

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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II : vanishing theorems**

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^e série, tome 26,
n° 4 (1972), p. 747-806*

http://www.numdam.org/item?id=ASNSP_1972_3_26_4_747_0

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E. E. LEVI CONVEXITY
AND THE HANS LEWY PROBLEM.
PART II : VANISHING THEOREMS

by ALDO ANDREOTTI and C. DENSON HILL (*)

In Part I of this work [6] we introduced certain cohomology groups, and showed how they are connected with the Cauchy problem (or H. Lewy problem) and the Riemann-Hilbert problem, for cohomology classes and the operator $\bar{\partial}$. In this paper we give local vanishing theorems for these groups at boundary points of locally closed regions, partially bounded by a smooth hypersurface in \mathbb{C}^n . These theorems (Theorems 1 and 2) show that it is possible to decide, according to the E. E. Levi convexity of the hypersurface, whether or not the problems mentioned above are locally well posed.

Global vanishing theorems or theorems of finiteness can be derived directly from the local vanishing theorems (n. 18). Also, global Cauchy or Riemann-Hilbert problems can be treated once the global finiteness theorems have been acquired (cf. n. 19). Therefore we have refrained from setting up all the machinery necessary for a direct proof of global vanishing theorems or theorems of finiteness; we have concentrated our attention on the local situation. This can be handled with some simplifications with respect to the general case.

We have followed partly [4] and partly the paper of L. Hörmander [7]. It should be pointed out that essential use is made (as is required by the nature of the cohomology groups we treat) of the regularity theorem of J. J. Kohn and L. Nirenberg [8].

These results have been announced at the International Congress of Mathematicians, in Nice (1970), by one of the authors.

Pervenuto alla Redazione il 12 Maggio 1971.

(¹) Research supported by the Office of Scientific Research of the United States Air Force under Contract AF F 44620-72-C-0031, and by the North Atlantic Treaty Organization during the term of a Postdoctoral Fellowship in Science.

1. *Annali della Scuola Norm. Sup. di Pisa.*

The reader should refer to Part I [6] for any notation not explained here, for the definition of the cohomology groups we employ, and for the motivation behind this work.

§ 1. Préliminaires.

1. Sequences of Hilbert spaces and densely defined operators.

a) For the sake of completeness, we first recall some well-known facts. Let H_1 and H_2 denote two Hilbert spaces and let D be a dense subspace of H_1 .

Let $t: D \rightarrow H_2$ be a linear map and let

$$G_t(D) = \{(x, y) \in H_1 \times H_2 \mid y = t(x), x \in D\}$$

be the graph of t . Let $\overline{G_t(D)}$ be the closure of $G_t(D)$ in the product topology of H_1 and H_2 . Then $\overline{G_t(D)}$ is a linear subspace of $H_1 \times H_2$.

If $\overline{G_t(D)}$ is the graph of a (necessarily linear) map T from the space $D_T = \text{pr}_{H_1} \overline{G_t(D)}$ we say that t defines a *densely defined closed linear operator*

$$T: H_1 \dots \rightarrow H_2$$

with domain D_T and graph $G_T = \overline{G_t(D)}$. We will call T the closure of t with respect to D .

EXAMPLE: Let t be a differential operator $t \equiv P(x, D)$ with C^∞ coefficients defined on an open set $A \subset \mathbb{R}^m$. Let $\mathcal{D}(A)$ be the space of C^∞ functions with compact support in A and let $H_1 = H_2 = L^2(A, d\mu)$, where $d\mu$ is Lebesgue measure. Let $D \supset \mathcal{D}(A)$ be a space of square integrable C^∞ functions on A such that t defines a linear operator $t: D \rightarrow H_2$. The closure of G_t is still the graph of a linear map T . For this it is enough to show that if $\{x_\nu\}_{\nu \in \mathbb{N}}, \{x'_\nu\}_{\nu \in \mathbb{N}}$ are two sequences in D both converging to an element $x_0 \in H_1$, and if $t(x_\nu) \rightarrow y_0, t(x'_\nu) \rightarrow y'_0$, then we must have $y_0 = y'_0$. Denoting by $(,)$ the scalar product in $L^2(A, d\mu)$ we get for every $f \in \mathcal{D}(A)$:

$$(y_0 - y'_0, f) = \lim_{\nu \rightarrow \infty} (P(x, D)x_\nu - P(x, D)x'_\nu, f) = \lim_{\nu \rightarrow \infty} (x_\nu - x'_\nu, {}^tP(x, D)f) = 0,$$

where ${}^tP(x, D)$ denotes the formal adjoint of $P(x, D)$. The condition $(y_0 - y'_0, f) = 0, \forall f \in \mathcal{D}(A)$, implies $y_0 = y'_0$ since $\mathcal{D}(A)$ is dense in $L^2(A, d\mu)$. Of course the closure T of the operator t depends upon the choice of the dense subspace D .

b) Let $T: H_1 \dots \rightarrow H_2$ be a closed densely defined operator. We set

$$D_{T^*} = \{g \in H_2 \mid \exists g^* \in H_1 \text{ such that } \forall f \in D_T, (Tf, g)_{H_2} = (f, g^*)_{H_1}\}.$$

Since D_T is dense in H_1 the (adjoint) linear map $T^*: D_{T^*} \rightarrow H_1$, which associates $T^*g = g^*$ to g , is well defined.

If we consider in $H_1 \times H_2$ the hermitian form

$$Q(U, V) = (u_1, v_1)_{H_1} - (u_2, v_2)_{H_2},$$

for $U = u_1 \times u_2$ and $V = v_1 \times v_2$, then we can describe the graph of T^* as follows:

$$G_{T^*} = \bigcap_{v \in G_T} \{x \in H_1 \times H_2 \mid Q(v, x) = 0\}.$$

This shows that T^* is a closed operator, its graph being the Q orthogonal complement in $H_1 \times H_2$ of the graph G_T of T . From this it follows that G_T is, in turn, the Q orthogonal complement of G_{T^*} and therefore we deduce that

T^* is densely defined with domain D_{T^*} and that $T^{**} = T$.

The following are obvious properties for the closed densely defined operator T :

1. $\text{Ker } T = \text{pr}_{H_1}(G_T \cap H_1 \times \{0\})$ is closed and contained in D_T .
2. $\text{Ker } T^* = (\text{Im } T)^\perp$; thus $\text{Ker } T = (\text{Im } T^*)^\perp$.
3. $(\text{Ker } T^*)^\perp = \overline{\text{Im } T}$; thus $(\text{Ker } T)^\perp = \overline{\text{Im } T^*}$.

In particular,

$$D_T = \text{Ker } T \oplus D_T \cap \overline{(\text{Im } T^*)}$$

$$D_{T^*} = \text{Ker } T^* \oplus D_{T^*} \cap \overline{(\text{Im } T)}.$$

THEOREM A. Let $T: H_1 \dots \rightarrow H_2$ be a closed densely defined operator. The following conditions are equivalent:

- a) $\text{Im } T$ is closed.
- b) There exists a constant $c > 0$ such that

$$\|f\|_{H_1} \leq c \|Tf\|_{H_2} \quad \forall f \in D_T \cap \overline{(\text{Im } T^*)} = D_T \cap (\text{Ker } T)^\perp.$$

- c) $\text{Im } T^*$ is closed.

d) *There exists a constant $c > 0$ such that*

$$\|g\|_{H_1} \leq c \|T^*g\|_{H_1} \quad \forall g \in D_{T^*} \cap \overline{(\text{Im } T)} = D_{T^*} \cap (\text{Ker } T^*)^\perp.$$

The best constants in b) and d) are the same. We have borrowed this theorem from Hörmander [7] (p. 91, Theorem 1.1.1).

c) **DEFINITION.** *A complex of Hilbert spaces and closed densely defined operators is a sequence*

$$\dots \rightarrow H_i \dots \xrightarrow{T_i} H_{i+1} \dots \xrightarrow{T_{i+1}} H_{i+2} \dots \rightarrow$$

of Hilbert spaces $\{H_i\}_{i \in \mathbb{Z}}$ and closed densely defined operators $T_i : H_i \rightarrow H_{i+1}$ such that, for each $i \in \mathbb{Z}$, we have $\text{Im } T_i \subset \text{Ker } T_{i+1}$. In particular, a short complex is a sequence

$$(\alpha) \quad H_1 \dots \xrightarrow{T} H_2 \dots \xrightarrow{S} H_3$$

with H_1, H_2, H_3 Hilbert spaces and T and S closed densely defined and such that $\text{Im } T \subset \text{Ker } S$.

Replacing the operators in (α) by their adjoints we get a new complex (the adjoint complex)

$$(\alpha^*) \quad H_1 \xleftarrow{T^*} \dots H_2 \xleftarrow{S^*} \dots H_3.$$

Indeed, $\text{Ker } T^ = (\text{Im } T)^\perp \supset (\text{Ker } S)^\perp = \overline{\text{Im } S^*}$. The following theorem gives a criterion for acyclicity, i. e., for having*

$$H((\alpha)) = \text{Ker } S / \text{Im } T = 0.$$

THEOREM B. *Assume that for the complex (α) we can find a constant $c > 0$ such that*

$$(1) \quad \|g\|_{H_2}^2 \leq c^2 \{ \|T^*g\|_{H_1}^2 + \|Sg\|_{H_3}^2 \}, \quad \forall g \in D_{T^*} \cap D_S.$$

Then

- (a) *$\text{Im } T, \text{Im } S, \text{Im } T^*, \text{Im } S^*$ are all closed.*
- (b) *The complex (α) is acyclic, $\text{Im } T = \text{Ker } S$; for $g \in \text{Ker } S$ we can find $u \in D_T \cap \text{Im } T^*$ such that*

$$Tu = g, \quad \|u\|_{H_1} \leq c \|g\|_{H_2}.$$

(c) The adjoint complex (α^*) is acyclic, $\text{Im } S^* = \text{Ker } T^*$; for $g \in \text{Ker } T^*$ we can find $u \in D_{S^*} \cap \text{Im } S$ such that

$$S^*u = g, \quad \|u\|_{H_3} \leq c \|g\|_{H_1}.$$

(d) If $h \in \text{Im } T^*$ we can find $v \in \text{Ker } S$ such that

$$T^*v = h, \quad \|v\|_{H_2} \leq c \|h\|_{H_1}.$$

(e) If $k \in \text{Im } S$ we can find $w \in \text{Ker } T^*$ such that

$$Sw = k, \quad \|w\|_{H_2} \leq c \|k\|_{H_3}.$$

Conversely if the complex (α) is acyclic and $\text{Im } S$ is closed, then for some constant $c > 0$ inequality (1) must hold.

PROOF. (a) If $g \in D_S \cap (\text{Ker } S)^\perp \subset D_S \cap \text{Ker } T^* \subset D_S \cap D_{T^*}$, it follows from (1) that $\|g\|_{H_2}^2 \leq c^2 \|Sg\|_{H_3}^2$. Thus by Theorem A, $\text{Im } S$ is closed and $\text{Im } S^*$ is closed.

If $g \in D_{T^*} \cap (\text{Ker } T^*)^\perp \subset D_{T^*} \cap \overline{(\text{Im } T)} \subset D_{T^*} \cap D_S$, from (1) we get $\|g\|_{H_2}^2 \leq c^2 \|T^*g\|_{H_1}^2$. Thus by Theorem A, $\text{Im } T^*$ is closed and $\text{Im } T$ is closed.

(b) Let $g \in \text{Ker } S$. To show that $g \in \text{Im } T$, it suffices to show that there exists a $u \in H_1$ such that

$$(u, T^*f) = (g, f), \quad \forall f \in D_{T^*}.$$

Indeed, then $u \in D_{T^{**}} = D_T$ and therefore $(u, T^*f) = (Tu, f)$, $\forall f \in D_{T^*}$. Since D_{T^*} is dense in H_2 , it follows from $(Tu, f) = (g, f)$ that $g = Tu$.

Since $(\text{Ker } S)^\perp \subset (\text{Im } T)^\perp = \text{Ker } T^*$, we can write $D_{T^*} = (\text{Ker } S)^\perp \oplus \text{Ker } S \cap D_{T^*}$. Now if $f \in \text{Ker } S^\perp$, we have $(g, f) = 0$ and $T^*f = 0$ since $(\text{Ker } S)^\perp \subset \text{Ker } T^*$.

If $f \in \text{Ker } S \cap D_{T^*}$, then by (1) we have

$$\|f\|_{H_2} \leq c \|T^*f\|_{H_1}.$$

This shows that if $f \in \text{Ker } T^*$, then $(g, f) = 0$ ⁽¹⁾ and therefore we can con-

⁽¹⁾ We can write $f = f_1 + f_2$ with $f_1 \in (\text{Ker } S)^\perp$ and $f_2 \in \text{Ker } S \cap D_{T^*}$. Since f and f_1 are in $\text{Ker } T^*$, so is f_2 . We then have $(g, f) = (g, f_1) + (g, f_2) = 0$ because $(g, f_1) = 0$, $f_1 \in \text{Ker } S^\perp$, and for f_2 we have $\|f_2\| \leq c \|T^*f_2\|$.

sider (g, f) as a linear functional on $\text{Im } T^*$. Moreover we have

$$|(g, f)| \leq c \|g\|_{H_2} \cdot \|T^*f\|_{H_1},$$

from what we have just said. Thus (g, f) as a linear functional on $\text{Im } T^*$ is continuous; hence there exists a $u \in \text{Im } T^*$ with $\|u\|_{H_1} \leq c \|g\|_{H_2}$ such that $(g, f) = (u, T^*f)$.

(c) is proved as in (b).

(d) From (b) we have now that $\text{Im } T = \text{Ker } S$ and $\text{Ker } T^* = (\text{Ker } S)^\perp$. Thus $D_{T^*} = \text{Ker } T^* \oplus \text{Ker } S \cap D_{T^*}$. If $h = T^*u$, $u \in D_{T^*}$, we can write $u = \alpha + v$ with $\alpha \in \text{Ker } T^*$ and $v \in \text{Ker } S \cap D_{T^*}$. Then we have $h = T^*v$ and from (1) $\|v\|^2 \leq c^2 \|T^*v\|^2 = c^2 \|h\|^2$.

(e) is the same as (d).

It remains to prove the converse part. First we remark that from $\text{Im } T = \text{Ker } S$ it follows that $(\text{Im } T)^\perp = (\text{Ker } S)^\perp$. Hence, using Theorem A and the fact that $\text{Im } S$ is closed, we get that the adjoint complex is acyclic: $\text{Im } S^* = \text{Ker } T^*$.

Secondly, we have

$$\begin{aligned} D_S &= \text{Ker } S \oplus (\text{Ker } S)^\perp \cap D_S = \text{Ker } S \oplus \text{Im } S^* \cap D_S \\ &= \text{Ker } S \oplus \text{Ker } T^* \cap D_S \end{aligned}$$

$$\begin{aligned} D_{T^*} &= \text{Ker } T^* \oplus (\text{Ker } T^*)^\perp \cap D_{T^*} = \text{Ker } T^* \oplus \text{Im } T \cap D_{T^*} \\ &= \text{Ker } T^* \oplus \text{Ker } S \cap D_{T^*}. \end{aligned}$$

Thus $D_S \cap D_{T^*} = \text{Ker } T^* \cap D_S \oplus \text{Ker } S \cap D_{T^*}$ (indeed, if $u + v \in D_{T^*}$ with $u \in \text{Ker } T^*$, $v \in \text{Ker } S \cap D_{T^*}$ and if $u + v \in D_S$, then, since $v \in D_S$, we must have $u \in \text{Ker } T^* \cap D_S$). Let $g \in D_S \cap D_{T^*}$, $g = u + v$ according to the orthogonal decomposition given above. We have $\|g\|^2 = \|u\|^2 + \|v\|^2$. Also by Theorem A we must have

$$\|u\|^2 \leq c^2 \|Su\|^2 = c^2 \|Sg\|^2$$

$$\|v\|^2 \leq c^2 \|T^*v\|^2 = c^2 \|T^*g\|^2.$$

Adding these two inequalities we obtain inequality (1).

REMARK. An estimate of type (1) is called a *W*-ellipticity estimate (cf. [4], [5]). The validity of a *W*-ellipticity estimate implies for H_1, H_2, H_3

the following direct sum decomposition into closed subspaces

$$\{\text{Ker } T \oplus \text{Im } T^*\} \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{T^*} \end{array} \{(\text{Im } T = \text{Ker } S) \oplus (\text{Ker } T^* = \text{Im } S^*)\}.$$

$$\cdot \begin{array}{c} \xrightarrow{S} \\ \xleftarrow{S^*} \end{array} \{\text{Im } S \oplus \text{Ker } S^*\}.$$

COROLLARY. Assume that a *W-ellipticity estimate* (1) holds. Then given $f \in H_2$ with $f \in \text{Ker } S$, there exists a unique u in H_1 with $u \in D_T \cap \text{Im } T^*$ and a unique x in H_2 with $x \in D_{T^*} \cap \text{Ker } S$ such that

- | | |
|---------------|--------------------------|
| 1. $Tu = f$ | 3. $Sx = 0$ |
| 2. $u = T^*x$ | 4. $(TT^* + S^*S)x = f.$ |

Moreover, we have the estimates

$$\|u\|_{H_1} \leq c \|f\|_{H_2}; \quad \|x\|_{H_2} \leq c \|u\|_{H_1};$$

and therefore

$$\|x\|_{H_2} \leq c^2 \|f\|_{H_2}.$$

An analogous (adjoint) statement is valid, in which the S, T, S^*, T^* are replaced, respectively, by T^*, S^*, T, S .

PROOF. By looking at the above direct sum decomposition we see: since $f \in \text{Ker } S$, then $f \in \text{Im } T$; hence for some $u \in D_T$, $Tu = f$. The choice of u is uniquely determined by requiring that $u \in (\text{Ker } T)^\perp \cap D_T$, and from the previous theorem (b) we get the estimate $\|u\|_{H_1} \leq c \|f\|_{H_2}$. But $(\text{Ker } T)^\perp = \text{Im } T^*$. Thus there exists an $x \in D_{T^*}$ such that $u = T^*x$. The choice of x is uniquely determined by requiring that $x \in (\text{Ker } T^*)^\perp \cap D_{T^*} = \text{Ker } S \cap D_{T^*}$. Hence $Sx = 0$ and by the previous theorem (d) we get the estimate

$$\|x\|_{H_2} \leq c \|u\|_{H_1}.$$

Now $S^*Sx = 0$ since $Sx = 0$; therefore

$$TT^*x + S^*Sx = Tu = f.$$

2. *Notation and preliminaries.* a) Let U be an open set in \mathbb{C}^n , where $z = (z_1, \dots, z_n)$ will denote the usual holomorphic coordinates.

For u, v in $C_{(r,s)}^\infty(U)$,

$$u = \sum'_{I, \bar{J}} u_{I, \bar{J}} dz^I \wedge d\bar{z}^{\bar{J}}, \quad v = \sum'_{I, \bar{J}} v_{I, \bar{J}} dz^I \wedge d\bar{z}^{\bar{J}}, \quad (2)$$

we define the pointwise euclidean inner product

$$\langle u, v \rangle = \sum'_{I, \bar{J}} u_{I, \bar{J}} \bar{v}_{I, \bar{J}},$$

and the pointwise euclidean norm

$$|u|^2 = \sum'_{I, \bar{J}} |u_{I, \bar{J}}|^2 = \langle u, u \rangle.$$

If we want to emphasize the system of coordinates we have chosen, we will write $|u|_z^2$ instead of $|u|^2$.

