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A. M. BRUCKNER
CHARLES L. THORNE

Density covers

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DENSITY COVERS

A. M. BRUCKNER and CHARLES L. THORNE [1]

1. Introduction.

Let A be a subset of $[0, 1]$ having upper density $\overline{d}_A$ at the origin. Under certain circumstances it is possible to extend A to a set $A \cup B$ having density $d_{A \cup B} = \overline{d}_A$ at the origin. Under other circumstances, any extension of A which has density at the origin must have density equal to 1. The main purpose of this article is to determine conditions under which these phenomena arise. After a short preliminary section, we look at these problems in a slightly broader context, obtaining Theorem 1, whose two corollaries give the conditions we seek. The problem of extending a set to one having density at a given point arises in a natural way in connection with certain problems concerning the boundary behavior of harmonic functions. Such problems are outgrowths of the problems studied in [1]. We wish to thank Professor Max Weiss for calling our attention to these problems and for some discussions while our work on the problem was in its early stages.

2. Preliminaries.

Before stating and proving our theorems, we shall collect some of the notions, definitions and notations which appear frequently in the sequel. If A is a measurable set contained in the interval $[0, 1]$, we shall denote its upper density at the origin by $\overline{d}_A$; that is, $\overline{d}_A = \overline{\lim_{b \to 0^+}} \dfrac{\lambda (A \cap [0, b])}{b}$, where λ denotes Lebesgue measure. The corresponding lower limit is denoted by

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$\underline{d}_A$. If $\overline{d}_A = \underline{d}_A$, we denote this common value by d_A and call it the density of A at the origin. It will be convenient to use the function g_A given by $g_A(x) = x^{-1}\lambda(A \cap [0, x])$ to denote the relative measure of the set A in the interval $[0, x]$.

3. Main results.

Theorem 1, below, arises in a natural way from the following observation. Suppose a set A has density d_A at the origin. Then A is uniformly distributed near the origin in the sense that if the interval $[0, b]$ is divided into k equal subintervals and b is sufficiently small, then A has relative measure approximately equal to d_A in each of these intervals. The converse is also true. Thus given a set A, we ask for conditions under which we can find a set B such that $A \cup B$ is uniformly distributed with $d_{A \cup B} = \overline{d}_A$. The problem is to add B in such a way that we raise the lower density without raising the upper density. Roughly speaking, Theorem 1 tells us that this can be done if and only if A is « subuniformly distributed » in the sense that if the interval $[0, b]$ is divided into k equal subintervals and b is sufficiently small, then none of these subintervals has relative measure more than approximately $\overline{d}_A$.

THEOREM 1. For a measurable set $A \subset [0, 1]$ and r a real number such that $0 \leq r \leq 1$, there exists a measurable set $B \subset [0, 1]$ disjoint from A such that $A \cup B$ has (one-sided) density $d_{A \cup B} = r$ at 0 if and only if

$$\overline{\lim_{b \to 0+}} \frac{\lambda\left(A \cap \left(\frac{k-1}{k}b, b\right)\right)}{b/k} \leq r \leq 1$$

for all natural numbers $k \geq 1$.

PROOF. If $d_{A \cup B}$ exists and equals r then

$$\overline{\lim_{b \to 0+}} \frac{\lambda\left(A \cap \left(\frac{k-1}{k}b, b\right)\right)}{b/k} \leq \lim_{b \to 0+} \frac{\lambda\left((A \cup B) \cap \left(\frac{k-1}{k}b, b\right)\right)}{b/k}$$

$$= \lim_{b \to 0+} \left[k \frac{\lambda((A \cup B) \cap (0, b))}{b} - (k-1) \frac{\lambda\left((A \cup B) \cap \left(0, \frac{k-1}{k}b\right)\right)}{\frac{k-1}{k}b} \right]$$

$$= r.$$

Conversely, suppose $\overline{\lim\limits_{b\to 0+}}\ \dfrac{\lambda\left(A\cap\left(\frac{k-1}{k}b,\ b\right)\right)}{b/k}\leq r\leq 1$ for all k. For each $k\geq 1$, let $I_k=\left\{x:\text{ if } \xi\leq x \text{ then } \lambda\left(A\cap\left(\frac{k-1}{k}\xi,\ \xi\right)\right)<\frac{\xi}{k}\left(r+\frac{1}{k}\right)\right\}$. From the assumption we see that I_k contains a nondegenerate interval of the form $(0, x)$.

