

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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**“Farfield” behavior of solutions to partial differential equations :
asymptotic expansions and maximal rates of decay along a ray**

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^e série, tome 26,
n° 4 (1972), p. 807-827*

http://www.numdam.org/item?id=ASNSP_1972_3_26_4_807_0

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**« FARFIELD » BEHAVIOR OF SOLUTIONS
TO PARTIAL DIFFERENTIAL EQUATIONS :
ASYMPTOTIC EXPANSIONS AND MAXIMAL RATES
OF DECAY ALONG A RAY**

by ORAZIO ARENA and WALTER LITTMAN

1. Introduction.

If $u = u(x, t)$ is a solution of the Klein-Gordon equation $u_{tt} - \Delta u + u = 0$ for all $x = (x_1, \dots, x_n)$ and t , having finite energy and vanishing in the forward light cone $|x| < t$, $t > 0$, then u must vanish identically. This result, due to I. SEGAL [1] for $n = 1$, and to R. GOODMAN [1], [2] and C. MORAWETZ [1] for $n \geq 1$, has led to a number of generalizations.

It was shown in W. LITTMAN [2] that the condition that u vanish in the forward light cone could be replaced, for a large class of equations

with constant coefficients, by the condition that $u = O(r^{\frac{1-N}{2}})$ (where $r^2 = |x|^2 + t^2$, $N = n + 1$) uniformly in the analog of the light cone.

However it is of interest to know how fast u may be allowed to decay in a semi-infinite cylinder in space-time, say one parallel to the t -axis. In that direction K. MASUDA [1], [2] has shown for a class of equations including the Klein-Gordon equation that, if a solution u is defined for all t and decays exponentially in a semi-infinite cylinder as $t \rightarrow \infty$, then u vanishes identically. In [2] K. MASUDA shows, by an explicit example in \mathbb{R}^1 , that the exponential decay assumption is necessary for the conclusion of his theorem.

One of the aims of the present paper is to show that, for a large class of problems for partial differential equations $P(D)u = f$ in \mathbb{R}^N , the exponential decay in a semi-infinite cylinder can be replaced by decay faster

than any negative power, provided f has compact support. For a related problem, see R. S. STRICHARTZ [1].

In Section 2 the appropriate assumptions on the operator are spelled out and the conditions at infinity, insuring existence and uniqueness are stated. The notion of « ξ -local Cauchy problem » is introduced. These problems have the feature that their solutions are expressed as a sum of integrals which are Fourier transforms of surface carried measures and of distributions involving Cauchy type singularities.

In section 3 a far field asymptotic expansion for these integrals is carried out, in negative powers of the distance to the origin, which has the important property that its coefficients are (real) *analytic* functions of the angle. (A variant of MORSE'S lemma, involving analytic dependence on a parameter, is used in obtaining the last result). These results yield immediately corresponding expansions for the solutions to the problems discussed in section 2.

In section 4 we prove a lemma (4.1) according to which a function, having an asymptotic expansion of the type just described in a cone, cannot vanish faster than any negative power of $|x|$ as $|x| \rightarrow \infty$ in a semi-infinite cylinder (contained in the cone) without vanishing faster than any negative power of $|x|$ in the whole cone. A sharpened version of this lemma follows, in which the semi-infinite cylinder is replaced by spike-like region which, after rotation about the origin, can be described by $x_1^2 + \dots + x_{N-1}^2 \leq c x_N^{-2\alpha}$, $x_N \geq b$, for some positive α .

These lemmas are applied to yield corresponding results (theorem 4.1) for solutions to « ξ -local Cauchy problems » introduced earlier. Applications to more traditional type problems such as the Sommerfeld radiation problem for a class of hyperbolic equations are stated as theorems 4.2 and 4.3. We state these here in the special case of the reduced wave equation and the Klein-Gordon equation respectively :

I. Suppose that u is a solution to the reduced wave equation $\Delta u + k^2 u = 0$ ($k > 0$) in the exterior of a sphere, satisfying a Sommerfeld type condition at infinity. If u goes to zero faster than any power of x_N^{-1} as $|x| \rightarrow \infty$ uniformly in the cylinder $x_1^2 + \dots + x_{N-1}^2 \leq R^2$, $x_N \geq a$, then u vanishes identically.

II. Suppose that u satisfies the Klein-Gordon equation $u_{tt} - \Delta u + u = 0$ for all $x \equiv (x_1, \dots, x_{N-1})$ and $t \geq 0$ and has Cauchy data with compact support. If u goes to zero faster than any negative power of t as $t \rightarrow \infty$ uniformly in a cylinder $x_1^2 + \dots + x_{N-1}^2 \leq R^2$, $t \geq 0$, then u vanishes identically for $t \geq 0$.

Results are also obtained where the above cylindrical domains in I. and II. are replaced by « spike-like » domains, given by inequalities of the

form

$$x_1^2 + \dots + x_{N-1}^2 \leq c x_N^{-2\alpha}, \quad x_N \geq b,$$

for some positive α .

Finally, a remark concerning systems of differential equations is made.

In this paper the spaces \mathcal{S} , \mathcal{S}' , C_0^∞ etc. will have their usual meanings as explained, for example, in HÖRMANDER [1].

2. ξ -local Cauchy problems.

We consider a class of problems for partial differential equations with constant coefficients in \mathbb{R}^N which, although superficially appearing to be problems of a different nature, are really variants of the same problem.

Consider the following two problems:

- (1) The non-homogeneous reduced wave equation

$$\Delta u + u = f \in \mathcal{S}$$

with a Sommerfeld type condition at infinity.

- (2) The Cauchy problem for non-homogeneous Klein-Gordon equation

$$u_{tt} - \Delta u + u = f(x, t) \in \mathcal{S}.$$

Both these problems can be brought under one roof, namely what we will call, for lack of a better name, *ξ -local Cauchy problems*.

Let $P(\xi) = P(\xi_1, \dots, \xi_N)$ be a polynomial with real coefficients.

We will be interested in the partial differential operator

$$P(D) \equiv P\left(\frac{1}{i} \frac{\partial}{\partial x}\right) \equiv P\left(\frac{1}{i} \frac{\partial}{\partial x_1}, \dots, \frac{1}{i} \frac{\partial}{\partial x_N}\right).$$

Let us assume that:

- (i) the set S of real solutions of $P(\xi) = 0$ is non-empty;
- (ii) $\text{grad } P(\xi) \neq 0$ in S and therefore S is a smooth $N - 1$ dimensional surface; to be more precise S is the finite union of smooth $N - 1$ dimensional (connected) surfaces;
- (iii) the Gaussian curvature of S , i. e. the product of $N - 1$ principal curvatures, is different from zero.