Another choice of holomorphic coordinates $\zeta = (\zeta_1, \dots, \zeta_n)$ on U will produce an equivalent norm on any compact subset K of U ; i. e., we can find positive constants $C_1(K), C_2(K)$ such that for any point of K ,

$$C_1(K) |u|_z^2 \leq |u|_\zeta^2 \leq C_2(K) |u|_z^2.$$

We fix the volume element

$$d\mu = (2i)^n dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n,$$

and define the $*$ operator, $*$: $C_{(r,s)}^\infty(U) \rightarrow C_{(n-s, n-r)}^\infty(U)$, by requiring that

$$(u, v) d\mu = u \wedge *v.$$

One has

$$**u = (-1)^{r+s} u.$$

b) Let $h: U \rightarrow \mathbb{R}$ be a C^∞ function; we set

$$\Omega = \{x \in U \mid h(x) \leq 0\},$$

$$S = \{x \in U \mid h(x) = 0\},$$

and we assume that $dh \neq 0$ on S , so that S is a smooth hypersurface. Note that Ω may be partly open and partly closed.

(2) The prime denotes summation over strictly increasing multi-indices.

Let u, v be forms in $C_{(r,s)}^\infty(\Omega)$ such that

$$\{\text{support of } u\} \cap \{\text{support of } v\}$$

is a compact subset of Ω . We then define

$$(u, v) \equiv (u, v)_\Omega = \int_\Omega \langle u, v \rangle d\mu.$$

If $u \in C_{(r,s-1)}^\infty(\Omega)$, $v \in C_{(r,s)}^\infty(\Omega)$, and if the intersection of their supports is compact in Ω , we obtain from Stokes formula the relation

$$(1) \quad (\bar{\partial}u, v)_\Omega = (u, \vartheta v)_\Omega + \int_S u \wedge \bar{*}v$$

where

$$\vartheta = - * \bar{\partial} *.$$

We want to investigate under which conditions the boundary integral in (1) vanishes.

For that purpose, we introduce the differential ideal $\mathcal{I}^\infty = \bigoplus \mathcal{I}_{(r,s)}^\infty$, where

$$\mathcal{I}_{(r,s)}^\infty(\Omega) = \{\varphi \in C_{(r,s)}^\infty(\Omega) \mid \varphi = h\alpha + \bar{\partial}h \wedge \beta \text{ for some } \alpha \in C_{(r,s)}^\infty(\Omega)$$

$$\text{and some } \beta \in C_{(r,s-1)}^\infty(\Omega)\},$$

and we denote by $\mathcal{I}_{0(r,s)}^\infty(\Omega)$ the subspace of $\mathcal{I}_{(r,s)}^\infty(\Omega)$ consisting of those elements with compact support in Ω . We may drop the superscript ∞ when it is obvious from the context.

LEMMA 1. Consider the equation

$$(2) \quad \int_S f \wedge g = 0$$

for $f \in C_{0(r,s-1)}^\infty(\Omega)$ and $g \in C_{(n-r,n-s)}^\infty(\Omega)$.

a) The necessary and sufficient condition for (2) to hold for any choice of f is that $g \in \mathcal{I}_{(n-r,n-s)}^\infty(\Omega)$.

b) For g to belong to $\mathcal{I}_{(n-r,n-s)}^\infty(\Omega)$, it is enough to verify that equation (2) holds for all $f \in \overline{*}\mathcal{I}_{0(n-r,n-s+1)}^\infty(\Omega)$.

PROOF. *Sufficiency.* Let $g = h\alpha + \bar{\partial}h \wedge \beta$ with $\alpha \in C_{(n-r, n-s)}^\infty(\Omega)$ and $\beta \in C_{(n-r, n-s-1)}^\infty(\Omega)$. Since $h = 0$ on S , we get

$$\int_S f \wedge g = \int_S f \wedge \bar{\partial}h \wedge \beta = \int_S f \wedge dh \wedge \beta \quad (\text{for reasons of degree}).$$

Now if $\iota : S \rightarrow U$ is the natural injection, $\iota^*dh = d\iota^*(h) = d0 = 0$. Thus the last integral is zero because dh on S is zero.

Necessity. It is enough to prove statement (b); for that, it is enough to show that $g \in \mathcal{T}_{(n-r, n-s)}^\infty$ in a small neighborhood ω of each point $z_0 \in S$ in U (by using a partition of unity). If sufficiently small, we may assume that h can be taken as a coordinate in a system of local C^∞ coordinates on ω .

Let f be compactly supported in ω , and set, for ε small and positive,

$$F(\varepsilon) = \int_{-\varepsilon \leq h \leq 0} f \wedge g \wedge dh = \int_{-\varepsilon \leq h \leq 0} f \wedge g \wedge \bar{\partial}h.$$

Then $F(\varepsilon)$ is a C^1 function in a small interval $0 \leq \varepsilon \leq \varepsilon_0$, and we have

$$F(0) = 0, \quad F'(\tilde{\varepsilon}) = - \int_{\{h=\tilde{\varepsilon}\}} f \wedge g.$$

From the mean value theorem we derive:

$$\frac{F(\varepsilon)}{\varepsilon} = \frac{F(\varepsilon) - F(0)}{\varepsilon} = F'(\tilde{\varepsilon}) \quad \text{for some } \tilde{\varepsilon} \text{ with } 0 < \tilde{\varepsilon} < \varepsilon;$$

moreover,

$$\lim_{\tilde{\varepsilon} \searrow 0} \int_{\{h=\tilde{\varepsilon}\}} f \wedge g = \int_{\{h=0\}} f \wedge g = 0 \text{ by assumption if } f \in {}^*\bar{\mathcal{T}}_{0(n-r, n-s+1)}(\Omega \cap \omega)$$

Therefore, if f is of the above specified type, we must have

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} F(\varepsilon) = 0.$$

Let ζ be a real C^∞ function, compactly supported in ω , with $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ in a neighborhood $\omega' \subset \subset \omega$ of z_0 . Then

$$f = {}^*\overline{(\zeta g \wedge \bar{\partial}h)} \in {}^*\bar{\mathcal{T}}_{0(n-r, n-s+1)}(\Omega \cap \omega),$$

and therefore we must have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{-\varepsilon \leq h \leq 0} \zeta |g \wedge \bar{\partial} h|^2 d\mu = 0,$$

because $f \wedge g \wedge \bar{\partial} h = \zeta \overline{(g \wedge \bar{\partial} h)} \wedge g \wedge \bar{\partial} h = \pm \zeta |g \wedge \bar{\partial} h|^2 d\mu$. Again by the mean value theorem we have, for some ε' with $0 < \varepsilon' < \varepsilon$,

$$\frac{1}{\varepsilon} \int_{-\varepsilon \leq h \leq 0} \zeta |g \wedge \bar{\partial} h|^2 d\mu = \int_{\{h=\varepsilon'\}} \zeta |g \wedge \bar{\partial} h|^2 dS_{\varepsilon'}.$$

When $\varepsilon \rightarrow 0$, then $\varepsilon' \rightarrow 0$, and we deduce that

$$0 = \lim_{\varepsilon' \rightarrow 0} \int_{\{h=\varepsilon'\}} \zeta |g \wedge \bar{\partial} h|^2 dS_{\varepsilon'} = \int_S \zeta |g \wedge \bar{\partial} h|^2 dS.$$

Therefore, $\zeta |g \wedge \bar{\partial} h|^2 = 0$ on S and consequently in a neighborhood of z_0 ,

$$g \wedge \bar{\partial} h|_S = 0.$$

According to Proposition 1.2 of [6], this implies that $g \in \mathcal{F}_{(n-r, n-s)}(\Omega \cap \omega')$.

As a corollary, we deduce the following

PROPOSITION 1. (a) *If $v \in C_{(r, s)}^\infty(\Omega)$, and if for every $u \in C_{0(r, s-1)}^\infty(\Omega)$ we have*

$$(\bar{\partial}u, v) = (u, \partial v),$$

then $v \in \overline{\mathcal{G}_{(n-r, n-s)}^\infty}(\Omega)$.*

(b) *If $u \in C_{(r, s-1)}^\infty(\Omega)$, and if, for every $v \in C_{0(r, s)}^\infty(\Omega)$, we have*

$$(\bar{\partial}u, v) = (u, \partial v),$$

then $u \in \mathcal{F}_{(r, s-1)}^\infty(\Omega)$.

REMARKS 1. The condition $v \in \overline{*\mathcal{F}}$ is already implied by the validity of $(\bar{\partial}u, v) = (u, \partial v)$ for all $u \in \overline{*\mathcal{F}}$.

2. The condition $u \in \mathcal{F}$ is already implied by the validity of $(\bar{\partial}u, v) = (u, \partial v)$ for all $v \in \mathcal{F}$.

c) Let $\Phi: \Omega \rightarrow R$ be a C^∞ function; for u, v in $C_{(r, s)}(\Omega)$ with $\{\text{support of } u\} \cap \{\text{support of } v\}$ compact, we define the weighted scalar product

$$(u, v)_\Phi = \int_\Omega e^\Phi \langle u, v \rangle d\mu,$$

and for compactly supported u we set $\|u\|_{\Phi}^2 = (u, u)_{\Phi}$. We call Φ a weight on the space $C_{(r,s)}(\Omega)$. The formal adjoint with respect to this scalar product of the operator $\bar{\partial}_j \equiv \partial_{\bar{j}} = \frac{\partial}{\partial z_j}$ is the operator $\delta_j = e^{-\Phi} \frac{\partial}{\partial z_j} e^{\Phi} = e^{-\Phi} \partial_j e^{\Phi}$, and the commutator is given by

$$(\delta_k \bar{\partial}_j - \bar{\partial}_j \delta_k) u = - \frac{\partial^2 \Phi}{\partial z_j \partial z_k} \cdot u \quad (\text{Ricci identity}).$$

If we introduce the weight Φ_1 on the space $C_{(r,s-1)}^{\infty}(\Omega)$ and the weight Φ_2 on the space $C_{(r,s)}^{\infty}(\Omega)$, we get as formal adjoint of the operator $\bar{\partial} : C_{(r,s-1)}^{\infty}(\Omega) \rightarrow C_{(r,s)}^{\infty}(\Omega)$ the differential operator:

$$-\partial_{\Phi_1} \partial_{\Phi_2} = e^{-\Phi_1} \partial e^{\Phi_1}.$$

d) Given the functions Φ and h on Ω , for every $z \in \Omega$, we will make use of the following notations:

$$\mathcal{L}(\Phi)(z) \equiv \sum_{j,k=1}^n \frac{\partial^2 \Phi}{\partial z_j \partial z_k}(z) w_j \bar{w}_k \quad w = (w_1, \dots, w_n) \in \mathbb{C}^n.$$

For $u \in C_{(r,s)}^{\infty}(\Omega)$, we set, when $s \geq 1$,

$$\mathcal{L}(\Phi)\{u, u\}(z) \equiv \sum_{I, \bar{K}} \sum_{j,k=1}^n \frac{\partial^2 \Phi}{\partial z_j \partial z_k}(z) u_{I, \bar{j}} \bar{u}_{I, \bar{k}},$$

and

$$\widehat{\mathcal{L}}(h)\{u, u\}(z) \equiv \frac{1}{|\text{grad } h|} \mathcal{L}(h)\{u, u\}(z)$$

where $|\text{grad } h| = 2 \left(\sum_{j=1}^n |\partial h / \partial z_j|^2 \right)^{\frac{1}{2}}$ is the euclidean norm of the gradient of h , calculated at z .

e) We end this paragraph with some explicit formulas we need in the proof of the basic estimates:

Let

$$u = \sum'_{A \bar{B}} u_{A \bar{B}} dz^A \wedge d\bar{z}^{\bar{B}} \in C_{(r,s)}(U)$$

where $A = (\alpha_1, \dots, \alpha_r)$, $\bar{B} = (\bar{\beta}_1, \dots, \bar{\beta}_s)$.

We denote by ∂_{ν} the partial derivative $\partial/\partial z_{\nu}$, and by $\partial_{\bar{\mu}}$ the partial derivative $\partial/\partial \bar{z}_{\mu}$.

We have

$$(1) \quad \bar{\partial} u = (-1)^r \sum'_{A, \bar{C}} \left(\sum_h (-1)^{h-1} \partial_{\bar{r}_h} u_{A, \bar{r}_1 \dots \hat{r}_h \dots \bar{r}_{s+1}} \right) dz^A \wedge d\bar{z}^{\bar{C}}$$

where $\bar{C} = (\bar{r}_1, \dots, \bar{r}_{s+1})$. If D denotes a multi-index of length $s - 1$, we have

$$(2) \quad -_{\bar{\Phi}_1} \partial_{\bar{\Phi}_1} u = (-1)^{r-1} \sum'_{A, \bar{D}} \sum_j (e^{-\bar{\Phi}_1} \partial_j e^{\bar{\Phi}_1}) u_{A, \bar{j} \bar{D}} dz^A \wedge d\bar{z}^{\bar{D}}.$$

In particular,

$$(3) \quad -_{\bar{\Phi}} \partial_{\bar{\Phi}} u = (-1)^{r-1} \sum'_{A, \bar{D}} \sum_j \delta_j u_{A, \bar{j} \bar{D}} dz^A \wedge d\bar{z}^{\bar{D}},$$

and if we set $\bar{\Phi}_1 = \bar{\Phi} - 2\psi$, $\bar{\Phi}_2 = \bar{\Phi} - \psi$, we get

$$(4) \quad -_{\bar{\Phi}} \partial_{\bar{\Phi}} u = e^{-\psi} -_{\bar{\Phi}_1} \partial_{\bar{\Phi}_1} u + (-1)^{r-1} \sum'_{A, \bar{D}} \sum_j \partial_j \psi \cdot u_{A, \bar{j} \bar{D}} dz^A \wedge d\bar{z}^{\bar{D}}.$$

Finally, the condition $u \in * \bar{\mathcal{J}}$ is equivalent to the condition

$$(5) \quad \sum'_{A, D} \sum_j (\partial_j h) u_{A, \bar{j} \bar{D}} |_{h=0} = 0.$$

§ 2. The Estimates.

This section is devoted to establishing W -ellipticity estimates for the $\bar{\partial}$ -operator and boundary estimates needed for regularization of L_2 -solutions.

3. *Stretching the coordinates.* Let $\bar{\Phi}$ be a C^∞ function defined in some neighborhood U of a point $z_0 \in \mathbb{C}^n$. We assume that at z_0 the Levi form $\mathcal{L}(\bar{\Phi})(z_0)$ has at least p positive eigenvalues.

By a unitary transformation we can assume that $\mathcal{L}(\bar{\Phi})$ is diagonalized at the point z_0 with eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n, \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p > 0,$$

so that

$$\mathcal{L}(\bar{\Phi})(z_0) = \sum_{j=1}^n \lambda_j |w_j|^2.$$

We now change the coordinates, first bringing z_0 to the origin by a translation and then applying the transformation $z \rightarrow \zeta$:

$$\begin{cases} z_1 = \eta \zeta_1 \\ \cdot \quad \cdot \quad \cdot \\ z_p = \eta \zeta_p \end{cases} \quad \begin{cases} z_{p+1} = \zeta_{p+1} \\ \cdot \quad \cdot \quad \cdot \\ z_n = \zeta_n \end{cases} \quad \text{with } \eta > 0.$$

In the new coordinates we get

$$\mathcal{L}(\Phi)(z_0) = \sum_{j=1}^p \eta^2 \lambda_j |w_j|^2 + \sum_{j=p+1}^n \lambda_j |w_j|^2$$

and therefore

$$\begin{aligned} \mathcal{L}(\Phi)\{u, u\}(z_0) &= \sum'_{I, K} \left\{ \sum_{j=1}^p \eta^2 \lambda_j |u_{I, jK}|^2 + \sum_{j=p+1}^n \lambda_j |u_{I, jK}|^2 \right\} \\ &\geq (\eta^2 \lambda_p - n |\lambda_n|) |u|^2 \end{aligned}$$

provided u is of type (r, s) with $s > n - p$; in fact, every component $u_{I, J}$, when J contains more than $n - p$ indices, must have one of the indices $1, \dots, p$ among the indices appearing in J .

We can therefore find $\eta_0 > 0$ and $\varepsilon_0 > 0$ such that if $\eta > \eta_0$, then, in the ball $B(z_0) \equiv \{\sum_j |\zeta_j|^2 \leq \varepsilon_0\}$, we have for any $u \in C_{(r, s)}^\infty(U)$, with $s > n - p$,

$$\mathcal{L}(\Phi)\{u, u\}(\zeta) > |u(\zeta)|_\zeta^2, \quad \forall \zeta \in B(z_0).$$

Now let μ be a C^∞ function defined on \mathbb{R} with the properties :

$$\mu'(t) \geq 0, \quad \mu''(t) \geq 0, \quad \forall t \in \mathbb{R}.$$

If we set $\tilde{\Phi} = \mu(\Phi)$ we get :

$$\begin{aligned} \mathcal{L}(\tilde{\Phi}) &= \mu'(\Phi) \mathcal{L}(\Phi) + \mu''(\Phi) \left| \sum_j \frac{\partial \Phi}{\partial z_j} w_j \right|^2 \\ &\geq \mu'(\Phi) \mathcal{L}(\Phi). \end{aligned}$$

We thus obtain the following

PROPOSITION 1. *Let Φ be a C^∞ function defined in an open neighborhood U of z_0 in \mathbb{C}^n , and such that $\mathcal{L}(\Phi)(z_0)$ has at least p positive eigenvalues. Then we can find*

- i) *a neighborhood ω of z_0 with $\bar{\omega} \subset U$,*
- ii) *a system of holomorphic coordinates ζ in U , such that for any C^∞ function $\mu : \mathbb{R} \rightarrow \mathbb{R}$ with $\mu' \geq 0, \mu'' \geq 0$ and any $u \in C_{(r, s)}^\infty(U)$ with $s > n - p$, we have*

$$\mu'(\Phi(\zeta)) |u|_\zeta^2 \leq \mathcal{L}(\mu(\Phi))\{u, u\}(\zeta), \quad \forall \zeta \in \omega.$$

In particular, for $\mu(t) \equiv t$ we have

$$|u|_\zeta^2 \leq \mathcal{L}(\Phi)\{u, u\}(\zeta), \quad \forall \zeta \in \omega.$$

4. *Choice of the weight functions and basic inequality.* a) We assume in this section that Ω is compact, so that $S = \partial\Omega$ is a compact smooth hypersurface.

