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A PSEUDOCONVEX - PSEUDOCONCAVE GENERALIZATION OF GRAUERT'S DIRECT IMAGE THEOREM

YUM-TONG SIU

In this paper we prove the following pseudoconvex-pseudoconcave generalization of Grauert's direct image theorem.

MAIN THEOREM. *Suppose* X *is a (not necessarily reduced) complex space,* S *is a nonsingular (or perfect) complex space of pure dimension* $n \geq 1$, *and* $\pi : X \rightarrow S$ *is a holomorphic map. Suppose* $\varphi : X \rightarrow (a, b)$ *is a* C^2 *function (where* $a \in \{-\infty\} \cup \mathbf{R}$ *and* $b \in \mathbf{R} \cup \{\infty\}$) *such that*

(i) *the restriction of* π *to* $\{a' \leqq \varphi \leqq b'\}$ *is proper for every* $a < a' < b' < b$,

(ii) *for some* $a < a^* < b^* < b$, φ *is strictly* p *pseudoconvex on* $\{\varphi > b^*\}$ *and is strictly* q*-pseudoconvex on* $\{\varphi < a^*\}$, *and*

(*iii*) $\{\varphi \leqq c\}$ *is the topological closure of* $\{\varphi < c\}$ *for every* $b^* < c < b$.

Suppose $\mathcal{F}$ *is a coherent analytic sheaf on* X *such that* $\mathcal{F}$ *is* π*-flat on* $\{\varphi < a^*\}$ *and* codh $\mathcal{F} \geqq r$ *on* $\{\varphi < a^*\}$. *Then*

(*a*) *for* $a < c' \leqq c < a^* < b^* < d \leqq d' < b$ *and* $p \leqq l < r - q - 3n + 1$ *the* l^{th} *direct image* $(\pi_c^d)_l(\mathcal{F})$ *of* $\mathcal{F} \mid \{c < \varphi < d\}$ *under* $\pi \mid \{c < \varphi < d\}$ *is coherent on* S *and the sheaf-homomorphism* $(\pi_{c'}^{d'})_l(\mathcal{F}) \rightarrow (\pi_c^d)_l(\mathcal{F})$ *induced by restriction is a sheaf-isomorphism*; *and*

(*b*) *for* $p < l < r - q - 3n + 1$ *the* l^{th} *direct image* $\pi_l(\mathcal{F})$ *of* $\mathcal{F}$ *under* π *is coherent on* S *and the sheaf-homomorphism* $\pi_l(\mathcal{F}) \rightarrow (\pi_c^d)_l(\mathcal{F})$ *induced by restriction is a sheaf-isomorphism for*

$$a < c < a^* < b^* < d < b.$$

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The Main Theorem slightly falls short of proving the general conjecture in generalizing Grauert's direct image theorem [3] to the pseudoconvex-pseudoconcave case. The general conjecture predicts the coherence of $\pi_l(\mathcal{F})$ for $p \leqq l < r - q - n$ and assumes neither the π-flatness of $\mathcal{F}$ near the pseudoconcave boundary nor any condition on S other than dim $S = n$.

For the Main Theorem, the case where $(S, \mathcal{O})$ is perfect (i. e. codh $\mathcal{O}_s = \dim_s S$ for every $s \in S$) is no more general than the case where S is nonsingular, because a perfect space can be locally represented as a branched covering over a nonsingular space with a *flat* covering map and because an analytic sheaf on a branched covering is coherent if and only if its zeroth direct image under the covering map is coherent [12, Lemma (1.1)].

The condition that $\{\varphi \leqq c\}$ is the topological closure of $\{\varphi < c\}$ is put in as an assumption of the Main Theorem because of a recent observation made by Fischer [2] on [1]. It is conjectured that the condition is unnecessary.

The proof of the Main Theorem has two steps. The first step is to prove an analog of Grauert's Hauptlemma [3, p. 47] (see (i) and (ii) in the proof of the Main Theorem of this paper). The second step is to derive the coherence of the direct images from that analog of Grauert's Hauptlemma. Since the first step is trivially analagous to the corresponding one in [8] (Proposition $(14.1)_n$ and its Corollary), we do not carry out the first step here. The second step depends on the following general abstract theorem on direct images whose proof occupies the major portion of this paper:

COHERENCE THEOREM. *Suppose $\tilde{X}$ is a complex space, G is an open subset of $\mathbf{C}^n$, and $\tilde{\pi}: \tilde{X} \to G$ is a holomorphic map. Suppose X is an open subset of $\tilde{X}$ such that the restriction of $\tilde{\pi}$ to X^- is proper.*

Let $\pi = \tilde{\pi} \mid X$. Suppose l is a nonnegative integer and $\tilde{\mathcal{F}}$ is a coherent analytic sheaf on $\tilde{X}$. Let $\mathcal{F} = \tilde{\mathcal{F}} \mid X$. Suppose the followiug two conditions are satisfied.

(i) *For every $t \in G$, $\pi_l(\mathcal{F})_t$ is finitely generated over ${}_n\mathcal{O}_t$ (where ${}_n\mathcal{O}$ is the structure sheaf of $\mathbf{C}^n$).*

(ii) *For every $t \in G$ and every open neighborhood U of t in G there exists an open neighborhood U' of t in U such that for $l \leqq q \leqq l + n$ the $\Gamma(U', {}_n\mathcal{O})$-submodule of $H^q(\pi^{-1}(U'), \mathcal{F})$ generated by the image of $H^q(\pi^{-1}(U), \mathcal{F}) \to H^q(\pi^{-1}(U'), \mathcal{F})$ is finitely generated over $\Gamma(U', {}_n\mathcal{O})$.*

Then $\pi_l(\mathcal{F})$ is coherent on G.

The proof of the Coherence Theorem is essentially algebraic in nature. At the end of this paper the result of [12] is coupled with the Main Theorem to yield the semi-continuity of dimensions of sheaf cohomology groups of pseudoconvex-pseudoconcave spaces under holomorphic deformation.

A number of other partial results on the general conjecture of generalizing Grauert's direct image theorem have been obtained by Ling [7], Markoe-Rossi [8], and the author [9], [10], [11]. However, the Main Theorem in this paper does not imply the other partial results because of the loss of the coherence of some direct images with dimensions close to the bounds imposed by the pseudoconvexity and pseudoconcavity. Kiehl [4] has obtained a different kind of generalization of Grauert's direct image theorem, namely to the case where the map remains proper but the spaces are not complex-analytic and the sheaves are pseudocoherent instead of coherent.

After I finished this manuscript, I received a preprint from Knorr [6] in which he obtained a pure pseudoconvex generalization of Grauert's direct image theorem.

In this paper, $\mathbf{N}$ denotes the set of all positive integers and $\mathbf{N}_*$ denotes the set of all nonnegative integers. $\mathbf{R}_+$ denotes the set of all positive numbers. $t_1, \ldots, t_n$ denote the coordinates of $\mathbf{C}^n$.

The following lemma is a trivial modification of [9, Proposition 1.1].

LEMMA 1. (*Coherence Criterion*). *Suppose G is an open subset of $\mathbf{C}^n$ and $\mathcal{G}$ is an analytic sheaf on G. Then $\mathcal{G}$ is coherent at a point t^0 of G if and only if there exist Stein open neighborhoods $U' \subset U$ of t^0 and $\xi_1, \ldots, \xi_k \in \Gamma(U, \mathcal{G})$ satisfying the following two conditions:*

(i) $\mathcal{G} = \sum_{i=1}^k {}_n\mathcal{O}\, \xi_i$ *on* U'.