Let us now assign a unit normal ν to each point on S , varying smoothly with position. Given a unit vector ω , the set of points p_i on S at which the normal to S is parallel to ω is finite in number. Moreover this number will be constant and the p_i 's will be analytic functions of ω as long as ω stays sufficiently close to a non-characteristic direction.

By ξ -local Cauchy problem we mean the following problem:
Find a solution in $\mathcal{S}'(\mathbb{R}^N)$ to

$$P(D)u = f \in \mathcal{S}$$

such that, given any point p on S , for any $\varphi \in C_0^\infty$ with small enough support near p ,

$$|\nu(p) \cdot x|^k |u * \widehat{\varphi}| \leq \text{constant}$$

in the half space $\nu(p) \cdot x \leq 0$, for arbitrary positive k .

In other words the solution, if suitably localized in ξ -space, is to approach zero faster than any power uniformly in the $-\nu(p)$ direction.

It follows from results in Littman [2] that this problem is uniquely solvable and that the solution is in fact given by

$$(2.1) \quad u(x) = P. V. \int e^{ix \cdot \xi} \frac{\widehat{f}(\xi)}{P(\xi)} d\xi + \pi\sigma \int_S e^{ix \cdot \xi} \frac{\widehat{f}(\xi)}{|\text{grad } P(\xi)|} dS_\xi,$$

where $\widehat{f}(\xi)$ stands for the Fourier transform of f and where σ is the sign of $\nu \cdot \text{grad } P(\xi)$ on S . Here « P. V. » means « Cauchy principal value ».

Precisely, existence follows from lemma 2 of Littman [2]. To obtain uniqueness suppose $f \equiv 0$. We note that $u * \widehat{\varphi}$ is again a solution to the homogeneous equation, which, applying a modification of theorem I⁽¹⁾ of Littman [2], must vanish identically. By varying φ we deduce that \widehat{u} has support away from S . Since $P(\xi) \widehat{u}(\xi) = 0$ we conclude $\widehat{u}(\xi) \equiv 0$, hence $u \equiv 0$.

In lemma 2 of Littman [2] it was assumed that $\widehat{f} \in C_0^\infty$. This may be weakened to $f \in \mathcal{S}$ arguing as in lemma 3.3 of the present paper.

It is of interest to mention two problems which can be stated as ξ -local Cauchy problems.

A. Sommerfeld type problems. Given a partial differential operator $Q(\omega, D)$, with C^∞ coefficients depending only on ω , such that

$$Q(\omega, p_i) = 0 \quad \text{and} \quad Q(-\omega, p_i) \neq 0,$$

(1) Namely, in the statement of that theorem, \mathcal{K}_ε may be replaced by $\mathcal{K}_\varepsilon \cap \mathcal{K}_1$, where \mathcal{K}_1 is the cone generated by a neighborhood of the spherical set $\nu(S \cap \text{supp } \widehat{u})$.

consider the problem

$$\left\{ \begin{array}{l} P(D)u = f \in \mathcal{S} \quad \text{in } \mathbb{R}^N \\ u(x) = 0 \quad (|x|^{-\frac{1-N}{2}}) \\ Q(\omega, D)u = o(|x|^{-\frac{1-N}{2}}), \quad \text{uniformly as } |x| \rightarrow \infty. \end{array} \right.$$

Under assumptions (i)-(iii) there exists a unique solution of this problem, given by formula (2.1).

For hypoelliptic $P(D)$ this problem or its equivalent was treated by GRUSIN [1] and VAINBERG [1]. For general $P(D)$ this problem can also be shown to be equivalent to a corresponding ξ -local Cauchy problem and its solution is given by formula (2.1). However we shall not present a proof of this fact here.

B. *Hyperbolic Cauchy-type problems in \mathbb{R}^N* . Assume that:

(iv) $P(D)$ is strictly hyperbolic with respect to $x_N = t$;

(v) the closure of the cone generated by all directed normals ν to S does not intersect the set $x_N \leq 0$, except at the origin.

Under assumptions (i)-(v) the problem

$$\left\{ \begin{array}{l} P(D)u = f \in \mathcal{S} \quad \text{in } \mathbb{R}^N \\ u = 0 \quad (|t|^{-k}) \quad \text{for all } k \geq 0, \quad \text{uniformly as } t \rightarrow -\infty \end{array} \right.$$

has a unique solution $u \in \mathcal{S}'(\mathbb{R}^N)$, again given by formula (2.1).

The uniqueness part of the above assertion follows from Theorem I in Littman [2]. The existence part for $\widehat{f} \in C_0^\infty$ follows from lemma 2 of that paper. For $f \in \mathcal{S}$ an argument similar to the one used in lemma 3.3 of the present paper is applicable.

REMARK: In B. the case $f \equiv 0$ for $t \leq T$ is equivalent to the usual Cauchy problem, with zero data assigned on $t = T$. The general case may be thought of as a Cauchy problem with Cauchy data zero at $t = -\infty$.

Finally, let us remark that for purposes of the present paper the only fact we need is that the solution of the problems under consideration is given by formula (2.1).

3. Asymptotic expansions.

In the following \mathcal{N}_x and \mathcal{N}_α will denote neighborhoods of the origin in x -space and α -space respectively. « Analytic » will mean real analytic.

We shall prove the following

MORSE'S LEMMA (with an analytic parameter). *Let*

$$\varphi(x, \alpha) \equiv \varphi(x_1, \dots, x_m; \alpha_1, \dots, \alpha_\mu)$$

be an analytic function of x and α , defined in the neighborhood $\mathcal{N}_x \times \mathcal{N}_\alpha$ of the point $x = 0, \alpha = 0$. Suppose that

$$\varphi(0; \alpha) = 0, \quad \varphi_{x_i}(0; \alpha) = 0 \quad i = 1, \dots, m,$$

and that the hessian matrix $[h_{ij}] \equiv [\varphi_{x_i x_j}(0; 0)]$ is non-degenerate.