Let us consider a C^∞ vector field (complex-valued) on U :

$$\xi \equiv \sum_{\beta} \xi^{\beta} \frac{\partial}{\partial z_{\beta}} + \sum_{\beta} \xi^{\bar{\beta}} \frac{\partial}{\partial \bar{z}_{\beta}}.$$

Setting $dz_{\beta}^{\wedge} = dz_1 \wedge \dots \wedge \widehat{dz_{\beta}} \wedge \dots \wedge dz_n$, $d\bar{z}_{\beta}^{\wedge} = d\bar{z}_1 \wedge \dots \wedge \widehat{d\bar{z}_{\beta}} \wedge \dots \wedge d\bar{z}_n$ and $dz = dz_1 \wedge \dots \wedge dz_n$, $d\bar{z} = d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n$ we have the formula

$$\begin{aligned} d \left(\sum_{\beta} (-1)^{\beta-1} \xi^{\beta} dz_{\beta}^{\wedge} \wedge d\bar{z} + (-1)^n \sum_{\beta} (-1)^{\beta-1} \xi^{\bar{\beta}} dz \wedge d\bar{z}_{\beta}^{\wedge} \right) = \\ = \sum_{\beta} (\partial_{\beta} \xi^{\beta} + \partial_{\bar{\beta}} \xi^{\bar{\beta}}) dz \wedge d\bar{z}. \end{aligned}$$

On S we define the quantity

$$\xi_n = \frac{1}{2} (\sum_j |\partial_j h|^2)^{-\frac{1}{2}} \{ \sum_{\beta} (\xi^{\beta} \partial_{\beta} h + \xi^{\bar{\beta}} \partial_{\bar{\beta}} h) \}.$$

Then we have the formula

$$(1) \quad \int_{\Omega} \sum_{\beta} (\partial_{\beta} \xi^{\beta} + \partial_{\bar{\beta}} \xi^{\bar{\beta}}) d\mu = \int_{\partial\Omega} \xi_n dS,$$

where dS denotes the element of area on $S = \partial\Omega$, oriented in such a way that $d\mu = dS \wedge dg$, where $g = \frac{1}{2} (\sum_j |\partial_j h|^2)^{-\frac{1}{2}} h$. We denote by $\text{div } \xi$ the quantity $\sum_{\beta} (\partial_{\beta} \xi^{\beta} + \partial_{\bar{\beta}} \xi^{\bar{\beta}})$.

b) Given the form $u \in C_{(r,s)}^{\infty}(\Omega)$ with $s \geq 1$, we construct the vector field on Ω

$$\theta \equiv (\xi^1, \dots, \xi^n, \eta^{\bar{1}}, \dots, \eta^{\bar{n}})$$

where

$$(3) \quad \begin{aligned} \xi^k &= \sum_j \sum'_{A, \bar{D}} e^{\Phi} (\partial_{\bar{j}} u_{A, k\bar{D}}) \overline{(u_{A, j\bar{D}})} \\ \eta^{\bar{k}} &= - \sum_j \sum'_{A, \bar{D}} e^{\Phi} (\partial_j u_{A, j\bar{D}}) \overline{(u_{A, k\bar{D}})}. \end{aligned}$$

A direct calculation establishes the following *basic identity* (cf. [5], p. 113).

$$(4) \quad \operatorname{div} \theta = e^{\Phi} \mathcal{L}(\Phi) \{u, u\} + e^{\Phi} \sum_j |\partial_{\bar{j}} u|^2 - e^{\Phi} |\bar{\partial} u|^2 - e^{\Phi} |_{-\Phi} \partial_{\Phi} u|^2,$$

valid for any choice of Φ and for any $u \in C_{(r,s)}^{\infty}(\Omega)$ with $s \geq 1$.

The calculation is based on the computation of the divergence of the vector field θ , making use of the Ricci identity (n. 2), and the identity

$$|\bar{\partial} u|^2 = \sum_j |\partial_{\bar{j}} u|^2 - \sum_{k,j} \sum'_{A, \bar{D}} (\partial_{\bar{k}} u_{A, j\bar{D}}) \overline{(\partial_{\bar{j}} u_{A, k\bar{D}})}.$$

LEMMA 2. *If $u \in {}^* \overline{\mathcal{G}} \cap C_{(r,s)}^{\infty}(\Omega)$, then the vector field θ has the following properties:*

$$(a) \quad \eta_n = 0 \quad (\text{on } S)$$

$$(b) \quad \int_{\partial \Omega} \xi_n dS = - \int_{\partial \Omega} e^{\Phi} \widehat{\mathcal{L}}(h) \{u, u\} dS.$$

PROOF. Let $c = \frac{1}{2} (\sum |\partial_{\alpha} h|^2)^{-\frac{1}{2}}$. Then we have

$$(a) \quad \eta_n = c \sum_k \eta^{\bar{k}} \partial_{\bar{k}} h = -ce^{\Phi} \sum'_{A, \bar{D}} (\sum_j \delta_j u_{A, j\bar{D}}) \overline{(\sum_k u_{A, k\bar{D}} \partial_{\bar{k}} h)}$$

$$= 0 \quad \text{by formula (5) of n. 2.}$$

$$(b) \quad \xi_n = c \sum_k \xi^k \partial_k h = ce^{\Phi} \sum_k \sum_j \sum'_{A, \bar{D}} \partial_k h (\partial_{\bar{j}} u_{A, k\bar{D}}) \overline{(u_{A, j\bar{D}})}$$

$$= ce^{\Phi} \sum_j \sum'_{A, \bar{D}} \partial_{\bar{j}} (\sum_k \partial_k h u_{A, k\bar{D}}) \overline{(u_{A, j\bar{D}})}$$

$$- ce^{\Phi} \sum_{\bar{j}, k} \sum'_{A, \bar{D}} \frac{\partial^2 h}{\partial z_k \partial z_j} u_{A, k\bar{D}} \overline{u_{A, j\bar{D}}}$$

$$= -e^{\Phi} \widehat{\mathcal{L}}(h) \{u, u\} \quad (\text{again by virtue of formula (5) of n. 2}).$$

Making use of this lemma, and the previous basic identity (4), we obtain

PROPOSITION 2. *For any $u \in {}^* \overline{\mathcal{G}}(\Omega) \cap C_{(r,s)}^{\infty}(\Omega)$, we have the following identity:*

$$\int_{\Omega} e^{-\Phi} \mathcal{L}(\Phi) \{u, u\} d\mu + \int_{\Omega} e^{-\Phi} \sum_j |\partial_{\bar{j}} u|^2 d\mu + \int_{\partial\Omega} e^{-\Phi} \widehat{\mathcal{L}}(h) \{u, u\} dS =$$

$$= \int_{\Omega} e^{-\Phi} |\bar{\partial} u|^2 d\mu + \int_{\Omega} e^{-\Phi} |{}_{\Phi}\partial_{-\Phi} u|^2 d\mu.$$

Here we have changed the weight Φ into $-\Phi$, so as to have the first term on the left with the positive sign.

c) For the spaces

$$F_1 \equiv C_{(r, s-1)}^{\infty}(\Omega), \quad F_2 \equiv C_{(r, s)}^{\infty}(\Omega), \quad F_3 \equiv C_{(r, s+1)}^{\infty}(\Omega)$$

we now select the weights as follows :

on F_3 we choose the weight $\Phi_3 \equiv \Phi$,

on F_2 we choose the weight $\Phi_2 = \Phi - \psi$,

on F_1 we choose the weight $\Phi_1 = \Phi - 2\psi$,

where ψ is a C^{∞} function on U . From Proposition 2 we derive

PROPOSITION 3. For any $u \in C_{(r, s)}^{\infty}(\Omega)$ we have the following inequality :

$$\int_{\Omega} e^{-\Phi} \mathcal{L}(\Phi) \{u, u\} d\mu + \int_{\Omega} e^{-\Phi} \sum_j |\partial_{\bar{j}} u|^2 d\mu + \int_{\partial\Omega} e^{-\Phi} \widehat{\mathcal{L}}(h) \{u, u\} dS \leq$$

$$\leq 2 \|{}_{\Phi_1}\partial_{-\Phi_2} u\|_{-\Phi_1}^2 + \|\bar{\partial} u\|_{-\Phi_3}^2 + \int_{\Omega} e^{-\Phi} 2n |\partial\psi|^2 |u|^2 d\mu.$$

PROOF. We have to compare ${}_{\Phi}\partial_{-\Phi} u$ with ${}_{\Phi_1}\partial_{-\Phi_2} u$. From formula (4) of n. 2 we obtain

$${}_{\Phi}\partial_{-\Phi} u = e^{\psi} {}_{\Phi_1}\partial_{-\Phi_2} u - (-1)^{r-1} \sum'_{A, \bar{D}} \sum_{j=1}^n \frac{\partial\psi}{\partial z_j} u_{A, j\bar{D}} dz^A \wedge \bar{d}z^{\bar{D}}.$$

From the inequality $|a + b|^2 \leq 2|a|^2 + 2|b|^2$ we derive the estimate

$$|{}_{\Phi}\partial_{-\Phi} u|^2 \leq 2 \cdot e^{2\psi} |{}_{\Phi_1}\partial_{-\Phi_2} u|^2 + 2n |\partial\psi|^2 |u|^2.$$

Multiplying by $e^{-\Phi}$ and integrating over Ω we obtain

$$\| \vartheta_{-\Phi} u \|_{-\Phi}^2 \leq 2 \| \vartheta_{-\Phi_1} u \|_{-\Phi_1}^2 + \int_{\Omega} e^{-\Phi} 2n | \partial \psi |^2 | u |^2 d\mu.$$

This proves Proposition 3.

5. *Local estimates in the interior.* Let U and Ω be as before, with Ω compact.

Select a function Φ that is strongly plurisubharmonic in Ω . Replacing Φ by $\Phi = \mu(\Phi)$, where $\mu: \mathbb{R} \rightarrow \mathbb{R}$ is C^∞ increasing ($\mu' \geq 0$) and convex ($\mu'' \geq 0$), we may assume that for every $u \in C_{(r,s)}^\infty(\Omega)$ with $s \geq 1$, we have

$$\mathcal{L}(\Phi)\{u, u\} \geq 2n | u |^2.$$

From Proposition 3 we thus derive the following

PROPOSITION 4. *We can choose a strongly plurisubharmonic function Φ on Ω such that:*

for any choice of a C^∞ increasing convex function μ on \mathbb{R} , and

for any choice of $u \in C_{(r,s)}^\infty(\overset{\circ}{\Omega})$, with $s \geq 1$,

we have

$$\int_{\Omega} e^{-\mu(\Phi)} \mu'(\Phi) | u |^2 d\mu \leq \| \vartheta_{-\Phi_1} u \|_{-\Phi_1}^2 + \| \bar{\partial} u \|_{-\Phi_3}^2 + \int_{\Omega} e^{-\mu(\Phi)} | \partial \psi |^2 | u |^2 d\mu.$$

Here

$$\Phi_3 = \mu(\Phi), \quad \Phi_2 = \mu(\Phi) - \psi, \quad \Phi_1 = \mu(\Phi) - 2\psi.$$

6. *Local estimates at the boundary.* Let U be open in \mathbb{C}^n and let $\Omega = \{x \in U \mid h(x) \leq 0\}$, where h is C^∞ and $dh \neq 0$ on $S = \{x \in U \mid h(x) = 0\}$.

We want to prove the following:

PROPOSITION 5. *Let $z_0 \in S$ and assume that $\mathcal{L}(h)(z_0)$ has at least p positive eigenvalues. Then there exists*

i) *an open neighborhood ω of z_0 in U ,*

ii) *a system of holomorphic coordinates in U ,*

iii) *a strongly plurisubharmonic function Φ on ω , such that: For any choice of an increasing convex C^∞ function μ on \mathbb{R} , and for any $u \in \overline{*}\mathcal{T}(\omega \cap \Omega) \cap C_{(r,s)}^\infty(\omega \cap \Omega)$ with $s > n - p$, we have on $\omega \cap \Omega$,*

$$\int_{\omega \cap \Omega} e^{-\mu(\Phi)} \mu'(\Phi) |u|^2 d\mu + \int_{\omega \cap S} e^{-\mu(\Phi)} |u|^2 dS \leq \\ \leq \| \vartheta_{-\Phi_1} u \|_{-\Phi_1}^2 + \| \bar{\partial} u \|_{-\Phi_3}^2 + \int_{\omega \cap \Omega} e^{-\mu(\Phi)} |\partial\psi|^2 |u|^2 d\mu.$$

Here $\Phi_3 = \mu(\Phi)$, $\Phi_2 = \mu(\Phi) - \psi$, $\Phi_1 = \mu(\Phi) - 2\psi$, and the above expression is computed in the coordinate system given by ii).

PROOF. We apply Proposition 1 to the function h ; there is an open set $\omega \ni z_0$, and there are holomorphic coordinates ζ on U such that, on ω ,

$$\widehat{\mathcal{L}}(h) \{u, u\} \geq |u|^2,$$

for any $u \in \overline{*}\mathcal{J} \cap C_{0(r,s)}^\infty(\omega \cap \Omega)$ with $s > n - p$.

Now choose Φ as was done in Proposition 4 (replacing Ω by ω there) so that

$$\mathcal{L}(\Phi) \{u, u\} \geq 2n |u|^2.$$

The estimate then follows from Proposition 3.

REMARK. We may select ω to be a domain of holomorphy and Φ strongly plurisubharmonic in ω such that the sets $\{x \in \omega \mid \Phi(x) \leq c\}$ are compact in ω for every $c \in \mathbb{R}$ (i. e., $\Phi : \omega \rightarrow \mathbb{R}$ is a proper map).

COROLLARY. Under the assumptions of Proposition 5, if ψ is a C^∞ function in ω we can choose the increasing convex function μ such that for $\Phi_3 = \mu(\Phi)$, $\Phi_2 = \mu(\Phi) - \psi$, $\Phi_1 = \mu(\Phi) - 2\psi$ and for any $u \in \overline{*}\mathcal{J}(\omega \cap \Omega) \cap C_{0(r,s)}^\infty(\omega \cap \Omega)$ with $s > n - p$, we have

$$(\alpha) \quad \|u\|_{-\Phi_2}^2 + \int_{\omega \cap S} e^{-\Phi_2} |u|^2 dS \leq \| \vartheta_{-\Phi_1} u \|_{-\Phi_1}^2 + \| \bar{\partial} u \|_{-\Phi_3}^2.$$

Moreover, if ω and Φ are as in the previous remark, given any $f \in C_{(r,s)}^\infty(\omega \cap \Omega)$, we can select μ such that

$$(\beta) \quad \int_{\omega \cap \Omega} e^{-\Phi_2} |f|^2 d\mu < \infty.$$

PROOF. For the validity of the inequality (α) , μ must be chosen such that

$$e^{-\psi} (\mu'(\Phi) - |\partial\psi|^2) \geq 1$$

or

$$\mu'(\Phi) \geq e^\psi + |\partial\psi|^2.$$

This is possible since the second member is a bounded function on any set $\{k \leq \Phi \leq k + 1\}$ for $k = 1, 2, \dots$

For condition (β) it is enough to choose μ increasing convex such that

$$e^{\mu(k)} \geq 2^k \int_{\omega \cap \Omega \cap \{k \leq \Phi \leq k+1\}} e^\psi |f|^2 d\mu.$$

7. *Symmetrization of the weights.* Since the paper of Kohn and Nirenberg employs only L^2 -norms without weights, we show how to write the basic estimates using only L^2 -norms and putting an equal share of the weight on each of the operators $\bar{\partial}$ and ∂ .

For notational convenience we replace the weight functions Φ_1, Φ_2, Φ_3 by $2\Phi_1, 2\Phi_2, 2\Phi_3$. For any point $z_0 \in S$ where $\mathcal{L}(h)(z_0)$ has at least p positive eigenvalues, we can select an open neighborhood ω of z_0 which is a domain of holomorphy, a C^∞ function ψ on ω , and a strongly plurisubharmonic proper function $\Phi: \omega \rightarrow \mathbb{R}$ so that we have the estimate (for $2\Phi_3 = \Phi, 2\Phi_2 = \Phi - \psi, 2\Phi_1 = \Phi - 2\psi$)

$$(1) \quad \|u\|_{-2\Phi_2}^2 + \int_S e^{-2\Phi_3} |u|^2 dS \leq \|_{2\Phi_1} \partial_{-2\Phi_2} u\|_{-\Phi_1}^2 + \|\bar{\partial}u\|_{-2\Phi_3}^2$$

for all $u \in \mathcal{F}(\omega \cap \Omega) \cap C_{0(r,s)}^\infty(\omega \cap \Omega)$ with $s > n - p$.

Let us denote by D the set $\omega \cap \Omega$. If α and β are C^∞ functions on D , we will use the following notation:

$${}_{\alpha}\bar{\partial}\beta = e^\alpha \bar{\partial} e^\beta, \quad {}_{\alpha}\partial\beta = e^\alpha \partial e^\beta.$$

We then have the commutative diagrams:

$$\begin{array}{ccccc} C_{(r,s-1)}^\infty(D) & \xrightarrow{\bar{\partial}} & C_{(r,s)}^\infty(D) & \xrightarrow{\bar{\partial}} & C_{(r,s+1)}^\infty(D) \\ \downarrow e^{-\Phi_1} & & \downarrow e^{-\Phi_1} & & \downarrow e^{-\Phi_3} \\ C_{(r,s-1)}^\infty(D) & \xrightarrow{-\Phi_1 \bar{\partial}} & C_{(r,s)}^\infty(D) & \xrightarrow{-\Phi_3 \bar{\partial}} & C_{(r,s+1)}^\infty(D) \end{array}$$

and

$$\begin{array}{ccccc}
 C_{(r, s-1)}^\infty(D) & \xleftarrow{2\varphi_1 \bar{\partial} - 2\varphi_2} & C_{(r, s)}^\infty(D) & \xleftarrow{2\varphi_2 \bar{\partial} - 2\varphi_3} & C_{(r, s+1)}^\infty(D) \\
 \downarrow e^{-\varphi_1} & & \downarrow e^{-\varphi_2} & & \downarrow e^{-\varphi_3} \\
 C_{(r, s-1)}^\infty(D) & \xleftarrow{\varphi_1 \bar{\partial} - \varphi_2} & C_{(r, s)}^\infty(D) & \xleftarrow{\varphi_2 \bar{\partial} - \varphi_3} & C_{(r, s+1)}^\infty(D)
 \end{array}$$

Applying the isomorphisms

$$\begin{aligned}
 e^{-\varphi_1} : C_{(r, s-1)}^\infty(D) &\rightarrow C_{(r, s-1)}^\infty(D) \\
 e^{-\varphi_2} : C_{(r, s)}^\infty(D) &\rightarrow C_{(r, s)}^\infty(D) \\
 e^{-\varphi_3} : C_{(r, s+1)}^\infty(D) &\rightarrow C_{(r, s+1)}^\infty(D)
 \end{aligned}$$

we deduce from (1) the following inequality on $D = \omega \cap \Omega$:

$$(2) \quad \|u\|^2 + \int_S e^{-\psi} |u|^2 dS \leq \| \varphi_1 \bar{\partial} - \varphi_2 u \|^2 + \| -\varphi_2 \bar{\partial} - \varphi_3 u \|^2.$$

This is valid for all $u \in \overline{*G}(D) \cap C_{0(r, s)}^\infty(D)$ with $s > n - p$.

Moreover, given any $f \in C_{(r, s)}^\infty(D)$ the choice of Φ can be made such that for $f' = e^{-\varphi_2} f$ we have

$$(3) \quad \|f'\|^2 = \int_D |f'|^2 d\mu < \infty.$$

We note that the problem of finding a solution to

$$\bar{\partial}u = f \quad \text{when} \quad \bar{\partial}f = 0$$

is equivalent to the problem of finding a solution to

$$-\varphi_2 \bar{\partial} - \varphi_1 u' = f' \quad \text{when} \quad -\varphi_3 \bar{\partial} - \varphi_2 f' = 0.$$

§ 3. Vanishing theorem for $H^{r, s}(D)$.

8. *Preliminaries.* a) The situation we are considering in the following : On an open set U of \mathbb{C}^n we have a C^∞ function $h : U \rightarrow \mathbb{R}$; we set

$$\Omega = \{x \in U \mid h(x) \leq 0\}, \quad S = \{x \in U \mid h(x) = 0\}$$

and we assume that $dh \neq 0$ on S , so that S is a smooth hypersurface.

Let $z_0 \in S$; we assume that $\mathcal{L}(\hbar)(z_0)$ has at least p positive eigenvalues. Then according to Proposition 5 we can choose the coordinates in \mathbb{C}^n , we can find an open neighborhood ω of z_0 in U which is a domain of holomorphy, and we can choose a C^∞ strongly plurisubharmonic function $\Phi: \omega \rightarrow \mathbb{R}$ which is proper and such that:

for any choice of a C^∞ function ψ on ω

for any choice of an increasing convex C^∞ function μ on \mathbb{R} we have an estimate

$$\int_{\omega \cap \Omega} e^{-\mu(\Phi)} \mu'(\Phi) |u|^2 d\mu + \int_{\omega \cap S} e^{-\mu(\Phi)} |u|^2 dS \leq \leq \|_{2\Phi_1} \vartheta_{-2\Phi_2} u \|_{-\Phi_1}^2 + \| \bar{\partial} u \|_{-2\Phi_2}^2 + \int_{\omega \cap \Omega} e^{-\mu(\Phi)} |\partial\psi|^2 |u|^2 d\mu$$

for any $u \in \overline{*G}(\omega \cap \Omega) \cap C_{0(r,s)}^\infty(\omega \cap \Omega)$ with $s > n - p$.