(ii) *For every $t \in U'$ and every $\nu \in \mathbf{N}$ there exists $d \geqq \nu$ in $\mathbf{N}$ such that for every $\xi \in \sum_{i=1}^k \Gamma(U, {}_n\mathcal{O})\, \xi_i$ with $\xi_t \in (\mathfrak{m}(t)^d \mathcal{G})_t$ (where $\mathfrak{m}(t)$ is the maximum sheaf of ideals on $\mathbf{C}^n$ whose zero-set is $\{t\}$), there exist $\alpha_1, \ldots, \alpha_k \in \Gamma(U', \mathfrak{m}(t)^\nu)$ with $\xi = \sum_{i=1}^k \alpha_i \xi_i$ in U'.*

For $\varrho = (\varrho_1, \ldots, \varrho_n) \in \mathbf{R}_+^n$ we denote by $K(\varrho)$ the open polydisc with polyradius ϱ, i.e.

$$K(\varrho) = \{(t_1, \ldots, t_n) \in \mathbf{C}^n \mid |t_i| < \varrho_i \text{ for } 1 \leqq i \leqq n\}.$$

If $\varrho' = (\varrho_1', \ldots, \varrho_n') \in \mathbf{R}_+^n$, then we say that $\varrho' < \varrho$ if $\varrho_i' < \varrho_i$ for $1 \leqq i \leqq n$, and we say that $\varrho' \leqq \varrho$ if $\varrho_i' \leqq \varrho_i$ for $1 \leqq i \leqq n$.

If $(d_1, \ldots, d_n) \in \mathbf{N}_*^n$ and $(d_1', \ldots, d_n') \in \mathbf{N}_*^n$, then we say that $(d_1', \ldots d_n') \leqq (d_1, \ldots, d_n)$ if $d_i' \leqq d_i$ for $1 \leqq i \leqq n$.

LEMMA 2. *Suppose $\varrho' < \varrho$ in $\mathbf{R}_+^n$, I is a finitely generated $\Gamma(K(\varrho), {}_n\mathcal{O})$-module, and I' is a $\Gamma(K(\varrho'), {}_n\mathcal{O})$-module. Suppose $\alpha: I \to I'$ is a homomorphism over $\Gamma(K(\varrho), {}_n\mathcal{O})$ when I' is naturally regarded as a $\Gamma(K(\varrho), {}_n\mathcal{O})$-module. If J is a $\Gamma(K(\varrho), {}_n\mathcal{O})$-submodule of I, then the $\Gamma(K(\varrho'), {}_n\mathcal{O})$-submodule of I' generated by $\alpha(J)$ is finitely generated over $\Gamma(K(\varrho'), {}_n\mathcal{O})$.*

PROOF: Since I is finitely generated over $\Gamma(K(\varrho), {}_n\mathcal{O})$, there exists a $\Gamma(K(\varrho), {}_n\mathcal{O})$-epimorphism $\beta : \Gamma(K(\varrho), {}_n\mathcal{O}^p) \longrightarrow I$. Let $\mathcal{G}$ be the analytic subsheaf of ${}_n\mathcal{O}^p \mid K(\varrho)$ generated by $\beta^{-1}(J)$. Being generated by global sections, $\mathcal{G}$ is coherent.

Since $\alpha : I \longrightarrow I'$ is a $\Gamma(K(\varrho), {}_n\mathcal{O})$-homomorphism, β induces uniquely a $\Gamma(K(\varrho'), {}_n\mathcal{O})$-homomorphism $\sigma : \Gamma(K(\varrho'), {}_n\mathcal{O}^p) \longrightarrow I'$ such that $\alpha\beta = \sigma\tau$, where $\tau : \Gamma(K(\varrho), {}_n\mathcal{O}^p) \longrightarrow \Gamma(K(\varrho'), {}_n\mathcal{O}^p)$ is the restriction map.

Let $\nu : \Gamma(K(\varrho'), \mathcal{G}) \longrightarrow \Gamma(K(\varrho'), {}_n\mathcal{O}^p)$ be induced by $\mathcal{G} \hookrightarrow {}_n\mathcal{O}^p$. Since by Theorem B of Cartan-Oka $\Gamma(K(\varrho'), \mathcal{G})$ is finitely generated over $\Gamma(K(\varrho'), {}_n\mathcal{O})$ and $\operatorname{Im} \sigma\nu$ and $\alpha(J)$ generate the same $\Gamma(K(\varrho'), {}_n\mathcal{O})$-submodule of I, the lemma follows. Q.E.D.

In Lemma 2 through Proposition 4 we assume the following:

(i) $\tilde{\pi} : \tilde{X} \longrightarrow K(\tilde{\varrho})$ is a holomorphic map, where $\tilde{\varrho} \in \mathbf{R}^n_+$ and $\tilde{X}$ is a complex space.

(ii) X is an open subset of $\tilde{X}$ such that the restriction of $\tilde{\pi}$ to X^- is proper.

(iii) $\tilde{\mathcal{F}}$ is a coherent analytic sheaf on $\tilde{X}$.

Let $\pi = \tilde{\pi} \mid X$ and $\mathcal{F} = \tilde{\mathcal{F}} \mid X$. For $\varrho \leqq \tilde{\varrho}$ in $\mathbf{R}^n_+$ let $X(\varrho) = \pi^{-1}(K(\varrho))$. Suppose $\varrho^0 \in \mathbf{R}^n_+$ and $\varrho^0 < \tilde{\varrho}$.

We introduce the following statement.

$(*)_q$ *For every* $\varrho \leqq \varrho^0$ *in* $\mathbf{R}^n_+$ *there exists* $\varrho' = \gamma_q(\varrho) \leqq \varrho$ *in* $\mathbf{R}^n_+$ *such that the* $\Gamma(K(\varrho'), {}_n\mathcal{O})$*-submodule of* $H^q(X(\varrho'), \mathcal{F})$ *generated by the image of* $H^q(X(\varrho), \mathcal{F}) \longrightarrow H^q(X(\varrho'), \mathcal{F})$ *is finitely generated over* $\Gamma(K(\varrho'), {}_n\mathcal{O})$.