We select a sequence of points c_k inductively. Let $c_1 \in I_1$. Note that for any $k\geq 1$, $\left(\frac{k-1}{k}\right)^n\to 0$ as $n\to\infty$. By induction pick $c_{k+1}\in I_{k+1}$ such that $c_{k+1}=\left(\frac{k-1}{k}\right)^{n_k}c_k$ for some integer $n_k\geq 1$ and also such that $c_{k+1}\leq\frac{1}{k+1}$.

For each $k\geq 1$ and each n such that $0\leq n\leq n_k$ let $a_k^n=\left(\frac{k-1}{k}\right)^n c_k$. Note that $a_k^n\in I_k$ for $0\leq n\leq n_k$ and $a_k^n=\left(\frac{k-1}{k}\right)a_k^{n-1}$ for $1\leq n\leq n_k$. Since $0\leq r\leq 1$ we can pick $B_k^n\subset(a_k^n,\ a_k^{n-1})$ disjoint from A such that $r\,(a_k^{n-1}-a_k^n)\leq\lambda\,[(A\cup B_k^n)\cap(a_k^n,\ a_k^{n-1})]\leq\left(r+\frac{1}{k}\right)(a_k^{n-1}-a_k^n)$ for $1\leq n\leq n_k$.

Now let $B=\bigcup\limits_{\substack{1\leq n\leq n_k\\ k\geq 1}}B_k^n$. Since the c_k decrease to 0 as $k\to\infty$, it suffices to show that for $x\in[c_{k+1},\ c_k]$, $\frac{k-1}{k}r\leq g_{A\cup B}(x)\leq r+\frac{2}{k}$. But if $x\in[c_{k+1},\ c_k]$ then $x\in[a_k^n,\ a_k^{n-1}]$ for some n such that $1\leq n\leq n_k$. So

$$g_{A\cup B}(x)=\frac{1}{x}\{\sum_{\substack{1\leq m\leq n_j\\ k<j}}\lambda\,[(A\cup B_j^m)\cap(a_j^m,\ a_j^{m-1})]$$

$$+\sum_{n+1\leq m\leq n_k}\lambda\,[(A\cup B_k^m)\cap(a_k^m,\ a_k^{m-1})]+\lambda\,[(A\cup B_k^n)\cap(a_k^n,\ x)]\}.$$

Hence

$$r\left(\frac{k-1}{k}\right)=r\,\frac{a_k^n}{a_k^{n-1}}\leq r\,\frac{a_k^n}{x}$$

$$=\frac{1}{x}\{\sum_{\substack{1\leq m\leq n_j\\ k<j}}r\,(a_j^{m-1}-a_j^m)+\sum_{n+1\leq m\leq n_k}r\,(a_k^{m-1}-a_k^m)\}$$

$$\leq g_{A\cup B}(x)$$

$$\leq \frac{1}{x}\left\{\sum_{\substack{1\leq m\leq n_j\\ k<j}}\left(r+\frac{1}{j}\right)(a_j^{m-1}-a_j^m)+\sum_{n+1\leq m\leq n_k}\left(r+\frac{1}{k}\right)(a_k^{m-1}-a_k^m)+(x-a_k^n)\right\}$$

$$\leq\left(r+\frac{1}{k}\right)\frac{a_k^n}{x}+\frac{x-a_k^n}{x}$$

$$\leq\left(r+\frac{1}{k}\right)+\frac{a_k^{n-1}-a_k^n}{a_k^{n-1}}=r+\frac{2}{k}.$$

COROLLARY 1. For a measurable set $A\subset[0,1]$ there exists a measurable set $B\subset[0,1]$ disjoint from A such that $d_{A\cup B}$ exists and equals $\overline{d}_A$ if and only if

$$\overline{\lim_{b\to 0+}}\frac{\lambda\left(A\cap\left(\frac{k-1}{k}b,\ b\right)\right)}{b/k}\leq\overline{d}_A,\quad\text{for all}\quad k\geq 1.$$

We remark in passing that for any measurable set A

$$\overline{\lim_{b\to 0+}}\frac{\lambda\left(A\cap\left(\frac{k-1}{k}b,\ b\right)\right)}{b/k}\geq\overline{d}_A,\quad\text{for all}\quad k\geq 1.$$

Thus the condition of the corollary is actually one of equality for all $k\geq 1$.

COROLLARY 2. For a measurable set $A\subset[0,1]$ the following two conditions are equivalent.

(1) If $B\subset[0,1]$ is measurable, disjoint from A and $d_{A\cup B}$ exists, then $d_{A\cup B}=1$;

(2) $$\sup_{k\geq 1}\ \overline{\lim_{b\to 0+}}\frac{\lambda\left(A\cap\left(\frac{k-1}{k}b,\ b\right)\right)}{b/k}=1.$$

Another way of stating Corollary 1 is contained in Theorem 2, below. In essence, it states that A can be extended to a set having density equal to $\overline{d}_A$ if and only if intervals near the origin can have « more than their share of A » only if they are very short.