Then there exists a non-singular analytic transformation $x \rightarrow y(x, \alpha)$ of the variables x in a neighborhood $\mathcal{N}'_x \subset \mathcal{N}_x$, under which

$$\varphi(x_1, \dots, x_m; \alpha) = -y_1^2 - \dots - y_h^2 + y_{h+1}^2 + \dots + y_m^2.$$

The transformation depends analytically on $\alpha \in \mathcal{N}'_\alpha$, a sufficiently small subset of \mathcal{N}_α , and takes $x = 0$ into $y = 0$. Moreover the inverse transformation $y \rightarrow x (= x(y, \alpha))$ exists in a fixed neighborhood \mathcal{N}_y of the origin (independent of α) for $\alpha \in \mathcal{N}'_\alpha$ and is analytic in $\mathcal{N}_y \times \mathcal{N}'_\alpha$.

PROOF. The proof is a slight variant of the original proof due to MORSE [1].

Since $[h_{ij}]$ is symmetric, it has a set of m orthonormal eigenvectors. We may assume that these point in the positive coordinate x_i axes; otherwise we may achieve this situation by a rotation of axes.

Now by Taylor's theorem with integral remainder, we can write

$$\varphi(x; \alpha) = \sum_{i,j=1}^m a_{ij}(x; \alpha) x_i x_j,$$

where

$$a_{ij}(x; \alpha) = \int_0^1 (1-u) \varphi_{x_i x_j}(ux_1, \dots, ux_m; \alpha) du.$$

We note that the a_{ij} 's are analytic in $\mathcal{N}''_x \times \mathcal{N}_\alpha$, for some \mathcal{N}''_x , are symmetric in i and j , and

$$a_{ij}(0; \alpha) = \frac{1}{2} \varphi_{x_i x_j}(0; \alpha), \quad \alpha \in \mathcal{N}_\alpha.$$

Furthermore the matrix $[a_{ij}(0; 0)]$ is diagonal with all its eigenvalues different from zero. From this it follows that, for x sufficiently small and α in some sufficient small compact subset of \mathcal{N}_α , $a_{11}(x; \alpha) \neq 0$.

Hence we may define:

$$\begin{cases} z_1 = \sum_j a_{1j}(x; \alpha) x_j / |a_{11}(x; \alpha)|^{1/2} & j = 1, \dots, m \\ z_j = x_j & j = 2, \dots, m \end{cases}$$

and obtain

$$\varphi(x; \alpha) \equiv \pm z_1^2 + Q$$

where $Q = \sum_{i, j \geq 2} b_{ij}(z; \alpha) z_i z_j$ and the matrix $[b_{ij}]$ is symmetric analytic near $(0; 0)$ and diagonal at $(0; 0)$ with all its eigenvalues different from zero.

Thus Q is a quadratic form in the variables z_2, \dots, z_m , with coefficients depending analytically on $z_1, \dots, z_m; \alpha_1, \dots, \alpha_\mu$, such that it is diagonalized when these arguments vanish. Applying the previous argument again, the parameter variables this time being $z_1, \alpha_1, \dots, \alpha_\mu$, we obtain

$$\varphi(x; \alpha) \equiv \pm \bar{z}_1^2 \pm \bar{z}_2^2 + \bar{Q},$$

where \bar{Q} is a quadratic form in the variables $\bar{z}_3, \dots, \bar{z}_m$, with coefficients depending on $\bar{z}_1, \bar{z}_2, \dots, \bar{z}_m, \alpha_1, \dots, \alpha_\mu$. By repeated application of this procedure and possibly reordering the variables, we eventually obtain the desired form for φ . The transformations involved enjoy all the properties required in the statement. Therefore the lemma is proved.

Now we will carry out an asymptotic expansion for the integrals appearing in formula (2.1).

Let S be a compact real analytic $N - 1$ surface (possibly with boundary), with gaussian curvature different from zero, embedded analytically in \mathbb{R}^N and let g be a smooth mass density defined on S with compact support on S .

Suppose ν is a unit normal on S varying smoothly on S and let us denote the Gauss normal map $\xi \in S \rightarrow \nu(\xi)$ also by ν .

Suppose that, in a neighborhood \mathcal{N}_1 of the support of g , ν is 1:1, and that the inverse image $\nu^{-1}(\mathcal{N}_2)$ of a neighborhood \mathcal{N}_2 of a given direction ω_0 ($\mathcal{N}_1 \supset \nu^{-1}(\mathcal{N}_2)$) contains the support of g . Assume further that g is analytic in $\nu^{-1}(\bar{\mathcal{N}}_3)$, \mathcal{N}_3 being a neighborhood of ω_0 such that $\bar{\mathcal{N}}_3 \subset \mathcal{N}_2$.

We shall often make use of the notation: $e(\cdot) \equiv e^{i(\cdot)}$.

LEMMA 3.1. Under the above mentioned assumptions the integral

$$I = \int_S e^{ix \cdot \xi} g(\xi) dS_\xi$$

has the asymptotic expansion as $|x| \rightarrow \infty$

$$I \sim |x|^{-\frac{N-1}{2}} e^{ix \cdot \xi_\omega} \sum_{j=0}^{\infty} |x|^{-j} a_j(\omega),$$

where the coefficients are analytic functions of ω for ω sufficiently close to ω_0 and the expansion is uniform there. Here ξ_ω denotes the unique point in $\mathcal{N}_1 \cap S$ at which $\nu(\xi_\omega) = \omega$.

PROOF OF LEMMA 3.1. First, let us suppose that at the point $p_0 = \nu^{-1}(\omega_0)$ the second fundamental form of the surface S has distinct eigenvalues. Then this will also be the case for points p in a neighborhood of p_0 .

For each such point p on S , let us introduce a new coordinate system $\sigma_1, \dots, \sigma_n, \tau$, centered at $\xi = p$, obtained from the original coordinate system by an Euclidean transformation (that is, a rotation followed by a translation), such that the distinct eigenvectors of the second fundamental form at p are the new $\sigma_1, \dots, \sigma_n$ axes, respectively, and in such a way that the (algebraically) smallest eigenvalue corresponds to σ_1 , etc. and the τ -axis is ν .

In case the second fundamental form of S at p_0 does not have distinct eigenvalues, we may by an appropriate non-singular affine transformation of ξ -space, close to the identity and centered at p_0 , convert the integral over S into one over a surface having the desired property at p_0 . To be more precise, $\nu(p_0)$ is left invariant and the transformation takes place in the tangent plane to S at p_0 .