LEMMA 2. *Suppose* $(*)_q$ *holds. If* $d_1, \dots, d_k \in \mathbf{N}_*$, $t^0 = (t^0_1, \dots, t^0_n) \in K(\varrho^0)$, *and* $\varrho \leqq \varrho^0$ *in* $\mathbf{R}^n_+$, *then there exists* $g = \Pi^{t^0}_{k,q}(d_1, \dots, d_k; \varrho) \in \mathbf{N}_*$ *such that for any* $d \in \mathbf{N}_*$ *with* $d \geqq g$ *one has*

$$\sigma\left(\alpha^{-1}\left(\Sigma^k_{i=1}(t_i - t^0_i)^{d_i} H^q(X(\varrho), \mathcal{F})\right)\right) \subset \beta^{-1}\left(\Sigma^k_{i=1}(t_i - t^0_i)^{d_i} H^q(X(\varrho'), \mathcal{F})\right),$$

where $\varrho' = \frac{1}{2}\gamma_q(\varrho)$, $\alpha : H^q(X(\varrho), \mathcal{F}) \longrightarrow H^q(X(\varrho), \mathcal{F})$ *is defined by multiplication by* $(t_{k+1} - t^0_{k+1})^d$, $\beta : H^q(X(\varrho'), \mathcal{F}) \longrightarrow H^q(X(\varrho'), \mathcal{F})$ *is defined by multiplication by* $(t_{k+1} - t^0_{k+1})^g$, *and* $\sigma : H^q(X(\varrho), \mathcal{F}) \longrightarrow H^q(X(\varrho'), \mathcal{F})$ *is induced by restriction.*

PROOF: Let $\varrho^* = \gamma_q(\varrho)$ and let I be the $\Gamma(K(\varrho^*)\, {}_n\mathcal{O})$-submodule generated by the image of $H^q(X(\varrho), \mathcal{F}) \longrightarrow H^q(X(\varrho^*), \mathcal{F})$. For $d \in \mathbf{N}_*$ let

$$J_d = \{\xi \in I \mid (t_{k+1} - t^0_{k+1})^d\, \xi \in \Sigma^k_{i=1}(t_i - t^0_i)^{d_i} I\}.$$

Let $J = \bigcup_{d \in \mathbf{N}_*} J_d$. By Lemma 1 the $\Gamma(K(\varrho'), {}_n\mathcal{O})$-submodule generated by the image of $J \longrightarrow H^q(X(\varrho'), \mathcal{F})$ (induced by the restriction map $H^q(X(\varrho^*), \mathcal{F}) \longrightarrow H^q(X(\varrho'), \mathcal{F})$) is finitely generated over $\Gamma(K(\varrho'), {}_n\mathcal{O})$. Hence there exists $g \in \mathbf{N}_*$ such that for every $d \geqq g$ in $\mathbf{N}_*$ the image of $J_d \longrightarrow H^q(X(\varrho'), \mathcal{F})$ and the image of $J_g \longrightarrow H^q(X(\varrho'), \mathcal{F})$ generate the same $\Gamma(K(\varrho'), {}_n\mathcal{O})$-submodule of $H^q(X(\varrho'), \mathcal{F})$. It is easily verified that g satisfies the requirement. Q. E. D.

We call a k-tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ an *integral echelon function of order* k *if* $\lambda_1 \in \mathbf{N}_*$ and λ_i is a map from $\mathbf{N}_*^{i-1}$ to $\mathbf{N}_*$ for $1 < i \leqq n$. Denote by Λ_k the set of all integral echelon functions of order k. For $(d_1, \ldots, d_k) \in \mathbf{N}_*^k$ we say that $(d_1, \ldots, d_k) \geqq \lambda$ if $d_1 \geqq \lambda_1$ and $d_i \geqq \lambda_i(d_1, \ldots, d_{i-1})$ for $1 < i \leqq n$.

We consider the following three statements.

$(\mathrm{I})_q^k$ *If* $\varrho \leqq \varrho^0$ *in* $\mathbf{R}_+^n$, $t^0 \in K(\varrho^0)$, *and* $d_1, \ldots, d_k \in \mathbf{N}_*$, *then there exist* $(e_1, \ldots, e_k) = \Phi_{k,q}^{t^0}(d_1, \ldots, d_k; \varrho) \leqq (d_1, \ldots, d_k)$ *in* $\mathbf{N}_*^k$ *and* $\varrho' = \varphi_{k,q}(\varrho) \leqq \varrho$ *in* $\mathbf{R}_+^n$ *such that*

(i) *for every* $\lambda \in \Lambda_k$ *there exists* $\lambda' \in \Lambda_k$ *(which depends on* λ, ϱ, t^0, k, *and* q*) satisfying the condition that* $\Phi_{k,q}^{t^0}(d_1', \ldots, d_k'; \varrho) \geqq \lambda$ *for* $(d_1', \ldots, d_k') \geqq \lambda'$; *and*

(ii) *one has* $\mathrm{Im}\ \beta \subset \mathrm{Im}\ \alpha$ *in*

$$H^q(X(\varrho'), \mathcal{F}) \xrightarrow{\alpha} H^q(X(\varrho'), \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{e_i} \mathcal{F}) \xleftarrow{\beta} H^q(X(\varrho), \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F}),$$

where α *comes from the quotient map* $\mathcal{F} \longrightarrow \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{e_i} \mathcal{F}$ *and* β *comes from the inclusion* $X(\varrho) \hookrightarrow X(\varrho)$ *and the quotient map* $\mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F} \longrightarrow \mathcal{F}/\Sigma_{i=1}^k (t_t - t_i^0)^{e_i} \mathcal{F}$.

$(\mathrm{II})_q^k$ *If* $\varrho \leqq \varrho^0$ *in* $\mathbf{R}_+^n$, $t^0 \in K(\varrho^0)$, *and* $d_1, \ldots, d_k \in \mathbf{N}_*$, *then there exist* $(e_1, \ldots, e_k) = \Psi_{k,q}^{t^0}(d_1, \ldots, d_k; \varrho) \leqq (d_1, \ldots, d_k)$ *in* $\mathbf{N}_*^k$ *and* $\varrho' = \psi_{k,q}(\varrho) \leqq \varrho$ *in* $\mathbf{R}_+^n$ *such that*

(i) *for every* $\lambda \in \Lambda_k$ *there exists* $\lambda' \in \Lambda_k$ *(which depends on* λ, ϱ, t^0, k, *and* q*) satisfying the condition that* $\Psi_{k,q}^{t^0}(d_1', \ldots, d_k'; \varrho) \geqq \lambda$ *for* $(d_1', \ldots, d_k') \geqq \lambda'$; *and*

(ii) *one has* $\alpha(\mathrm{Ker}\ \beta) \subset \Sigma_{i=1}^k (t_i - t_i^0)^{e_i} H^q(X(\varrho'), \mathcal{F})$ *in*

$$H^q(X(\varrho'), \mathcal{F}) \xleftarrow{\alpha} H^q(X(\varrho), \mathcal{F}) \xrightarrow{\beta} H^q(X(\varrho), \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F}),$$

where α *comes from the inclusion* $X(\varrho') \hookrightarrow X(\varrho)$ *and* β *comes from the quotient map* $\mathcal{F} \longrightarrow \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F}$.