THEOREM 2. For a measurable set $A\subset[0,1]$ there exists a measurable set $B\subset[0,1]$ disjoint from A for which $d_{A\cup B}$ exists and $d_{A\cup B}=\overline{d}_A$ if and only if for any $0 < \alpha \leq 1$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 \leq \leq a < b \leq \delta$ and $\frac{\lambda(A \cap (a, b))}{b - a} > \bar{d}_A + \varepsilon$ then $\frac{a}{b} > 1 - \alpha$.

PROOF. Suppose there exists $B \subset [0, 1]$ disjoint from from A for which $d_{A \cup B}$ exists and equals $\bar{d}_A$. Let $0 < \alpha \leq 1$ and $\varepsilon > 0$. Pick $\eta > 0$ such that $\frac{1 + (1 - \alpha)}{1 - (1 - \alpha)} \eta < \varepsilon$. Pick $\delta > 0$ such that $\bar{d}_A - \eta < \frac{\lambda((A \cap B) \cup (0, b))}{b} < < \bar{d}_A + \eta$ for all $0 < b \leq \delta$. Now if $0 \leq a < b \leq \delta$ and $\frac{a}{b} \leq 1 - \alpha$ then

$$\frac{\lambda(A \cap (a, b))}{b - a} \leq \frac{\lambda((A \cup B) \cap (a, b))}{b - a} = \frac{\lambda((A \cup B) \cap (0, b))}{b} \frac{b}{b - a} -$$

$$- \frac{\lambda((A \cup B) \cap (0, a))}{a} \frac{a}{b - a} \leq (\bar{d}_A + \eta) \frac{b}{b - a} - (\bar{d}_A - \eta) \frac{a}{b - a} =$$

$$= \bar{d}_A + \eta \left(\frac{b + a}{b - a}\right) \leq \bar{d}_A + \left[\frac{1 + (1 - \alpha)}{1 - (1 - \alpha)}\right] \eta < \bar{d}_A + \varepsilon.$$

Conversely, assuming the above α, ε, δ condition on A, we have for each $k \geq 1$ and $\varepsilon > 0$ that there exists $\delta > 0$ such that if $0 \leq a < b \leq \delta$ and $\frac{a}{b} = 1 - \frac{1}{k}$ then $\frac{\lambda(A \cap (a, b))}{b - a} \leq \bar{d}_A + \varepsilon$. This means that

$$\overline{\lim_{b \to 0+}} \frac{\lambda\left(A \cap \left(\frac{k - 1}{k} b, b\right)\right)}{b/k} \leq \bar{d}_A .$$

Thus Corollary 1 applies and provides the existence of B with the necessary properties.

Theorem 3, below, gives several equivalent ways, with varying geometric content, of stating that a set A can be extended in the desired fashion.

THEOREM 3. For a measurable set $A \subset [0, 1]$ the following are equivalent.

(1) There exists a measurable set $B \subset [0, 1]$ disjoint from A such that $d_{A \cup B}$ exists and equals $\bar{d}_A$.

(2) For any $0 < \alpha \leq 1$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 \leq \leq a < b \leq \delta$ and $\frac{\lambda(A \cap (a, b))}{b - a} > \bar{d}_A + \varepsilon$ then $\frac{a}{b} > 1 - \alpha$.

(3) For each $0 < \alpha \leq 1$, $\overline{\lim\limits_{b \to 0+}} \dfrac{\lambda(A \cap ((1-r)\,b, b))}{rb} \leq \bar{d}_A$ uniformly for $r \geq \alpha$.

(4) For each $k \geq 1$ and every $0 \leq j < k$, $\overline{\lim\limits_{b \to 0+}} \dfrac{\lambda\left(A \cap \left(\dfrac{j}{k}\,b,\ \dfrac{j+1}{k}\,b\right)\right)}{b/k} \leq \bar{d}_A$.

(5) For each $k \geq 1$, $\overline{\lim\limits_{b \to 0+}} \dfrac{\lambda\left(A \cap \left(\dfrac{k-1}{k}\,b,\ b\right)\right)}{b/k} \leq \bar{d}_A$.

PROOF. We have already shown the equivalence of conditions 1 and 2 and 1 and 5. Condition 3 is merely a restatement of condition 2. Condition 4 is that the set A be « subuniformly distributed ». In this context, condition 4 is seen to be a variation of condition 5 with the special « j » cases of 4 being merely repetitions of the general « k » cases of condition 5.