With each direction ω (close to ω_0), there is thus associated a change of variables $\xi \rightarrow (\sigma, \tau)$ depending analytically on ω . Introducing the σ variables into the integral for I , we have

$$I(\omega, x) = \int_{\sigma \in \mathbb{R}^N} e^{ix \cdot \xi(\sigma, \omega)} g(\xi(\sigma, \omega)) \frac{dS}{d\sigma} d\sigma$$

where $\sigma = \sigma_1, \dots, \sigma_n$ ($n = N - 1$) and $\xi(\sigma, \omega)$ represents S parametrically.

Since we wish to study the asymptotic behavior of I as $|x| \rightarrow \infty$ in the ω direction, we set $x = r\omega$, thus obtaining

$$I(\omega, r) = e(r\omega \cdot \xi_\omega) \int_{\sigma \in \mathbb{R}^n} e(r\tau(\sigma, \omega)) g(\xi(\sigma, \omega)) J_1(\sigma, \omega) d\sigma,$$

where $J_1(\sigma, \omega)$ is the appropriate Jacobian and $\tau(\sigma, \omega)$ expresses the surface S in the σ, τ -coordinates.

Applying MORSE'S lemma with the parameter, we get for each ω , close to ω_0 , a change of variables $\sigma \rightarrow s$ such that $\tau(\sigma, \omega) = \sum_{i=1}^n \varepsilon_i s_i^2$ ($\varepsilon_i = \pm 1$), thus yielding

$$I(\omega, r) = e(x \cdot \xi_\omega) \int_{j=1}^n \prod e(r\varepsilon_j s_j^2) g(\xi(s, \omega)) J_2(s, \omega) ds,$$

where $J_2(s, \omega) = \frac{dS}{d\sigma} \frac{d\sigma}{ds}$ is the Jacobian of the composite map. $\xi(s, \omega)$ as well as $J_2(s, \omega)$ are analytic near the origin, while g is analytic in a neighborhood of the origin and has compact support in a somewhat larger neighborhood of the origin $s = 0$.

Hence we may write

$$I(\omega, r) = e(x \cdot \xi_\omega) \int_{j=1}^n \prod e(r\varepsilon_j s_j^2) h(s, \omega) ds,$$

where h is analytic in a neighborhood of $s = 0$, $\omega = \omega_0$ but has compact support for s in a larger neighborhood of $s = 0$.

Let us introduce, now, the C^∞ function of one variable

$$\begin{cases} \beta(t^2) = 1 & \text{for } t^2 \leq \frac{a}{2}, \\ \beta(t^2) = 0 & \text{for } t^2 \geq a. \end{cases}$$

(a being sufficiently small) and $\beta(s_1^2) \beta(s_2^2) \dots \beta(s_n^2) \equiv B(s)$. Then we split I into the sum $I' + I''$, obtained by replacing, in I , h by Bh and $(1 - B)h$ respectively.

The integral I'' may be shown to be $O(r^{-k})$ for any k , by making the substitution $s_j = \pm \sqrt{\eta_j}$ and integrating by parts an appropriate number of times.

Now we turn our attention to

$$I' = e(x \cdot \xi_\omega) \int_{j=1}^n \prod e(r\varepsilon_j s_j^2) h(s, \omega) B(s) ds.$$

Expanding h in a Taylor series in power of s , with integral remainder,

$$h(s, \omega) = \sum_{0 \leq |\alpha| \leq m-1} b_\alpha(\omega) s^\alpha + \sum_{|\alpha|=m} \tilde{b}_\alpha(s, \omega) s^\alpha,$$

where \tilde{b}_α is as smooth as h , we obtain as in Littman [1]:

$$(3.1) \quad e^{-ix \cdot \xi} \omega I'(r, \omega) = \sum_{0 \leq |\alpha| \leq m-1} b_\alpha(\omega) \prod_{j=1}^n \int e(\varepsilon_j s_j^2 r) s_j^{\alpha_j} \beta(s_j^2) ds_j \\ + \sum_{|\alpha|=m} \int e(\sum \varepsilon_j s_j^2 r) \tilde{b}_\alpha(s, \omega) s^\alpha B(s) ds.$$

We remark that, in the first group of terms, the terms with any odd α_j must vanish leaving only terms with all even powers in s .

Next, letting $s_j = \pm \sqrt{\eta_j}$ and $\alpha_j = 2\gamma_j$, we see that

$$(3.2) \quad 2 \int_0^{+\infty} e(\varepsilon_j s_j^2 r) s_j^{2\gamma_j} \beta(s_j^2) ds_j = \int_0^\infty e(\varepsilon_j \eta_j r) \eta_j^{\gamma_j - \frac{1}{2}} \beta(\eta_j) d\eta_j = \\ = \frac{1}{\varepsilon_j^{\gamma_j} r^{\gamma_j}} \int_0^\infty e(\varepsilon_j \eta_j r) \eta_j^{-\frac{1}{2}} \beta(\eta_j) d\eta_j + 0(r^{-\infty}),$$

where $0(r^{-\infty})$ denotes a sum of terms approaching zero faster than any power of $1/r$. Now, by integrating by parts, it follows that

$$\int_0^\infty e(\varepsilon_j \eta_j r) \eta_j^{-\frac{1}{2}} (1 - \beta(\eta_j)) d\eta_j = 0(r^{-\infty}),$$

enabling us to remove the factor $\beta(\eta_j)$ in the last integral appearing in (3.2). With this factor removed that integral is in fact equal to $r^{-\frac{1}{2}}$ within a multiplicative constant.

Hence the first group of terms in (3.1) gives rise to the expansion

$$e^{-ix \cdot \xi} \omega I'(r, \omega) = r^{-\frac{N-1}{2}} \sum_{q=0}^{m-1} a_q(\omega) r^{-q} + R_m,$$

the rate of decay of R_m being governed by the last sum of terms in (3.1).

We wish to show that, by appropriate choice of m , R_m can be made to approach zero faster than r^{-k} for prescribed k .

To this end we let $m \geq 2k + n$, make the substitution $s_j = \pm \sqrt{\eta_j}$ and integrate by parts $\left[\frac{\alpha_j}{2}\right]$ times with respect to η_j , for each η_j ; that yields an estimate $O(r^{-k})$ for any k for the last group of terms in (3.1).

Therefore the stated asymptotic expansion for I holds with coefficients analytic functions of ω (ω sufficiently close to ω_0).

A careful analysis of the proof shows that the asymptotic expansion is valid uniformly with respect to all directions in a neighborhood of ω_0 .

LEMMA 3.2. *Suppose that all the assumptions preceding Lemma 3.1 are satisfied.*

Let us define

$$J = P. V. \int e^{ix \cdot \xi} \frac{g(\xi)}{P(\xi)} d\xi.$$

Then J has the same type of asymptotic expansion as was stated for I in Lemma 3.1.

