imbedding of $H^0(\mathcal{A}, \mu, \Omega)$ into $L_2(g, \Omega)$ is compact.*

PROOF. Since $\lambda^{-1} \in L_1(\Omega)$, $H^0(\mathcal{A}, \mu, \Omega)$ is compactly imbedded in $L_1(\Omega)$ by virtue of the Sobolev imbedding theorem [16]. The result is then a consequence of inequality (1.24). ||

We remark here that the conclusion of Lemma 1.6 is in fact equivalent to the compact boundedness of g. Consequently the concept of compact boundedness will be roughly a minimal condition under which the Riesz-Schauder theory of compact operators may be applied.

§ 2. A Fredholm alternative.

With regard to the operator $\mathcal{L}$, given by

$$\mathcal{L}u = -\frac{\partial}{\partial x_i}(a^{ij}(x)\, u_{x_j} + a^i(x)\, u) + b^i(x)\, u_{x_i} + a(x)\, u, \tag{2.1}$$

we will assume throughout this paper that $\mathcal{L}$ is elliptic and that the functions λ^{-1}, Λ defined by (1.6) are integrable on Ω. Let us define now

$$g(x) = b_{ij}(x)(a^i(x)\, a^j(x) + b^j(x)\, b^j(x)) + |a(x)|, \tag{2.2}$$

where $\mathcal{B} = [b_{ij}] = \mathcal{A}^{-1}$ is the inverse matrix of $\mathcal{A}$, and also assume that g is integrable on Ω. A further condition on the matrix $\mathcal{A}$ is required, namely that there exists a constant $K_{\mathcal{A}}$ such that

$$(a^{ij}\xi_i\eta_j)^2 \leq K_{\mathcal{A}}^2\,(a^{ij}\xi_i\xi_j)(a^{ij}\eta_i\eta_j) \tag{2.3}$$

for all ξ, η in E^n and x in Ω. Observe that if $\mathcal{A}$ is symmetric, then (2.3) is satisfied with $K_{\mathcal{A}} = 1$. Also, if $\mathcal{A}$ is strictly elliptic and bounded in Ω, then (2.3) is satisfied for large enough $K_{\mathcal{A}}$. Condition (2.3) is to be understood to hold throughout this paper.

LEMMA 2.1 *Let u and φ belong to $W_2^1(\mathcal{A}, g, \Omega)$ and $H^0(\mathcal{A}, g, \Omega)$ respectively. Then $\mathcal{L}(u, \varphi)$ is finite and*

$$|\mathcal{L}(u, \varphi)| \leq C(K_{\mathcal{A}}) \, \| u \|_{\mathcal{A}, g, \Omega} \| \varphi \|_{\mathcal{A}, g, \Omega} \tag{2.4}$$

where $C(K_{\mathcal{A}})$ is a constant depending on $K_{\mathcal{A}}$.

PROOF. We have

$$|\mathcal{L}(u, \varphi)| = \left| \int_{\Omega} (a^{ij} u_{x_j} \varphi_{x_i} + a^i u \varphi_{x_i} + b^i \varphi u_{x_i} + a u \varphi) \, dx \right|$$

$$\leq K_{\mathcal{A}} \int_{\Omega} \{ (a^{ij} u_{x_i} u_{x_j})^{1/2} (a^{ij} \varphi_{x_i} \varphi_{x_j})^{1/2} + g^{1/2} |u| (a^{ij} \varphi_{x_i} \varphi_{x_j})^{1/2}$$

$$+ g^{1/2} |\varphi| (a^{ij} u_{x_i} u_{x_j})^{1/2} + |a| \, |u \varphi| \} \, dx$$

$$\leq C K_{\mathcal{A}} \| u \|_{\mathcal{A}, g, \Omega} \| \varphi \|_{\mathcal{A}, g, \Omega} \cdot \; \|$$

As an obvious corollary of Lemma 2.1 now follows

LEMMA 2.2. *$\mathcal{L}(u, \varphi)$ is a bounded bilinear form on $H^0(\mathcal{A}, g, \Omega)$ and for fixed u in $W_2^1(\mathcal{A}, g, \Omega)$ a bounded linear functional on $H^0(\mathcal{A}, g, \Omega)$.*

The next lemma provides a bound from below for $\mathcal{L}(u, u)$.

LEMMA 2.3. *There exist constants $\nu > 0$ and σ_0 depending on $K_{\mathcal{A}}$ such that for any u belonging to $H^0(\mathcal{A}, g, \Omega)$*

$$\mathcal{L}(u, u) \geq \nu \| u \|^2_{\mathcal{A}, \Omega} - \sigma_0 \int_{\Omega} g u^2 \, dx. \tag{2.5}$$

PROOF. We have

$$\mathcal{L}(u, u) = \int_{\Omega} (a^{ij} u_{x_i} u_{x_j} + (a^i + b^i) u u_{x_i} + a u^2) \, dx$$

$$\geq \int_{\Omega} \{ a^{ij} u_{x_i} u_{x_j} - 2 K_{\mathcal{A}} g^{1/2} |u| (a^{ij} u_{x_i} u_{x_j})^{1/2} - |a| u^2 \} \, dx$$

$$\geq \nu \int_{\Omega} a^{ij} u_{x_i} u_{x_j} - \sigma_0 \int g u^2 \, dx. \; \|$$

It follows from Lemma 2.3 that the operator

$$\mathcal{L}_\sigma(g)\,u = \mathcal{L}u + \sigma g u \tag{2.6}$$

is *coercive* on $H^0(\mathcal{A}, \Omega)$ if $\sigma \geq \sigma_0$ and *coercive* on $H^0(\mathcal{A}, g, \Omega)$ if $\sigma > \sigma_0$.

A preliminary existence result now folllows from Lemmas 2.2 and 2.3, by means of the Lax-Milgram theorem [34].

LEMMA 2.4. *For $\sigma > \sigma_0$, the operator $\mathcal{L}_\sigma(g)$ is a bijective mapping from $H^0(\mathcal{A}, g, \Omega)$ to $H^{-1}(\mathcal{A}^{-1}, g^{-1}, \Omega)$ and if g is bounded on $H^0(\mathcal{A}, \Omega)$, $\mathcal{L}_\sigma(g)$ is a bijective mapping from $H^0(\mathcal{A}, \Omega)$ to $H^{-1}(\mathcal{A}^{-1}, \Omega)$.*

The Fredholm alternative for $\mathcal{L}$ is now a consequence of Lemma 2.4. Let us assume that g is compactly bounded on $H^0(\mathcal{A}, \Omega)$ so that, by Lemma 1.6, the mapping C defined by

$$Cu(\varphi) = \int_\Omega g u \varphi \, dx \tag{2.7}$$

is a compact mapping from $H^0(\mathcal{A}, \Omega)$ into $H^{-1}(\mathcal{A}^{-1}, \Omega)$. Then for u in $H^0(\mathcal{A}, \Omega)$, T in $H^{-1}(\mathcal{A}^{-1}, \Omega)$, the equation

$$\mathcal{L}u + \sigma g u = T \tag{2.8}$$

is equivalent to the equation

$$u + (\sigma - \sigma_0)\, G_{\sigma_0}(g)\, Cu = G_{\sigma_0}(g)\, T, \tag{2.9}$$

where $G_\sigma(g) = \mathcal{L}_\sigma(g)^{-1}$ is called the *Green's operator* for $\mathcal{L}_\sigma(g)$ whenever it exists. Next, we may define the *formal adjoint* of $\mathcal{L}$, $\mathcal{L}^*$ by

$$\mathcal{L}^* u = -\frac{\partial}{\partial x_i}\left(a^{ji}(x)\, u_{x_j} + b^i(x)\, u\right) + a^i(x)\, u_{x_i} + a(x)\, u \tag{2.10}$$

and observe that $\mathcal{L}^*$ satisfies the same conditions as imposed above on $\mathcal{L}$. Applying the Riesz-Schauder theory of compact operators in a Hilbert space, we consequently obtain the following alternative

THEOREM 2.1. *Let g be compactly bounded on $H^0(\mathcal{A}, \Omega)$. Then there exists a countable, isolated set of real numbers Σ such that if $\sigma \notin \Sigma$, the operator $\mathcal{L}_\sigma(g)$ is a bijective mapping from $H^0(\mathcal{A}, \Omega)$ to $H^{-1}(\mathcal{A}^{-1}, \Omega)$. For $\sigma \in \Sigma$, the null spaces of $\mathcal{L}_\sigma(g)$, $\mathcal{L}_\sigma^*(g)$ are of positive, finite dimension and the range of $\mathcal{L}_\sigma$ in $H^{-1}(\mathcal{A}^{-1}, \Omega)$ is the orthogonal complement of the null space of $\mathcal{L}_\sigma^*$.*

In order to make Theorem 2.1 appear more concrete, let us remark here that a sufficient condition for g to be compactly bounded on $H^0(\mathcal{A}, \Omega)$ is given in Lemma 1.4 and that the dual space $H^{-1}(\mathcal{A}^{-1}, \Omega)$ is characterized in Proposition 1.2. In the following section, we will establish a natural condition from the classical standpoint, namely that

$$a - \frac{\partial a^i}{\partial x^i} \geq 0, \qquad \text{for } x \text{ in } \Omega, \tag{2.11}$$

to guarantee $\Sigma \subset (0, \infty)$.

Normally, a Fredholm alternative is established for the operators of the form $\mathcal{L}_\sigma$ where

$$\mathcal{L}_\sigma u = \mathcal{L} u + \sigma u. \tag{2.12}$$

Clearly, if the function $1 + g$ is compactly bounded on $H^0(\mathcal{A}, \Omega)$, then $\mathcal{L}_\sigma$ is bounded and coercive on $H^0(\mathcal{A}, \Omega)$ for sufficiently large σ and hence the conclusions of Theorem 2.1 apply to $\mathcal{L}_\sigma$. We will express this observation as

Corollary 2.1. *Suppose that the function $1 + g$ is compactly bounded on $H^0(\mathcal{A}, \Omega)$. Then the operators $\mathcal{L}_\sigma$ defined by* (2.12) *satisfy the Fredholm alternative stated in Theorem* 2.1. *In particular if $\lambda^{-1} \in L_t(\Omega)$, $g \in L_s(\Omega)$ where $s^{-1} + t^{-1} = 2n^{-1}$ if $n > 2$ and $g \in L \log L(\Omega)$ if $n = 2$, $t = \infty$ the conclusions of Theorem* 2.1 *apply to $\mathcal{L}_\sigma$.*

§. 3. The weak maximum principle, existence and uniqueness.