We remark that any sufficiently small domain of holomorphy ω which contains z_0 will have this property. In any case, ω will be assumed bounded.

b) We fix ω and set $D = \omega \cap \Omega$. Let $\{\eta_\nu\}_{\nu \in N}$ be a sequence of real-valued C^∞ functions on ω such that:

i) η_ν has compact support in ω , $\forall \nu \in N$

ii) for any compact subset $K \subset \omega$ there is an integer $\nu_0(K)$ such that $\eta_\nu \equiv 1$ on K if $\nu \geq \nu_0(K)$.

We choose now the function ψ to be C^∞ on ω and such that:

$$\sum_{k=1}^n \left| \frac{\partial \eta_\nu}{\partial z_k} \right|^2 = |\bar{\partial} \eta_\nu|^2 \leq e^\psi$$

for $\nu = 1, 2, 3, \dots$.

This function ψ exists, since on any compact set $K \subset \omega$, there is only a finite set of ν 's such that $|\bar{\partial} \eta_\nu|^2 \neq 0$ on K .

We then choose the function Φ as stated above and we take the weights

$$2\Phi_3 = \mu(\Phi)$$

$$2\Phi_2 = \mu(\Phi) - \psi$$

$$2\Phi_1 = \mu(\Phi) - 2\psi.$$

Let $f \in C_{(r,s)}^\infty(D)$ be given. Then we can select the function μ in such a way that we have, according to the corollary of Proposition 5 and (n. 7). the

Basic estimate on D :

$$\|u\|^2 + \int_{\bar{S}} e^{-\psi} |u|^2 dS \leq \|_{\Phi_1} \partial_{-\Phi_2} u\|^2 + \|_{-\Phi_2} \bar{\partial}_{\Phi_2} u\|^2$$

for all $u \in \overline{*C}(D) \cap C_{0(r,s)}^\infty(D)$ with $s > n - p$;
 moreover,

$$\int_{\bar{D}} e^{-2\Phi_2} |f|^2 d\mu = \int_{\bar{D}} |f'|^2 d\mu < \infty, \quad \text{where } f' = e^{-\Phi_2} f.$$

c) We now define the Hilbert spaces

$$H_1 = L_{(r,s-1)}^2(D, d\mu)$$

$$H_2 = L_{(r,s)}^2(D, d\mu)$$

$$H_3 = L_{(r,s+1)}^2(D, d\mu)$$

where $L_{(h,k)}^2(D, d\mu)$ denotes the closure of $C_{0(h,k)}^\infty(D)$ with respect to the usual L^2 -norm (without weight).

Next we define the densely defined linear operators

$$T : H_1 \dots \rightarrow H_2$$

$$S : H_2 \dots \rightarrow H_3$$

as follows :

T is the closure of the operator

$$_{-\Phi_2} \bar{\partial}_{\Phi_1} : C_{0(r,s-1)}^\infty(D) \rightarrow H_2 = L_{(r,s)}^2(D, d\mu) ;$$

S is the closure of the operator

$$_{-\Phi_3} \bar{\partial}_{\Phi_2} : C_{0(r,s)}^\infty(D) \rightarrow H_3 = L_{(r,s+1)}^2(D, d\mu).$$

Since

$$_{-\Phi_3} \bar{\partial}_{\Phi_2} \cdot _{-\Phi_2} \bar{\partial}_{\Phi_1} : C_{0(r,s-1)}^\infty(D) \rightarrow 0 \in H_3,$$

we obtain, when closing the graphs, that

$$\text{Im } T \subset \text{Ker } S.$$

In conclusion we get a short complex of Hilbert spaces and densely defined closed operators :

$$H_1 \dots \xrightarrow{T} H_2 \dots \xrightarrow{S} H_3 .$$

PROPOSITION 6. *For the above defined short complex we have*

- i) $D_T \cap C_{0(r, s-1)}^\infty(D) = C_{0(r, s-1)}^\infty(D)$
- ii) $D_S \cap C_{0(r, s)}^\infty(D) = C_{0(r, s)}^\infty(D)$
- iii) $D_{T^*} \cap C_{0(r, s)}^\infty(D) = \overline{*}\mathcal{J}(D) \cap C_{0(r, s)}^\infty(D)$
- iv) $D_{S^*} \cap C_{0(r, s+1)}^\infty(D) = \overline{*}\mathcal{J}(D) \cap C_{0(r, s+1)}^\infty(D)$
- v) $D_{T^*} \cap D_S \cap C_{0(r, s)}^\infty(D) = \overline{*}\mathcal{J}(D) \cap C_{0(r, s)}^\infty(D)$.

PROOF. By definition $C_{0(r, s-1)}^\infty(D) \subset D_T$ and $C_{0(r, s)}^\infty(D) \subset D_S$; hence i) and ii) follow.

Let $v \in D_{T^*} \cap C_{0(r, s)}^\infty(D)$; this means that for every $u \in C_{0(r, s-1)}^\infty(D)$ we have $(Tu, v) = (u, T^*v)$, in particular for every $u \in C_{0(r, s-1)}^\infty(\overset{\circ}{D})$.

Now on $C_{0(r, s-1)}^\infty(D)$ T is given by the differential operator $-\varphi_2 \overline{\partial} \varphi_1$, so that we have, for $u \in C_{0(r, s-1)}^\infty(D)$, $v \in C_{0(r, s)}^\infty(D)$,

$$(\alpha) \quad (Tu, v) = (u, \varphi_1 \vartheta_{-\varphi_2} v) + \int_S e^{\frac{1}{2}\psi} u \wedge * \bar{v}.$$

In particular, for every $u \in C_{0(r, s-1)}^\infty(\overset{\circ}{D})$ we obtain

$$(u, \varphi_1 \vartheta_{-\varphi_2} v - T^*v) = 0.$$

This shows that if $v \in D_{T^*}$ and $v \in C_{0(r, s)}^\infty(D)$, we have

$$(\beta) \quad T^*v = \varphi_1 \vartheta_{-\varphi_2} v.$$

But then from (α) it follows that we must have

$$\int_S e^{\frac{1}{2}\psi} u \wedge * \bar{v} = 0 \quad \text{for every } u \in C_{0(r, s-1)}^\infty(D).$$

According to Lemma 1, this implies that $v \in {}^*\overline{\mathcal{J}}$. Conversely, because of (β) , if $v \in {}^*\overline{\mathcal{J}}$, then $v \in D_{T^*}$, again by virtue of that lemma. This proves iii).

The proof of iv) is by the same argument. Finally, v) follows from ii) and iii).

REMARK. We have shown that

- (a) On ${}^*\overline{\mathcal{J}}(D) \cap C_{0(r,s)}^\infty(D)$ we have $T^* = \phi_1 \partial - \phi_2$
- (b) On ${}^*\overline{\mathcal{J}}(D) \cap C_{0(r,s+1)}^\infty(D)$ we have $S^* = \phi_2 \partial - \phi_3$.

9. Density theorem. a) Consider the densely defined operator

$$T^* \times S: H_2 \rightarrow H_1 \times H_3.$$

For $u \in D_{T^*} \cap D_S$ we define

$$\| \| u \| \| = (\| u \|^2 + \| T^* u \|^2 + \| Su \|^2)^{\frac{1}{2}}$$

as the « graph norm » for $T^* \times S$.

We have the following density theorem :

THEOREM 0. The space ${}^*\overline{\mathcal{J}}(D) \cap C_{0(r,s)}^\infty(D)$ is dense in $D_{T^*} \cap D_S$ in the graph norm.

PROOF. (a) Consider the sequence η_ν defined in the previous section. Let $u \in D_{T^*} \cap D_S$. Then $\eta_\nu u \in D_{T^*} \cap D_S$ for each ν . For this it is enough to show that

- i) if $u \in D_S$, then $\eta_\nu u \in D_S$,
- ii) if $u \in D_{T^*}$, then $\eta_\nu u \in D_{T^*}$.

For $u \in C_{0(r,s)}^\infty(D)$, we have $u \in D_S$ and

$$(\alpha) \quad S(\eta_\nu u) = \eta_\nu Su + e^{-\frac{1}{2}\psi} \overline{\partial} \eta_\nu \wedge u.$$

For $v \in C_{0(r,s-1)}^\infty(D)$, we have $v \in D_{T^*}$ and

$$(\beta) \quad T(\eta_\nu v) = \eta_\nu Tv + e^{-\frac{1}{2}\psi} \overline{\partial} \eta_\nu \wedge v.$$

Let $u \in D_S$. We can find a sequence $u_k \in C_{0(r,s)}^\infty(D)$ with $\| u - u_k \| \rightarrow 0$ and $\| Su - Su_k \| \rightarrow 0$. Then (α) remains valid also in the limit; thus $\eta_\nu u \in D_S$.

Let $u \in D_{T^*}$. Then, for any $v \in C_{0(r, s-1)}^\infty(D)$ we have

$$\begin{aligned} (\eta_\nu u, Tv) &= (u, \eta_\nu Tv) = (u, T(\eta_\nu v)) - (u, e^{-\frac{1}{2}\psi} \bar{\partial} \eta_\nu \wedge v) \\ &= (T^*u, \eta_\nu v) - (u, e^{-\frac{1}{2}\psi} \bar{\partial} \eta_\nu \wedge v). \end{aligned}$$

From this we obtain, for $v \in D_T$

$$|(\eta_\nu u, Tv)| \leq c(u) \|v\|.$$

Hence the functional $l(v) = (\eta_\nu u, Tv)$ is continuous, extends to all of H_1 , and we can write

$$(\eta_\nu u, Tv) = (g, v)$$

for some $g \in H_1$. This shows that $\eta_\nu u \in D_{T^*}$ (and $T^*(\eta_\nu u) = g$).

(b) From the last argument, because of the choice of ψ , we have

$$\begin{aligned} (\eta_\nu T^*u, v) - (T^*\eta_\nu u, v) &= (u, e^{-\frac{1}{2}\psi} \bar{\partial} \eta_\nu \wedge v), \\ e^{-\frac{1}{2}\psi} |\bar{\partial} \eta_\nu| &\leq 1. \end{aligned}$$

Hence there exists a constant $c > 0$, independent of ν , such that

$$|(\eta_\nu T^* - T^*\eta_\nu)u, v| \leq c \left(\int_{\text{supp } |\bar{\partial} \eta_\nu|} |u|^2 d\mu \right)^{\frac{1}{2}} \|v\|,$$

for every $u \in D_{T^*}$ and $v \in C_{0(r, s-1)}^\infty(D)$.

Consequently we must have

$$(\gamma) \quad \|(\eta_\nu T^* - T^*\eta_\nu)u\| \leq c \left(\int_{\text{supp } |\bar{\partial} \eta_\nu|} |u|^2 d\mu \right)^{\frac{1}{2}}.$$

By a similar and simpler argument, we deduce from (α) that

$$(\delta) \quad \|(\eta_\nu S - S\eta_\nu)u\| \leq c \left(\int_{\text{supp } |\bar{\partial} \eta_\nu|} |u|^2 d\mu \right)^{\frac{1}{2}},$$

with another constant $c > 0$ independent of ν .

(c) We now show that $\| \eta_\nu u - u \| \rightarrow 0$ as $\nu \rightarrow \infty$.

1. $\eta_\nu u - u \rightarrow 0$ pointwise on D , $|\eta_\nu u - u|^2 \leq |u|^2$ and $\|u\|^2 < \infty$. Therefore by the Lebesgue-dominated convergence theorem we obtain that

$$\|\eta_\nu u - u\| \rightarrow 0 \quad \text{as} \quad \nu \rightarrow \infty.$$

2. $\|S(\eta_\nu u) - Su\| \leq \|S(\eta_\nu u) - \eta_\nu Su\| + \|\eta_\nu Su - Su\|.$

The second term on the right tends to zero by the same argument given for 1. The first goes to zero as $\nu \rightarrow \infty$, because $\text{vol}(\text{supp} |\bar{\partial} \eta_\nu|) \rightarrow 0$.

3. $\|T^*(\eta_\nu u) - T^*u\| \leq \|T^*(\eta_\nu u) - \eta_\nu T^*u\| + \|\eta_\nu T^*u - T^*u\|.$

Both terms tend to zero as $\nu \rightarrow \infty$ by the same arguments as those given in 2.

(d) We can therefore suppose that $u \in D_{T^*} \cap D_S$ has compact support in D . We want to show that we can find a sequence $\{u_k\}$ with

$$u_k \in {}^*\bar{\mathcal{F}}(D) \cap C_{0(r,s)}^\infty(D)$$

such that $\|u - u_k\| \rightarrow 0$ as $k \rightarrow \infty$.

Let D' be a subset of D , open in D , relatively compact in D , and such that $\text{supp } u \subset D'$. Let ω' be an open set in ω such that $D' = D \cap \omega'$, and therefore $D' = \{x \in \omega' \mid h(x) \leq 0\}$. We now consider on D' the system of equations

$$(\varepsilon) \quad \begin{cases} Su = f \\ T^*u = g, \end{cases}$$

where $f = Su$ and $g = T^*u$ by definition. If we extend u, f, g to $\tilde{u}, \tilde{f}, \tilde{g}$ on ω' by setting them equal to zero on $\omega' - D'$, then since $u \in D_{T^*}$, for any $v \in C_{(r,s-1)}^\infty(\omega')$ we have

$$\begin{aligned} (\tilde{g}, v)_{\omega'} &= (T^*u, v)_{D'} = (u, Tv)_{D'} \\ &= (\tilde{u}, -\phi_2 \bar{\partial} \phi_1 v)_{\omega'}. \end{aligned}$$

Now the system (ε) is of the type studied by Hörmander [7]⁽³⁾ and «the Cauchy data of u with respect to the operator T^* vanish on $h=0$ ».

⁽³⁾ Note that the system has the form

$$\begin{cases} e^{-\frac{1}{2}\psi} \bar{\partial} u + Bu = f \\ e^{-\frac{1}{2}\psi} \partial u + Cu = g, \end{cases}$$

with B and C of order zero with C^∞ coefficients that are bounded on D' ; ψ, B, C can be extended in a C^∞ fashion, keeping them bounded on ω' .

Therefore, we can apply to (ε) the proposition 1.2.4 of his paper and deduce that there is a sequence $u_k \in C_{0(r,s)}^\infty(D')$, ($k = 1, 2, 3, \dots$) such that

$$\|u - u_k\| \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and moreover $u_k \in D_{T^*} \cap C_{0(r,s)}^\infty(D')$; i. e., $u_k \in {}^*\bar{\mathcal{F}}(D') \cap C_{0(r,s)}^\infty(D')$. This completes the proof of the theorem.

b) As an immediate consequence of the density theorem we derive from the basic estimates the following important

COROLLARY 1. *For the short complex of Hilbert spaces and closed densely defined operators, we have a W-ellipticity estimate:*

$$\|u\|^2 \leq \|T^*u\|^2 + \|Su\|^2, \quad \forall u \in D_{T^*} \cap D_S$$

whenever $s > n - p$.

From this corollary we obtain that if $s > n - p$ the graph norm $\| \cdot \|$ is equivalent to the « Dirichlet norm »

$$d(u)^2 = \|T^*u\|^2 + \|Su\|^2.$$

Now let us remark that under the graph norm the domain of a closed densely defined operator is a complete space. In particular, by applying these remarks to the operator $T^* \times S$, whose domain is $D_{T^*} \cap D_S$, we get the following

COROLLARY 2. *For any $u \in C_{0(r,s)}^\infty(D)$ set*

$$Q_0(u, u) = \|\varphi_1 \vartheta_{-\varphi_2} u\|^2 + \|_{-\varphi_3} \bar{\partial}_{\varphi_2} u\|^2$$

$$B = {}^*\bar{\mathcal{F}}(D) \cap C_{0(r,s)}^\infty(D).$$

Then if $s > n - p$, the completion K of the space B with respect to the form $Q_0(u, u)^{\frac{1}{2}}$ is the space $K = D_{T^*} \cap D_S$.

c) Now let $p \in S \cap D$ and consider any neighborhood ω' of p in U such that $\bar{\omega}' \cap D$ is compact. Set $D' = D'(p) = \omega' \cap D$.

Then from the basic estimate we deduce the following

COROLLARY 3. *Let a point $p \in S \cap D$ and a relatively compact neighborhood $D'(p)$ of p in D be given. Then there exists a constant $c(D') > 0$*

such that

$$\int_S |u|^2 dS \leq c(D') Q_0(u, u)$$

for all $u \in B(D') = \overline{*G}(D') \cap C_{0(r, s)}^\infty(D')$ with $s > n - p$.

10. *The vanishing theorem.* a) We formulate the vanishing theorem as follows :

THEOREM 1. *Let U be open in \mathbb{C}^n , let $h : U \rightarrow \mathbb{R}$ be a C^∞ function with $dh \neq 0$ on $S = \{x \in U \mid h(x) = 0\}$. Let $z_0 \in S$ be a point of S at which $\mathcal{L}(h)$, the Levi form of h , has at least p positive eigenvalues. Then there exists a neighborhood W of z_0 in U such that for any domain of holomorphy ω contained in W , setting $D = \{x \in \omega \mid h(x) \leq 0\}$, we have*

$$H^{r, s}(D) = 0 \quad \text{for} \quad s > n - p.$$

REMARK. What really has a geometric meaning is the number π of positive eigenvalues of the Levi form restricted to the holomorphic tangent space HT at z_0 :

$$\mathcal{L}(h)|_{HT(z_0)} \equiv \begin{cases} \sum_{j, k=1}^n \frac{\partial^2 h(z_0)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \\ \sum_{j=1}^n \frac{\partial h(z_0)}{\partial z_j} w_j = 0. \end{cases}$$

But then by replacing h by e^{ch} with $c > 0$ sufficiently large, we may always achieve that $\mathcal{L}(h)(z_0)$ has at least $\pi + 1$ positive eigenvalues.

PROOF. To prove Theorem 1 we have to show that given $f \in C_{(r, s)}^\infty(D)$, with $\bar{\partial}f = 0$ and $s > n - p$, we can find a $u \in C_{(r, s-1)}^\infty(D)$ such that

$$(1) \quad \bar{\partial}u = f.$$

We may assume that f is the element we have chosen in n. 8 b), so that $f' = e^{-\phi_2} f \in L_{(r, s)}^2(D)$ and $-\phi_2 \bar{\partial} \phi_2 f' = 0$. Equation (1) is therefore equivalent to the equation

$$(2) \quad -\phi_2 \bar{\partial} \phi_1 u' = f',$$

wherein we want to show the existence of a solution $u' \in \hat{O}_{(r, s-1)}^\infty(D)$, knowing that the compatibility condition $-\phi_2 \bar{\partial} \phi_1 f' = 0$ is satisfied.

b) *Weak solution of Equation (2).* Equation (2) admits an L^2 -solution by virtue of Theorem B and Corollary 1 of Theorem 0. Indeed, according to the corollary of Theorem B, we see that we can find $u' \in D_T \cap \text{Im } T^* \subset H_1$ and $x' \in D_{T^*} \cap \text{Ker } S \subset H_2$ such that

$$(2') \quad Tu' = f', \quad \|u'\| \leq \|f'\|$$

and

$$u' = T^*x', \quad \|x'\| \leq \|u'\|$$

$$Sx' = 0.$$

Moreover,

$$(3) \quad (TT^* + S^*S)x' = f'.$$

Equation (2') exhibits the weak solution of Equation (2), since T is the closure of the operator $-\bar{\varphi}_2 \partial_{\varphi_1}$.

c) *Regularity of the solution in the interior of D .* The weak solution we have produced is of the form $u' = T^*x'$ where x' is a solution of Equation (3).