$(\mathrm{III})_q^k$ *If* $\varrho \leqq \varrho^0$ *in* $\mathbf{R}_+^n$, $t^0 \in K(\varrho^0)$, *and* $d_1, \dots, d_k \in \mathbf{N}_*$, *then there exist* $(e_1, \dots, e_k) = \Theta_{k,q}^{t^0}(d_1, \dots, d_k; \varrho) \leqq (d_1, \dots, d_k)$ *in* $\mathbf{N}_*^k$, $h = \Delta_{k,q}^{t^0}(d_1, \dots, d_k; \varrho) \in \mathbf{N}_*$, *and* $\varrho' = \theta_{k,q}(\varrho) \leqq \varrho$ *in* $\mathbf{R}_+^n$ *such that*

(i) *for every* $\lambda \in \Lambda_k$ *there exists* $\lambda' \in \Lambda_k$ *(which depends on* λ, ϱ, t^0, k, *and* q*) satisfying the condition that* $\Theta_{k,q}^{t^0}(d_1', \dots, d_k'; \varrho) \geqq \lambda$ *for* $(d_1', \dots, d_k') \geqq \lambda'$; *and*

(ii) *for any* $d \geqq h$ *in* $\mathbf{N}$ *one has* $\operatorname{Ker} \alpha_d \subset \operatorname{Ker}(\beta\alpha_h)$ *where*

$$\alpha_\nu : H^q(X(\varrho), \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F}) \longrightarrow H^q(X(\varrho), \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F})$$

is defined by multiplication by $(t_{k+1} - t_{k+1}^0)^\nu$ *and*

$$\beta : H^q(X(\varrho), \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F}) \longrightarrow H^q(X(\varrho'), \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{e_i} \mathcal{F})$$

comes from the inclusion $X(\varrho') \hookrightarrow X(\varrho)$ *and the quotient map*

$$\mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F} \longrightarrow \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{e_i} \mathcal{F}.$$

We observe that, since $X(\varrho^0)$ is relatively compact in $\widetilde{X}$ and $\mathcal{F} = \widetilde{\mathcal{F}} | X$, for $t^0 \in K(\varrho^0)$ and $d_1, \dots, d_k \in \mathbf{N}_*$ there exists $f = \Xi_k^{t^0}(d_1, \dots, d_k)$ such that $(t_{k+1} - t_{k+1}^0)$ is not a zero-divisor for $(t_{k+1} - t_{k+1}^0)^f (\mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F})_x$ for $x \in X(\varrho^0)$.

PROPOSITION 1. $(\mathrm{I})_{k-1}^q$ and $(\mathrm{II})_{k-1}^q \Longrightarrow (\mathrm{II})_k^q$.

PROOF. Let $f = \Xi_{k-1}^{t^0}(d_1, \dots, d_{k-1})$. We can assume without loss of generality that $d_k \geqq f$, because we can always set $e_i = 0$ $(1 \leqq i \leqq k)$ and $\varrho' = \varrho$ if this condition is not satisfied. Let $\varrho^* = \varphi_{k-1,q}(\varrho)$ and $\varrho' = \psi_{k-1,q}(\varrho^*)$. Let

$$(d_1', \dots, d_{k-1}') = \Phi_{k-1,q}^{t^0}(d_1, \dots, d_{k-1}; \varrho),$$

$$(e_1, \dots, e_{k-1}) = \Psi_{k-1,q}^{t^0}(d_1', \dots, d_{k-1}'; \varrho^*),$$

and $e_k = d_k - f$. We are going to prove that $e_1, \dots, e_k$ and ϱ' satisfy the requirements. Condition (i) is clearly satisfied.

Let $\mathcal{G} = \mathcal{F}/\Sigma_{i=1}^{k-1} (t_i - t_i^0)^{d_i} \mathcal{F}$ and $\mathcal{R} = \mathcal{F}/\Sigma_{i=1}^{k-1} (t_i - t_i^0)^{d_i'} \mathcal{F}$.

Consider the following commutative diagram with an exact second row:

$$
\begin{array}{ccccc}
H^q(X(\varrho), (t_k - t_k^0)^f \mathcal{G}) & \xrightarrow{\ \varkappa\ } & H^q(X(\varrho), \mathcal{G}) & & \\
\tau\downarrow & & \varepsilon\downarrow & & \\
H^q(X(\varrho), (t_k - t_k^0)^{d_k} \mathcal{G}) & \xrightarrow{\ c\ } & H^q(X(\varrho), \mathcal{G}) & \xrightarrow{\ b\ } & H^q(X(\varrho), \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F}) \\
 & & a\uparrow & \nearrow \beta & \\
 & & H^q(X(\varrho), \mathcal{F}) & &
\end{array}
$$

where τ, ε are defined by multiplication by $(t_k - t_k^0)^{e_k}$, $\varkappa$ comes from the inclusion $(t_k - t_k^0)^f \mathcal{G} \hookrightarrow \mathcal{G}$, c and b come from the exact sequence

$$0 \longrightarrow (t_k - t_k^0)^{d_k} \mathcal{G} \hookrightarrow \mathcal{G} \longrightarrow \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F} \longrightarrow 0,$$

a comes from the quotient map $\mathcal{F} \longrightarrow \mathcal{G}$, and β comes from the quotient map $\mathcal{F} \longrightarrow \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i} \mathcal{F}$. Since $(t_k - t_k^0)$ is not a zero-divisor for $(t_k - t_k^0)^f \mathcal{G}_x$ for $x \in X(\varrho^0)$, τ is an isomorphism.

Hence

$$(\#) \qquad a(\operatorname{Ker} \beta) \subset (t_k - t_k^0)^{e_k} H^q(X(\varrho), \mathcal{G}).$$

Next consider the commutative diagram

$$
\begin{array}{ccc}
H^q(X(\varrho), \mathcal{F}) & \xrightarrow{\ \mu\ } & H^q(X(\varrho^*), \mathcal{F}) \\
a\downarrow & & \nu\downarrow \\
H^q(X(\varrho), \mathcal{G}) & \xrightarrow{\ \sigma\ } & H^q(X(\varrho^*), \mathcal{R}),
\end{array}
$$

where μ comes from the inclusion $X(\varrho) \hookrightarrow X(\varrho^*)$, ν comes from the quotient map $\mathcal{F} \longrightarrow \mathcal{R}$, and σ comes from the inclusion $X(\varrho^*) \hookrightarrow X(\varrho)$ and the quotient map $\mathcal{G} \longrightarrow \mathcal{R}$.

By the choice of $d_1', \dots, d_k'$ and ϱ^*, we have $\operatorname{Im} \sigma \subset \operatorname{Im} \nu$. It follows from $(\#)$ that

$$\nu\mu(\operatorname{Ker} \beta) = \sigma a(\operatorname{Ker} \beta) \subset (t_k - t_k^0)^{e_k} \operatorname{Im} \nu.$$

Hence

$$(\dagger)\qquad \mu\,(\operatorname{Ker}\beta) \subset (t_k - t_k^0)^{e_k}\, H^q(X(\varrho^*), \mathcal{F}) + \operatorname{Ker}\nu.$$

Let $\alpha^*: H^q(X(\varrho^*), \mathcal{F}) \to H^q(X(\varrho'), \mathcal{F})$ be induced by the inclusion $X(\varrho') \hookrightarrow X(\varrho^*)$. By the choice of $e_1, \dots, e_{k-1}$ and ϱ', we have

$$\alpha^*(\operatorname{Ker}\nu) \subset \Sigma_{i=1}^{k-1} (t_i - t_i^0)^{e_i}\, H^q(X(\varrho'), \mathcal{F}).$$

Since the restriction map $\alpha: H^q(X(\varrho), \mathcal{F}) \to H^q(X(\varrho'), \mathcal{F})$ equais $\alpha^*\mu$, it follows from $(\dagger)$ that

$$\alpha\,(\operatorname{Ker}\beta) \subset \Sigma_{i=1}^{k} (t_i - t_i^0)^{e_i}\, H^q(X(\varrho'), \mathcal{F}).$$

Q. E. D.

PROPOSITION 2. $(\mathrm{I})_{k-1}^q$ *and* $(\mathrm{III})_{k-1}^{q+1} \Longrightarrow (\mathrm{I})_k^q$.