The purpose of the present section is to derive a weak maximum principle for subsolution of $\mathcal{L}u = 0$ from which follow a uniqueness theorem and hence by Theorem 2.1 an existence theorem for the generalized Dirichlet problem for $\mathcal{L}$. The condition which we add to those imposed previously on $\mathcal{L}$, is that $a(x) - a^i_{x_i}(x)$ is non-negative in the sense of $\mathcal{D}'(\Omega)$, that is

$$\int_\Omega (a^i(x)\, \varphi_{x_i} + a(x)\, \varphi)\, dx \geq 0 \tag{3.1}$$

for all nonnegative φ in $\mathcal{D}(\Omega)$. Note that condition (3.1) is equivalent to the property that the positive constants are supersolutions of $\mathcal{L}u = 0$.

A generalized notion of inequality on $\partial\Omega$ is also required here. Let us say that a function u belonging to $W^1_2(\mathcal{A}, \mu, \Omega)$ satisfies $u \leq 0$ on $\partial\Omega$ if the function $u^+ = \sup(u, 0)$ belongs to $H^0(\mathcal{A}, \mu, \Omega)$. Consequently, we may define for u in $W_2^1(\mathcal{A}, \mu, \Omega)$

$$\tag{3.2} \begin{aligned} \sup_{\partial\Omega} u &= \inf\{L \in \mathbb{R}\,;\ u - L \leq 0 \quad \text{on} \quad \partial\Omega\} \\ \inf_{\partial\Omega} u &= \sup\{L \in \mathbb{R}\,;\ L - u \leq 0 \quad \text{on} \quad \partial\Omega\}. \end{aligned}$$

Note that if u is the limit in $H_2^1(\mathcal{A}, \mu, \Omega)$ of a sequence of smooth functions that are ≤ 0 on $\partial\Omega$, then $u \leq 0$ on $\partial\Omega$. We can now state.

THEOREM 3.1. *Let u be a $W_2^1(\mathcal{A}, g, \Omega)$ subsolution of $\mathcal{L}u = 0$ in Ω and suppose that g as given by* (2.2) *is compactly bounded on $H^0(\mathcal{A}, \Omega)$. Then*

$$\tag{3.3} \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ .$$

PROOF. From Lemma 2.2, it follows that

$$\tag{3.4} \mathcal{L}(u, \varphi) \leq 0$$

for all $\varphi \geq 0$, lying in $H^0(\mathcal{A}, \Omega)$. For k satisfying

$$\tag{3.5} L = \sup_{\partial\Omega} u^+ \leq k < \sup_{\Omega} u^+ \leq \infty$$

we introduce the function

$$\tag{3.6} v = v_k = (u - k)^+.$$

Then, defining the operator $\mathcal{L}'$ by

$$\tag{3.7} \mathcal{L}' u = -\frac{\partial}{\partial x_i}(a^{ij}(x)\, u_{x_j}) + (b^i(x) - a^i(x))\, u_{x_i},$$

we obtain from (3.1) and (3.4) that

$$\tag{3.8} \mathcal{L}'(v, \varphi) \leq 0$$

for all $\varphi \geq 0$ lying in $H^0(\mathcal{A}, \Omega)$ and satisfying $\operatorname{supp} \varphi \subset \operatorname{supp} v$. (In fact it is easily shown that v is a subsolution of $\mathcal{L}' u = 0$). Theorem 3.1 will be established by showing that (3.8) implies v is identically zero for $k = L$. Let us show first that v is bounded. This is accomplished simply by inserting the test function $\varphi = v$ in (3.8). Therefore

$$(3.9)\qquad \int_\Omega a^{ij} v_{x_i} v_{x_j}\, dx \leq \int_\Omega |(b^i - a^i) v_{x_i}|\, v dx$$

$$\leq K_{\mathscr{A}}^2/2 \int_\Omega g v^2\, dx + \frac{1}{2}\int_\Omega a^{ij}\, v_{x_i}\, v_{x_j}\, dx$$

by (2.3) and consequently

$$(3.10)\qquad \int_\Omega a^{ij} v_{x_i} v_{x_j}\, dx \leq K_{\mathscr{A}}^2 \int_\Omega g v^2\, dx.$$

Using Lemma 1.1 and the fact that g is compactly bounded on $H^0(\mathscr{A}, \Omega)$ we have accordingly

$$(3.11)\qquad \| v \|^2_{L_{n'}(\Omega)} \leq C(n)\, \| \lambda^{-1} \|_{L_1(\Omega)} \int_\Omega a^{ij} v_{x_i} v_{x_j}\, dx$$

$$\leq \varepsilon C(n)\, \| \lambda^{-1} \|_{L_1(\Omega)}\, K_{\mathscr{A}}^2 \int_\Omega a^{ij} v_{x_i} v_{x_j}\, dx$$

$$+ K(\varepsilon)\, C(n)\, \| \lambda^{-1} \|_{L_1(\Omega)} K_{\mathscr{A}}^2 \left(\int_\Omega v\, dx\right)^2$$

and hence by appropriate choice of $\varepsilon > 0$, we obtain

$$(3.12)\qquad \| v \|_{L_{n'}(\Omega)} \leq C\, \| v \|_{L_1(\Omega)}$$

where C depends on $K, n, \| \lambda^{-1} \|_{L_1(\Omega)}$ and $K_{\mathscr{A}}$. From (3.12), by Hölder's inequality

$$(3.13)\qquad \| v \|_{L_{n'}(\Omega)} \leq C\, |\operatorname{supp} v|^{1/n}\, \| v \|_{L_{n'}(\Omega)}$$

and hence if $k < \sup_\Omega u^+$, we have

$$(3.14)\qquad |\operatorname{supp} v| \geq C^{-n}$$

from which we conclude that u^+ must attain its supremum on a set of positive measure and consequently be bounded.

To complete the proof, we set $M = \sup_\Omega u^+$ and consider as a test function in (3.8)

$$(3.15)\qquad \varphi = \frac{v}{M + \varepsilon - v}$$

for $k = L$ and some $\varepsilon > 0$. We obtain thus

$$\int_\Omega \frac{a^{ij} v_{x_i} v_{x_j}}{(M+\varepsilon-v)^2}\,dx \leq \int_\Omega |(b^i - a^i) v_{x_i}| \frac{v}{M(M+\varepsilon-v)}\,dx \tag{3.16}$$

$$\leq K_{\mathcal{A}}^2 \int \frac{g v^2}{M^2}\,dx \quad \text{by (2.3)}.$$

Now, we set

$$w = w_\varepsilon = \log \frac{M+\varepsilon}{M+\varepsilon-v} \tag{3.17}$$

so that by Lemma 1.1 and (3.16)

$$\| w \|^2_{L_{n'}(\Omega)} \leq \mathcal{C} K^2_{\mathcal{A}} \int_\Omega g\,dx. \tag{3.18}$$

Letting ε tend to zero, we conclude that either the function $w_0 = -\log\left(1 - \frac{v}{M}\right)$ is integrable or that v is identically zero. The integrability of w_0 implies that v can only coincide with M on a set of zero measure contradicting our previous conclusion. Hence $v = 0$ and Theorem 3.1 is proved. ||

We proceed to draw the readily apparent conclusions from Theorem 3.1. We have first, the automatic extensions to supersolutions and solutions of $\mathcal{L}u = 0$.

COROLLARY 3.1. *Let u be a $W_2^1(\mathcal{A}, g, \Omega)$ supersolution of $\mathcal{L}u = 0$ and suppose that g is compactly bounded on $H^0(\mathcal{A}, \Omega)$. Then*

$$\inf_\Omega u \geq \inf_{\partial\Omega} u^- \tag{3.19}$$

where $u^- = \inf(u, 0)$.

COROLLARY 3.2. *Let u be a $W_2^1(\mathcal{A}, g, \Omega)$ solution of $\mathcal{L}u = 0$ in Ω with g compactly bounded on $H^0(\mathcal{A}, \Omega)$. Then*

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u|. \tag{3.20}$$

COROLLARY 3.3. *Let u be a $H^0(\mathcal{A}, \Omega)$ solution of $\mathcal{L}u = 0$ in Ω with g compactly bounded on $H^0(\mathcal{A}, \Omega)$. Then $u = 0$ in Ω.*

Applying Theorem 2.1, we obtain an existence and uniqueness theorem for $\mathcal{L}$ from the previous Corollary.

COROLLARY 3.4. *Let g be compactly bounded on $H^0(\mathscr{A}, \Omega)$. Then $\mathcal{L}$ is a bijective, bicontinuous mapping from $H^0(\mathscr{A}, \Omega)$ onto $H^{-1}(\mathscr{A}^{-1}, \Omega)$.*

The fact that $\mathcal{L}$ is bicontinuous follows from Lemma 2.2 and the closed graph theorem. We proceed now to deduce from Corollary 3.4 the solvability of a generalized Dirichlet problem for $\mathcal{L}$. Let u and v be functions in $W_2^1(\mathscr{A}, g, \Omega)$. Consistent with our earlier definition of inequality on $\partial\Omega$, we say that $u = v$ on $\partial\Omega$ if the function $u - v$ lies in $H^0(\mathscr{A}, g, \Omega)$. We can now state

THEOREM 3.2. *Assume that the operator $\mathcal{L}$ satisfies the conditions of Theorem 3.1. Let v be a given function in $W_2^1(\mathscr{A}, g, \Omega)$ and f^i, $i = 1, \dots, n$, measurable functions on Ω satisfying*

$$\int_\Omega b_{ij} f^i f^j \, dx < \infty \tag{3.21}$$

where $\mathcal{B} = [b_{ij}]$ is the inverse matrix of $\mathscr{A}$. Then there exists a unique function u in the class $W_2^1(\mathscr{A}, g, \Omega)$ satisfying the equation

$$\mathcal{L}u = \frac{\partial}{\partial x_i} f^i \tag{3.22}$$

in Ω and agreeing with v on $\partial\Omega$. Furthermore there exists a constant N, independent of u, for which

$$\| u \|_{\mathscr{A}, g, \Omega} \leq \| v \|_{\mathscr{A}, g, \Omega} + N \left[\int_\Omega b_{ij} f^i f^j \, dx \right]^{1/2}. \tag{3.23}$$

PROOF. Setting $w = u - v$, we see that (3.22) is equivalent to the equation

$$\mathcal{L}(w, \varphi) = - \mathcal{L}(v, \varphi) - \int_\Omega f^i \varphi_{x_i} \, dx \tag{3.24}$$

fos all φ in $\mathcal{D}(\Omega)$. The conclusion of Theorem 3.2 then follows from Corollary 3.4, Lemma 2.2 and Proposition 1.2. ||

An interesting estimate for *subsolutions* of equation (3.22) now arises jointly from Theorems 3.1 and 3.2.