PROOF. The proof of this lemma follows from the proof of lemma 2 in Littman [2], where it is shown that the asymptotic behavior of integral of the type J are governed within $O(r^{-\infty})$ by the behavior of a corresponding integral of type I .

LEMMA 3.3. *Let ω_0 be a non-characteristic direction for $P(D)$. Suppose that $g(x) \in \mathcal{S}$ is real analytic near S . Then, for ω in a sufficiently small neighborhood of ω_0 ,*

$$I = \int_S e^{ix \cdot \xi} g(\xi) dS_\xi$$

has the uniform asymptotic expansion as $r = |x| \rightarrow \infty$

$$I \sim r^{-\frac{N-1}{2}} \sum_{i=1}^k e(r \omega \cdot p_i) \sum_{j=0}^{\infty} a_{ij}(\omega) r^{-j},$$

where $p_i = p_i(\omega)$ are the k points p on S at which $\nu(p) = \pm \omega$; the $p_i(\omega)$ as well as the coefficients $a_{ij}(\omega)$ vary analytically with ω .

A similar result holds for the Cauchy type integral J .

PROOF. We decompose g into a sum $\sum_{i=0}^k g_i$ such that, for $i \geq 1$, g_i belongs to C_0^∞ and is analytic near $p_i(\omega_0)$, while g_0 belongs to \mathcal{S} and vanishes near the points p_i ($i = 1, \dots, k$).

Since Lemma 3.1 can be used to deal with the integrals involving g_i ($i = 1, \dots, k$), it remains to investigate the integral involving g_0 .

First, we note that on $S \cap (\text{supp } g_0) \setminus \nu$ is bounded away from a whole neighborhood of directions close to ω_0 . Next, we note that for the case of $g_0(\xi)$ infinitely differentiable with support in a unit sphere with $\max_x |D^j g_0| \leq K_m$ ($j \leq m$) we have, by section 2 of Littman [1]

$$(1 + |x|)^m I(x, g_0) \leq \text{const}_m \cdot K_m.$$

For g_0 not having compact support, we introduce a partition of unity $\Sigma \varphi_h$ of appropriately chosen C^∞ functions with compact support in unit spheres, (necessarily) having the property that the distance from the support of φ_h to the origin goes to infinity as a positive power of the index h . Let $g_{oh} = g_0 \varphi_h$. Since g_0 and any fixed number of derivatives tend to zero faster than any power of $|x|$ as $|x| \rightarrow \infty$, it follows that for a fixed $m \geq 0$

$$\sup_x (1 + |x|)^m I(x, g_{oh}) \rightarrow 0$$

faster than any negative power of h as $h \rightarrow \infty$.

Thus

$$\sum_{h=1}^{\infty} \sup_x (1 + |x|)^m |I(x, g_{oh})| \leq C_m$$

implying that

$$|\sup_x (1 + |x|)^m I(x, g_0)| \leq C_m,$$

and hence $I(x, g_0) = O(r^{-\infty})$.

The proof for the Cauchy-type integral J follows similar lines and in addition uses the fact that $\text{grad } P(\xi)$ on S cannot approach zero faster than some negative power of $|x|$ as $|x| \rightarrow \infty$ ⁽²⁾.

REMARK. In lemmas 3.1, 3.2, 3.3 all x derivatives of I and J have similar expansions. That follows simply by multiplying $g(\xi)$ by the appropriate monomial in ξ .

It can be shown that the asymptotic expansion may be formally differentiated; however we shall not need this fact here.

We are now able to state the following:

THEOREM 3.1. *The (unique) solutions to the ξ -local Cauchy problems of section 2, or to problems A and B of that section, have asymptotic expansions of the type described in lemma 3.3. Similar expansions hold for the derivatives of the solutions.*

⁽²⁾ Here the role of sec. 2 of Littman [1] is played by sec. 6 of reference [5] in Littman [1].

This theorem follows by simply applying lemma 3.3 to the solution formulas (2.1) as well as taking into account the previous remark.

Let us mention that, for the case of the reduced wave equation, *convergent* expansions were obtained by C. H. WILCOX [1] for $N=3$ and by S. N. KARP [1] for $N=2$. (The latter was not in powers of r).

Asymptotic expansions near infinity of solution of elliptic partial differential equations with no lower order terms are due to N. MEYERS [1] (second order) and PAZY [1] (higher order). The latter results may also be found in A. FRIEDMAN's book [1]. The condition of no lower order terms, although necessary, seems to be omitted from the statement of the theorems in the last two references.

REMARK. The coefficients of the asymptotic expansions of lemma 3.3 may actually be computed explicitly by the methods used here.

For instance, for $j=0$, we obtain :

$$\begin{aligned}
 I &= \left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} r^{-\frac{N-1}{2}} \sum_{i=1}^k e(r\omega \cdot p_i) (1 + \sqrt{-1})^{l_i^+} (1 - \sqrt{-1})^{l_i^-} \cdot |K(p_i)|^{-\frac{1}{2}} g(p_i) \\
 &\quad + 0(r^{-\frac{N+1}{2}}), \\
 J &= \pi \left(\frac{\pi}{2}\right)^{\frac{N-1}{2}} r^{-\frac{N-1}{2}} \sum_{i=1}^k e(r\omega \cdot p_i) (1 + \sqrt{-1})^{l_i^+} (1 - \sqrt{-1})^{l_i^-} \\
 &\quad \cdot |K(p_i)|^{-\frac{1}{2}} g(p_i) \left[\frac{\partial P}{\partial \omega}(p_i)\right]^{-1} + 0(r^{-\frac{N+1}{2}}),
 \end{aligned}$$

where $l_i^+ \equiv l_i^+(\omega)$ and $l_i^- \equiv l_i^-(\omega)$ denote the number of positive and negative principal curvatures at p_i respectively. $K(p_i)$ denotes the Gaussian curvature at p_i , i. e., the product of the principal curvatures at p_i , while $\frac{\partial P}{\partial \omega}$ stands for the directional derivative of $P(\xi)$ in the ω direction.