PROOF. Let $f = \Xi_{k-1}^{t^0}(d_1, \dots, d_{k-1})$ and $h = \Delta_{k-1,\,q+1}^{t^0}(d_1, \dots, d_{k-1}; \varrho)$. We can assume without loss of generality that $d_k \geqq f + h$, because we can always set $e_i = 0$ $(1 \leqq i \leqq k)$ and $\varrho' = \varrho$ if this condition is not satisfied. Let $\varrho^* = \theta_{k-1,\,q+1}(\varrho)$ and $\varrho' = \varphi_{k-1,\,q}(\varrho^*)$. Let

$$(d_1', \dots, d_{k-1}') = \Theta_{k-1,\,q+1}^{t^0}(d_1, \dots, d_{k-1}; \varrho),$$

$$(e_1, \dots, e_{k-1}) = \Phi_{k-1,\,q}^{t^0}(d_1', \dots, d_{k-1}'; \varrho^*),$$

and $e_k = d_k - f - h$. We are going to prove that $e_1, \dots e_k$ and ϱ' satisfy the requirements. Clearly condition (i) is satisfied.

Let

$$\mathcal{G} = \mathcal{F}/\Sigma_{i=1}^{k-1} (t_i - t_i^0)^{d_i}\, \mathcal{F} \text{ and } \mathcal{R} = \mathcal{F}/\Sigma_{i=1}^{k-1} (t_i - t_0^i)^{d_i'}\, \mathcal{F}.$$

Consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & H^{q+1}(X(\varrho^*), (t_k - t_k^0)^{e_k}\mathcal{R}) & & \\
 & \overset{\sigma}{\nearrow} & & \overset{\tau}{\nwarrow} & \\
H^{q+1}(X(\varrho), (t_k - t_k^0)^{d_k}\mathcal{G}) & \xrightarrow{b} & H^{q+1}(X(\varrho), \mathcal{G}) & & H^{q+1}(X(\varrho^*), \mathcal{R}) \\
\varkappa \uparrow & & \uparrow \nu & & \uparrow \eta \\
H^{q+1}(X(\varrho), (t_k - t_k^0)^{f}\mathcal{G}) & \xrightarrow{\mu} & H^{q+1}(X(\varrho), \mathcal{G}) & \xrightarrow{\xi} & H^{q+1}(X(\varrho), \mathcal{G}),
\end{array}$$

where b is induced by $(t_k - t_k^0)^{d_k} \mathcal{G} \hookrightarrow \mathcal{G}$, μ is induced by $(t_k - t_k^0)^f \mathcal{G} \hookrightarrow \mathcal{G}$, $\varkappa$ and ν are defined by multiplication by $(t_k - t_k^0)^{d_k - f}$, ξ is defined by multiplication by $(t_k - t_k^0)^h$, τ is defined by multiplication by $(t_k - t_k^0)^{e_k}$, η is induced by the inclusion $X(\varrho^*) \hookrightarrow X(\varrho)$ and the quotient map $\mathcal{G} \to \mathcal{R}$, and σ is induced by the inclusion $X(\varrho^*) \hookrightarrow X(\varrho)$ and the map $(t_k - t_k^0)^{d_k} \mathcal{G} \to (t_k - t_k^0)^{e_k} \mathcal{R}$ which is the composite of the quotient map $(t_k - t_k^0)^{e_k} \mathcal{G} \to (t_k - t_k^0)^{e_k} \mathcal{R}$ and the inclusion map $(t_k - t_k^0)^{d_k} \mathcal{G} \hookrightarrow (t_k - t_k^0)^{e_k} \mathcal{G}$. Since $(t_k - t_k^0)$ is not a zero-divisor of $(t_k - t_k^0)^f \mathcal{G}_x$ for $x \in X(\varrho^0)$, $\varkappa$ is an isomorphism. By the choice of $d_1', \dots d_{k-1}', h$, and ϱ^*, we have $\operatorname{Ker} \nu \subset \operatorname{Ker} \eta\xi$.

Hence

$$\sigma(\operatorname{Ker} b) = \sigma(\operatorname{Ker} \nu\mu\varkappa^{-1}) \subset \sigma(\operatorname{Ker} \eta\xi\mu\varkappa^{-1}) \subset$$
$$\subset \sigma(\operatorname{Ker} \tau\eta\xi\mu\varkappa^{-1}) \subset \sigma(\operatorname{Ker} o) = 0.$$

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
H^q(X(\varrho), \mathcal{G}) \to & H^q(X(\varrho), \mathcal{G}/(t_k - t_k^0)^{d_k} \mathcal{G}) & \to & H^{q+1}(X(\varrho), (t_k - t_k^0)^{d_k} \mathcal{G}) & \xrightarrow{b} & H^{q+1}(X(\varrho), \mathcal{G}) \\
& \downarrow \beta^* & & \downarrow \sigma & & \\
H^q(X(\varrho), \mathcal{R}) \xrightarrow{\alpha^*} & H^q(X(\varrho^*), \mathcal{R}/(t_k - t_k^0)^{e_k} \mathcal{R}) & \to & H^{q+1}(X(\varrho^*), (t_k - t_k^0)^{e_k} \mathcal{R}), & &
\end{array}$$

where the first row comes from the exact sequence

$$0 \to (t_k - t_k^0)^{d_k} \mathcal{G} \hookrightarrow \mathcal{G} \to \mathcal{G}/(t_k - t_k^0)^{d_k} \mathcal{G} \to 0,$$

the second row from the exact sequence

$$0 \to (t_k - t_k^0)^{e_k} \mathcal{R} \hookrightarrow \mathcal{R} \to \mathcal{R}/(t_k - t_k^0)^{e_k} \mathcal{R} \to 0,$$

and β^* is induced by the inclusion $X(\varrho^*) \hookrightarrow X(\varrho)$ and the quotient map $\mathcal{G}/(t_k - t_k^0)^{d_k} \mathcal{G} \to \mathcal{R}/(t_k - t_k^0)^{e_k} \mathcal{R}$. Since $\sigma(\operatorname{Ker} b) = 0$, it follows that $\operatorname{Im} \beta^* \subset \subset \operatorname{Im} \alpha^*$.

Consider the following commutative diagram

$$
\begin{array}{ccccc}
 & & H^q(X(\varrho), \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{d_i}\mathcal{F}) & & \\
 & & \downarrow \beta & \searrow \beta^* & \\
H^q(X(\varrho')\mathcal{F}) & \xrightarrow{\alpha} & H^q(X(\varrho'), \mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{e_k}\mathcal{F}) & \xleftarrow{\zeta} & H^q(X(\varrho^*), \mathcal{R}/(t - t_k^0)^{e_k}\mathcal{R}) \\
 & \searrow \alpha' & \uparrow \varepsilon & & \uparrow \alpha^* \\
 & & H^q(X(\varrho'), \mathcal{F}/\Sigma_{i=1}^{k-1} (t_i - t_i^0)^{e_i}\mathcal{F}) & \xleftarrow{\beta'} & H^q(X(\varrho^*), \mathcal{R}),
\end{array}
$$

where all maps are induced by quotient maps (and inclusion maps).

By the choice of $e_1, \dots, e_{k-1}$ and ϱ', we have Im $\beta' \subset$ Im α'.