COROLLARY 3.5. Let u be a $W_2^1(\mathscr{A}, g, \Omega)$ subsolutions of equation (3.22) in Ω satisfying $u \leq 0$ on $\partial\Omega$ and suppose that h is a bounded function on

$H^0(\mathcal{A}, \Omega)$. Then

$$\| u^+ \|_{L_2(h, \Omega)} \leq KN \left(\int_\Omega b_{ij} f^i f^j \, dx \right)^{1/2} \tag{3.25}$$

where K is defined by the expression (1.23).

PROOF. We define a function v to be the $H^0(\mathcal{A}, \Omega)$ solution of equation (3.22) in Ω. Consequently the function $w = u - v$ will be a subsolution of the equation $\mathcal{L} w = 0$ in Ω and hence by Theorem 3.1 we have $u \leq v$ in Ω. The estimate (3.25) then follows from the estimates (3.23) and (1.23) for v. ||

Note that the above technique that is the combination of a maximum principle for subsolutions and an existence theorem for solutions provides a fairly general means of deducing certain types of subsolution estimates from the corresponding solution estimates.

Note also by duality that Theorem 3.2 will also hold if a^i is replaced by b^i in condition (3.1).

§ 4. Global regularity and estimates.

In the previous section we derived an existence and uniqueness theorem for the generalized Dirichlet problem for the equation (3.22) under the assumption of integrability of the function

$$h^2 = b_{ij} f^i f^j \tag{4.1}$$

where $\mathcal{B} = [b_{ij}]$ denoted the inverse matrix of $\mathcal{A}$. Our purpose in this section is to examine the global behaviour of solutions, subsolutions and supersolutions of equation (3.22) with respect to the L_p behaviour of h and λ^{-1}. Let us maintain the previously stated conditions on $\mathcal{L}$, notably conditions (2.3), (3.1) and the compact boundedness of g as defined by the inequality (2.2). As we shall indicate below, the condition (3.1) may be dropped for some estimates provided a^i and a are further restricted.

We will consider equations of an apparent more general form than equation (3.22), viz.

$$\mathcal{L} u = -\frac{\partial}{\partial x_i}(a^{ij} u_{x_j} + a^i u) + b^i u_{x_i} + au = -\frac{\partial f^i}{\partial x_i} + f \tag{4.2}$$

where f is an integrable function on Ω. By invoking the Newtonian potential of f, equation (4.2) may be written in the form (3.21) [27]. Our approach however will be to treat the full equation (4.2) directly. Let us note here that when λ^{-1} lies in $L_t(\Omega)$, $t \geq 1$, then $\Phi \to \int_\Omega f \Phi \, dx$ is a member of $H^{-1}(\mathcal{A}^{-1}, \Omega)$ provided f belongs to $L_r(\Omega)$ where

$$\frac{1}{r} = \frac{n+2}{2n} - \frac{1}{2t}. \tag{4.3}$$

A byproduct of the main theorem which follows will be an existence theorem for a generalized Dirichlet problem for equation (4.2) when the right hand side is merely assumed integrable in Ω.

THEOREM 4.1. *Let u be a $W_2^1(\mathcal{A}, g, \Omega)$ subsolution of equation* (4.2) *in Ω, satisfying $u \leq 0$ on $\partial\Omega$. Assume that the functions λ^{-1}, h, f lie in the spaces $L_t(\Omega)$, $L_s(\Omega)$, $L_r(\Omega)$ respectively where $r, s, t \geq 1$. Then we have the following results:*

I *if*

$$\frac{1}{s} + \frac{1}{2t} < \frac{1}{n}, \qquad \frac{1}{r} + \frac{1}{t} < \frac{2}{n} \tag{4.4}$$

the function u is bounded from above in Ω;

II *if*

$$\frac{1}{s} + \frac{1}{2t} = \frac{1}{n}, \qquad \frac{1}{r} + \frac{1}{t} = \frac{2}{n} \tag{4.5}$$

the function u^+ belongs to the Orlicz space $L_\Phi(\Omega)$ where $\Phi(t) = e^{|t|} - 1$;

III *if*

$$\frac{1}{s} + \frac{1}{2t} > \frac{1}{n}, \qquad \frac{1}{r} + \frac{1}{2t} = \frac{1}{n} + \frac{1}{s}, \qquad s \geq 2 \tag{4.6}$$

then u^+ belongs to $L_p(\Omega)$ where

$$\frac{1}{p} = \frac{1}{s} + \frac{1}{2t} - \frac{1}{n} = \frac{1}{r} + \frac{1}{t} - \frac{2}{n}. \tag{4.7}$$

Furthermore in each case we have a estimate

$$\| u^+ \| \leq C (\| h \|_{L_s(\Omega)} + \| f \|_{L_r(\Omega)}) \tag{4.8}$$

where C is a constant independent of u and $\| u^+ \|$ denotes $\sup_\Omega u^+$ *in case* I, $\| u^+ \|_{L_\Phi(\Omega)}$ in case II, $\| u^+ \|_{L_p(\Omega)}$ in case III.

PROOF. Let $\mathcal{L}'$ be defined by the expression (3.7) so that the function $v = u^+$ will satisfy

$$\mathcal{L}^1 (v, \Phi) \leq \int_\Omega (f^i \Phi_{x_i} + f \Phi)\, dx \tag{4.9}$$

for all $\Phi \geq 0$, lying in $H^0(\mathcal{A}, \Omega)$ and satisfying $\operatorname{supp} \Phi \subset \operatorname{supp} v$. The proof of Theorem 4.1 will now be accomplished by means of a so-called « one blow method ». Let $\mathcal{F}$ be a convex, even, uniformly Lipschitz continuous function on E' satisfying $\mathcal{F}(0) = 0$. By Lemma 1.3, the test function

$$\Phi(x) = \mathcal{F}(v(x)) \tag{4.10}$$

qualifies as a legitimate test function in the integral inequality (4.9). On substituting for Φ, we obtain

$$\int_\Omega \mathcal{F}'(v)\, a^{ij} v_{x_i} v_{x_j}\, dx \leq \int_\Omega \mathcal{F}(v)\, |(b^i - a^i)\, v_{x_i}|\, dx + \int_\Omega \mathcal{F}'(v)\, |f^i v_{x_i}|\, dx + \int_\Omega \mathcal{F}(v)\, |f|\, dx. \tag{4.11}$$

Using the condition (2.3) we have

$$\begin{aligned} |(b^i - a^i)\, v_{x_i}| &\leq K_{\mathcal{A}} (g\, a^{ij} v_{x_i} v_{x_i})^{1/2} \\ |f^i v_{x_i}| &\leq K_{\mathcal{A}}\, h\, (a^{ij} v_{x_i} v_{x_j})^{1/2} \end{aligned} \tag{4.12}$$

and hence we obtain from (4.11)

$$\int_\Omega \mathcal{F}'(v)\, a^{ij} v_{x_i} v_{x_i}\, dx \leq C \left\{ \int_{\Omega^+} \frac{\mathcal{F}^2(v)}{\mathcal{F}'(v)}\, g\, dx + \int_\Omega \mathcal{F}'(v)\, h^2\, dx + \int_\Omega \mathcal{F}(v)\, |f|\, dx \right\} \tag{4.13}$$

where C is a constant depending on $K_{\mathcal{A}}$, and Ω^+ denotes the support of $\mathcal{F}(v)$. Note that by virtue of the properties of $\mathcal{F}$, the derivative $\mathcal{F}'(v)$ is positive a.e (Ω^+). Let us now introduce a further function on E^1, $\mathcal{S}$, defined by

$$\mathcal{S}(|t|) = \int_0^{|t|} (\mathcal{F}(s))^{1/2}\, ds \tag{4.14}$$

and note that by the assumed properties of $\mathcal{F}$, $\mathcal{S}$ will be a function of the same type satisfying for $t \geq 0$

$$\mathcal{F}' = (\mathcal{S}')^2, \qquad \mathcal{F}^2 \leq \mathcal{F}' \mathcal{S}^2. \tag{4.15}$$

Hence substituting for $\mathcal{S}$ in the estimate (4.13) we get

$$\int_\Omega a^{ij} \mathcal{S}_{x_i} \mathcal{S}_{x_j} \, dx \leq C \left\{ \int_\Omega g \, \mathcal{S}^2 \, dx + \int_\Omega h^2 (\mathcal{S}')^2 \, dx + \int_\Omega |f| \, \mathcal{S}\mathcal{S}' \, dx \right\}. \tag{4.16}$$

For the remainder of this proof C will designate an arbitrary constant independent of u. So far, apart from establishing the formulation (4.9), which required the boundedness of g on $H^0(\mathcal{A}, \Omega)$, we haven't used the assumed conditions on f, g and h. Let us now apply the compact boundedness inequality (1.24) for g with $\varepsilon = (2C)^{-1}$ to the term in g in estimate (4.16) and also Hölder's inequality to the f and h terms there. We obtain, consequently

$$\int_\Omega a^{ij} \mathcal{S}_{x_i} \mathcal{S}_{x_j} \, dx \leq C \{ \| \mathcal{S} \|_1^2 + \| h \|_s^2 \, \| \mathcal{S}' \|^2_{\frac{s}{s-2}} + \| f \|_r \, \| \mathcal{S}\mathcal{S}' \|_{\frac{r}{r-1}} \} \tag{4.17}$$

where we have abbreviated $\|\cdot\|_{L_p(\Omega)}$ as $\|\cdot\|_p$. As application of the Sobolev inequality. Lemma 1.1, hence yields

$$\| \mathcal{S} \|_{t^*}^2 \leq C \{ \| \mathcal{S} \|_1^2 + \| h \|_s^2 \, \| \mathcal{S}' \|^2_{\frac{s}{s-2}} + \| f \|_r \, \| \mathcal{S}\mathcal{S}' \|_{\frac{r}{r-1}} \} \tag{3.18}$$

where

$$\frac{1}{t^*} = \frac{1}{2}\left(1 + \frac{1}{t}\right) - \frac{1}{n} > 0. \tag{4.19}$$

We will defer temporarily the case $n = 2$, $t = \infty$.