Now the operator $TT^* + S^*S$ is an extension of the (system) of partial differential operators (defined on $C_{0(r,s)}^\infty(\overset{\circ}{D})$)

$$-\bar{\varphi}_2 \partial_{\varphi_1} \varphi_1 \partial_{-\varphi_2} + \varphi_2 \partial_{-\varphi_3} - \bar{\varphi}_3 \partial_{\varphi_3}$$

whose principal part is

$$e^{-\psi} \{\bar{\partial} \partial + \partial \bar{\partial}\}.$$

This is an elliptic system; therefore, from the interior regularity theorem for such systems, we get

$$x' \in C_{(r,s)}^\infty(\overset{\circ}{D}).$$

d) *Regularity up to the boundary.* Consider the space $K = D_{T^*} \cap D_S$ that was introduced in Corollary 2 to Theorem 0. We have

$$x' \in D_{T^*} \cap \text{Ker } S \subset D_{T^*} \cap D_S = K.$$

For any $v \in K$ we also have

$$\begin{aligned} (f', v) &= (TT^*x', v) + (S^*Sx', v) \\ &= (T^*x', T^*v) + (Sx', Sv). \end{aligned}$$

Thus x' is the unique solution $x' \in K$ of the equation

$$Q_0(x', v) = (f', v) \quad \text{for all } v \in B,$$

when Q_0 is extended as a bilinear form on K . (Indeed, by the W -ellipticity inequality, Q_0 defines on K an equivalent norm to the graph norm under which K is complete).

Now we make use of the estimate given in Corollary 3 to Theorem 0 and apply the results of the paper by Kohn and Nirenberg [8]. First we remark that their assumptions $a)$, $b)$, $c)$ on page 451 of that paper are verified; also their assumptions $i)$, $ii)$, $iii)$ on page 452 and 453 are verified because in our case $Q'(u, v) \equiv 0$. The bilinear form $Q_0(u, v)$ is associated to an elliptic system with principal part $e^{-\psi} \{\bar{\partial}\partial + \partial\bar{\partial}\}$. In $\omega' \subset\subset \omega$ we have $e^{-\psi} \geq \text{const.} > 0$. Thus the system is strongly elliptic in D' . Also $x' \in K$ as required in their Theorem 4 (page 458). Then Theorem 5 on page 459 can be applied since $x' \in K$ and by virtue of Corollary 3 of n. 9⁽⁴⁾.

It follows that $x' \in C_{(r,s)}^\infty(D')$; this being true for any choice of D' , it follows that $x' \in C_{(r,s)}^\infty(D)$. But then we have

$$x' \in D_{T^*} \cap \text{Ker } S \cap C_{(r,s)}^\infty(D) = {}^*\bar{\mathcal{J}}(D) \cap C_{(r,s)}^\infty(D) \cap \text{Ker } S.$$

Restricted to this space, the operator $T^* = \varphi_1\partial - \varphi_2\bar{\partial}$. Therefore, $u' = \varphi_1\partial - \varphi_2\bar{\partial} x'$ is in $C_{(r,s-1)}^\infty(D)$. This achieves the proof of Theorem 1.

REMARK. Note that $\varphi_1\partial - \varphi_2\bar{\partial}$ sends ${}^*\bar{\mathcal{J}}(D) \cap C_{(r,s)}^\infty(D)$ into ${}^*\bar{\mathcal{J}}(D) \cap C_{(r,s-1)}^\infty(D)$. Therefore, the solution $u' = \varphi_1\partial - \varphi_2\bar{\partial} x'$ of (2') is in ${}^*\bar{\mathcal{J}}(D) \cap C_{(r,s)}^\infty(D)$, and the same is true for the corresponding solution u of (2) which we have found; i. e., u satisfies some « natural boundary conditions ».

§ 4. Vanishing theorem for $H^{r,s}(D, \mathcal{J})$.

11. *Basic estimate.* $a)$ For an open set $U \subset \mathbb{C}^n$, let $h: U \rightarrow \mathbb{R}$ be C^∞ , $\Omega = \{x \in U \mid h(x) \leq 0\}$ and $S = \{x \in U \mid h(x) = 0\}$ be smooth ($dh \neq 0$ on S).

We assume that

- i) Ω is compact;

(4) In the statement of their Theorem 5 the inequality of Corollary 3 is postulated for all $u \in B$. But in the proof (see page 476, line 20) the inequality is only used for all $u \in B(D')$.

ii) an open subset Σ of S is given, and there is a constant $c_1 > 0$ such that, for any $u \in C_{(r,s)}^\infty(\Omega)$ with $s > n - p$, we have:

$$\widehat{\mathcal{L}}(h)\{u, u\} \geq c_1 |u|^2 \quad \text{at any point of } \Sigma,$$

$$\widehat{\mathcal{L}}(h)\{u, u\} \geq 0 \quad \text{at any point of } S - \Sigma.$$

Let $\Phi: \Omega \rightarrow \mathbb{R}$ be a fixed strongly plurisubharmonic function on Ω . Then there exists a constant $c_2 > 0$ such that, for $u \in C_{(r,s)}^\infty(\Omega)$ with $s \geq 1$, we have

$$\mathcal{L}(\Phi)\{u, u\} \geq c_2 |u|^2$$

at each point of Ω . From Proposition 3, we obtain, for any $u \in {}^*\bar{\mathcal{J}}(\Omega) \cap C_{(r,s)}^\infty(\Omega)$ with $s > n - p$, the estimate:

$$c_2 \|u\|_{-\Phi}^2 + c_1 \int_{\Sigma} e^{-\Phi} |u|^2 dS \leq \|\bar{\partial}u\|_{-\Phi}^2 + \|\Phi\bar{\partial}_{-\Phi}u\|_{-\Phi}^2.$$

Replacing Φ by 2Φ and symmetrizing the weight (here $\psi = 0$), we can rewrite the estimate as

$$c_2 \|u\|^2 + c_1 \int_{\Sigma} |u|^2 dS \leq \|_{-\Phi}\bar{\partial}_{\Phi}u\|^2 + \|\Phi\bar{\partial}_{-\Phi}u\|^2.$$

b) Let us write $u \in {}^*\bar{\mathcal{J}}(\Omega) \cap C_{(r,s)}^\infty(\Omega)$ as

$$u = {}^*\bar{v},$$

with $v \in \mathcal{J}(\Omega) \cap C_{(n-r, n-s)}^\infty(\Omega)$. Then we find that

$$|{}^*\bar{v}|^2 = |v|^2 \quad \text{so that} \quad \|u\|^2 = \|v\|^2.$$

Also,

$$_{-\Phi}\bar{\partial}_{\Phi} {}^*\bar{v} = \pm {}^*\bar{_{-\Phi}\bar{\partial}_{\Phi}v},$$

$$\Phi\bar{\partial}_{-\Phi} {}^*\bar{v} = \pm {}^*\bar{\Phi\bar{\partial}u}.$$

We thus obtain the following

PROPOSITION 7. *Let $\Omega; S, \Sigma$ satisfy the assumptions (i) and (ii) specified above. Then there exists a positive constant c such that, for any $v \in \mathcal{J}(\Omega)$*

$\cap C_{(r, s)}^\infty(\Omega)$ with $s < p$, we have the basic estimate :

$$\|v\|^2 + \int_{\bar{\Sigma}} |v|^2 dS \leq c \{ \|\bar{\partial}_{-\Phi} v\|^2 + \|\partial_{-\Phi} v\|^2 \}$$

(where Φ is any fixed strongly plurisubharmonic function on Ω).

c) Given $s < p$, we consider the Hilbert spaces

$$H_1 = L_{(r, s-1)}^2(\Omega, d\mu),$$

$$H_2 = L_{(r, s)}^2(\Omega, d\mu),$$

$$H_3 = L_{(r, s+1)}^2(\Omega, d\mu),$$

where $L_{(h, k)}^2(\Omega, d\mu)$ denotes the closure of $C_{(h, k)}^\infty(\Omega)$ with respect to the usual L^2 -norm.

We define a densely defined operator $T: H_1 \rightarrow H_2$ as the closure of the differential operator

$$\bar{\partial}_{-\Phi}: \mathcal{F}_{(r, s-1)}^\infty(\Omega) \rightarrow \mathcal{F}_{(r, s)}^\infty(\Omega) \subset H_2.$$

Similarly, we define a densely defined operator $S: H_2 \rightarrow H_3$ as the closure of the differential operator

$$\bar{\partial}_{-\Phi}: \mathcal{F}_{(r, s)}^\infty(\Omega) \rightarrow \mathcal{F}_{(r, s+1)}^\infty(\Omega) \subset H_3.$$

Since $\bar{\partial}_{-\Phi} \circ \bar{\partial}_{-\Phi} = 0$, we have $\text{Im } T \subset \text{Ker } S$. Thus we obtain a short complex of Hilbert spaces and closed densely defined operators :

$$H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3.$$

PROPOSITION 8. For the above-defined short complex, we have

- i) $D_T \cap C_{(r, s-1)}^\infty(\Omega) = \mathcal{F}_{(r, s-1)}^\infty(\Omega),$
- ii) $D_S \cap C_{(r, s)}^\infty(\Omega) = \mathcal{F}_{(r, s)}^\infty(\Omega),$
- iii) $D_{T^*} \cap C_{(r, s)}^\infty(\Omega) = C_{(r, s)}^\infty(\Omega),$
- iv) $D_{S^*} \cap C_{(r, s+1)}^\infty(\Omega) = C_{(r, s+1)}^\infty(\Omega),$
- v) $D_{T^*} \cap D_S \cap C_{(r, s)}^\infty(\Omega) = \mathcal{F}_{(r, s)}^\infty(\Omega).$

PROOF. By definition, $\mathcal{J}_{(r,s-1)}^\infty(\Omega) \subset D_T$. Let $u \in D_T \cap C_{(r,s-1)}^\infty(\Omega)$. Then there exists a sequence $\{u_m\}$ with $u_m \in \mathcal{J}_{(r,s-1)}^\infty(\Omega)$ such that $\|u_m - u\| \rightarrow 0$ and $\|\bar{\varphi}_{\partial-\varphi} u_m - Tu\| \rightarrow 0$. In particular, for any $w \in C_{(r,s)}^\infty(\Omega)$, we must have

$$(\bar{\varphi}_{\partial-\varphi} u_m, w) \rightarrow (Tu, w).$$

Since $u_m \in \mathcal{J}_{(r,s-1)}^\infty(\Omega)$, we have $\int_S u_m \wedge * \bar{w} = 0$; hence

$$(\bar{\varphi}_{\partial-\varphi} u_m, w) = (u_m, -\varphi \vartheta_\varphi w).$$

It follows that

$$(a) \quad (u, -\varphi \vartheta_\varphi w) = (Tu, w), \quad \forall w \in C_{(r,s)}^\infty(\Omega).$$

But if $u \in D_T \cap C_{(r,s-1)}^\infty(\Omega)$, the action of T on u is the operator $\bar{\varphi}_{\partial-\varphi}$ (as follows from (a) by taking $w \in C_{(r,s)}^\infty(\overset{\circ}{\Omega})$). This shows that $\int_S u \wedge * \bar{w} = 0$ for all $w \in C_{(r,s)}^\infty(\Omega)$; hence by Lemma 1, we must have $u \in \mathcal{J}_{(r,s-1)}^\infty(\Omega)$. This proves (i). The proof of (ii) is similar.

The proof of (iii) follows from the fact that, for any $u \in C_{(r,s)}^\infty(\Omega)$ and any $v \in \mathcal{J}_{(r,s-1)}^\infty(\Omega)$, we have

$$(Tv, u) = (v, -\varphi \vartheta_\varphi u).$$

This must hold in the limit also for any $v \in D_T$. Thus $u \in D_{T^*}$. The proof of (iv) is similar. Finally, (v) follows from (ii) and (iii).

12. *Density theorem.* For the operator

$$T^* \times S: H_2 \dots \rightarrow H_1 \times H_3$$

defined for $u \in D_{T^*} \cap D_S$, we introduce the graph norm

$$\| \| u \| \| = (\| u \|^2 + \| T^* u \|^2 + \| Su \|^2)^{\frac{1}{2}}.$$

We have the following density theorem:

THEOREM 0'. *The space $\mathcal{J}_{(r,s)}^\infty(\Omega)$ is dense in $D_{T^*} \cap D_S$ with respect to the graph norm.*

PROOF. Let $u \in D_{T^*} \cap D_S$. We consider in Ω the system of equations

$$(\alpha) \begin{cases} Su = f \\ T^*u = g, \end{cases}$$

where $f = Su$ and $g = T^*u$ by definition. Extend u, f, g to $\tilde{u}, \tilde{f}, \tilde{g}$ on U by setting them equal to zero outside of Ω . Since $u \in D_S$, for any $v \in C_{(r, s+1)}^\infty(U)$, we have

$$(\tilde{f}, v)_U = (Su, v)_\Omega = (u, S^*v)_\Omega = (\tilde{u}, -\bar{\partial}_\Phi v)_U.$$

The system (α) is thus of the type studied by Hörmander ([5], Proposition 1.2.4). That proposition can be applied because the « Cauchy data of u with respect to the operator S vanishes on $h = 0$ ». Therefore, there exists a sequence $u_k \in \mathcal{F}_{(r, s)}^\infty(\Omega)$ such that $\|u - u_k\| \rightarrow 0$. Note that we have essentially used the condition that S be a smooth hypersurface.

COROLLARY 1'. For the short complex defined above, we have a *W-ellipticity estimate*

$$\|u\|^2 \leq c \{ \|T^*u\|^2 + \|Su\|^2 \}, \quad \forall u \in D_{T^*} \cap D_S,$$

provided $s < p$.

Arguing as we did in 9 (b), we also deduce

COROLLARY 2'. For $u \in \mathcal{F}_{(r, s)}^\infty(\Omega)$, set

$$Q_0(u, u) = \|-\bar{\partial}_\Phi u\|^2 + \|\bar{\partial}_\Phi u\|^2,$$

$$B = \mathcal{F}_{(r, s)}^\infty(\Omega).$$

Then if $s < p$, the completion K of B with respect to the norm $Q_0(u, u)^{\frac{1}{2}}$ is the space $D_{T^*} \cap D_S$.

Finally, from the basic estimate we derive

COROLLARY 3'. For any $u \in B = \mathcal{F}_{(r, s)}^\infty(\Omega)$ with $s < p$, we have

$$\int_\Sigma |u|^2 dS \leq c Q_0(u, u).$$

13. *Vanishing theorem (preliminary form)*. Let Ω be compact in an open subset $U \subset \mathbb{C}^n$ with smooth boundary S , and let Σ be an open subset of S .

We assume that $\Omega = \{h(x) \leq 0\}$ where h is C^∞ on U , $S = \{h = 0\}$, and $dh \neq 0$ on S . Moreover, we assume that h is chosen so that we can find a constant $c_1 > 0$ with the property

$$\begin{aligned} \widehat{\mathcal{L}}(h) \{u, u\} &\geq c_1 |u|^2 && \text{on } \Sigma, \\ \widehat{\mathcal{L}}(h) \{u, u\} &\geq 0 && \text{on } S - \Sigma \end{aligned}$$

for every $u \in C_{(r,s)}^\infty(\Omega)$ with $s > n - p$.

Let us denote by $\mathcal{F}_{(r,s)}^\infty(\overset{\circ}{\Omega} \cup \Sigma)$ the space of C^∞ forms on $\overset{\circ}{\Omega} \cup \Sigma$ which belong to the differential ideal \mathcal{I} , generated by h and $\bar{\partial}h$, in a neighborhood of any point of Σ . Under these assumptions, one can prove the following

PROPOSITION 9. *Let $f \in \mathcal{F}_{(r,s)}^\infty(\Omega)$ with $0 < s < p$, and suppose that $\bar{\partial}f = 0$ on Ω . Then we can find a $g \in \mathcal{F}_{(r,s-1)}^\infty(\overset{\circ}{\Omega} \cup \Sigma)$ such that*

$$\bar{\partial}g = f.$$

PROOF. Because of the W-ellipticity estimate given by Corollary 1' to Theorem 0', we can apply the corollary to Theorem A. Precisely: if $f' = e^\varphi f$, we have $f' \in \mathcal{F}_{(r,s)}^\infty(\Omega)$ and ${}_{\varphi} \bar{\partial}_{-\varphi} f' = 0$. Hence we can find $u' \in D_T \cap \text{Im } T^*$, $x' \in D_{T^*} \cap \text{Ker } S$ such that

$$Tu' = f', \quad u' = T^*x', \quad Sx' = 0,$$

and therefore

$$(TT^* + S^*S)x' = f'.$$

Since the operator $TT^* + S^*S$ is an extension of the differential operator

$${}_{\varphi} \bar{\partial}_{-\varphi} - {}_{\varphi} \partial_{\varphi} + -{}_{\varphi} \partial_{\varphi} \quad {}_{\varphi} \bar{\partial}_{-\varphi},$$

whose principal part is $\bar{\partial}\partial + \partial\bar{\partial}$, we obtain from the interior regularity theorem for elliptic operators that $x' \in C_{(r,s)}^\infty(\overset{\circ}{\Omega})$.

We have to prove that x' is of class C^∞ also at points of Σ , and that $u' = T^*x'$ belongs to the ideal generated by h and $\bar{\partial}h$ at points of Σ .

Consider a point $z_0 \in \Sigma$, and let ω be a small neighborhood of z_0 is U such that $\omega \cap S = \omega \cap \Sigma$. We set $D = \omega \cap \Omega$ and

$$B(D) = \mathcal{F}_{(r,s)}(\Omega) \cap C_{0(r,s)}^\infty(\omega \cap \Omega).$$

We have

$$x' \in D_{T^*} \cap \text{Ker } S \subset D_{T^*} \cap D_S = K,$$

since $K = D_{T^*} \cap D_S$ by Corollary 2' to Theorem 0'. For any $v \in K$, we have

$$(f', v) = (TT^*x', v) + (S^*Sx', v) = (T^*x', T^*v) + (Sx', Sv).$$

Therefore, $x' \in K$ is the unique solution in K of the equation

$$Q_0(x', v) = (f', v) \quad \text{for all } v \in B.$$

Now the system we are dealing with is strongly elliptic, and we have the inequality

$$\int_{\Sigma} |u|^2 dS \leq cQ_0(u, u)$$

for all $u \in B(D)$. By the theorem of Kohn and Nirenberg, it follows that x' is C^∞ in D . Thus $x' \in C_{(r,s)}^\infty(\overset{\circ}{\Omega} \cup \Sigma)$. Therefore, $u' \in C_{(r,s-1)}^\infty(\overset{\circ}{\Omega} \cup \Sigma)$ because, on $\overset{\circ}{\Omega} \cup \Sigma$, $u' = -\varphi \partial_{\bar{\varphi}} x'$.

Now $u' \in D_T$; hence $u' \in D_T \cap C_{(r,s-1)}^\infty(\overset{\circ}{\Omega} \cup \Sigma)$. Applying the argument used in Proposition 8 (i), with $v \in C_{0(r,s)}^\infty(\overset{\circ}{\Omega} \cup \Sigma)$, we get $u' \in \mathcal{F}_{(r,s-1)}^\infty(\overset{\circ}{\Omega} \cup \Sigma)$. Hence $u = e^{-\Phi} u' \in \mathcal{F}_{(r,s-1)}^\infty(\overset{\circ}{\Omega} \cup \Sigma)$. Therefore, u has the desired properties and is a solution of the equation $\bar{\partial}u = f$; the proof of Proposition 9 is complete.