It follows that

$$\operatorname{Im} \beta = \zeta(\operatorname{Im} \beta^*) \subset \zeta(\operatorname{Im} \alpha^*) = \varepsilon(\operatorname{Im} \beta')$$

$$\subset \varepsilon(\operatorname{Im} \alpha') \subset \operatorname{Im} \alpha.$$

Q. E. D.

PROPOSITION 3. $(\mathrm{I})_k^q, (\mathrm{II})_k^q$, *and* $(*)_q \Longrightarrow (\mathrm{III})_k^q$.

PROOF. Let $\varrho^* = \varphi_{k,q}(\varrho)$, $\varrho^\# = \psi_{k,q}(\varrho^*)$ and $\varrho' = \dfrac{1}{2}\gamma_q(\varrho^\#)$.

Let

$$(d_1', \dots, d_k') = \Phi_{k,q}^{t^0}(d_1, \dots, d_k; \varrho),$$

$$(e_1, \dots, e_k) = \Psi_{k\,q}^{t^0}(d_1', \dots, d_k'; \varrho^*),$$

$$h = \Pi_{k,q}^{t^0}(e_1, \dots, e_k; \varrho^\#).$$

We are going to prove that $e_1, \dots, e_k, h$ and ϱ' satisfy the requirements. Clearly condition (i) is satisfied.

Take $d \geqq h$ in N_*. Let

$$r: H^q(X(\varrho^*), \mathcal{F}) \to H^q(X(\varrho^\#), \mathcal{F}),$$

$$r': H^q(X(\varrho^*), \mathcal{F}) \to H^q(X(\varrho'), \mathcal{F})$$

be restriction maps. Let

$$\nu: H^q(X(\varrho^*), \mathcal{F}) \to H^q(X(\varrho^*), \mathcal{F})$$

be defined by multiplication by $(t_{k+1}-t_{k+1}^0)^d$. Let $\mathcal{G}=\mathcal{F}/\Sigma_{i=1}^k(t_i-t_i^0)^{d_i}\mathcal{F}$, $\mathcal{R}=\mathcal{F}/\Sigma_{i=1}^k(t_i-t_i^0)^{d_i'}\mathcal{F}$, and $\mathcal{S}=\mathcal{F}/\Sigma_{i=1}^k(t_i-t_i^0)^{e_i}\mathcal{F}$. Let

$$\varkappa : H^q(X(\varrho^*)\mathcal{F}) \rightarrow H^q(X(\varrho^*), \mathcal{R})$$

be induced by the quotient map $\mathcal{F}\rightarrow\mathcal{R}$. Let

$$\mu : H^q(X(\varrho^*), \mathcal{F}) \rightarrow H^q(X(\varrho'), \mathcal{S})$$

be induced by the inclusion $X(\varrho')\hookrightarrow X(\varrho^*)$ and the quotient map $\mathcal{F}\rightarrow\mathcal{S}$. By the choice of $e_1, \dots e_k$ and $\varrho^\#$, we have

$$(t_{k+1}-t_{k+1}^0)^d\, r(\nu^{-1}(\operatorname{Ker}\varkappa)) \subset r(\operatorname{Ker}\varkappa) \subset \Sigma_{i=1}^k(t_i-t_i^0)^{e_i} H^q(X(\varrho^\#), \mathcal{F}).$$

By the choice of h and ϱ', it follows that

$$(t_{k+1}-t_{k+1}^0)^h\, r'(\nu^{-1}(\operatorname{Ker}\varkappa)) \subset \Sigma_{i=1}^k(t_i-t_i^0)^{e_i} H^q(X(\varrho'), \mathcal{F}).$$

Hence

$$(t_{k+1}-t_{k+1}^0)^h\, \mu(\nu^{-1}(\operatorname{Ker}\varkappa)) = 0.$$

Let $\varepsilon : H^q(X(\varrho), \mathcal{G}) \rightarrow H^q(X(\varrho^*), \mathcal{R})$ be induced by the inclusion $X(\varrho^*)\hookrightarrow X(\varrho)$ and the quotient map $\mathcal{G}\rightarrow\mathcal{R}$. Let $\alpha_d : H^q(X(\varrho), \mathcal{G}) \rightarrow H^q(X(\varrho), \mathcal{G})$ be defined by multiplication by $(t_{k+1}-t_{k+1}^0)^d$. Let $\beta : H^q(X(\varrho), \mathcal{G}) \rightarrow H^q(X(\varrho'), \mathcal{S})$ be induced by the inclusion $X(\varrho')\hookrightarrow X(\varrho)$ and the quotient map $\mathcal{G}\rightarrow\mathcal{S}$. By the choice of $d_1', \dots, d_k'$ and ϱ^*, we have $\operatorname{Im}\varepsilon \subset \operatorname{Im}\varkappa$.

It follows that

$$\varepsilon(\operatorname{Ker}\alpha_d) \subset \varkappa(\operatorname{Ker}\varkappa\nu) = \varkappa(\nu^{-1}(\operatorname{Ker}\varkappa)).$$

Hence

$$(t_{k+1}-t_{k+1}^0)^h\, \beta(\operatorname{Ker}\alpha_d) \subset (t_{k+1}-t_{k+1}^0)^h\, \mu(\nu^{-1}(\operatorname{Ker}\varkappa)) = 0.$$

Q. E. D.

PROPOSITION 4. *If* $l \in \mathbf{N}_*$ *and* $(*)_q$ *holds for* $l \leqq q \leqq l+n$, *then* $(\mathrm{I})_k^r$, $(\mathrm{II})_k^r$, *and* $(\mathrm{III})_k^r$ *hold for* $0 \leqq k \leqq n$ *and* $l \leqq r \leqq l+n-k$.

PROOF. Since $(\mathrm{I})_0^q$ and $(\mathrm{II})_0^q$ hold trivially for any $q \in \mathbf{N}_*$, the proposition follows readily from Propositions 1, 2, 3 und induction on k.

Q. E. D.

PROOF OF COHERENCE THEOREM.

We can assume that $K(\varrho^0)$ is a relatively compact subset of G for some $\varrho^0 \in \mathbf{R}_+^n$ and we need only prove the coherence of $\pi_l(\mathcal{F})$ at 0. Since $(*)_l$ holds and $\pi_l(\mathcal{F})_0$ is finitely generated over ${}_n\mathcal{O}_0$, according to [9, Proposition 2.3], after shrinking ϱ^0, we can find $\widetilde{\xi}_1, \dots, \widetilde{\xi}_k \in H^l(X(\varrho_0), \mathcal{F})$ such that for every $\varrho \leqq \varrho^0$ in $\mathbf{R}_+^n$ there exists $\gamma(\varrho) \leqq \varrho$ in $\mathbf{R}_+^n$ satisfying the condition that the $\Gamma(K(\gamma(\varrho)), {}_n\mathcal{O})$-submodule of $H^l(X(\gamma(\varrho)), \mathcal{F})$ generated by the images of $\widetilde{\xi}_1, \dots, \widetilde{\xi}_k$ in $H^l(X(\gamma(\varrho)), \mathcal{F})$ contains the image of $H^l(X(\varrho), \mathcal{F}) \longrightarrow H^l(X(\gamma(\varrho)), \mathcal{F})$. Choose $\varrho' \in \mathbf{R}_+^n$ such that $\varrho' \leqq \gamma(\varrho^0)$, $\varrho' \leqq \varphi_{n,l}(\varrho^0)$, and $\varrho' \leqq \psi_{n,l}(\varrho^0)$, where $\varphi_{n,l}$ and $\psi_{n,l}$ are respectively from $(\mathrm{I})_n^l$ and $(\mathrm{II})_n^l$ (obtained from Proposition 4). Let $\varrho'' = \gamma(\varrho')$. Let $\eta : H^l(X(\varrho^0), \mathcal{F}) \longrightarrow \Gamma(K(\varrho^0), \pi_l(\mathcal{F}))$ be the natural map. Let $\xi_i = \eta(\widetilde{\xi}_i)$, $1 \leqq i \leqq k$. We are going to prove that $\xi_1, \dots, \xi_k$ satisfy conditions (i) and (ii) of Lemma 1 with $\mathcal{G} = \pi_l(\mathcal{F})$ and with $U' = K(\varrho'')$ and $U = K(\varrho^0)$ as the two Stein open neighborhoods of 0.