There are two established techniques whose implementation at this juncture could lead us from the fundamental estimates (4.17) or (4.18) to our goal. These are the technique of iteration of L_p norms introduced by Moser [17] and utilized for example in [22] and [29] and the technique of variable truncation initiated by De Giorgi [5] and employed for example in [11] and [27]. We will delineate now a further technique which effects the desired estimates through the judicious choice of a single function which satisfies certain differential inequalities arising from the estimate (4.18). This so-called « one blow method » although conceptually simple does require some, possibly unaesthetic, technicalities in its realization.

First, let us note a simple interpolation inequality which holds for $\mathcal{S}$. For any $\varepsilon > 0$, there exists a constant N_ε such that for all values $v \geq 0$,

$$\mathcal{S}(v) \leq \varepsilon (\mathcal{S}(v))^{t^*} + N_\varepsilon v. \tag{4.20}$$

The constant N_ε must, of course, depend on $\mathcal{S}$. Next, if k is a positive constant, then it is clear that the function v/k will satisfy the integral inequality (4.9) with f and f_i replaced by f/k and f_i/k respectively and consequently also the inequality (4.18) with h replaced by h/k. Let us now fix a number k by defining

$$k = \inf \left\{ l \leq 0 \, ; \left\| \mathcal{S}\left(\frac{v}{l}\right) \right\|_{L_{t^*}(\Omega)} \leq 1 \right\}. \tag{4.21}$$

Then, assuming that v is not identically zero, we must have, by virtue of the continuity of $\left\| \mathcal{S}\left(\frac{v}{l}\right) \right\|_{t^*}$ with respect to l, that

$$\left\| \mathcal{S}\left(\frac{v}{k}\right) \right\|_{t^*} = 1. \tag{4.22}$$

Therefore, using (4.22) together with (4.20) in (4.18) for v/k, we are led to an estimate for k, viz

$$k^2 \leq C \left\{ \| v \|_1^2 + \| h \|_s^2 \, \| \mathcal{S}' \|^2_{\frac{s}{s-2}} + k \, \| f \|_r \, \| \mathcal{S}\mathcal{S}' \|_{\frac{r}{r-1}} \right\}. \tag{4.23}$$

Now suppose that $\mathcal{S}$ is restricted to satisfy differential inequalities of the following forms

$$\mathcal{S}' \leq a_0 + a_1 \mathcal{S}^\alpha \tag{4.24}$$

$$\mathcal{S}' \leq b_0 + b_1 \mathcal{S}^\beta$$

where a_0, a_1, b_0, b_1 are non-negative constants and

$$\alpha = t^* \left(1 - \frac{2}{s}\right), \qquad \beta = t^* \left(1 - \frac{1}{r}\right) - 1. \tag{4.25}$$

Then it follows from (4.22) and (4.2) that

$$k \leq C \{ \| v \|_1 + \| h \|_r + \| f \|_r \} \tag{4.26}$$

where the constant C would also depend on the contants in (4.24). Only an approximation argument now separates us from the conclusion. Let N satisfy $0 < N \leq \infty$ and let $\mathcal{H}$ denote a positive, convex function on the interval $(0, N)$, satisfying $\mathcal{H}(0) = 0$, $\mathcal{H}(N) = \infty$ together with the differential inequalities (4.24). If N is not infinite, then $\mathcal{H}$ is extended to $(0, \infty)$ by defining $\mathcal{H}(t) = \infty$ for $t \geq N$. It is then clearly possible to choose a sequence of functions $\mathcal{F}_m$, $m = 1, 2, \ldots$ satisfying the same properties as $\mathcal{F}$ such that the resulting functions $\mathcal{S}_m$ satisfy (i) $\mathcal{S}_m(v) \leq 0$, (ii) $\mathcal{S}_m(v) = \mathcal{H}(v)$ for $\mathcal{H}(v) \leq m$, (iii) the differential inequalities (4.24) and (iv) the interpolation inequality (4.20) with N_ε independent of m. Consequently the resulting sequence k_m satisfies the estimate (4.26), which is hence valid for $k = \sup k_m$. Therefore we have

$$\int_\Omega \left\{ \mathcal{H}\left(\frac{v}{k}\right) \right\}^{t^*} dx \leq \overline{\lim} \int_\Omega \left\{ \mathcal{S}_m\left(\frac{v}{k}\right) \right\}^{t^*} dx \leq 1 \tag{4.27}$$

so that the integrability of v on Ω is now reduced to the question of choosing $\mathcal{H}$ satisfying (4.24). We now separate the three cases. The exponent conditions in case I imply that $\alpha, \beta > 1$. By adjusting either r or s we can insure that $\alpha = \beta$. Then the function

$$\mathcal{H}(v) = (1 - v)^{-\frac{1}{\alpha - 1}} - 1 \tag{4.28}$$

satisfies the inequalities (4.24) with $a_0 = a_1 = b_0 = b_1 = \dfrac{1}{\alpha - 1}$ and in this case $N = 1$. From the estimates (4.26) and (4.27) we must have

$$\sup_\Omega v \leq k \leq C \{ \| v \|_1 + \| h \|_s + \| f \|_r \} \tag{4.29}$$

When N is infinite and $\mathcal{H}$ is the defining function for an Orlicz space then clearly k majorizes the norm of v in the space. Bearing this in mind cases II and III fall out of the inequalities (4.24). For in case II, $\alpha = \beta = 1$ and hence $H(v) = e^v - 1$ satisfies (4.24) with $a_0 = a_1 = b_0 = b_1 = 1$. Finally in case III, $\alpha = \beta < 1$ and $\mathcal{H}(v) = v^{\frac{1}{1-\alpha}} = v^p$ satisfies (4.24) with $a_0 = b_0 = 0$, $a_1 = b_1 = p$. Thus we have proved that in all cases

$$\| v \| \leq C \{ \| v \|_1 + \| h \|_s + \| f \|_r \} \tag{4.30}$$

where $\| v \|$ denotes the appropriate norm of v. The estimate (4.8) then follows by Corollary 3.5. It remains solely to consider the situation $n = 2$,

$t = \infty$. For case I, a sufficiently large choice of t^* in (4.18) enables the above proof to go through. In case II, the second part of Lemma 1.1 provides us with a stronger result on taking $\mathcal{S}(v) = \mathcal{H}(v) = v$, that is for $\|v\|$ we may take the norm of v in the Orlicz space $L_\Phi(\Omega)$ where $\Phi(t) = e^{t^2} - 1$. Finally, case III is clearly void. Theorem 4.1 is thus proved ‖.

As in the case of Theorem 3.1, the above theorem spawns many corollaries. Let v be a $W_2^1(\mathcal{A}, g, \Omega)$ function. Then an extension of Theorem 4.1 to a boundary inequality $u \leq v$ follows by considering the differential inequality for $w = u - v$, viz

$$\mathcal{L}w \leq -\frac{\partial}{\partial x_i}(f^i - a^{ij} v_{x_j} - a^i v) + f - b^i v_{x_i} - av \tag{4.31}$$

which also has the form (4.2). In particular we have

Corollary 4.1. *Let u be a $W_2^1(\mathcal{A}, g, \Omega)$ subsolution of equation* (4.2) *in Ω and v a $W_2^1(\mathcal{A}, g, \Omega)$ supersolution of the equation $\mathcal{L}u = 0$ in Ω such that $u \leq v$ on $\partial\Omega$. Then the function $(u - v)^+$ satisfies the conclusions of Theorem* 4.1. *In particnlar, the result holds if $v = \sup_\Omega u^+$.*

The extensions to solutions and supersolutions of equation (4.2) are of course obvious. We just mention the following

Corollary 4.2. *Let u be a $W_2^1(\mathcal{A}, g, \Omega)$ supersolution (solution) of equation* (4.2) *in Ω. Then the statements of Theorem* 4.1 *are applicable to the function $v = -u$ $(|u|)$.*

Some remarks concerning the weakening of our hypotheses are in order here.

Remark 1. If we drop the positivity condition (3.2), then Theorem 4.1 is not true in general. However, under the stronger assumption that the functions $b_{ij} a^i a^j$ and a lie in $L_q(\Omega)$ where q satisfies $\frac{1}{q} + \frac{1}{t} < \frac{2}{n}$, the statement of Theorem 4.1 holds provided the estimate (4.8) is replaced by the estimate (4.30) for v. The extra terms arising in the fundamental inequality (4.18) due to the presence of the coefficients a^i and a are handled by an interpolation inequality similar to the inequality (4.20). In general the dependance of the estimate (4.30) on $\|v\|_{L_1(\Omega)}$ may be removed if the operator $\mathcal{L}^{-1}$ is a continuous mapping from $H^{-1}(\mathcal{A}, \Omega)$ onto $H^0(\mathcal{A}, \Omega)$.

Remark 2. The ε^ν-compact boundedness or h^2 and f on $H^0(\mathcal{A}, \Omega)$ for $\nu > 0$ would in fact be sufficient for the validity of case 1 of Theorem 4.1.

Likewise, the same condition on the quantities $b_{ij} a^i a^j$ and a is enough to guarantee the previous remark. Since the method of proof in this situation seems considerably different from the one blow method illustrated above, it will be omitted. It may however be readily effected by means of the L_p iteration method.

An alternative proof of case I of Theorem 4.1.