REMARKS 1. If the assumptions of Proposition 9 are satisfied with $\Sigma = S$, we get, for $s < p$, the vanishing theorem

$$H^{r,s}(\Omega, \mathcal{F}) = 0.$$

2. We have the estimate

$$\|e^\Phi g\|^2 \leq c^2 \|e^\Phi f\|^2$$

for the solution g of the equation $\bar{\partial}u = f$ (with $g \in \mathcal{F}_{(r,s-1)}^\infty(\overset{\circ}{\Omega} \cup \Sigma)$) that we have constructed.

3. We have chosen Φ to be strongly plurisubharmonic. However, the only requirement we have made on Φ is that there should exist a constant $c > 0$ such that, for all $u \in C_{(r,s)}^\infty(\Omega)$ with $s > n - p$, we have an estimate

$$\mathcal{L}(\Phi)\{u, u\} \geq c|u|^2 \quad \text{at each point of } \Omega.$$

14. a) We now restrict our attention to the following situation: We are given an open bounded set U in \mathbb{C}^n and two C^∞ function h and g on \bar{U} with the following properties:

- i) on $\bar{S} = \{x \in \bar{U} \mid h(x) = 0\}$, $dh \neq 0$, set $S = \bar{S} \cap U$,
- ii) on $\bar{\Gamma} = \{x \in \bar{U} \mid g(x) = 0\}$, $dg \neq 0$, set $\Gamma = \bar{\Gamma} \cap U$,
- iii) on $S \cap \Gamma$, $dh \wedge dg \neq 0$.

Let

$$D = \{x \in U \mid h(x) \leq 0\} \cap \{x \in U \mid g(x) < 0\}$$

be non-empty. Because of assumption (iii),

$$\bar{D} = \{x \in U \mid h(x) \leq 0\} \cap \{x \in U \mid g(x) \leq 0\}.$$

We will assume that

iv) \bar{D} is a compact set, and that every connected component of D meets S .

Let $\mathcal{J}_{(r,s)}^\infty(D)$ denote the subspace of $C_{(r,s)}^\infty(D)$ consisting of those forms of the type $\alpha h + \beta \wedge \bar{\partial}h$, with $\alpha \in C_{(r,s)}^\infty(D)$ and $\beta \in C_{(r,s-1)}^\infty(D)$. We set $\mathcal{J}_{0(r,s)}^\infty(D) = \mathcal{J}_{(r,s)}^\infty(D) \cap C_{0(r,s)}^\infty(D)$.

LEMMA 3. *Assumptions:*

$\alpha)$ *There exists a constant $c_1 > 0$ such that*

$$\widehat{\mathcal{L}}(h) \{u, u\} \geq c_1 |u|^2$$

on $S \cap D$ for all $u \in C_{(r,s)}^\infty(U)$ with $s > n - p$.

$\beta)$ $\mathcal{L}(g)$ is positive semidefinite.

Contention:

Let $\eta > 0$ be given, and set

$$D_\eta = \{h(x) \leq 0\} \cap \{g(x) + \eta < 0\}.$$

Then if $f \in \mathcal{J}_{0(r,s)}^\infty(D)$, $\bar{\partial}f = 0$, and $0 < s < p$, we can find $u \in \mathcal{J}_{(r,s-1)}^\infty(D_\eta)$ such that, on D_η , we have

$$\bar{\partial}u = f.$$

PROOF. Consider the graph of the function $y = |t|$ for $t \in \mathbb{R}$, and let $\varepsilon > 0$ be given. We can construct a convex function $\lambda_\varepsilon = \lambda_\varepsilon(t)$, which rounds the corner of the graph $y = |t|$ in the interval $-\varepsilon \leq t \leq \varepsilon$. We may as-

sume that

- i) $\lambda_\varepsilon \in C^\infty(\mathbb{R})$,
- ii) $\lambda_\varepsilon(t) = |t|$ if $|t| \geq \varepsilon$,
- iii) $\lambda_\varepsilon''(t) \geq 0$,
- iv) $0 \leq \lambda_\varepsilon'(t) \leq 1$ for $0 \leq t \leq \varepsilon$,
- v) $-1 \leq \lambda_\varepsilon'(t) \leq 0$ for $-\varepsilon \leq t \leq 0$.

Consider the function

$$k_\varepsilon = \frac{1}{2} \{h + g + \lambda_\varepsilon(h - g)\};$$

it is a C^∞ function on U . We define

$$\Omega_\varepsilon = \{x \in U \mid k_\varepsilon \leq 0\}, \quad \partial\Omega_\varepsilon = \{x \in U \mid k_\varepsilon = 0\}.$$

Then we have

$$(\alpha) \quad \Omega_\varepsilon \cap \{|h - g| \geq \varepsilon\} = \bar{D} \cap \{|h - g| \geq \varepsilon\}.$$

Indeed, $D = \{x \in U \mid \sup(h, g) \leq 0\}$, and, moreover,

$$k_\varepsilon \geq \sup(h, g) \quad \text{since} \quad \lambda_\varepsilon(h - g) \geq |h - g|,$$

with the equality sign holding if and only if $|h - g| \geq \varepsilon$.

We also have

$$(\beta) \quad \Omega_\varepsilon \subset \bar{D} \quad \text{since} \quad k_\varepsilon \geq \sup(h, g),$$

$$(\gamma) \quad dk_\varepsilon \neq 0 \quad \text{on} \quad \partial\Omega_\varepsilon \quad \text{if} \quad \varepsilon \text{ is sufficiently small.}$$

Indeed, if $k_\varepsilon = 0$, then $h \leq 0$ and $g \leq 0$. If $|h - g| \geq \varepsilon$, then we are on S (if $h - g \geq \varepsilon$) or Γ (if $h - g < -\varepsilon$); there k_ε equals h or g , respectively, and $dk_\varepsilon \neq 0$. If $|h - g| < \varepsilon$, then we are in the region $\{x \in \bar{D} \mid h < \varepsilon\} \cap \{x \in \bar{D} \mid g < \varepsilon\}$, which is a neighborhood N_ε of $S \cap \Gamma$. If ε is small, this neighborhood N_ε is small, because $S \cap \Gamma$ is compact⁽⁵⁾. Now

$$dk_\varepsilon = \frac{1}{2} (1 + \lambda_\varepsilon') dh + \frac{1}{2} (1 - \lambda_\varepsilon') dg,$$

⁽⁵⁾ Let W be an open neighborhood of $S \cap \Gamma$. Consider $\Phi: \bar{U} \rightarrow \mathbb{R}^2$ given by $\Phi(x) = (\sup(h(x), g(x)), \sup(h(x), g(x)))$. Then $\Phi(\bar{U} - W)$ is compact and does not contain the origin. Therefore, there exists an $\varepsilon > 0$ such that $\{h < \varepsilon\} \cap \{g < \varepsilon\} \subset W$.

and either $1 + \lambda'_\varepsilon \neq 0$ or $1 - \lambda'_\varepsilon \neq 0$. Then either $dk_\varepsilon \wedge dg \neq 0$ or $dk_\varepsilon \wedge dh \neq 0$ because, if ε is small, we have $dh \wedge dg \neq 0$ on N_ε .

Finally, we have

$$(\delta) \quad \mathcal{L}(k_\varepsilon) = \frac{1}{2} \{ (1 + \lambda'_\varepsilon) \mathcal{L}(h) + (1 - \lambda'_\varepsilon) \mathcal{L}(g) + \lambda''_\varepsilon (h - g) | \partial h - \partial g |^2 \}.$$

Since $\mathcal{L}(g) \geq 0$, we obtain therefore that

$$\widehat{\mathcal{L}}(k_\varepsilon) \{u, u\} \geq \frac{1}{2} (1 + \lambda'_\varepsilon) \widehat{\mathcal{L}}(h) \{u, u\} \geq \frac{c_1}{2} (1 + \lambda'_\varepsilon) |u|^2$$

if $u \in C_{(r,s)}^\infty(U)$ and $s > n - p$.

Let $\Sigma_\varepsilon = \Omega_\varepsilon \cap S$. We can then apply Proposition 9 to Ω_ε and Σ_ε , as the required assumptions are verified for ε sufficiently small. Moreover, for small ε , $f \in \mathcal{F}_{(r,s)}^\infty(\Omega_\varepsilon)$. We thus obtain a $u_\varepsilon \in \mathcal{F}_{(r,s-1)}^\infty(\overset{\circ}{\Omega}_\varepsilon \cup \Sigma_\varepsilon)$ such that

$$(*) \quad \bar{\partial} u_\varepsilon = f.$$

Now if ε is sufficiently small, we must have

$$D_\eta \subset \Omega_\varepsilon \quad \text{and} \quad \Sigma_\varepsilon \cap D_\eta = S \cap D_\eta.$$

Restricting equation (*) to D_η , we get a solution $u = u_\varepsilon \in \mathcal{F}_{(r,s-1)}^\infty(D_\eta)$.

b) Let $B \subset A$ be open sets in \mathbb{C}^n . We say that (A, B) is a « Runge pair in dimensions $\leq q$ » if the natural map

$$H_k^{r,s}(B) \rightarrow H_k^{r,s}(A)$$

is an injection for $s \leq q$.

LEMMA 4. *Assumptions :*

$$\left. \begin{array}{l} \alpha) \\ \beta) \end{array} \right\} \text{ as in Lemma 3;}$$

$\gamma)$ there exist $\varepsilon_0 > 0, \eta_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0, 0 < \eta \leq \eta_0$, if we set

$$B(\varepsilon, \eta) = \{h < \varepsilon\} \cap \left\{ -\frac{1}{2} \eta < g < 0 \right\},$$

$$A(\varepsilon) = \{h < \varepsilon\} \cap \{g < 0\},$$

then $(A(\varepsilon), B(\varepsilon, \eta))$ is a Runge pair in dimension $\leq p$.

Contention :

$$H^{r,s}(D, \mathcal{G}) = 0 \quad \text{if} \quad s < p.$$

PROOF. (α) We first consider the case $s = 0$. If $f \in H^{r_0}(D, \mathcal{G})$, then f is a holomorphic form whose coefficients vanish on $S \cap D$. Since we have assumed that every connected component of D meets S , we deduce that $f \equiv 0$. Therefore, the lemma is true for $s = 0$.

(β) Let $0 < s < p$ and $f \in \mathcal{F}_{(r,s)}^\infty(D)$ with $\bar{\partial}f = 0$ be given. According to Lemma 2.1 of [6], we can find a $v \in \mathcal{F}_{(r,s-1)}^\infty(D)$ such that all the coefficients of $f - \bar{\partial}v$ vanish on $S \cap D$ to infinite order. It is therefore not restrictive to assume that all the coefficients of f vanish to infinite order on $S \cap D$. We extend f to \tilde{f} , defined on $A(\varepsilon)$, by defining \tilde{f} to be zero outside of D .

Let χ be a C^∞ function on $A(\varepsilon)$ such that

$$\chi = \begin{cases} 1 & \text{on } \{g < -\eta/3\} \\ 0 & \text{on } \{g > -\eta/4\}. \end{cases}$$

Then $\chi\tilde{f} \in C_{0(r,s)}^\infty(A(\varepsilon))$, and $\bar{\partial}(\chi\tilde{f}) = \bar{\partial}\chi \wedge \tilde{f}$ has its support in the region

$$\{h \leq 0\} \cap \{-\eta/3 \leq g \leq -\eta/4\} \subset B(\varepsilon, \eta).$$

By assumption (γ), we can find a $\varphi \in C_{0(r,s)}^\infty(B(\varepsilon, \eta))$ such that

$$\bar{\partial}\chi \wedge \tilde{f} = \bar{\partial}\varphi.$$

We define

$$f_1 = \chi\tilde{f} - \varphi.$$

Then f_1 has the following properties :

- i) $\bar{\partial}f_1 = 0$;
- ii) the support of f_1 is compact in $A(\varepsilon)$ and contained in $D_{\eta/3} \cap B(\varepsilon, \eta)$;
- iii) $f_1 = f$ on $D_{\eta/2}$.

Let ϱ be a C^∞ function on \bar{U} with $1 \geq \varrho \geq 0$, and

$$\varrho = \begin{cases} 1 & \text{on } \{g \geq -\eta/2\} \\ 0 & \text{on } \{g \leq -\eta\}. \end{cases}$$

Set

$$D'(\varepsilon, \eta) = \{h \leq \varepsilon\varrho\} \cap \{g < 0\}.$$

Then

$$D'(\varepsilon, \eta) \supset D \cap B(\varepsilon, \eta),$$

$$D'(\varepsilon, \eta) \cap DD_\eta = D_\eta.$$

Let η be fixed. If ε is sufficiently small, the hypersurface

$$S(\varepsilon, \eta) = \{h = \varepsilon \varrho\}$$

is smooth and transversal to L . Then $f_1 \in \mathcal{J}_{0(r, s)}^\infty(D'(\varepsilon, \eta))$ and, for small ε , Lemma 3 can be applied to $D'(\varepsilon, \eta)$ and f_1 .

We conclude that, given $f \in \mathcal{J}_{(r, s)}^\infty(D)$ with $\bar{\partial}f = 0$, and given $\eta > 0$, we can find $u \in \mathcal{J}_{(r, s-1)}^\infty(D_\eta)$ such that

$$f = \bar{\partial}u \text{ on } D_\eta.$$

(γ) Now consider a sequence $\eta > \eta_0 > \eta_1 > \eta_2 > \dots > 0$ with $\eta_\nu \rightarrow 0$. If η is sufficiently small, each connected component of $D_{\eta'}$, with $0 < \eta' \leq \eta$, meets the hypersurface S . Indeed, $dg \neq 0$ on $\bar{D} - D_\eta$, for $\eta > 0$ small, g has no local minima in $D - D_\eta$. Therefore, for $0 < \eta' < \eta$, we have that each connected component of $D_{\eta'}$ meets and contains a connected component of D_η . The number of connected components of D_η can only decrease as η tends to zero. Thus it remains constant $= k$ when η is sufficiently small, and, if η is small, each of these components meets S ⁽⁶⁾.

According to (β), we can find a $\mu_\nu \in \mathcal{J}_{(r, s-1)}^\infty(D_{\eta_\nu})$ such that

$$f = \bar{\partial}\mu_\nu \text{ on } D_{\eta_\nu}.$$

Therefore,

$$\bar{\partial}(\mu_\nu - \mu_{\nu-1}) = 0 \text{ on } D_{\eta_{\nu-1}}.$$

If $s = 1$, then $\mu_2 = \mu_1$ on D_{η_1} , $\mu_3 = \mu_2$ on D_{η_2} , Therefore, by setting $\mu = \mu_\nu$ on D_{η_ν} , we define an element $\mu \in \mathcal{J}_{(r, s)}^\infty(D)$ with

$$\bar{\partial}\mu = f.$$

⁽⁶⁾ In fact, D is arcwise-locally connected. The same is true for \bar{D}_η if η is small. Thus there are finitely many components of \bar{D}_η , say A_1, \dots, A_k . Let D_1, D_2, \dots be the connected components of D . If $A_i \subset D_j$, we can find a path connecting a point of A_i with a point of S in D_j . If η is small, the component A_i will contain that path.

If $s > 1$, then, because of (β) , we can find a

$$\sigma_1 \in \mathcal{G}_{(r, s-2)}^\infty(D_{\eta_0})$$

such that

$$\mu_2 - \mu_1 = \bar{\partial} \sigma_1 \text{ on } D_{\eta_0}.$$

Let τ_1 be C^∞ on D with

$$\tau_1 = \begin{cases} 1 & \text{on } \{g \leq -\eta\} \\ 0 & \text{on } \left\{g \geq -\left(\eta_0 + \frac{\eta - \eta_0}{2}\right)\right\}. \end{cases}$$

Then $\tau_1 \sigma_1$ is compactly supported in D_{η_0} , and we have

$$\mu'_2 = \mu_2 - \bar{\partial}(\tau_1 \sigma_1) = \mu_1 \text{ on } D_\eta.$$

Thus we can replace μ_2 by μ'_2 . Again, we can find a

$$\sigma_2 \in \mathcal{G}_{(r, s-2)}^\infty(D_{\eta_1})$$

such that

$$\mu_3 - \mu'_2 = \bar{\partial} \sigma_2 \text{ on } D_{\eta_1}.$$

Let τ_2 be C^∞ on D with

$$\tau_2 = \begin{cases} 1 & \text{on } \{g \leq -\eta_0\} \\ 0 & \text{on } \left\{g \leq -\left(\eta + \frac{\eta_0 - \eta_1}{2}\right)\right\}. \end{cases}$$

Then $\tau_2 \sigma_2$ is compactly supported in D_{η_1} , and

$$\mu'_3 = \mu_3 - \bar{\partial}(\tau_2 \sigma_2) = \mu'_2 \text{ on } D_{\eta_0}.$$

Proceeding in this way, we may successively replace μ_ν by μ'_ν , and we will have

$$f = \partial \mu'_\nu \text{ on } D_{\eta_\nu},$$

$$\mu'_\nu = \mu'_{\nu-1} \text{ on } D_{\eta_{\nu-3}}.$$

Setting $\mu = \mu'_\nu$ on $D_{\eta_{\nu-2}}$, we get a well-defined element $\mu \in \mathcal{G}_{(r, s-1)}^\infty(D)$ for which

$$\bar{\partial} \mu = f.$$

This achieves the proof of the lemma.

15. *The vanishing theorem.* We formulate the vanishing theorem as follows :

THEOREM 2. *Let U be open in \mathbb{C}^n , let $h : U \rightarrow \mathbb{R}$ be a C^∞ function with $dh \neq 0$ on $S = \{x \in U \mid h(x) = 0\}$. Let $z_0 \in S$ be a point on S at which $\mathcal{L}(h)$, the Levi form of h , has at least p positive eigenvalues. Then there exists a fundamental sequence of open neighborhoods $\{\omega\}$ of z_0 which are domains of holomorphy, and such that, for $D = \{x \in \omega \mid h(x) \leq 0\}$, we have*

$$H^{r,s}(D, \mathcal{F}) = 0 \quad \text{for } s < p.$$

Note that a remark similar to the one made after Theorem 1 applies here also.

In the proof, we will make use of the following criterion for Runge pairs.

CRITERION : *Let X, Y be complex manifolds with $Y \subset X$. Suppose that, for every compact set $K \subset Y$, we can find a C^∞ function $\Phi : X \rightarrow \mathbb{R}$ with the following properties :*

- i) *for all $c \in \mathbb{R}$, the sets $\{x \in X \mid \Phi(x) \leq c\}$ are compact ;*
- ii) *at every point of X , $\mathcal{L}(\Phi)$ has at least p positive eigenvalues ;*
- iii) *$K \subset \{x \in X \mid \Phi(x) \leq \sup \Phi\} \subset Y$.*

Then for any locally-free sheaf \mathcal{F} , the natural map

$$H_k^s(Y, \mathcal{F}) \rightarrow H_k^s(X, \mathcal{F})$$

is injective if $s \leq p$.

For the proof of this criterion, we refer to [5] (Lemma 29, p. 122). The statement can also be derived via Serre-duality from Theorem 12 of [1] (p. 248).

PROOF OF THE THEOREM. It is enough to find a sequence $\{\omega\}$ of Stein neighborhoods of z_0 such that the corresponding D satisfy conditions (α) , (β) , and (γ) , as required in Lemma 3 and 4.