Let the maps in the diagram

$$\begin{array}{ccc} H^l(X(\varrho^0), \mathcal{F}) & \xrightarrow{\;r'\;} & H^l(X(\varrho'), \mathcal{F}) \\ {\scriptstyle r}\downarrow & \swarrow{\scriptstyle r''} & \\ H^l(X(\varrho''), \mathcal{F}) & & \end{array}$$

be restriction maps. Fix arbitrary $t^0 \in K(\varrho'')$. Let $\mathfrak{m}(t^0)$ be the maximum sheaf of ideals on $\mathbf{C}^n$ with zero set $\{t^0\}$.

(*a*) To verify condition (i) of Lemma 1, take arbitrarily $\xi \in \pi_l(\mathcal{F})_{t^0}$. For some open neighborhood W of t^0 in $K(\varrho'')$, ξ is induced by some $\widetilde{\xi} \in H^l(\pi^{-1}(W), \mathcal{F})$. By using a coordinate system of $\mathbf{C}^n$ centered at t^0 and applying $(\mathrm{II})_n^l$ (obtained from Proposition 4), we obtain an open neighborhood W' of t^0 in W and

$$(d_1', \dots, d_n'), (e_1, \dots, e_n) \in \mathbf{N}_*^n$$

such that $d_i' \geqq e_i \geqq 1$ $(1 \leqq i \leqq n)$ and one has

$$\alpha(\operatorname{Ker} \beta) \subset \Sigma_{i=1}^n (t_i - t_i^0)^{e_i} H^l(\pi^{-1}(W'), \mathcal{F}),$$

where

$$\alpha : H^l(\pi^{-1}(W), \mathcal{F}) \longrightarrow H^l(\pi^{-1}(W') \mathcal{F})$$

is induced by the inclusion $\pi^{-1}(W') \hookrightarrow \pi^{-1}(W)$ and

$$\beta : H^l(\pi^{-1}(W), \mathcal{F}) \longrightarrow H^l(\pi^{-1}(t^0), \mathcal{F}/\Sigma_{i=1}^n (t_i - t_i^0)^{d_i'} \mathcal{F})$$

is induced by the quotient map $\mathcal{F} \longrightarrow \mathcal{F}/\Sigma_{i=1}^n (t_i - t_i^0)^{d_i'} \mathcal{F}$. By using $(\mathrm{I})_n^l$ (obtained from Proposition 4), we obtain

$$(d_1, \dots, d_n), (d_1^*, \dots, d_n^*) \in \mathbf{N}_*^n$$

such that $d_i \geqq d_i^* \geqq d_i' \ (1 \leqq i \leqq n)$ and one has $\operatorname{Im} \beta' \subset \operatorname{Im} \alpha'$, where

$$\alpha' : H^l(X(\varrho'), \mathcal{F}) \longrightarrow H^l(\pi^{-1}(t^0), \mathcal{F}/\Sigma_{i=1}^n (t_i - t_i^0)^{d_i^*} \mathcal{F})$$

and

$$\beta' : H^l(\pi^{-1}(t^0), \mathcal{F}/\Sigma_{i=1}^n (t_i - t_i^0)^{d_i} \mathcal{F}) \longrightarrow H^l(\pi^{-1}(t^0), \mathcal{F}/\Sigma_{i=1}^n (t_i - t_i^0)^{d_i^*} \mathcal{F})$$

are induced respectively by the quotient maps $\mathcal{F} \longrightarrow \mathcal{F}/\Sigma_{i=1}^n (t_i - t_i^0)^{d_i^*} \mathcal{F}$ and

$$\mathcal{F}/\Sigma_{i=1}^n (t_i - t_i^0)^{d_i} \mathcal{F} \longrightarrow \mathcal{F}/\Sigma_{i=1}^n (t_i - t_i^0)^{d_i^*} \mathcal{F}.$$

Let $\mu: H^l(X(\varrho'), \mathcal{F}) \longrightarrow H^l(\pi^{-1}(W), \mathcal{F})$ and $\mu': H^l(X(\varrho'), \mathcal{F}) \longrightarrow H^l(\pi^{-1}(W'), \mathcal{F})$ be induced respectively by $\pi^{-1}(W) \hookrightarrow X(\varrho')$ and $\pi^{-1}(W') \hookrightarrow X(\varrho')$.

Let ξ^* be the image of $\tilde{\xi}$ in $H^l(\pi^{-1}(t^0), \mathcal{F}/\Sigma_{i=1}^n (t_i - t_i^0)^{d_i} \mathcal{F})$. There exists $\zeta \in H^l(X(\varrho'), \mathcal{F})$ such that $\beta'(\xi^*) = \alpha'(\zeta)$. Hence $\beta(\tilde{\xi} - \mu(\zeta)) = 0$. It follows that

$$\alpha(\tilde{\xi}) - \mu'(\zeta) \in \Sigma_{i=1}^n (t_i - t_i^0)^{e_i} H \ (\pi^{-1}(W'), \mathcal{F}).$$

Since $\varrho'' = \gamma(\varrho')$, there exist $a_1, \dots, a_k \in \Gamma(K(\varrho''), \mathcal{F})$ such that $r''(\zeta) = \Sigma_{i=1}^k a_i r(\tilde{\xi}_i)$.

Consequently

$$\xi - (\Sigma_{i=1}^k a_i \xi_i)_{t^0} \in (\mathfrak{m}(t^0) \pi_l(\mathcal{F}))_{t^0}.$$

Since ξ is an arbitrary element of $\pi_l(\mathcal{F})_{t^0}$, we have

$$\pi_l(\mathcal{F})_{t^0} \subset (\Sigma_{i=1}^n {}_n\mathcal{O} \xi_i)_{t^0} + \mathfrak{m}(t^0)_{t^0} \pi_l(\mathcal{F})_{t^0}.$$

By Nakayama's lemma $\pi_l(\mathcal{F})_{t^0} = (\Sigma_{i=1}^n {}_n\mathcal{O} \xi_i)_{t_0}$. Condition (i) of Lemma 1 is proved.