It is desirable to have a proof of Theorem 4.1 which provides a priori estimates for subsolutions or solutions without recourse to the uniqueness results of Section 3. The following proof accomplishes this and is significant as it may consequently be extended to the nonlinear situation (see [33]). The method is to deduce directly from the estimate (4.30), an estimate independent of $\| v \|_{L_1(\Omega)}$. Let us put

$M = \sup\limits_{\Omega} v$ and $N = M + \|f\|_r + \| h \|_s$ and assume that $M < N$. Otherwise the situation is described by Theorem 3.1. The function $\mathcal{F}$ defined by

$$\mathcal{F}(v) = \frac{v}{N - v} \tag{4.32}$$

is then clearly admissible in the inequality (4.13). Therefore we have

$$\int_{\Omega} \frac{N}{(N-v)^2} a^{ij} v_{x_i} v_{x_j} \, dx \leq C \left\{ \int_{\Omega} \frac{gv^2}{N} \, dx + \int_{\Omega} \frac{Nh^2}{(N-v)^2} \, dx + \int_{\Omega} \frac{v \, |f|}{N-v} \, dx \right\} \tag{4.33}$$

and hence defining

$$w(x) = \mathcal{S}(v) = \log \frac{N}{N-v}, \tag{4.34}$$

we obtain from (4.33)

$$\int_{\Omega} a_{ij} w_{x_i} w_{x_j} \, dx \leq C \int_{\Omega} \left(g + \frac{h^2}{\| h \|_s^2} + \frac{|f|}{\| f \|_r} \right) dx \leq C_0 \tag{4.35}$$

where C_0 is a constant depending on $K_{\mathcal{A}}$, $\| g \|_1$, $|\Omega|$ and r, s. Since w belongs to $H^0(\mathcal{A}, \Omega)$, an estimate for $\| w \|_{L_1(\Omega)}$ follows from (4.35). Now let us replace the function $\mathcal{S}(v)$ in the inequality (4.11) by the function

$$\mathcal{F}_0(v) = \frac{\mathcal{F}(w)}{N - v}. \tag{4.36}$$

Using the formula

$$(4.37) \qquad \mathcal{F}_0'(v) = \frac{\mathcal{F}'(w) + \mathcal{F}(w)}{(N-v)^2}$$

we obtain thus

$$(4.38) \qquad \int_\Omega (\mathcal{F}'(w) + \mathcal{F}(w))\, a^{ij} w_{x_i} w_{x_j}\, dx$$

$$\leq \int_\Omega \mathcal{F}(w)\, |(b^i - a^i) w_{x_i}|\, dx + \int_\Omega (\mathcal{F}(w) + \mathcal{F}'(w)) \frac{|f^i w_{x_i}|}{\| h \|\, s} dx$$

$$+ \int_\Omega \mathcal{F}(w) \frac{|f|}{\| f \|_r} dx.$$

Making use of the second of the inequalities (4.12), we may reduce (4.38) to

$$(4.39) \qquad \int_\Omega \mathcal{F}'(w)\, a^{ij} w_{x_i} w_{x_j}\, dx \leq \int_\Omega \mathcal{F}(w)\, |(b^i - a^i) w_{x_i}|\, dx$$

$$\int_\Omega \mathcal{F}'(w) \frac{|f^i w_{x_i}|}{\| h \|\, s} dx + \int_\Omega \mathcal{F}(w) \left(\frac{|f|}{\| f \|_r} + \frac{h}{\| h \|_s} \right) dx.$$

That is, we obtain an inequality of the form (4.11) again but with v, f^i and f replaced respectively by $w, f^i / \| h \|\, s$ and $|f| / \| f \|_r + h / \| h \|\, s$. Consequently the estimate (4.30) holds for these functions and coupling this with the estimate for $\| w \|_1$ derived above, we obtain

$$(4.40) \qquad \sup w \leq C_1$$

where the constant C_1 now depends on n, r, s, t, $|\Omega|$, $K_{\mathcal{A}}$, $\| g \|_1$, $\| \lambda^{-1} \|_t$ and the constant K_s in (1.24). But the estimate (4.40) is equivalent to

$$(4.41) \qquad \log \frac{M + \| f \|_r + \| h \|_s}{\| f \|_r + \| h \|_s} < C_1$$

and hence

$$(4.40) \qquad \sup_\Omega u^+ \leq C (\| f \|_r + \| h \|_s)$$

where $C = e^{C_1} - 1 \, \|$.

It is possible to derive from the estimate (4.40) an estimate which exhibits more explicitly the dependence of the constant, C, on the quantities $\| \lambda^{-1} \|_t$ and $| \Omega |$. To this end, we set $R = | \Omega |^{1/n}$ and make a coordinate transformation to coordinates y_i given by

$$y_i = \frac{x_i}{R} . \tag{4.41}$$

The transformed domain, $\widetilde{\Omega}$, obviously satisfies $| \widetilde{\Omega} | = 1$. Next we set $T = \| \lambda^{-1}(y) \|_{L_t(\widetilde{\Omega})}$ and multiply the equation (4.2) through by $R^2 T$. We obtain thus an equivalent equation in the coordinates, y_i

$$\mathcal{L}u = -\frac{\partial}{\partial y_i}(\widetilde{a}^{ij}(y)\, u_{y_j} + \widetilde{a}^i(y)) + \widetilde{b}^i(y)\, u_{y_i} + \widetilde{a}(y)\, u = \widetilde{f}(y) - \frac{\partial}{\partial y_i}\widetilde{f}^i(y) \tag{4.42}$$

where the new coefficients are retated to the original ones by

$$\widetilde{a}^{ij}(y) = Ta^{ij}(x), \quad \widetilde{a}^i(y) = RT\, a^i(x), \quad \widetilde{b}^i(y) = RT\, b^i(x), \tag{4.43}$$

$$\widetilde{a}(y) = TR^2\, a(x), \quad \widetilde{f}(y) = TR^2 f(x), \quad \widetilde{f}^i(y) = TRf^i(x).$$

The transformed quantity, $\widetilde{g}$, corresponding to g, then satisfies the inequality for compact boundedness

$$\int_{\widetilde{\Omega}} \widetilde{g}\, u^2\, dy \leq \varepsilon \int_{\widetilde{\Omega}} \widetilde{a}^{ij}(y)\, u_{y_i}\, u_{y_i}\, dy + TR^{n+2} K_\varepsilon \left(\int_{\widetilde{\Omega}} u\, dy \right)^2 \tag{4.44}$$

for all u in $\mathcal{D}(\widetilde{\Omega})$. Therefore applying the estimate (4.40) we obtain

$$\sup_{\widetilde{\Omega}} u(y) \leq C(\| \widetilde{f} \|_{L_r(\widetilde{\Omega})} + \| \widetilde{h} \|_{L_s(\widetilde{\Omega})}) \tag{4.45}$$

where $\widetilde{h}(y) = (\widetilde{b}_{ij}(y)\, \widetilde{f}^i(y)\, \widetilde{f}^j(y))^{1/2} = T^{1/2} Rh(x)$, and the constant C now depends only on $n, r, s, t, K_{\mathcal{A}}, \| \widetilde{g} \|_{L_1(\widetilde{\Omega})}$ and the quantity $TR^{n+2} K_\varepsilon$ for small enough ε depending on $K_{\mathcal{A}}$. We remark here the dependence on $\| \widetilde{g} \|_{L_1(\widetilde{\Omega})}$ may, in fact, be eliminated if one uses inequality (1.24), for g and v, together with the estimate (4.16), for $\mathcal{S} = v$, to estimate the right hand side in the inequelity (4.33). Transforming back to the original coordinates, x_i, and noting more carefully the dependence of C on its arguments, we thus obtain

THEOREM 4.2. *Let u, $\mathcal{L}$, f and f^i satisfy the hypotheses of case* I *of Theorem* 4.1. *Then we have the estimate*

$$(4.46) \qquad \sup_{\Omega} u \leq C \{ \| \lambda^{-1} \|_{L_t(\Omega)}^{1/2} \| g \|_{L_s(\Omega)} R^{1-\frac{n}{s}-\frac{n}{2t}} + \| \lambda^{-1} \|_{L_t(\Omega)} \| f \|_{L_r(\Omega)} R^{2-\frac{n}{r}-\frac{n}{t}} \}$$

where the constant C is a non-decreasing function of the quantities n, r^{-1}, s^{-1}, t^{-1}, $K_{\mathcal{A}}$, $\| \lambda^{-1} \|_{L_t(\Omega)} R^{n+2-\frac{n}{t}} K_\varepsilon$ for a fixed ε depending on $K_{\mathcal{A}}$ only.

The estimate (4.46) will of course hold for supersolutions, (resp. solutions), when u is replaced by $-u$, (resp. $|u|$). Furthermore if we drop the inequality $u \leq 0$ on $\partial\Omega$ we must add the term $\sup_{\partial\Omega} u^+$ to the right hand side of (4.46).

The special case $a^i = b^i$.

An observation of the preceding estimates reveals that the hypothesis of compact boundedness of g could hace been replaced by only the boundedness of g provided the quantity $b_{ij} c^i c^j$ was compactly bounded, where $c^i = b^i - a^i$. In the case, $a^i = b^i$, which corresponds to a mild extension of the self adjoint situation, the previous proofs simplify considerably. In fact the uniqueness and existence arguments of Section 3 become trivial and in the proof of Theorem 4.1 we may take g and consequently K_ε to be zero. We point out briefly that the technicalities of the one blow method may also be reduced somewhat in this case. For since there is then no need to assume the convexity of the functions $\mathcal{F}$, we may define for fixed m satisfying $0 < m < \sup v$ and for $k > m$,

$$(4.47) \qquad \mathcal{G}\left(\frac{v}{k}\right) = \mathcal{H}\left(\frac{v_m}{k}\right)$$

where $v_m = \inf(v, m)$, and subsequently choose as a test function in (4.9), $\Phi(x) = \mathcal{F}(v/k)$ where $\mathcal{F}$ corresponds to the function $\mathcal{G}$ defined by (4.47). The desired estimates appear on letting m approach $\sup_{\Omega} u$ in the estimate (4.26) derived for k. This last technique will be treated more fully in the paper [33].

A further existence theorem.

Through a dualization process, further existence and uniqueness theorems arise from Theorem 4.1. In order to formulate these, we introduce classes of Banach spaces which extend the Hilbert spaces $H^0(\mathscr{A}, \Omega)$ and $H^{-1}(\mathscr{A}, \Omega)$ introduced in Section 1. For $p \geq 1$, let $\mathscr{A} = [a^{ij}(x)]$ be an $n \times n$ positive matrix valued function on Ω satisfying $\lambda^{-1/2} \in L_{p'}(\Omega)$, $\lambda^{1/2} \in L_p(\Omega)$ where as before λ denotes the minimum eigenvalue of $\mathscr{A}^s$, the symmetric part of $\mathscr{A}$. A norm given by

$$\| u \|_{H_p^0(\mathscr{A}, \Omega)} = \left\{ \int_\Omega (a^{ij} u_{x_i} u_{x_j} \, dx)^{p/2} \right\}^{1/p} \tag{4.48}$$

may then be defined on $\mathcal{D}(\Omega)$ and the Banach space $H_p^0(\mathscr{A}, \Omega)$ obtained by completion of $\mathcal{D}(\Omega)$ under (4.48) will consist of strongly differentiable functions. Analogously to the definition of $H^{-1}(\mathscr{A}, \Omega)$ we can introduce a further Banach space $H_p^{-1}(\mathscr{A}, \Omega)$ as the space of distributions, T representable in the form

$$T\varphi = T_{f_i}(\varphi) = \int f_i \varphi_{x_i} \, dx \qquad \text{for } \varphi \in \mathcal{D}(\Omega) \tag{4.49}$$

where f_i satisfies

$$\| T_{f_i} \|_{H_p^{-1}(\mathscr{A}, \Omega)} = \left\{ \int_\Omega (a^{ij} f_i f_j)^{p/2} \, dx \right\}^{1/p} < \infty. \tag{4.50}$$

It follows then that the dual space of $H_0^p(\mathscr{A}, \Omega)$ is $H_{p'}^{-1}(\mathscr{A}^{-1}, \Omega)$, $p' = p/(p-1)$, the proof of this assertion being an obvious extension of Proposition 1.2. But now having these spaces at our disposal, we may reexpress the conclusion of Corollary 4.2 for solutions as follows.