4. Consequences of the asymptotic expansion.

LEMMA 4.1. Let $u(r, \omega)$ ($r = |x|$, ω a point on the unit sphere S^{N-1} : $|x| = 1$) be defined in an open acute cone $\mathcal{K}: |\omega - \omega_0| < K$. Assume in \mathcal{K} u has an asymptotic expansion as $r \rightarrow \infty$, uniformly in \mathcal{K} ,

$$u(r, \omega) \asymp \sum_{s=0}^{\infty} a_s(\omega) r^{-s}$$

(as parametrized for example by projection in the plane orthogonal to ω_0) with

each a_s analytic in ω . Suppose further that in some semi-infinite cylinder \mathcal{C} (contained in \mathcal{K})

$$x_1^2 + \dots + x_{N-1}^2 \leq R^2, \quad x_N \geq a$$

$u(r, \omega) \rightarrow 0$ uniformly faster than any negative power of x_N as $r \rightarrow \infty$.
Then all the terms $a_s(\omega)$ vanish in \mathcal{K} identically.

PROOF. We observe that it suffices to consider the two dimensional case. Namely, choosing a plane π containing the origin and the point ω_0 , we apply the two dimensional theorem to the restriction u_π of u to that plane and conclude that each coefficient $a_s(\omega)$ must vanish for directions ω parallel to π . Since π is arbitrary $a_s(\omega)$ must vanish in all of \mathcal{K} .

We prove the two dimensional result in the following form :
Suppose

$$(4.1) \quad u(x, y) \sim \sum_{s=0}^{\infty} a_s \left(\frac{x}{y} \right) y^{-s}$$

is an asymptotic expansion for u valid for $y \rightarrow \infty$ uniformly in $\mathcal{K} : \left| \frac{x}{y} \right| < \varepsilon$ and that the coefficients $a_s \left(\frac{x}{y} \right)$ are real analytic in \mathcal{K} . Assume further that $u(x, y) \rightarrow 0$ faster than any power of y^{-1} , as $y \rightarrow \infty$, uniformly in $|x| \leq \delta$. Then all $a_s = 0$ for $\left| \frac{x}{y} \right| < \varepsilon$.

Proof of the two dimensional case. Each a_s can be expanded in power series

$$(4.2) \quad a_s \left(\frac{x}{y} \right) = \sum_{j=0}^{\infty} \left(\frac{x}{y} \right)^j a_{sj}$$

converging in a neighborhood of zero possibly depending on s .

Thus we write

$$u(x, y) \sim \sum_{s=0}^{\infty} \left(\sum_{j=0}^{\infty} \left(\frac{x}{y} \right)^j a_{sj} \right) y^{-s}.$$

From the definition of asymptotic expansion, we know that

$$u(x, y) - \sum_{s=0}^{\sigma} \left(\sum_{j=0}^{\infty} \left(\frac{x}{y} \right)^j a_{sj} \right) y^{-s} = o(y^{-\sigma-1})$$

uniformly in \mathcal{K} , hence uniformly for $|x| \leq \delta$.

Therefore

$$\sum_{s=0}^{\sigma} \left(\sum_{j=0}^{\infty} \left(\frac{x}{y} \right)^j a_{sj} \right) y^{-s} = 0 (y^{-\sigma-1})$$

uniformly in $|x| \leq \delta$.

Now

$$\sum_{j=\sigma-s+1}^{\infty} \left(\frac{x}{y} \right)^j a_{sj} = \left(\frac{x}{y} \right)^{\sigma-s+1} \sum_{j=0}^{\infty} \left(\frac{x}{y} \right)^j a_{s, \sigma-s+1+j} = 0 (y^{-\sigma-1+s}).$$

Hence

$$\sum_{s=0}^{\sigma} \left(\sum_{j=\sigma-s+1}^{\infty} \left(\frac{x}{y} \right)^j a_{sj} \right) y^{-s} = 0 (y^{-\sigma-1}).$$

It follows that for each $\sigma \geq 0$:

$$\sum_{s=0}^{\sigma} \left(\sum_{j=0}^{\sigma-s} \left(\frac{x}{y} \right)^j a_{sj} \right) y^{-s} = 0 (y^{-\sigma-1})$$

that is the same as

$$\sum_{k=0}^{\sigma} \left(\sum_{j=0}^k a_{k-j, j} x^j \right) y^{-k} = 0 (y^{-\sigma-1}).$$

We observe that, for fixed x , the left hand side of the latter relation is a polynomial of degree $\leq \sigma$ in $\eta = y^{-1}$, vanishing at $\eta = 0$ to order $\sigma + 1$. Hence its coefficients

$$c_k(x) = \sum_{j+s=k} a_{sj} x^j, \quad 0 \leq k \leq \sigma$$

must vanish for all x , $|x| \leq \delta$.

But this implies that all the a_{sj} 's must vanish for $s + j \leq \sigma$. Since σ is arbitrary this yields the identical vanishing of all the coefficients a_{sj} , which in turn implies the identical vanishing of all the coefficients a_s in (4.1).

LEMMA 4.2. *Let G be the region given by rotating the two dimensional region G'*

$$|x_1| \leq \delta |x_2|^{-\alpha} \quad \alpha \text{ positive integer}$$

about the x_2 -axis.

Then in the statement of lemma 4.1 the semi-infinite cylinder \mathcal{C} may be replaced by G .

PROOF. Just as in the case of lemma 4.1 it suffices to consider the two-dimensional case.

We introduce new scalar variables

$$x = x_1 x_2^{\alpha} \quad \text{and} \quad y = x_2.$$

Thus we have, in place of the asymptotic expansion (4.1), the expansion

$$u(x, y) \asymp \sum_{s=0}^{\infty} a_s (x/y^\beta) y^{-s} \quad (\beta = \alpha + 1),$$

valid uniformly for $|x| \leq \delta$, where the coefficients a_s are analytic functions of their arguments (x/y^β) .

As before we expand a_s about zero

$$a_s \left(\frac{x}{y^\beta} \right) = \sum_{j=0}^{\infty} \left(\frac{x}{y^\beta} \right)^j a_{sj}$$

and first conclude that

$$\sum_{s=0}^{\sigma} \left(\sum_{j=0}^{\infty} \left(\frac{x}{y^\beta} \right)^j a_{sj} \right) y^{-s} = 0 (y^{-\sigma-1}).$$

Exactly as in the case of the previous lemma, we get

$$(4.3) \quad \sum_{k=0}^{\sigma} c_k(x) y^{-k} = 0 (y^{-\sigma-1})$$

where

$$c_k(x) = \begin{cases} \sum_{\substack{s+\beta j=k \\ s \geq 0, j \geq 0}} a_{sj} x^j, & \text{if the set of summation is not empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Arguing as before, for fixed x , the left hand side of (4.3) is a polynomial in $\eta = y^{-1}$ of degree $\leq \sigma$ having a zero at $\eta = 0$ of order $\sigma + 1$, hence vanishing identically. Thus $c_k(x)$ must vanish identically for all x , $|x| \leq \delta$. This implies all $a_{sj} = 0$ and consequently all a_s vanish identically.