(*b*) To prove condition (ii) of Lemma 1, take $\nu \in \mathbf{N}$. By using $(I)_n^l$ (obtained from Proposition 4), we can find

$$(d_1, \dots, d_n), (e_1, \dots, e_n) \in \mathbf{N}_*^n$$

such that $d_i \geqq e_i \geqq \nu \ (1 \leqq i \leqq n)$ and one has

$$r'(\operatorname{Ker} \sigma) \subset \Sigma_{i=1}^n (t_i - t_i^0)^{e_i} H^l(X(\varrho'), \mathcal{F}),$$

where

$$\sigma : H^l(X(\varrho^0), \mathcal{F}) \longrightarrow H^l(\pi^{-1}(t^0), \mathcal{F}/\Sigma_{i=1}^n (t_i - t_i^0)^{d_i} \mathcal{F})$$

is induced by the quotient map $\mathcal{F} \longrightarrow \mathcal{F}/\Sigma_{i=1}^n (t_i - t_i^0)^{d_i} \mathcal{F}$. Choose $d \in \mathbf{N}_*$ such that $d \geqq \nu$ and $\mathfrak{m}(t^0)^d \subset \Sigma_{i=1}^n (t_i - t_i^0)^{d_i}\, {}_n\mathcal{O}$.

Take $\xi = \Sigma_{i=1}^k a_i \xi_i$ where $a_i \in \Gamma(K(\varrho^0), {}_n\mathcal{O})$, $1 \leqq i \leqq k$, such that $\xi_{t^0} \in \in (\mathfrak{m}(t^0)^d \pi_l(\mathcal{F}))_{t^0}$. Let $\tilde{\xi} = \Sigma_{i=1}^n a_i \tilde{\xi}_i$. Then $\sigma(\tilde{\xi}) = 0$. Hence

$$r'(\tilde{\xi}) \in \Sigma_{i=1}^n (t_i - t_i^0)^{e_i} H^l(X(\varrho'), \mathcal{F}).$$

Since $e_i \geqq \nu \; (1 \leqq i \leqq n)$ and

$$\operatorname{Im} r'' \subset \Sigma_{i=1}^k \Gamma(K(\varrho''), {}_n\mathcal{O})\, r(\tilde{\xi}),$$

it follows that

$$r(\tilde{\xi}) \in \Sigma_{i=1}^k \Gamma(K(\varrho''), \mathfrak{m}(t_0)^\nu)\, r(\tilde{\xi}_i).$$

Consequently

$$\xi \mid K(\varrho'') \in \Sigma_{i=1}^k \Gamma(K(\varrho''), \mathfrak{m}(t^0)^\nu)\, \xi_i \mid K(\varrho'').$$

Condition (ii) of lemma 1 is proved. Q. E. D.

PROOF OF MAIN THEOREM.

As remarked earlier, the case where S is perfect is no more general than the case where S is nonsingular. We can assume without loss of generality that S is an open polydisc in $\mathbf{C}^n$.

Fix arbitrarily $a < c < a^* < b^* < d < b$. Let $X_c^d = \{c < \varphi < d\}$ and $\pi_c^d = \pi \mid X_c^d$. By using the techniques of [1] and trivially modifying the arguments used in the proofs leading to [9, Proposition 12.1 and Corollary to Proposition $(14.1)_n$], we obtain the following two statements for $p \leqq l < r - q - 2n + 1$:

(i) $(\pi_c^d)_l(\mathcal{F})_t$ is finitely generated over ${}_n\mathcal{O}_t$ for $t \in S$.

(ii) For every $t \in S$ and every open neighborhood U of t in S there exists an open neighborhood U' of t in U such that the $\Gamma(U', {}_n\mathcal{O})$ submodule of $H^l((\pi_c^d)^{-1}(U'), \mathcal{F})$ generated by the image of $H^l((\pi_c^d)^{-1}(U), \mathcal{F}) \longrightarrow H^l((\pi_c^d)^{-1}(U'), \mathcal{F})$ is finitely generated over $\Gamma(U', {}_n\mathcal{O})$.

By the Coherence Theorem, $(\pi_c^d)_l(\mathcal{F})$ is coherent on S for $p \leqq l < r - q - 3n + 1$.

For $t^0 \in S$ and $e_1, \ldots, e_k \in \mathbf{N}$, by applying conclusion (i) above to the sheaf $\mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)^{e_i} \mathcal{F}$ and the map $\sigma\pi : X \longrightarrow \sigma(S)$ (where $\sigma : \mathbf{C}^n \longrightarrow \mathbf{C}^{n-k}$ is the projection onto the last $n-k$ coordinates), we conclude that $(\pi_c^d)_l(\mathcal{F}/\Sigma_{i=1}^k(t_i - t_i^0)^{e_i}\mathcal{F})_{t_0}$ is finitely generated over ${}_n\mathcal{O}_{t^0}$ for $p \leqq l < r-q-2n+1$.

By the methods and results of [1], for $t^0 \in S$ and $e_1, \dots, e_n \in \mathbf{N}_*$, the restriction map

$$H^l(X_{c'}^{d'}, \mathcal{F}/\Sigma_{i=1}^n(t_i - t_i)^{e_i}\mathcal{F}) \to H^l(X_c^d, \mathcal{F}/\Sigma_{i=1}^n(t_i - t_i^0)^{e_i}\mathcal{F})$$

is an isomorphism for $p \leqq l < r-q-n$ and $a < c' < c < d < d' < b$. By [9, Proposition 4.1] the sheaf homomorphism $(\pi_{c'}^{d'})_l(\mathcal{F}) \to (\pi_c^d)_l(\mathcal{F})$ is injective for $p \leqq l < r-2n+1$ and $a < c' \leqq c < d \leqq d' < b$.

By using the techniques of [1] we conclude easily that for any relatively compact Stein open subset G of S, the restriction map $H^l(\pi^{-1}(G), \mathcal{F}) \to H^l((\pi_c^d)^{-1}(G), \mathcal{F})$ is surjective for $p \leqq l < r-q-n$ and is bijective for $p < l < r-q-n$ (cf. the proof of [9, Proposition 11.12])

Q. E. D.

The following theorem is a consequence of the Main Theorem and the results of [12] (and [1]).

SEMI-CONTINUITY THEOREM. *Suppose the assumptions of the Main Theorem are satisfied and, in addition, S is nonsingular and $\mathcal{F}$ is π-flat on X. For $s \in S$ and $l \in \mathbf{N}_*$ let $d_l(s)$ be the dimension of $H^l(\pi^{-1}(s), \mathcal{F}/\mathfrak{m}(s)\mathcal{F})$ over* C, *where $\mathfrak{m}(s)$ is the maximum sheaf of ideals on S whose zero-set is $\{s\}$. Then the following three conclusions hold.*

(i) *For $p \leqq l < r-4n+1$ the function d_l is finite-valued upper semicontinuous on S and for every $k \in \mathbf{N}_*$ the subset of S where $d_l \leqq k$ is a subvariety of S.*

(ii) *If d_l is constant on S for some $p < l < r-4n+1$, then $\pi_l(\mathcal{F})$ is locally free on S.*

(iii) *If $p \leqq k \leqq l \leqq r-q-4$ and the functions d_k and d_l are constant on S, then the partial Euler-Poincaré characteristic $\Sigma_{i=k}^l(-1)^i d_i$ is locally constant on S.*

REMARK. In the Semi-Continuity Theorem, if $\{\varphi \leqq a^*\} = \emptyset$, then r can be taken to be ∞ and conclusions (i) and (iii) remain valid if S is only assumed to be an arbitrary complex space.

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