COROLLARY 4.3. *The operator $\mathcal{L}^{-1}$, which exists by virtue of Corollary* 3.4, *maps $H_s^{-1}(\mathscr{A}^{-1}, \Omega)$ continuously into $L_\infty(\Omega)$ if $1/s + 1/2t < 1/n$, $L_\Phi(\Omega)$ if $1/s + 1/2t = 1/r$, $L_p(\Omega)$ if $1/s + 1/2t > 1/n$, $s \geq 2$ where $1/p = 1/s + 1/2t - 1/n$.*

For s, satisfying $1/s + 1/2t < 1/n$, let $\mathcal{G}_s : H_s^{-1}(\mathscr{A}^{-1}, \Omega) \to L_\infty(\Omega)$ be the restriction of $\mathcal{L}^{-1}$ as given by the Corollary 4.3. Then the adjoint map $\mathcal{G}_s^*$ will be a continuous map from $L_1(\Omega)$ into $H_{s'}^0(\mathscr{A}, \Omega)$. The question then arises as to whether $\mathcal{G}_s^*$ corresponds to the inverse of the formal adjoint of $\mathcal{L}$, $\mathcal{L}^*$, since if this is so we have established the unique solvability of the equation $\mathcal{L}^* u = f$ for $f \in L_1(\Omega)$, $u \in H_{s'}^0(\mathscr{A}, \Omega)$. One can see that $\mathcal{G}_s^*$ will correspond with $(\mathcal{L}^*)^{-1}$ if the form $\mathcal{L}$ is bounded on $H_s^0(\mathscr{A}, \Omega) \times H_{s'}^0(\mathscr{A}, \Omega)$. A sufficient condition for this would be that g lies in $L_{s/2}(\Omega)$, which would also guarantee the dual result, namely that $\mathcal{L}$ is also bounded on $H_{s'}^0(\mathscr{A}, \Omega) \times H_s^0(\mathscr{A}, \Omega)$. We leave the checking of these statement to the reader, noting that they are of course, analogous to Lemma 2.1. By Lemma 1.4, if g lies in $L_{s/2}(\Omega)$, then g is compactly bounded on $H^0(\mathcal{A}, \Omega)$ and also by virtue of Remark 1 after Theocm 4.1, we may interchange $\mathcal{L}$ and $\mathcal{L}^*$ in Corollary 3.4. Thus, we are led to

THEOREM 4.3. *Let the operator $\mathcal{L}$ satisfy the condition* (3.1) *and suppose that $\lambda^{-1} \in L_t(\Omega)$, $g \in L_{s/2}(\Omega)$ where $1/s + 1/2t < 1/n$. Then for arbitrary integrable f, the equation $\mathcal{L}u = f$ is uniquely solvable for u in $\overset{0}{H}_{s'}(\mathcal{A}, \Omega)$. If f lies in $L_r(\Omega)$, and $1/s_0 = 1 + 1/n - 1/2t - 1/r \leq 1/2$, then u lies in $\overset{0}{H}_{s_0'}(\mathcal{A}, \Omega)$.*

Finally we note that the Theorems of this section have extended, by completely different arguments, the various global estimates to be found in the papers [20], [26] and [27], is well as those for linear equations in the book [11].

§ 5. Local regularity and estimates.

A notable feature of the preceding development of this paper has been the absence of restrictions on the maximum eigenvalue, Λ, of the coefficient matrix. This situation turns out to be a characteristic feature of the global theory since some control on Λ is certainly needed to work locally. A simple motivation of the type of condition required comes from desiring that the space $H^0(\mathcal{A}, \Omega)$ be stable under localization. According to Proposition 1.1, the boundedness of Λ on $H^0(\mathcal{A}, \Omega)$ would guarantee that $H^0(\mathcal{A}, \Omega)$ was a subspace of $H^{\text{loc}}(\mathcal{A}, \Omega)$. But for the estimates to be considered below a stricter condition on Λ appears necessary. Similarly, we will also improve our previous assumption on g, namely its compact boundedness on $H^0(\mathcal{A}, \Omega)$, although as we shall indicate at the appropriate points below, compact boundedness would suffice in some instances. Therefore, let us assume henceforth the following structure on the operator $\mathcal{L}$:

$$\lambda^{-2} \in L_t(\Omega),\ \Lambda,\ g \in L_s(\Omega),\ \frac{1}{t} + \frac{1}{s} < \frac{2}{n}. \tag{5.1}$$

For any subset S of Ω, we also make use of the notation

$$\Lambda(S) = |S|^{-\left(\frac{1}{t}+\frac{1}{s}\right)} \|\lambda^{-1}\|_{L_t(S)} \|\Lambda\|_{L_s(S)} \tag{5.2}$$

$$g(S) = |S|^{\frac{2}{n}-\left(\frac{1}{t}-\frac{1}{s}\right)} \|\lambda^{-1}\|_{L_t(S)} \|g\|_{L_s(S)}\,.$$

In dealing with local estimates, it is convenient to work in balls. Let

$B_R(x_0)$ denote the ball of radius R and center x_0. Our balls will be generally assumed concentric, in which case we simply write $B_R(x_0) = B_R$. Let us also use the notation

$$\Phi(p, R, u) = \left\{ \int_{B_R} |u|^p \, dx \right\}^{1/p} \tag{5.3}$$

for $p \neq 0$, so that for $p \geq 1$,

$$\Phi(p, R, u) = \| u \|_{L_p(B_R)}.$$

Although other methods are available we shall aproach interior pointwise estimates through the weak Harnack inequality. Our approach to the latter will follow the author's treatment in [31] of the special case $a^i = b^i = a = 0$, which stemmed originally from the paper of Moser [18]. The following properties of the functional, Φ, may be considered as partial motivation of the proofs.

$$\lim_{p \to \infty} \Phi(p, R, u) = \Phi(\infty, R, u) = \sup_{B_R} |u| \tag{5.4}$$

$$\lim_{p \to 0} \Phi(p, R, u) = \Phi(0, R, u) = e^{\frac{1}{|B_R|} \int_{B_R} \log |u| \, dx}.$$

Under the obove conditions on $\mathcal{L}$, we can establish the local boundedness of generalized solutions in $H(\mathcal{A}, \Omega)$. In fact we shall prove the following theorem for subsolutions.

THEOREM 5.1. *Let u be a $H(\mathcal{A}, \Omega)$ subsolution of $\mathcal{L}u = 0$ in Ω. Then for any ball $B_{2R}(x_0)$ in Ω and $p > s'$, $\frac{1}{s} + \frac{1}{s'} = 1$, we have*

$$\sup_{B_R(x_0)} u \leq C R^{-n/p} \| u^+ \|_{L_p(B_{2R}(x_0))} \tag{5.5}$$

where the constant C depends on n, s, t, p, $\Lambda(B_{2R})$ and $g(B_{2R})$.

But, more central to our particular means of treatment of further pointwise properties of solutions will be the following complementing supersolution estimate, which we have chosen (see [29]) to call a *weak Harnack inequality.*

THEOREM 5.2. *Let u be a $H(\mathcal{A}, \Omega)$ supersolution of $\mathcal{L}u = 0$ in Ω, non negative in a ball $B_{5R}(x_0) < \Omega$. Then for certain $p > s'$, we have*

$$R^{-n/p} \| u \|_{L_p(B_{2R}(x_0))} \leq C \inf_{B_R(x_0)} u \tag{5.6}$$

where C depends on n, s, t, p, $\Lambda(B_{5R})$ and $g(B_{5R})$.

PROOF. Assuming that u is bounded and non-negative in Theorem 2.7 it is convenient to prove Theorems 5.1 and 5.2 jointly. to attain the full strength of Theorem 5.1 merely requires modifications of our test functions, the essence of which has been demonstrated in the proof of Theorem 4.1. We will indicate the necessary extensions at the end of the proof. Further, by utilizing the coordinate transformation (4.41) and the formulae (4.42) and (4.43), it is euough to derive the estimates (5.5) and (5.6) for the case $R = 1$, $\| \lambda^{-1} \|_{L_t} = 1$. A final simplifying assumption is that we may take u bounded away from zero. For otherwise, we can replace u by $u + \varepsilon$, $\varepsilon > 0$ and let ε tend to zero in the final results. With these remarks behind us, let us set about the detailed proof, of which the first part will follows the Moser iteration technique.

Consider as test functions

$$\Phi = \eta^2 u^\beta, \quad \beta \neq 0, \tag{5.7}$$

where $\eta \geq 0$, lies in $C_0^1(\Omega)$. By Lemma 1.3 and Proposition 1.1, Φ lies in the space $H^0(\mathcal{A}, \Omega)$ and hence is a valid test function in the integral form (3.5). Also

$$D\Phi = 2\eta\, D\eta\, u^\beta + \beta \eta^2 u^{\beta-1} Du$$

so that substitution into (1.5) yields

$$\beta \int_\Omega \eta^2 u^{\beta-1} (a^{ij} u_{x_j} + a^i) u_{x_i}\, dx + \int_\Omega 2\eta\, u^\beta (a^{ij} u_{x_j} + a^i) \eta_{x_i}\, dx$$
$$+ \int_\Omega \eta^2 u^\beta (b^i u_{x_i} + au)\, dx \begin{cases} \leq 0 & \text{if } u \text{ is a subsolution} \\ \geq 0 & \text{if } u \text{ is a supersolution.} \end{cases} \tag{5.8}$$

We henceforth assume that $\beta > 0$ if u is a subsolution and $\beta < 0$ if u is a supersolution. Using the condition (2.3), we can then estimate from inequality (5.8),

$$\int_\Omega \eta^2 u^{\beta-1} a^{ij} u_{x_i} u_{x_j}\, dx \leq C(|\beta|) \int_\Omega (g\, \eta^2 + a^{ij} \eta_{x_i} \eta_{x_j}) u^{\beta+1}\, dx$$

where the constant C also depends on $K_{\mathcal{A}}$ and is bounded when $|\beta|$ is bounded away from zero. Hence we have

$$(5.9) \qquad \int_\Omega \lambda \eta^2 u^{\beta-1} |Du|^2 \leq C(|\beta|) \int_\Omega (g\eta^2 + A |D\eta|^2) u^{\beta+1} \, dx.$$