COROLLARY 1. *Suppose u is a finite linear combination with constant coefficients of integrals of the type appearing in (2.1), with functions $\widehat{f} = \widehat{f}_j \in \mathcal{S}(\mathbb{R}^N)$ assumed real analytic near S . Suppose that for some non-characteristic direction ω_0 ($|\omega_0| = 1$) $u \rightarrow 0$ faster than any negative power of $|x|$ as $|x| \rightarrow \infty$ uniformly in the set*

$$|x - x \cdot \omega_0| \leq \delta |x \cdot \omega_0|^{-a}, \quad x \cdot \omega_0 > a.$$

Then $u \rightarrow 0$ faster than any negative power of $|x|$ uniformly in a cone containing ω_0 .

PROOF. The proof follows by applying lemma 4.2 to the expansion theorems of section 3.

Let us now make the following assumption

(vi) each (complex) irreducible factor P_j of $P(\xi)$ has a $N - 1$ dimensional real solution set S_j .

It is then easy to prove the following

THEOREM 4.1. *Suppose $P(\xi)$ satisfies assumptions (i)-(iii) and (vi) and that $u(x)$ is the unique solution to the ξ -local Cauchy problem of section 2, with f of class C_0^∞ . Suppose that ω_0 is a non-characteristic direction and that it occurs as an « assigned normal direction » for each of the surfaces S_j .*

If $u \rightarrow 0$ faster than any negative power of $|x|$ as $|x| \rightarrow \infty$ uniformly in the set

$$|x - x \cdot \omega_0| \leq \delta |x \cdot \omega_0|^{-a}, \quad x \cdot \omega_0 > a,$$

then u has compact support.

PROOF. The proof follows directly from the previous corollary, the representation of the solution by integrals as in (2.1) and by theorem II of Littman [2].

As special cases we mention the next two theorems.

THEOREM 4.2. *Suppose $P(\xi)$ satisfies assumptions (i)-(iii) and (vi) and that ω_0 is a non-characteristic direction which occurs as an assigned normal direction for each surface S_j of assumption (vi). In addition let us assume that*

$$P(D)u = 0$$

in the exterior of a bounded set and that u satisfies the radiation condition as in problem A of section 2.

If $u \rightarrow 0$ faster than any negative power of $|x|$ as $|x| \rightarrow \infty$ uniformly in a cylindrical domain

$$|x - x \cdot \omega_0| \leq \delta, \quad x \cdot \omega_0 > a,$$

then u has compact support.

PROOF. First consider the case in which u is C^∞ . Extend u to be zero where it is not defined. Let $U = \varphi u$, φ being an appropriate function with compact support; we have $P(D)U = F \in C_0^\infty$.

Applying theorem 4.1 to the last equation, we conclude that U and hence u has compact support.

If u is not C^∞ , let u_ϵ be a smoothed out version of u such that $u_\epsilon \rightarrow u$. Applying the previous argument to u_ϵ , we obtain that u_ϵ has compact support which, by LIONS' theorem on the supports of convolutions

(J. L. LIONS [1]), must be contained in a fixed neighborhood of the support of F . Since $u_\epsilon \rightarrow u$, u must have compact support.

THEOREM 4.3. *In addition to properties (i)-(vi) let us assume that the x_N direction occurs as an assigned normal direction for each surface S_j of assumption (vi). Now, suppose that*

$$P(D)u = 0 \quad \text{for } x_N > T,$$

u assumes Cauchy data with compact support on $x_N = T$, and $u = O(|x_N|^{-k})$ uniformly in a semi-infinite cylinder $x_1^2 + \dots + x_{N-1}^2 \leq R^2$, $x_N > T$ for all positive k .

It then follows that $u \equiv 0$ for $x_N > T$.

PROOF. First suppose $u \in C^\infty$, thus insuring that u has Cauchy data in C_0^∞ .

It then follows (as in Littman [2], proof of theorem III) that $u = O(1 + |x|^\alpha)$ for $x_N \geq T$, and some α .

Extend u to all \mathbb{R}^N by defining it zero for $x_N < T$. Set $x' = (x_1, \dots, x_{N-1})$. From the hyperbolicity of $P(D)$, it follows that the support of u is contained in a certain truncated cone

$$\tilde{C}: |x'| \leq C|x_N| + a, \quad x_N \geq T.$$

Let $\varphi(x_N)$ be a C^∞ function which vanishes for $x_N - T < \epsilon$, lies between 0 and 1 for $\epsilon < x_N - T < 2\epsilon$ and equals one for $x_N > T + 2\epsilon$, and set $U = \varphi u$. Then the support of U will lie in the closed set $\tilde{C} \cap \{x_N \geq T + \epsilon\}$, while the support of $P(D)U \equiv F$ will lie in the set $\tilde{C}_\epsilon \equiv \tilde{C} \cap \{\epsilon \leq x_N - T \leq 2\epsilon\}$.

Applying theorem 4.1 to the equation

$$P(D)U = F,$$

we conclude that U has compact support which, again by Lions' theorem on the supports of convolutions, is contained in \tilde{C}_ϵ . Thus U and hence u vanishes for $x_N \geq T + 2\epsilon$. Since u is independent of ϵ , u must vanish for all $x_N \geq T$.

If u is not C^∞ , mollify u to obtain a C^∞ approximation $u_\epsilon \rightarrow u$. Applying the theorem (just proved) for the C^∞ case to u_ϵ , we conclude that $u_\epsilon \equiv 0$ in $x_N > T + \epsilon$. Since $u_\epsilon \rightarrow u$ we deduce $u \equiv 0$ for $x_N > T$.

Remarks concerning theorems 4.2 and 4.3.

a) If we are willing to assume that u is C^∞ we may replace the cylindrical region in these two theorems by a spike-like region as in theorem 4.1. On the other hand, for theorem 4.2, if $P(D)$ is hypoelliptic the assumption $u \in C^\infty$ becomes redundant.

b) More generally, let us mention that in either theorem, to be able to replace the cylindrical region by a spike-like region, it suffices to know that if a solution is C^∞ in the ω_0 direction, then it is C^∞ in all directions.