(α) By a suitable choice of local holomorphic coordinates at z_0 , we may assume that z_0 is the origin and

$$\begin{aligned}
 h(z) = & \frac{1}{2} (z_1 + \bar{z}_1) + a_{11} |z_1|^2 + 2 \operatorname{Re} \left(z_1 \sum_2^n a_{1j} \bar{z}_j \right) + \\
 & + \sum_2^p \lambda_j |z_j|^2 + \sum_{p+1}^n \mu_j |z_j|^2 + 0 (|z|^3)
 \end{aligned}$$

where $\lambda_j > 0$, $\mu_j \in \mathbb{R}$, $a_{11} \in \mathbb{R}$, and where the matrix

$$A = \begin{pmatrix} a_{12} & a_{12} & \dots & a_{1p} \\ a_{21} & \lambda_2 & \dots & 0 \\ \cdot & \cdot & & \\ a_{p1} & 0 & \dots & \lambda_p \end{pmatrix}$$

is hermitian and positive definite.

We set $z_j = x_j + iy_j$, $i \leq j \leq n$. Since $(\partial h / \partial x_1)_{z_0} = 1 \neq 0$, we may solve the equation $h(z) = 0$ with respect to x_1 in a neighborhood W of z_0 by

$$x_1 = \theta(y_1, z_2, \dots, z_n).$$

The Taylor expansion of θ at the origin begins with second order terms

$$\theta \equiv cy_1^2 + y_1 \cdot 2 \operatorname{Re} \left(\sum_2^n \alpha_j z_j \right) + 2 \operatorname{Re} \left(\sum_2^n s_{ij} z_i z_j \right) + \sum_2^n l_{ij} z_i \bar{z}_j + 0(|z|^3).$$

From $h(\theta, y_1, z_2, \dots, z_n) \equiv 0$, we deduce that $c + a_{11} = 0$, $\alpha_j - i \bar{a}_{1j} = 0$, $s_{ij} = 0$, and $l_{ij} + \delta_{ij} \lambda_j = 0$ or $l_{ij} + \delta_{ij} \mu_j = 0$. It follows that

$$-\theta = Q + \sum_{p+1}^n \mu_j |z_j|^2 + 0(|z|^3)$$

where

$$Q = a_{11} y_1^2 + 2 \operatorname{Re} \left\{ i y_1 \sum_2^n a_{1j} \bar{z}_j \right\} + \sum_2^p \lambda_j |z_j|^2$$

is positive and non-degenerate (indeed, we have $Q = (z A^t z)|_{x_1=0}$). In a neighborhood of the origin, we may replace h by $x_1 - \theta$; note that the region $\{h \leq 0\}$ corresponds to $\{x_1 \leq \theta\}$.

(β) Consider the last $n - p$ coordinates as parameters: $\zeta = (\zeta_{p+1}, \dots, \zeta_n) = (z_{p+1}, \dots, z_n)$. We let

$$\mathbb{C}^p(\zeta) = \{z_{p+1} = \zeta_{p+1}, \dots, z_n = \zeta_n\}$$

denote the p -dimensional space through the point $(0, \zeta)$ which is parallel to the coordinate space of z_1, \dots, z_p . On $\mathbb{G}^p(0)$, we have $-\theta_0 \equiv -\theta(y, z, 0) \equiv Q(y_1, x_2, z_2, \dots, x_p, y_p) + 0(|z|^3)$. Thus $-\theta_0$ has a non-degenerate critical point with positive definite Hessian at the origin. A C^∞ change of coordinates $y_1, x_2, y_2, \dots, x_p, y_p$ would give to $-\theta_0$ the form $-\theta_0 \equiv y_1^2 + \sum_2^p x_j^2 + \sum_2^p y_j^2$. This shows that we can find an $\varepsilon > 0$ and a $\delta > 0$ such that, for

any δ' with $0 < \delta' \leq \delta$, $V_0 = \{x_1 \leq \theta_0\} \cap \{x_1 \geq -\delta'\}$ is compact, connected, and contained in

$$\left\{ |y_1|^2 + \sum_2^p |z_j|^2 \leq \varepsilon \right\} \cap \{x_1 \geq -\delta\}.$$

Moreover, $dx_1 \wedge d\theta \neq 0$ on $\{x_1 = \theta_0\} \cap \{x_1 = -\delta'\}$.

Let W be an open bounded set in $\mathbb{C}^p(0)$ containing V_0 , and let $\sigma > 0$ be chosen so that, on the closure of $W \times \left\{ \sum_{p+1}^n |\zeta_j|^2 < \sigma \right\}$, θ is well-defined.

We claim that there exists an η with $\sigma > \eta > 0$ such that $\{\theta \geq x_1 \geq -\delta'\} \cap \mathbb{C}^p(\zeta)$ is compact in $W \times \mathbb{C}^p(\zeta)$ if

$$\sum_{p+1}^n |\zeta_j|^2 < \eta.$$

If this were not true, we could find a sequence $\zeta^r \rightarrow 0$ and, for each r , a point $z^r \in \partial W$ such that $\theta(z^r, \zeta^r) \geq x_1(z^r, \zeta^r) \geq -\delta'$. Since ∂W is compact, by selecting a subsequence we may assume that $z^r \rightarrow z \in \partial W$, so that, in the limit as $r \rightarrow \infty$, we get $\theta(z, 0) \geq x_1(z, 0) \geq -\delta'$ and $z \in \partial W$, which is absurd.

(γ) Consider the function

$$g' = -x_1 + K \sum_{p+1}^n |\zeta_j|^2.$$

In the (z, ζ) space, we clearly have $\mathcal{L}(g') \geq 0$ if $K > 0$. Let $m > 0$ be such that $\theta \leq m$ for all points in $W \times \left\{ \sum_{p+1}^n |\zeta_j|^2 < \eta \right\}$. Select

$$K \geq 2 \frac{\delta + m}{\eta}.$$

Then the region

$$\{x_1 \leq \theta\} \cap \{g' \leq \delta'\}, \quad \text{for } 0 < \delta' \leq \delta,$$

is compact because it is closed and contained in the compact set

$$\{\theta \geq x_1 \geq -\delta'\} \cap \left\{ \sum_{p+1}^n |\zeta_j|^2 \leq \frac{1}{2} \eta \right\}.$$

If η is sufficiently small, the piece of hypersurface

$$S_\delta \equiv \{x_1 = \theta\} \cap \{g' \leq \delta\}$$

is non singular, and $g' : S_\delta \rightarrow \mathbb{R}$ is C^∞ on S_δ . The set of critical values of g' is of measure zero. Therefore, we can select δ' with $0 < \delta' \leq \delta$ such that, setting

$$g = g' - \delta',$$

we have

$$dg \wedge d\theta \neq 0$$

on the set

$$\{x_1 = \theta\} \cap \{g = 0\}.$$

(δ) We take $U = W \times \left\{ \sum |\zeta_j|^2 < \frac{1}{2} \eta \right\}$, $h = x_1 - \theta$, $g = g' - \delta'$,

$$D = \{x \in U \mid h(x) \leq 0\} \cap \{x \in U \mid g(x) < 0\}.$$

It follows from (γ) that \bar{D} is compact. Moreover each connected component Δ of D meets $S = \{h(x) = 0\}$. For otherwise we would have $h(x) < 0$ on Δ and Δ would be open with a compact closure $\bar{\Delta}$. Then, since $g(x)$ tends to zero at each point of $\bar{\Delta} - \Delta$, g must have a minimum at a point $x_0 \in \Delta$; there $dg = 0$. But this is impossible because $\partial g / \partial x_1 = -1$.

Thus D satisfies the conditions specified at the beginning of n. 14 a). Moreover, by a linear change of coordinates we may assume (cf. Proposition 1) that assumption α) of Lemma 3 holds. We have also seen that g satisfies assumption β) of Lemma 3.

(ϵ) It remains to show that assumption γ) of Lemma 4 also holds. For this we will apply the criterion for Runge pairs given at the beginning. Let

$$B(\mu, \sigma) = \{h < \mu\} \cap \left\{ -\frac{1}{2} \sigma < g < 0 \right\}$$

with $0 < \mu \leq \mu_0$, $0 < \sigma \leq \sigma_0$, and μ_0 and σ_0 sufficiently small so that $B(\mu_0, \sigma_0) \subset U$.

Consider the three functions

$$\varphi_1 = h - \mu$$

$$\varphi_2 = g + \frac{1}{N} \sum_1^p |z_j|^2$$

$$\varphi_3 = -g + \frac{1}{N} \sum_1^p |z_j|^2 - \frac{1}{2} \sigma.$$

Then on $\mathbb{C}^p(\zeta)$, $\mathcal{L}(\varphi_1)$, $\mathcal{L}(\varphi_2)$, $\mathcal{L}(\varphi_3)$ are all positive definite. Set

$$B_N = \{\sup(\varphi_1, \varphi_2, \varphi_3) < 0\}.$$

We have

$$B_N \subset B_{N+1} \text{ for all } N \in \mathbb{N}$$

$$B_N \subset B(\mu, \sigma)$$

$$\bigcup_N B_N = B(\mu, \sigma).$$

Let K be compact, $K \subset B(\mu, \sigma)$. Then for some large N_0 ,

$$K \subset B_{N_0}.$$

Set

$$a_i = \inf_{B_{N_0}} \varphi_i, \quad \Phi_i = \frac{\varphi_i + |a_i|}{|a_i|} \geq 0$$

so that

$$B_{N_0} = \{\sup(\Phi_1, \Phi_2, \Phi_3) < 1\}.$$

We define

$$A_\nu = \left\{ \Phi_1^\nu + \Phi_2^\nu + \Phi_3^\nu < 1 - \frac{1}{\nu} \right\}, \quad \nu \in \mathbb{N}.$$

Then we have

$$A_\nu \subset A_{\nu+1}, \quad \bigcup_\nu A_\nu = B_{N_0}$$

and, moreover $\mathcal{L}(\Phi_1^\nu + \Phi_2^\nu + \Phi_3^\nu) < 0$ on $\mathbb{C}^p(\zeta)$. Hence we can find a ν_0 such that $K \subset A_{\nu_0}$.

Let

$$\eta_1 = \sup_{A_{\nu_0}} h, \quad \delta_1 = \sup_{A_{\nu_0}} g.$$

Select $\mu(t)$ and $\sigma(t)$ to be increasing convex C^∞ functions such that

$$\begin{cases} \mu(t) = 0 & \text{if } t \leq \eta_1 \\ \lim_{t \rightarrow \mu} \mu(t) = +\infty \end{cases} \quad \begin{cases} \sigma(t) = 0 & \text{if } t \leq \delta_1 \\ \lim_{t \rightarrow 0} \sigma(t) = +\infty. \end{cases}$$

Note that $\bar{A}_{\nu_0} \subset B(\mu, \sigma)$, so that $\eta_1 < \mu$ and $\delta < 0$. Consider the function.

$$\psi_\nu = \mu(h) + \sigma(g) + \Phi_1^\nu + \Phi_2^\nu + \Phi_3^\nu + \frac{1}{\nu} + \frac{1}{\nu} \sum_1^p |z_j|^2.$$

We have

$$\{\psi_\nu < \text{const}\} \subset A(\mu) = \{h < \mu\} \cap \{g < 0\}.$$

Also $\mathcal{L}(\psi_\nu) > 0$ on $\mathbb{C}^p(\zeta)$. Finally, since $\mu(h) = \sigma(g) = 0$ on K , we get $K \subset \{\psi_\nu < 1\}$ if $\nu > \nu_0$ is sufficiently large. But $\{\psi_\nu < 1\} \subset A_\nu \subset B(\mu, \sigma)$. Therefore, the criterion is applicable and we conclude that $(A(\mu), B(\mu, \sigma))$ is a Runge pair for all dimensions $\leq p$.

This proves that condition $\gamma)$ of Lemma 4 is satisfied.

Note that U can be taken to be an arbitrarily small domain of holomorphy, by taking δ sufficiently small. Then

$$\omega = \{x \in U \mid g(x) < 0\}$$

is also a domain of holomorphy, since $\mathcal{L}(g) \geq 0$. We set $D = \omega \cap \{h \leq 0\}$, and the proof of the theorem is complete.

16. Now we combine Theorems 1 and 2 with Theorem 7 of [6]. We obtain the following conclusion in which the eigenvalues of the Levi form are computed on the holomorphic tangent space.

Let U be an open set in \mathbb{C}^n and $h: U \rightarrow \mathbb{R}$ can be a C^∞ function such that, on $S = \{x \in U \mid h(x) = 0\}$, $dh \neq 0$. We set

$$U^+ = \{x \in U \mid h(x) \geq 0\}, \quad U^- = \{x \in U \mid h(x) \leq 0\}.$$

Let $z_0 \in S$ and consider

$$\mathcal{L}(h)|_{HT(z_0)} = \begin{cases} \sum_{j, k=1}^n \frac{\partial^2 h(z_0)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \\ \sum_{j=1}^n \frac{\partial h(z_0)}{\partial z_j} w_j = 0. \end{cases}$$

We assume that $\mathcal{L}(h)|_{HT(z_0)}$ has p positive and q negative eigenvalues, $p + q \leq n - 1$; then

THEOREM 3. *There exists a fundamental sequence of neighborhoods $\{\omega_\nu\}_{\nu \in \mathbb{N}}$ of z_0 in \mathbb{C}^n such that each ω_ν is a domain of holomorphy and moreover, setting*

$$\omega_\nu^- = \omega_\nu \cap U^-, \quad \omega_\nu^+ = \omega_\nu \cap U^+$$

we have

$$\left\{ \begin{array}{l} H^{r,s}(\omega_\nu^+) = 0 \text{ for} \\ \nu \text{ even} \end{array} \right\} \begin{cases} s > n - q - 1 \\ \text{or} \\ 0 < s < p \end{cases}, \quad \left\{ \begin{array}{l} H^{r,s}(\omega_\nu^-) = 0 \text{ for} \\ \nu \text{ odd} \end{array} \right\} \begin{cases} s > n - p - 1 \\ \text{or} \\ 0 < s < q \end{cases}$$

PROOF. We may replace h by $e^{ch} - 1$, with $c > 0$ sufficiently large, so that $\mathcal{L}(h)(z_0)$ has $p + 1$ positive eigenvalues at z_0 . Then we select the sequence ω_ν so that each ω_ν is a domain of holomorphy of the special type described in Theorem 2.

By Theorem 1 we have:

$$H^{r,s}(\omega_\nu^-) = 0 \text{ for } s > n - (p + 1) = n - p - 1.$$

From Theorem 2 we also have:

$$H^{r,s}(\omega_\nu^-, \mathcal{J}) = 0 \text{ for } s < p + 1.$$

Now by Theorem 7 of [6] we have the isomorphism

$$H^{r,s+1}(\omega_\nu^-, \mathcal{J}) \cong H^{r,s}(\omega_\nu^+) \text{ for } s \geq 1.$$

Hence we obtain:

$$H^{r,s}(\omega_\nu^+) = 0 \text{ for } 0 < s < p.$$

Changing the sign of h permutes ω_ν^+ with ω_ν^- , and p with q . In particular we have

$$H^{r,s}(\omega_\nu^+) = 0 \text{ for } s > n - q - 1.$$

From this the first half of the statement now follows, taking a sequence ω_ν where ν runs through the even numbers.

The second half of the statement is proved in the same way.

REMARKS: 1. For all $\nu \in \mathbb{N}$ we have $H^{r,s}(\omega_\nu^-) = 0$ if $s > n - q - 1$ and $H^{r,s}(\omega_\nu^-) = 0$ if $s > n - p - 1$. We do not know that we can find a sequence ω_ν such that, for all ν , $H^{r,s}(\omega_\nu^+) = 0$ if $0 < s < p$ and $H^{r,s}(\omega_\nu^-) = 0$ if $0 < s < q$.

2. For $s = 0$, we have

$$H^{r,0}(\omega_\nu) \xrightarrow{\sim} H^{r,0}(\omega_\nu^+) \text{ for } \nu \text{ even if } p > 0$$

$$H^{r,0}(\omega_\nu) \xrightarrow{\sim} H^{r,0}(\omega_\nu^-) \text{ for } \nu \text{ odd if } q > 0.$$

This is an immediate consequence of Proposition 4.3 of [6].

3. When the Levi form of S at z_0 is nondegenerate (i. e., when $q = n - p - 1$), then

$$H^{r,s}(\omega_v^+) = 0 \text{ for } s \neq 0, p \text{ (} v \text{ even)}$$

$$H^{r,s}(\omega_v^-) = 0 \text{ for } s \neq 0, q \text{ (} v \text{ odd).}$$

§ 5. Concluding remarks.

17. Construction of non-vanishing cohomology classes on S . a) We consider in $\mathbb{C}^n - \{0\}$ the differential forms

$$\psi_\alpha = \frac{\sum_j (-1)^j \bar{z}_j^{\alpha_j} d\bar{z}_1^{\alpha_1} \wedge \dots \wedge \widehat{d\bar{z}_j^{\alpha_j}} \wedge \dots \wedge d\bar{z}_n^{\alpha_n}}{(\sum_j z_j^{\alpha_j} \bar{z}_j^{\alpha_j})^n}$$

$$\omega = dz_1 \wedge \dots \wedge dz_n$$

where $(\alpha_1, \dots, \alpha_n) \in \mathbb{H}^n$. We set $\alpha + 1 = (\alpha_1 + 1, \dots, \alpha_n + 1)$.

LEMMA 5. Let U be an open set in $\mathbb{C}^n - \{0\}$ ($n \geq 2$) containing a closed half sphere

$$\Sigma = \{z \in \mathbb{C}^n \mid \sum_j |z_j|^2 = \epsilon, \text{ Re } z_1 \geq 0\}.$$

Then the forms $\psi_{\alpha+1}$ for $\alpha \in \mathbb{H}^n$ represent linearly independent classes of $H^{n-1}(U, \mathbb{C})$ over \mathbb{C} .

PROOF. Let $\eta = \sum c_\alpha \psi_{\alpha+1}$ be a finite linear combination of those forms with $c_\alpha \in \mathbb{C}$, and suppose that there exists a differential form μ of type $(0, n - 2)$ such that $\eta = \bar{\partial}\mu$ on U .