It is now convenient to introduce a function v by defining

$$v = \begin{cases} u^{\frac{\beta+1}{2}} & \text{if } \beta \neq -1 \\ \log u & \text{if } \beta = -1. \end{cases}$$

Putting $r = \beta + 1$, we may rewrite inequality (5.9)

$$(5.10) \qquad \int_\Omega \lambda \eta^2 |Dv|^2 \, dx \leq \begin{cases} C(|\beta|)^2 \int_\Omega (g\eta^2 + A|D\eta|^2) v^2 \, dx & \text{if } \beta \neq -1 \\ C \int_\Omega (g\eta^2 + A|D\eta|^2) \, dx & \text{if } \beta = -1. \end{cases}$$

The desired iteration process may now be developed from the first part of (5.10). For applying Lemma 1.1 to the function ηv we have, by (5.10),

$$(5.11) \qquad \| \eta v \|_{t^*}^2 \leq Cr^2 \int_\Omega (g\eta^2 + A |D\eta|^2 v^2) \, dx$$

so that by Hölder's inequality and the condition (5.1),

$$(5.12) \qquad \| \eta v \|_{t^*} \leq Cr \{ A(\operatorname{supp} \eta) + g(\operatorname{supp} \eta) \} \| (\eta + |D\eta|) v \|_{s^*}$$

where

$$\frac{2}{t^*} = 1 + \frac{1}{t} - \frac{2}{r} < \frac{2}{s^*} = 1 - \frac{1}{s}.$$

In the case, $n = 2$, $t = \infty$, it suffices to take arbitrary $t^* > s^*$ in the estimate (5.12). It is now appropriate to make more detailed specifications of the cut-off function η. Let ϱ_1, ϱ_2 satisfy $1 \leq \varrho_1 < \varrho_2 \leq 3$ and set $\eta = 1$ in B_{ϱ_1}, $\eta = 0$ outside B_{ϱ_2} with $|D\eta| \leq 2(\varrho_2 - \varrho_1)^{-1}$. We then obtain from the estimate (5.12)

$$(5.13) \qquad \| v \|_{L_{t^*}(B_{\varrho_1})} \leq \frac{Cr}{(\varrho_2 - \varrho_1)} \| v \|_{L_{s^*}(B_{\varrho_2})}$$

where C will now depend on the quantities listed in the theorem statements and be bounded when $|\beta|$ is bounded away from zero. Taking the $r/2^{th}$ root of the inequality (5.13), we obtain

$$(5.14)\qquad \begin{cases} \Phi(\chi q, \varrho_1, u) \leq \left(\dfrac{Cq}{\varrho_2 - \varrho_1}\right)^{\frac{2s^*}{q}} \Phi(q, \varrho_2, u) & \text{if} \quad q > 0 \\ \Phi(q, \varrho_2, u) \leq \left(\dfrac{C\,|q|}{\varrho_2 - \varrho_1}\right)^{\frac{2s^*}{|q|}} \Phi(\chi q, \varrho_1, u) & \text{if} \quad q < 0 \end{cases}$$

where $\chi = t^*/s^* > 1$ and $q = rs^*$. The inequalities (5.14) may now be iterated for different q values (see [17] or [29]) to give the following estimates. For subsolutions, we obtain

$$(5.15)\qquad \Phi(\infty, 1, u) \leq C\,\Phi(p, 2, u), \ \ p > s^*$$

whilst for supersolutions, we obtain for any $0 < p_0 < p < t^*$

$$(5.16)\qquad \Phi(p, 2, u) \leq C\,\Phi(p_0, 3, u), \quad \Phi(-p_0, 3, u) \leq C\,\Phi(-\infty, 1, u).$$

Hence Theorem 5.1 is proved for bounded, non-negative subsolutions and Theorem 5.2 will follow if we can show for some $p_0 > 0$,

$$(5.17)\qquad \Phi(p_0, 3) \leq C\Phi(-p_0, 3).$$

In the strictly, uniformly elliptic case the inequality (5.17) was a simple consequence of the second part of the inequalities (5.10) and Lemma 1.2. In the general case it seems impossible to proceed this way unless an additional condition is imposed on $\mathcal{L}$ which excludes non-uniformly elliptic $\mathcal{L}$ from consideration (see [9]). The method which we will employ now was developed in [31] for the purpose of overcoming this objection.

In the second of the inequalities (5.1) let us choose $\eta = 1$ in B_4 and vanishing outside B_5, with $|D\eta| \leq 1$. We thus obtain

$$\int_{B_4} \lambda\, |Dv|^2\, dx \leq C \int_{B_5} (g + \Lambda)\, dx$$
$$\leq C$$

where C depends on $g(B_5)$, $\Lambda(B_5)$ and $K_{\mathcal{A}}$. Let us now normalize u by replacing it by u/k where k is given by

$$k = \Phi(0, 4, u) = e^{\frac{1}{|B_4|}\int_{B_4} \log u \, du}.$$

Then $\int_{B_4} v\,dx = 0$ and so applying Lemma 1.1, we obtain

$$\| v \|_{L_{t^*}(B_4)} \leq C \tag{5.18}$$

where C depends on the quantities given in the statement of Theorem 5.2.

Now let us choose as test functions in (1.5)

$$\Phi(x) = \eta^2 u^{-1} (| v |^\beta + (2\beta)^\beta), \qquad \beta \geq 1 \tag{5.19}$$

where $\eta \geq 0$, lies in $C_0^1(B_4)$. By Lemma 1.3 and Proposition 1.1, v lies in $H^0(\mathscr{A}, \Omega)$. Furthermore

$$D\Phi = 2\eta\, D\eta\, u^{-1} (| v |^\beta + (2\beta)^\beta) + \eta^2 u^{-2} (\beta \operatorname{sign} v\, | v |^{\beta-1} - | v |^\beta - (2\beta)^\beta)\, Du.$$

With the aid of (2.3) and the simple inequality

$$2\beta\, | v |^{\beta-1} \leq | v |^\beta + (2\beta)^\beta \tag{5.20}$$

we obtain by substitution in (1.5) and some reduction, by now standard,

$$\beta \int_\Omega \eta^2 | v |^{\beta-1} a^{ij} v_{x_i} v_{x_j}\, dx \leq C \int_\Omega (g\eta^2 + a^{ij} \eta_{x_i} \eta_{x_j}) (| v |^{\beta+1} + (2\beta)^\beta)\, dx$$

where C depends on $K_{\mathscr{A}}$. Hence

$$\int_\Omega \lambda \eta^2 | v |^{\beta-1} | Dv |^2\, dx \leq C \int_\Omega (g\eta^2 + \Lambda\, | D\eta |^2) (| v |^{\beta+1} + (2\beta)^\beta)\, dx. \tag{5.21}$$

The inequality (5.21) resembles inequality (5.9) except for the term $(2\beta)^\beta$. But we can apply the same analysis to (5.21) arriving at an estimate

$$\Phi(\chi q, \varrho_1, v) \leq \left(\frac{Cq}{\varrho_2 - \varrho_1}\right)^{\frac{2s^*}{q}} (\Phi(q, \varrho_2, v) + \gamma q) \tag{5.22}$$

for arbitrary $3 \leq \varrho_1 < \varrho_2 \leq 4$ and χ, q as before, where γ is also a constant. Iterating the estimate (5.22) now yields

$$\Phi(p, 3, v) \leq C(p + \Phi(t^*, 4, v)) \tag{5.23}$$

for any $p \geq t^*$. Consequently by considering the power series expansion of the functiou $e^{p_0|v|}$ for $p_0 > 0$, we obtain

$$\int_{B_3} e^{p_0|v|}\, dx \leq c^{C\Phi(t^*, 4, v)}$$

for sufficiently small $p_0 > 0$. Hence we have

$$\int_{B_3} e^{p_0 v}\, dx \int_{B_3} e^{-p^0 v}\, dx \leq C$$

by (5.18), so that recalling the definition of v, the desired estimate (5.17) follows. Thus Theorem 5.2 is established. To prove Theorem 5.1 in its full generality merely requires replacement of the functions u^β by a sequence of functions u_N^β which agree with $(u^+)^\beta$ for $u \leq N$ and are linear for $u \geq N$. As N tends to infinity the desired estimates follow. A fuller proof is given in [31]. The proofs of Theorem 5.1 and 5.2 are now complete. Q.E.D.

Many interesting pointwise estimates may now be realized as consequences of the weak Harnack inequality. Let us consider first a *strong maximum principle* for subsolutions of $\mathcal{L}u = 0$.

COROLLARY 5.1. *Let u be a $H(\mathcal{A}, \Omega)$ subsolution of $\mathcal{L}u = 0$ in Ω and assume that condition* (3.1) *holds. Then, if for any ball $B_R(x_0)$ strictly contained in Ω, we have*

$$M = \sup_{B_R(x_0)} u = \sup_{\Omega} u > 0$$

u must be a constant in Ω and equality holds in condition (3.1).

PROOF. Clearly we may assume that $B_{5R}(x_0)$ lies in Ω. Applying the weak Harnack inequality (5.6), to the supersolution $v = M - u$, we obtain for $p = 1$,

$$R^{-n} \int_{B_{2R}(x_0)} (M - u)\, dx \leq C \inf_{B_R(x_0)} (M - u) = 0.$$

Consequently $u = M$ in $B_{2R}(x_0)$ and $u = M$ in Ω easily follows. Q. E. D.

Corollary 5.1 shows that a subsolution cannot possess, in a generalized sense, an interior positive maximum. For continuous subsolutions the statement reduces to the usual classical one. Note that the analogous principles for solutions and solutions will follow directly and also that a weak maximum principle is an immediate consequence.

By piecing Theorems 5.1 and 5.2 together, the full Harnack inequality for solutions obviously arises.

COROLLARY 5.2. *Let u be a $H(\mathcal{A}, \Omega)$ solution of $\mathcal{L}u = 0$ in Ω, non-negative in a ball $B_{5R}(x_0) \subset \Omega$. Then*

$$\sup_{B_R(x_0)} u \leq C \inf_{B_R(x_0)} u \tag{5.24}$$

where the constant C depends on n, s, t $\Lambda(B_{5R})$ and $g(B_{5R})$.

By chaining together a sequence of balls, in a standard fashion, one may prove from Corollary 5.2.

Corollary 5.3. *Let u be a non-negative, $H(\mathcal{A}, \Omega)$ solution of $\mathcal{L}u = 0$ in Ω. Then for any domain Ω' strictly contained in Ω, we have*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

where C depends on n, s, t, $\| \lambda^{-1} \|_{L_t(\Omega)}$, $\| \Lambda, g \|_{L_s(\Omega)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$.