We illustrate this remark in the case of theorem 4.3.

PROPOSITION. *If, in theorem 4.3, the strictly hyperbolic polynomial*

$$P(\xi_1, \dots, \xi_N) = \sum_{k=0}^m Q_k(\xi_1, \dots, \xi_{N-1}) \xi_N^k$$

is such that $Q_0(\xi_1, \dots, \xi_{N-1})$ is elliptic of order m in ξ_1, \dots, ξ_{N-1} , then the semi-infinite cylinder of theorem 4.3 may be replaced by the spike like region

$$x_1^2 + \dots + x_{N-1}^2 \leq c x_N^{-2\alpha}, \quad x_N \geq b, \quad \alpha \geq 1.$$

To prove this proposition we need the following

LEMMA. *Under the conditions imposed on $P(\xi)$ above, if $P(D)u = 0$ in an open set Ω and if $\frac{\partial^k}{\partial x_N^k} u \in L_{loc}^2(\Omega)$ for any $k \geq 0$, then $u \in C^\infty(\Omega)$.*

This lemma is a consequence of the differentiability properties of elliptic equations and follows by a straight-forward boot-strap argument applied to the equation

$$Q_0(D)u = - \sum_{k=1}^m Q_k(D) \frac{\partial^k}{\partial x_N^k} u.$$

PROOF OF THE PROPOSITION. We mollify u in the x_N direction (only) to obtain a solution u_ϵ which, by the previous lemma, is C^∞ with respect to all variables. We then apply theorem 4.3 for the spiked region to u_ϵ (taking into account the above remark a)) and let $u_\epsilon \rightarrow u$.

Remarks concerning systems.

Let us point out that the results of this paper carry over to systems of differential equations.

For example, consider the following system

$$\left\{ \begin{array}{l} \sum_j P_{kj}(D) u_j = f_k \quad \text{in } \mathbb{R}^N \\ u_j(x) = 0 \left(|x|^{\frac{1-N}{2}} \right) \\ Q(\omega, D) u_j = o \left(|x|^{\frac{1-N}{2}} \right), \quad \text{uniformly as } |x| \rightarrow \infty, \end{array} \right.$$

where $P_{kj}(D) \equiv P_{kj} \left(\frac{1}{i} \frac{\partial}{\partial x} \right)$ ($k = 1, \dots, r$ and $j = 1, \dots, r$) are partial differential operators with constant coefficients and $Q(\omega, D)$ is a suitable operator with coefficients depending only on the direction ω (see section 2).

In this case we are reduced to consider the problem

$$\left\{ \begin{array}{l} P(D) u_j = g \quad \text{in } \mathbb{R}^n \\ u_j(x) = 0 \left(|x|^{\frac{1-N}{2}} \right) \\ Q(\omega, D) u_j = o \left(|x|^{\frac{1-N}{2}} \right), \quad \text{uniformly as } |x| \rightarrow \infty, \end{array} \right.$$

where $P(D)$ is the determinant of the system and $g_j = \sum_{k=1}^r P_{kj}^*(D) f_k$, $P_{kj}^* =$ co-factor of P_{kj} .

Therefore, if $P(D)$ satisfies the assumptions required in this paper and g_j is in the class \mathcal{S} , then the above system is uniquely solvable and the solution (u_1, \dots, u_r) can be expanded in the form of the type described in lemma 3.3 of section 3.

Acknowledgement. The results described in this paper were obtained at the School of Mathematics, University of Minnesota. The work of O. Arena was supported by a grant from the Consiglio Nazionale delle Ricerche (Italy), while the work of W. Littman was supported by Air Force Grant AF-AFOSR-883-67.

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REFERENCES

- A. FRIEDMAN; [1] *Partial Differential Equations*, New York 1969.
- R. GOODMAN; [1] *Causal S-operators and domains of dependence for hyperbolic equations*, Thesis M. I. T. 1963.
- R. GOODMAN; [2] *One sided invariant subspace and domains of uniqueness for hyperbolic equations*, Proc. AMS **15** (1964), 653-660.
- V. V. GRUSIN; [1] *On Sommerfeld-type conditions*, Mat. Sb. (N. S.) **61** (103) (1963), 147-174 (Russian).
- L. HÖRMANDER; [1] *Linear partial differential operators*, Springer-Verlag, New York 1969.
- S. N. KARP; [1] *A convergent «far-field» expansion for two-dimensional radiation functions*, Comm. Pure Appl. Math. **14** (1961), 427-434.
- J. L. LIONS; [1] *Supports dans la transformation de Laplace*, J. Analyse Math. **2** (1953), 369-380.
- W. LITTMAN; [1] *Fourier transforms of surface-carried measures and differentiability of surface averages*, Bull. AMS **69** (1963), 766-770.
- W. LITTMAN; [2] *Maximal rates of decay of solutions of partial differential equations*, Archive Rat. Mech. Anal. **37** (1970), 11-20.
- K. MASUDA; [1] *On the exponential decay of solutions for some partial differential equations*, J. Math. Soc. Japan **19** (1967), 82-90.
- K. MASUDA; [2] *Asymptotic behavior in time of solutions for evolution equations*, J. Funct. Anal. **1** (1967), 84-92.
- N. MEYERS; [1] *An expansion about infinity of solutions of linear elliptic equations*, J. Math and Mech **12** (1963), 247-264.
- C. MORAWETZ; [1] *A uniqueness theorem for the relativistic wave equation*, Comm. Pure Appl. Math. **16** (1963), 353-362.
- M. MORSE; [1] *The calculus of variations in the large*, AMS Colloq. Publ. vol. 18, AMS Providence, R. I., 1934.
- A. PAZY; [1] *Asymptotic expansions of solutions of ordinary differential equations in Hilbert Space*, Archive Rat. Mech. Anal. **24** (1967), 193-218.
- I. E. SEGAL; [1] *Direct formulation of the causality requirements of the S-operator*, Phys. Rev. **109** (1958), 2191-2198.
- R. S. STRICHARTZ; [1] *The Stationary observer problem for $\square u = Mu$ and related equations*, Journal Diff. Eq., **9** (1971), 205-223.
- B. R. VAINBERG; [1] *Principles of radiation, limit absorption and limit amplitude in the general theory of partial differential equations*, Russian Math. Surveys, No. 3, **21** (1966), 115-193.
- C. H. WILCOX; [1] *A generalization of theorems of Rellich and Atkinson*, Proc. AMS. **7** (1956), 271-276.