REMARK. If $A, B \subset [0, 1]$ are disjoint measurable sets then the properties of $\underline{\lim}$ and $\overline{\lim}$ indicate that

$$\underline{d}_A + \underline{d}_B \leq \underline{d}_{A \cup B} \leq \begin{Bmatrix} \bar{d}_A + \underline{d}_B \\ \underline{d}_A + \bar{d}_B \end{Bmatrix} \leq \bar{d}_{A \cup B} \leq \bar{d}_A + \bar{d}_B .$$

So, in particular, if $d_{A \cup B}$ exists and equals $\bar{d}_A$ then $\underline{d}_B = 0$ and $\bar{d}_B = \bar{d}_A - \underline{d}_A$.

4. Open sets and interval sets.

In considering questions of the type examined in the previous section, it is sometimes desirable to replace an arbitrary measurable set with a simpler set having similar density properties. In this section, we show that arbitrary measurable sets can be replaced with open sets if one is willing to ignore null sets and with interval sets if one is willing to ignore sets of density zero.

THEOREM 4. Let $S \subset [0, 1]$ be a measurable set. There exists an open set V and a set Z of measure zero such that $V \supset S \setminus Z$, $\bar{d}_V = \bar{d}_S$, $\underline{d}_V = \underline{d}_S$.

PROOF. Almost every point of S is a point of density of S. Thus, if $Z_1 = \{x \in S : x \text{ is not a point of density of } S\}$, then $\lambda Z_1 = 0$. Let $S_1 = S \setminus Z_1$.

For each $x \in S_1$ pick an interval W_x with left endpoint x such that if V_x is an open interval with left endpoint x contained in W_x then $\lambda(S_1 \cap V_x) > (1-x)\lambda(V_x)$. Let $\mathcal{V}_x$ be the set of all such V_x. Let $\mathcal{V} = \bigcup_{x \in S} \mathcal{V}_x$. Then $\mathcal{V}$ forms a Vitali cover of S_1. Hence there exists a denumerable disjoint family $V_{x_1}, V_{x_2}, \ldots$ of intervals in $\mathcal{V}$ such that $\lambda(S_1 \setminus \bigcup_n V_{x_n}) = 0$. Let $V = \bigcup_n V_{x_n}$. Thus, for $Z = Z_1 \cup (S_1 \setminus \bigcup_n V_{x_n})$ we have $\lambda(Z) = 0$ and $V \supset S \setminus Z$.

It suffices to show that for $\varepsilon > 0$, $\overline{d}_V < \overline{d}_{S_1} + \varepsilon$ and $\underline{d}_V < \underline{d}_{S_1} + \varepsilon$. For this it suffices to show that if $0 < h < \varepsilon$ then $\lambda(S \cap (0,h)) > \lambda(V \cap (0,h)) - \varepsilon h$. But

$$\begin{aligned}
\lambda(S \cap (0,h)) &= \sum_{x_n < h} \lambda(S \cap V_{x_n} \cap (0,h)) \\
&> \sum_{x_n < h} (1-x_n)\,\lambda(V_{x_n} \cap (0,h)) \\
&= \sum_{x_n < h} \lambda(V_{x_n} \cap (0,h)) - \sum_{x_n < h} x_n \lambda(V_{x_n} \cap (0,h)) \\
&\geq \lambda(\bigcup_n V_{x_n} \cap (0,h)) - \int_0^h x\,dx \\
&= \lambda(V \cap (0,h)) - \frac{h^2}{2} \\
&> \lambda(V \cap (0,h)) - \varepsilon h.
\end{aligned}$$

Sometimes [2], it is convenient to replace an arbitrary open set with an interval set; that is, a set which is a disjoint union of open intervals converging to the origin. Theorem 4 does not remain valid if one replaces « open set » by « interval set » in the statement of the theorem. It is easy to verify, however, that the statement becomes valid if one relaxes the requirement that Z have zero measure to the requirement that Z have zero density at the origin. We observe that it follows immediately from Theosem 2, that if $A = \bigcup_{k=1}^{\infty}(a_k, b_k)$, $a_k \downarrow 0$, $b_k \downarrow 0$, is an interval set with $\overline{d}_A < 1$, then A can be extended to a set $A \cup B$ such that $d_{A \cup B} = \overline{d}_A$ only if $\lim_{k \to \infty} a_k/b_k = 1$. Thus, for example, if $0 < \alpha < 1$ and $a_k = \alpha b_k$ for all k, the set A cannot be extended in the desired fashion unless $\overline{d}_A = 1$.

University of California
Santa Barbara, California 93106

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