Let us now enlarge our scope to include the inhomogeneous equation (4.2). For k a positive constant and $u \geq 0$, we set $\bar{u} = u + k$ so that

$$\begin{aligned} \mathcal{L}u + \frac{\partial f^i}{\partial x_i} - f &= - \frac{\partial}{\partial x_i}(a^{ij} \bar{u}_{x_j} + \bar{a}^i \bar{u}) + b^i \bar{u}_{x_i} + \bar{a}\bar{u} \\ &= \bar{\mathcal{L}}\bar{u} \end{aligned}$$

where

$$\bar{a}^i = (1 - k\bar{u}^{-1}) a^i - f^i \bar{u}^{-1} \quad \text{and} \quad \bar{a} = (1 - k\bar{u}^{-1}) a - f \bar{u}^{-1}.$$

Therefore we have

$$|\bar{a}^i| \leq |a^i| + k^{-1} |f^i|$$

and

$$|a| \leq |a| + k^{-1} |f|.$$

Consequently under the assumption

$$h = (b_{ij} f^i f^j)^{1/2} \in L_{2s}(\Omega), \; f \in L_s(\Omega) \tag{5.25}$$

we obtain the following extension of our previous results.

COROLLARY 5.4 *Suppose that the equation* $\mathcal{L}u = 0$ *in Theorem* 5.1 *and* 5.2 *and Corollary* 5.2 *is replaced by the equation* $\mathcal{L}u = -\frac{\partial}{\partial x_i} f^i = f$. *Then the conclusion of these theorems apply to the function* $\bar{u} = u + k$ *where*

$$k = k_R = \| \lambda^{-1} \|_{L_t(\Omega)}^{1/2} \| h \|_{L_{2s}(\Omega)} R^{\delta} + \| \lambda^{-1} \|_{L_t(\Omega)} \| f \|_{L_s(\Omega)} R^{2\delta}$$

and $\delta = 1 - \frac{n}{2s} - \frac{n}{2t} > 0$.

We come now to the continuity properties of solutions. In order to extend the De Giorgi-Nash result, it seems necessary to impose an additional restriction on the matrix $\mathcal{A}$, namely

$$\Lambda = \sup_R \Lambda\,(\Omega \cap B_R) < \infty\,. \tag{5.26}$$

Observe that (5.26) implies that $\mathcal{L}$ is uniformly elliptic. Condition (5.26) also guarantees that the constant C in the weak Harnack inequality can be bounded independently of R. Hence the Hölder continuity of solutions follows in the usual way (see [29] or [31]). Thus we have proved

COROLLARY 5.5. *Let* u *be a* $H(\mathcal{A}, \Omega)$ *solution of equation* (4.2) *in* Ω *and suppose that conditions* (5.1), (5.25), (5.26) *hold. Then* u *is locally Hölder continuous in* Ω *and for any ball* $B_R(x_0)$ *lying in* Ω, *we have for any* $R < R_0$

$$\underset{B_R(x_0)}{\cos}\, u \leq C \left(\frac{R}{R_0}\right)^{\alpha} (\sup_{B_{R_0}} | u | + k_{R_0}) \tag{5.27}$$

where C *and* α *are positive constants depending on* $n, s, t, \Lambda, \| \lambda^{-1} \|_{L_t(\Omega)}, \| g \|_{L_s(\Omega)}$ *and* diam Ω.

But *semicontinuity* results for sub and supersolutions also arise from the weak Harnack inequality, as is shown in [31]. Let us just state the subsolution result. The reader may refer to [31] for the method of derivation.

COROLLARY 5.6. *Let* u *be a* $H(\mathcal{A}, \Omega)$ *subsolution of equation* (4.2) *in* Ω *and suppose that conditions* (5.1), (5.25), (5.26) *hold. Then* u *is locally lower semicontinuous in* Ω.

Note that the above continuity results are to be understood modulo sets of measure zero. We conclude this section with a series of comments concerning the possible weakening of hypotheses or strengthening of conclusions in the preceding results.

REMARK 1. In Theorems 5.1 and 5.2, condition (5.1) may be replaced by the $\mathcal{E}^{\nu}$-compact boundedness of A and g on $H^0(\mathcal{A}, \Omega)$. Furthermore this condition may be weakened still further in that the quantity $b_{ij} b^i b^j$ need only be assumed compactly bounded on $H^0(\mathcal{A}, \Omega)$.

REMARK 2. By means of a type of L_p interpolation inequality, Theorem 5.1 can be shown to hold for arbitrary $p > 0$, the norm being replaced, of course, by the functional Φ defined by the expression (5.3).

REMARK 3. The compact boundedness of g on $H^0(\mathcal{A}, \Omega)$ would suffice in the strong maximum principle, Corollary 5.1.

REMARK 4. The condition (5.26) may be slightly weakened with the result that Hölder continuity is replaced by continuity in Corollary 5.5 while Corollary 5.6 holds unchanged. That is *continuity* estimates will hold for a class of non-uniformly elliptic equations (see [31]).

Finally, we note that Corollaries 5.1, 5.5 and 5.6 may be obtained by alternate means. In particular they may be derived through Theorem 5.1. only, by consideration of certain logarithmic functions (see [17] and [33]). An alternate means of avoiding Lemma 1.2 in the Harnack inequality derivation has also been proposed by Bombieri, a simplified version being given in [19] by Moser. Corollary 5.5 turns out to be only a mild generalization of similar results in [9] and [20]. For previous Hölder estimates for strictly, uniformly elliptic equations see, for example, [5], [10], [11], [15], [17] and [24]. For semicontinuity results, see [3] and [13].

§ 6. Local estimates at the boundary.

Let S be a subset of the boundary of Ω, $\partial\Omega$ and let u lie in $H(\mathcal{A}, \Omega)$. Then we will say that u is nonpositive on S if u^+ is the limit in $H(\mathcal{A}, \Omega)$ of $C^1(\Omega)$ functions vanishing in neighbourhoods of S. When $S = \partial\Omega$, this agrees with our earlier definition in Section 3 (provided A is bounded on $H^0(\mathcal{A}, \Omega)$) and the other inequality definitions, on S will follows as indicated there. For a $H(\mathcal{A}, \Omega)$ function u, let us also define the following two functions for fixed real L,

$$(6.1) \qquad u_L^+ = \begin{cases} \sup(u, L), & x \in \Omega \\ L, & x \notin \Omega \end{cases} \qquad u_L^- = \begin{cases} \inf(u, L), & x \in \Omega \\ L, & x \notin \Omega, \end{cases}$$

whose domain are consequently all of E^n. Theorems 5.1 and 5.2 then admit the following extension to the boundary, proved we maintain the same conditions on $\mathcal{L}$. In fact we will state the extended versions of Corollary 5.4.

THEOREM 6.1. *Let u be a $H(\mathcal{A}, \Omega)$ subsolution of equation* (4.2) *in Ω. Then for any ball $B_{2R}(x_0)$ and $p > \frac{1}{2} s'$, we have*

$$\sup_{B_R(x_0)} u_L^+ \leq C\left(R^{-n/p} \| u_L^+ \|_{L_p(B_{2R}(x_0))} + k_R\right) \tag{6.2}$$

where C depends on $n, s, t, p, \Lambda(\Omega \cap B_{2R})$, $g(\Omega \cap B_{2R})$; f and f^i are extended to be zero outside of Ω and $L = \sup(0, \sup_{\Omega \cup B_{2R}} u)$.

THEOREM 6.2. *Let u be a $H(\mathcal{A}, \Omega)$ supersolution of equation* (4.2), *non-negative in $\Omega \cap B_{5R}(x_0)$ for some ball $B_{5R}(x_0)$. Then for $p < t^*/2$, we have*

$$R^{-n/p} \| u_l^- \|_{L_p(B_{2R}(x_0))} \leq C\left(\inf_{B_R(x_0)} u_l^- + k\right) \tag{6.3}$$

where C depends on $n, s, t, p, \Lambda(\Omega \cap B_{5R})$, $g(\Omega \cap B_{5R})$ and $l = \inf_{\Omega \cap B_{5R}} u$.

Theorems 6.1 and 6.2 may be proved by a simple reduction to the proofs of Theorems 5.1 and 5.2. The method for the strictly, uniformly elliptic case is given in [28], Section 2.7, and the same argument applies here. Note that Theorems 6.1 and 6.2 are in fact more general results than the interior estimates, Theorems 5.1 and 5.2, since the balls B_R involved can be arbitrary balls in E^n. As far as consequences of Theorems 6.1 and 6.2 go, the closest we can get to a Harnack inequality for solutions, would be the aggregate of the two theorems for the case $k = 0$, that is

$$\sup_{\Omega \cap B_R} u \leq C \inf_{\Omega \cap B_R} u \frac{\| u_L^+ \|_{L_p(B_{2R})}}{\| u_l^- \|_{L_p(R_{2R})}}, \qquad \frac{s'}{2} < p < \frac{t^*}{2}. \tag{6.4}$$

Modulus of continuity estimates in neighbourhoods of boundary points turn out to depend on some boundary smoothness which hither to has been an unrequired assumption. Following the customary procedure, we say that $\partial\Omega$ satisfies *condition A* at x^0 if

$$A_{x_0}(\partial\Omega) = \underline{\lim}_{R \to 0} \frac{|B_R(x_0) - \Omega|}{|B_R(x_0)|} > 0.$$

As a consequence of Theorem 6.2 now follows (see [28] or [29]).

COROLLARY 6.1. *Let u be a $H(\mathscr{A}, \Omega)$ solution of equation* (4.2) *in Ω and suppose that, as well as the hypotheses of Corollary* 5.5 *holding, $\partial\Omega$ satisfies condition A at x_0. Then for any ball $B_{R_0}(x_0)$ and $R < R_0$, we have*

$$\operatorname*{osc}_{\Omega \cap B_R} u \leq C\left\{\left(\frac{R}{R_0}\right)^{\alpha}\left(\sup_{\Omega \cap B_{R_0}} |u| + k_{R_0}\right) + \operatorname*{osc}_{\partial\Omega \cap B_{5\sqrt{RR_0}}} u\right\} \tag{6.5}$$

where C and α are positive constants depending on n, s, t, A, $\|\lambda^{-1}\|_{L_t(\Omega)}$, $\|g\|_{L_s(\Omega)}$, B_{R_0} and A_{x_0}.

Global continuity and Hölder continuity estimates for solutions of equation (4.2) then follow directly from the above corollary. Other methods for obtaining boundary Hölder estimates are given, for example, in [11] and [27].

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