Let $\sigma > 0$ be so small that the part $S_\sigma = \{\sum |z_j|^2 = \epsilon, \text{ Re } z_1 > -\sigma\}$ of the sphere $S = \{\sum |z_j|^2 = \epsilon\}$ is contained in U . For any function f holomorphic in \mathbb{C}^n we have

$$\int_S f \omega \wedge \eta = \frac{(2\pi i)^n}{(n-1)!} \sum_\alpha c_\alpha \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0).$$

Now

$$\int_{S_\sigma} f \omega \wedge \eta = \int_{S_\sigma} f \omega \wedge \bar{\partial} \mu = \int_{S_\sigma} d(f \omega \wedge \mu) = \int_{\partial S_\sigma} f \omega \wedge \mu,$$

therefore

$$\frac{(2\pi i)^n}{(n-1)!} \sum_\alpha c_\alpha \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) = \int_{\partial S_\sigma} f \omega \wedge \mu + \int_{S-S_\sigma} f \omega \wedge \eta.$$

For $\delta > 0$ the domain of integration of the right hand integrals is compact and contained in $D \equiv \{\sum |z_j|^2 < \varepsilon + \delta\} \cap \left\{ \operatorname{Re} z_1 < -\frac{\sigma}{2} \right\}$. If B is the ball, centered at the origin of radius $\frac{\sigma}{4}$, then B and D are Runge domains in \mathbb{C}^n and they are separated by $\operatorname{Re} \left(z_1 + \frac{\sigma}{2} \right)$. Hence $B \cup D$ is a Runge open set in \mathbb{C}^n . Let g be any holomorphic function in B and let K be a compact subset of B containing the origin in the interior. We can find a sequence $\{f_\nu\}$ of holomorphic functions in \mathbb{C}^n such that $f_\nu \rightarrow g$ uniformly on K and $f_\nu \rightarrow 0$ uniformly on $\partial S_\sigma \cup (S - S_\sigma)$. This gives, for any g , the relation

$$\frac{(2\pi i)}{(n-1)!} \sum_{\alpha} c_{\alpha} \frac{\partial^{|\alpha|} g}{\partial z^{\alpha}}(0) = 0.$$

This relation is only possible if all c_{α} 's are zero.

b) We resume the notations of n. 16. We have on an open set U in \mathbb{C}^n ($n > 1$) a C^{∞} function $h: U \rightarrow \mathbb{R}$ such that on

$$S = \{x \in U \mid h(x) = 0\}, \quad dh \neq 0.$$

Let $z_0 \in S$ and consider the Levi form of h restricted to the holomorphic tangent space to S at z_0 :

$$\mathcal{L}(h)|_{HT(z_0)} = \begin{cases} \sum_{j,k=1}^n \frac{\partial^2 h(z_0)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \\ \sum_{j=1}^n \frac{\partial h(z_0)}{\partial z_j} w_j = 0. \end{cases}$$

We assume it to be non-degenerate with p positive and $q = n - 1 - p$ negative eigenvalues. Then we have the following

THEOREM 4. *Under the above assumptions, there exists a neighborhood W of z_0 in U such that for any domain of holomorphy $\omega \ni z_0$ with $\omega \subset W$, the groups*

$$H^{r,p}(\omega^+), H^{r,q}(\omega^-), H^{r,p}(S \cap \omega), H^{r,q}(S \cap \omega)$$

are all infinite dimensional over \mathbb{C}

PROOF. (a) We first assume that $p = n - 1, q = 0$. It is enough to prove the statement of the theorem for the case $r = 0$. Replacing h by $e^{ch} - 1$ with $c > 0$ sufficiently large, we may assume that $\mathcal{L}(h)$ at z_0 is

positive definite. In a convenient neighborhood W of z_0 we can introduce holomorphic coordinates (with the origin at z_0) such that

$$h(z) = \operatorname{Re} z_1 + \sum a_{ij} z_i \bar{z}_j + 0(|z|^3),$$

with $\sum a_{ij} z_i \bar{z}_j$ positive definite. Now the hypersurface S is strongly convex in the elementary sense. For $a = (a_1, \dots, a_n)$, set $B(a, \epsilon) = \{z \in \mathbb{C}^n \mid \sum z_j - a_j \mid^2 < \epsilon\}$. If ϵ is sufficiently small, say $\epsilon < \epsilon_0$ with $\epsilon_0 > 0$, then

$$\partial B(0, \epsilon) \cap U^+ \text{ contains a closed half sphere.}$$

Then we can find a $\delta(\epsilon) > 0$ such that if $|a| < \delta(\epsilon)$, then

$$\partial B(a, \epsilon) \cap U^+ \text{ also contains a closed half sphere.}$$

Consider the coordinates $\zeta_1 = z_1 - a_1, \dots, \zeta_n = z_n - a_n$, and the forms $\psi_{\alpha+1}(\zeta)$. These forms are regular in $W \cap U^+ \cap W$ if $a \in \overset{\circ}{U}^-$; moreover, by the above lemma, they define linearly independent cohomology classes in $H^{n-1}(B(a, \epsilon) \cap U^+, \mathcal{O})$ provided $\epsilon < \epsilon_0$, $a \in \overset{\circ}{U}^-$, and $|a| < \delta(\epsilon)$.

Given a domain of holomorphy ω with $\omega \ni z_0$, we can find ϵ and a as above such that $B(a, \epsilon) \subset \omega$. Consider the commutative diagram

$$\begin{array}{ccc} H^{n-1}(W^+, \mathcal{O}) & \xrightarrow{\alpha} & H^{n-1}(B(a, \epsilon) \cap U^+, \mathcal{O}) \\ & \searrow \beta & \nearrow \gamma \\ & & H^{n-1}(\omega \cap U^+, \mathcal{O}) \end{array}$$

Since the image of α is infinite dimensional, then $\dim_{\mathbb{C}} H^{n-1}(\omega \cap U^+, \mathcal{O})$ must be infinite.

This achieves the proof since the rest of the contention is trivial. The statement for S follows from the Mayer-Vietoris sequence (cf. Theorem 1 of [6]).

(b) In the general case, we may take coordinates in a neighborhood W of z_0 such that

$$h(z) = \operatorname{Re} z_1 + a_{11} |z_1|^2 + 2 \operatorname{Re} \left(z_1 \sum_{j=2}^n a_j \bar{z}_j \right) + \\ + \sum_2^{p+1} \lambda_j |z_j|^2 - \sum_{p+2}^n \mu_j |z_j|^2 + 0(|z|^3)$$

with $\lambda_j > 0$, $\mu_j < 0$ and

$$a_{11} |z_1|^2 + 2 \operatorname{Re} \left(z_1 \sum_2^{p+1} a_j z_j \right) + \sum_2^{p+1} \lambda_j |z_j|^2 > 0$$

if $(z_1, \dots, z_{p+1}) \neq (0, \dots, 0)$.

By restricting W if necessary we may assume that

$$W \cap \{z_1 = \dots = z_{p+1} = 0\} \cap U^+ = \{z_0\}.$$

Also, we can find a sequence

$$a^{(k)} = (a_1^{(k)}, \dots, a_p^{(k)}, 0, \dots, 0), \quad k = 1, 2, \dots,$$

such that

i) $a^{(k)} \in \overset{\circ}{U}^-$

ii) $\lim_{k \rightarrow \infty} a^{(k)} = z_0$

iii) $W \cap \{z_1 = a_1^{(k)}, \dots, z_{p+1} = a_{p+1}^{(k)}\} \cap U^+ = \emptyset$.

Set $B(a, \epsilon) = \{z \in \mathbb{C}^n \mid \sum |z_j - a_j|^2 < \epsilon\}$, where $a = (a_1, \dots, a_n)$. For $\epsilon < \epsilon_0$, with $\epsilon_0 > 0$ conveniently chosen, we will have that $\partial B(0, \epsilon) \cap U^+ \cap \{z_{p+2} = \dots = z_n = 0\}$ must contain a closed half sphere. Also we can find a $\delta(\epsilon) > 0$ such that if $|a^{(k)}| < \delta(\epsilon)$, then $\partial B(a^{(k)}, \epsilon) \cap U^+ \cap \{z_{p+2} = \dots = z_n = 0\}$ contains a closed half sphere.

Introduce the coordinates

$$\zeta_1 = z_1 - a_1^{(k)}, \dots, \zeta_{p+1} = z_{p+1} - a_{p+1}^{(k)}, \zeta_{p+2} = z_{p+2}, \dots, \zeta_n = z_n.$$

We consider the forms ψ_{a+1} , in the coordinates $\zeta_1, \dots, \zeta_{p+1}$, as forms in W . Because of iii) the forms ψ_{a+1} are regular on $W^+ = W \cap U^+$. By Lemma 5 they define linearly independent elements in $H^p(W^+, \bar{O})$, since their restriction to $H^p(W^+ \cap \{z_{p+1} = \dots = z_n = 0\}, \bar{O})$ are such. The same is true for $H^p(B(a^{(k)}, \epsilon) \cap U^+, \bar{O})$, for $\epsilon < \epsilon_0$, and $|a^{(k)}| < \delta(\epsilon)$.

Arguing as before, we obtain the desired conclusion; the proof of the theorem is complete.

c) *Nonvalidity of the Poincaré lemma for $\bar{\partial}_s$.* Let us consider an integer p such that $H^{r,p}(\mathcal{S} \cap \Omega_\nu) \neq 0$ for a fundamental sequence of neighborhoods Ω_ν of a point $z_0 \in \mathcal{S}$, $\nu = 1, 2, 3, \dots$.

Consider the following diagram

$$\begin{array}{ccc}
 & \mathbf{0} & \mathbf{0} \\
 & \uparrow & \uparrow \\
 H^{r,p}(\Omega \cap S) & \xrightarrow{r_\nu} & H^{r,p}(\Omega_\nu \cap S) \\
 \uparrow j & & \uparrow j_\nu \\
 Z^{r,p}(\Omega \cap S) & \xrightarrow{r_\nu} & Z^{r,p}(\Omega_\nu \cap S) \\
 \uparrow i & & \uparrow i_\nu \\
 B^{r,p}(\Omega \cap S) & \xrightarrow{r_\nu} & B^{r,p}(\Omega_\nu \cap S) \\
 \uparrow & & \uparrow \\
 \mathbf{0} & & \mathbf{0}
 \end{array}$$

where $\Omega = \Omega_1$ and

$$Z^{r,p} = \ker \{\bar{\partial}_s: Q_{(r,p)}(\Omega_\nu \cap S) \rightarrow Q_{(r,p+1)}(\Omega_\nu \cap S)\},$$

$$B^{r,p} = \text{im} \{\bar{\partial}_s: Q_{(r,p-1)}(\Omega_\nu \cap S) \rightarrow Q_{(r,p)}(\Omega_\nu \cap S)\}.$$

We remark first that

$$Q_{(r,p)}(\Omega_\nu \cap S) = [C^\infty(\Omega_\nu \cap S)]^{\alpha(r,p)}$$

where $\alpha(r,p) = \binom{n}{r} \binom{n-1}{p}$. When $r = 0$, for example, we have $Q_{(0,p)}(\Omega_\nu \cap S) = \sum_{\alpha_1 < \dots < \alpha_p} a_{\alpha_1 \dots \alpha_p} \omega_{\alpha_1} \wedge \dots \wedge \omega_{\alpha_p}$ with $a_{\alpha_1 \dots \alpha_p} \in C^\infty(\Omega_\nu \cap S)$, and

where $\omega_1, \dots, \omega_{n-1}, \bar{\partial}Q$ is a basis for $C^\infty(0,1)$ -forms along S . Thus $Q_{(r,p)}(\Omega_\nu \cap S)$ is a Fréchet-Schwartz space. Moreover, the maps $i, j, i_\nu, j_\nu, r_\nu$ are continuous. Also, $Z^{r,p}(\Omega_\nu \cap S)$ is a closed subspace of $Q_{(r,p)}(\Omega_\nu \cap S)$ and therefore also a Fréchet-Schwartz space. The image $B^{r,p}(\Omega \cap S)$, being the image of a Fréchet space, is of the first category, as $B^{r,p} \neq Z^{r,p}$ by assumption. The sets $r_\nu^{-1} i_\nu(B^{r,p}(\Omega_\nu \cap S)) \subset Z^{r,p}(\Omega \cap S)$ have the following property :

$$r_\nu^{-1} i_\nu(B^{r,p}(\Omega_\nu \cap S)) = i r_\nu^{-1}(B^{r,p}(\Omega_\nu \cap S)).$$

Therefore,

α) either they coincide with the whole space $Z^{r,p}(\Omega \cap S)$,
 β) or else they are of the first category,
 because they are the continuous images of the spaces $r_\nu^{-1}(B^{r,p}(\Omega_\nu \cap S))$,
 endowed with the quotient topology of $Q_{(r,p-1)}(\Omega_\nu \cap S)$.

Now if the Levi form of z_0 , restricted to the holomorphic tangent space to S at z_0 , is nondegenerate and has either p positive or p negative eigenvalues, we have constructed elements $\varphi_\nu \in Z^{r,p}(\Omega \cap S)$ such that

$$r_\nu(\varphi_\nu) \not\subset i_\nu(B^{r,p}(\Omega_\nu \cap S)).$$

This rules out possibility (α) . Therefore, under the conditions specified above, $r_\nu^{-1}i_\nu(B^{r,p}(\Omega_\nu \cap S))$ is of the first category. Hence $\bigcup_{\nu=1}^\infty r_\nu^{-1}i_\nu(B^{r,p}(\Omega_\nu \cap S))$ is of the first category. This means that there exists a $g \in Z^{r,p}(\Omega \cap S)$ such that, for every ν , the equation

$$\bar{\partial}_s u_\nu = g$$

cannot be solved in every Ω_ν , although the integrability condition

$$\bar{\partial}_s g = 0 \quad \text{on} \quad \Omega \cap S$$

is satisfied.

CONCLUSION. Let S be a hypersurface in \mathbb{C}^n such that, at $z_0 \in S$, the Levi form of S is nondegenerate with p positive (and q negative) eigenvalues (along the holomorphic tangent space to S at z_0). Then for the complex of sheaves:

$$\mathbb{Q}^{r,0}(S) \rightarrow \mathbb{Q}^{r,1}(S) \rightarrow \dots \rightarrow \mathbb{Q}^{r,n-1}(S) \rightarrow 0,$$

the Poincaré lemma is not valid in dimensions p and q , but it holds in any other dimension.

Note that the nonsolvability in the example of H. Lewy, as discussed in [6], is included as a special case of the above result.

18. *Global theorems of finiteness.* Let X be a complex manifold of complex dimension n , and let $h: X \rightarrow \mathbb{R}$ be a C^∞ function on X . We set

$$U^- = \{x \in X \mid h(x) \leq 0\}$$

and we assume that:

- i) U^- is a compact set
- ii) on $S = \{x \in X \mid h(x) = 0\}$, we have $dh \neq 0$, so that $U^- - \overset{\circ}{U}^- = S = \partial U^-$

iii) at each point $z_0 \in S$ the Levi form $\mathcal{L}(h)|_{HT(z_0)}$ (the Levi form of h restricted to the holomorphic tangent space to S at z_0) has at least p positive and at least q negative eigenvalues ($p + q \leq n - 1$).

THEOREM 5. *Under the above assumptions we have*

$$H^{r,s}(U^-) \cong H^s(\overset{\circ}{U}^-, \Omega^r)$$

for $s < q$ and $s > n - p - 1$.

PROOF Using the customary argument via the bump lemma (cf. [1], p. 237) one establishes the existence of an $\epsilon > 0$ such that, if we set

$$U_\epsilon = \{x \in U \mid h(x) < \epsilon\},$$

we have that in the range $s < q$ and $s > n - p - 1$, the restriction maps

$$H^s(U_\epsilon, \Omega^r) \rightarrow H^s(\overset{\circ}{U}^-, \Omega^r)$$

$$H^s(U_\epsilon, \Omega^r) \rightarrow H^{r,s}(U^-)$$

are surjective. The first of these facts is proved in [1] ([1], Propositions 16 and 17). The second is proved in the same way, because at each boundary point we have the local vanishing theorem given by Theorem 3, or the extension theorems of Remark 2 to Theorem 3.

Moreover from [1] we also have that

$$H^s(U_\epsilon, \Omega^r) \rightarrow H^s(\overset{\circ}{U}^-, \Omega^r)$$

is an isomorphism ([1], Propositions 21 and 22). But this restriction map factors as follows:

$$\begin{array}{ccc} H^s(U_\epsilon, \Omega^r) & \xrightarrow{\alpha} & H^s(\overset{\circ}{U}^-, \Omega^r) \\ \searrow \beta & & \nearrow \gamma \\ & & H^{r,s}(U^-). \end{array}$$

Since β is surjective and α is an isomorphism, γ must be injective. But γ is also surjective. This proves our contention.

COROLLARY. *Under the same assumptions, and in the same ranges for s , we have*

$$\dim_{\mathbb{C}} H^{r,s}(U^-) < \infty.$$

Indeed the groups $H^s(\mathring{U}^-, \Omega^r)$ are finite dimensional ([1], Theorem 11). In a similar way, using cohomology with compact supports, one can prove the following

THEOREM 6. *Under the above assumptions we have*

$$H^{r,s}(U^-, \mathcal{J}) \cong H_k^s(\mathring{U}^-, \Omega^r)$$

for $s < p + 1$ or $s > n - q$. In particular, for s in this range, these cohomology groups are finite dimensional.

REMARK. Of course we also have vanishing theorems for $H^{r,s}(U^-)$ or $H^{r,s}(U^-, \mathcal{J})$ whenever the corresponding cohomology groups (according to Theorem 5 and 6) of \mathring{U}^- vanish. For instance: if h is such that $\mathcal{L}(h)$ has at least $p + 1$ positive eigenvalues in all of \mathring{U}^- , then we have

$$H^{r,s}(U^-) = 0 \quad \text{for } s > n - p - 1$$

$$H^{r,s}(U^-, \mathcal{J}) = 0 \quad \text{for } s < p + 1.$$

Consider in particular the case where, at every point $z_0 \in \mathcal{S}$, the Levi form $\mathcal{L}(h)|_{HT(z_0)}$ is non-degenerate with a fixed signature, say, p positive eigenvalues and $q = n - p - 1$ negative eigenvalues.

Then we get that $H^{r,s}(U^-)$ is finite dimensional for $s \neq q$. Moreover we have

THEOREM 7. *Under the assumptions specified above,*

$$\dim_{\mathbb{C}} H^{r,q}(U^-) = \infty.$$

PROOF. We can find an $\epsilon > 0$ sufficiently small for which the restriction map

$$\beta: H^q(U_\epsilon, \Omega^r) \rightarrow H^q(\mathring{U}^-, \Omega^r)$$

has a dense image (cf. [1], Proposition 19 and [3], Proposition 7). Moreover by [2], Theorem 2 we have that $\dim_{\mathbb{C}} H^q(\mathring{U}^-, \Omega^r) = \infty$ and that $H^q(\mathring{U}^-, \Omega^r)$ is a separated (Hausdorff) topological space ([4], Corollary 37 and the remark on page 82). Now β factors as follows:

$$\begin{array}{ccc} H^q(U_\epsilon, \Omega^r) & \xrightarrow{\beta} & H^q(\mathring{U}^-, \Omega^r) \\ \searrow \alpha & & \nearrow \gamma \\ & & H^{r,q}(U^-). \end{array}$$

Thus the image of γ is dense in $H^q(\overset{\circ}{U}^-, \Omega^r)$. If $\dim_{\mathbb{C}} H^{r,q}(U^-)$ were finite, then $\text{Im } \gamma$ would be a finite dimensional subspace of a separated topological vector space. Therefore $\text{Im } \gamma$ would be closed. But then it could not be dense because $\dim_{\mathbb{C}} H^q(\overset{\circ}{U}^-, \Omega^r) = \infty$. This shows that we must have $\dim_{\mathbb{C}} H^{r,q}(U^-) = \infty$.

19. *Global Cauchy and Riemann-Hilbert problems.* Let X be a complex manifold of complex dimension n . Let $h: X \rightarrow \mathbb{R}$ be a C^∞ function on X such that $dh \neq 0$ on $S = \{x \in X \mid h(x) = 0\}$. We set

$$U^- = \{x \in X \mid h(x) \leq 0\} \quad \text{and} \quad U^+ = \{x \in X \mid h(x) \geq 0\}.$$

We shall say that the

Riemann-Hilbert problem is *almost always solvable* in dimension q , if the natural map

$$\beta_q: H^q(U^+) \oplus H^q(U^-) \rightarrow H^q(S)$$

has finite dimensional kernels and cokernels.

We shall say that the

Cauchy problem is *almost always solvable* in dimension q from the side U^- , if the natural map

$$\beta_q: H^q(U^-) \rightarrow H^q(S)$$

has finite dimensional kernels and cokernels (analogous definition for U^+).

Assume now, for instance, that X is compact and that at every point $z_0 \in S$ the Levi form $\mathcal{L}(h)|_{HT}(z_0)$ is nondegenerate, with p positive and $q = n - p - 1$ negative eigenvalues. Then we have the following situation:

$H^s(U^-)$ is finite dimensional except for $s = q$, where it is infinite dimensional.

$H^s(U^+)$ is finite dimensional except for $s = p$, where it is infinite dimensional.

Using the Mayer-Vietoris sequence for $X = U^- \cup U^+$ ([6], Theorem 1) and the standard finiteness theorems for a compact X , we obtain further that:

If $p \neq q$, then the Cauchy problem is of interest in dimension q from the side U^- , and in dimension p from the side U^+ ; in these dimensions it is almost always solvable.

If $p = q = \frac{n-1}{2}$ (n must be odd), then the Riemann-Hilbert problem is of interest, and it is almost always solvable in dimension $p = q = \frac{n-1}{2}$.

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