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ON CENTROIDS OF GENERALIZED REGULAR RINGS

by NORMAN LANG

In [3], Fuchs and Halperin proved that any regular ring (in the sense of von Neumann) can be embedded as an ideal in a regular ring with unity and in [5], Funayama showed that the centroid of a regular ring is regular.

The corresponding questions can then be asked about the various types of generalized regular rings and, in fact, the first has been settled by Fuchs and Rangaswamy in [4].

The present paper is concerned with the centroids of generalized regular rings and the question of when these centroids are generalized regular. No set of necessary and sufficient conditions is obtained, but one necessary and some reasonably general sufficient conditions are found. The relevance of these is illustrated by examples and some properties of generalized regular rings are found along the way. The last section gives a result on semigroup rings which applies easily to matrix rings and many of their subrings.

§ **1.** All rings discussed are associative, An element a of a ring R is called *m-regular* (for an integer $m > 0$) if $a^m = a^m x a^m$ for some $x \in R$, and $\overline{m}$*-regular* if it is n-regular for all $n \geq m$. It is *left* (*right*) *m-regular* if $a^m = y a^{m+1}$ ($a^m = a^{m+1} y$) for some $y \in R$ and *strongly m-regular* if it is both left and right m-regular. Left, right and strong $\overline{m}$-regularity are similarly defined. The ring R is (left, right, strongly) *m-regular* ($\overline{m}$*-regular*) if every element has this property. If every element of R is n-regular, for some n which may depend on the element, R is said to be *π-regular.*

For a ring R, let $E(R)$ be the ring of all endomorphisms of the additive group of R and define $\widetilde{R} = \{\varrho \in E(R) \mid (xy)\varrho = (x\varrho) y = x (y\varrho)$ for all

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$x, y \in R\}$. $\widetilde{R}$ is then a ring with unity and will be called the *centroid* of R. In the case where R is an algebra over a field, it is not assumed that the elements of the centroid are linear transformations, although this will be the case if $R^2 = R$.

It is known that if $R^2 = R$ or if the two-sided annihilator A of R is zero, then $\widetilde{R}$ is a commutative ring. If R has unity, it is immediate that $\widetilde{R}$ is isomorphic to the centre of R and in [7], McCoy shows that the centre of a π-regular (m-regular) ring is π-regular (m-regular). Our problem is therefore trivial in the presence of an identity.

In [5], Funayama proved that, for $\varrho \in R$, the ring decomposition $\widetilde{R} = \operatorname{Im} \varrho \oplus \operatorname{Ker} \varrho$ holds if and only if the following two conditions are satisfied:

a) For all $x \in R$, $x\varrho^2 = 0$ implies $x\varrho = 0$.

b) For any $x \in R$, there exists $y \in R$ such that $x\varrho = y\varrho^2$.

It is evident that *a*) is equivalent to $\operatorname{Ker} \varrho = \operatorname{Ker} \varrho^2$, which in turn is equivalent to $\operatorname{Ker} \varrho \cap \operatorname{Im} \varrho = 0$. Similarly, either of the statements, $\operatorname{Im} \varrho = \operatorname{Im} \varrho^2$ or $R = \operatorname{Im} \varrho + \operatorname{Ker} \varrho$ is equivalent to *b*),

Funayama also shows that, for $\varrho \in \widetilde{R}$, $R = \operatorname{Im} \varrho \oplus \operatorname{Ker} \varrho$ implies that ϱ is regular in $\widetilde{R}$. It can then the observed that, if $\widetilde{R}$ is commutative, ϱ is regular in $\widetilde{R}$ if and only if conditions *a*) and *b*) hold.

The following results from [4] are used below and are given here for convenience.

(A). A π-regular ring R can be embedded as an ideal in a π-regular ring with identity if and only if the following two conditions hold.

Ai. The p-component, R_p, of the additive group of R is bounded, for all primes p (i.e. $p^k R_p = 0$ for some integer k).

Aii. R/R_t is a divisible abelian group, where R_t is the torsion part of the additive group of R.

(B). If R is a π regular ring with conditions Ai and Aii and if $p_1, \dots, p_h$ is any finite set of primes, then $R = R_{p_1} \oplus \dots \oplus R_{p_h} \oplus R_0$ (ring decomposition) where no element of R_0 has order divisible by any of the p_i.

(C). If R is a ring with a π regular ideal I and if, for all $a \in R$, there exists an $x \in R$ and an integer m such that $a^m x = xa^m$ and $a^m xa^m - a^m \in I$, then R is π regular.

§ **2.** The first result correlates some of the above facts to give a necessary condition.

THEOREM 1. If R is a π-regular ring such that $\widetilde{R}$ is also π-regular, then R satisfies conditions Ai and Aii.

PROOF. Let $\widetilde{Z}$ be the centre of $\widetilde{R}$. Then $\widetilde{Z}$ is a π-regular ring (with identity). Define $R^* = \{(a, \nu) \mid a \in R, \nu \in \widetilde{Z}\}$ with component-wise addition and multiplication defined by $(a, \nu)(b, \mu) = (ab + b\nu + a\mu, \nu\mu)$. R^* is then a ring with an ideal $(R, 0)$ which can be identified with R. Let $(a, \nu) \in R^*$. Since $\widetilde{Z}$ is π-regular, there exists a μ in $\widetilde{Z}$ and an integer m such that $\nu^m = \nu^m \mu \nu^m$. Then $(a, \nu)^m (0, \mu)(a, \nu)^m - (a, \nu)^m$ is an element of R and $(a, \nu)(0, \mu) = (0, \mu)(a, \nu)$. It follows by (C) that R^* is π-regular and since R is embedded as an ideal in R^*, we see from (A) that R satisfies the required conditions.

The conditions Ai and Aii are not, however, sufficient to ensure that $\widetilde{R}$ be π-regular. This is seen from Example 1 of § 5.

The following lemma is used repeatedly in the sequel.

LEMMA 1. Let $R = \bigoplus_{i \in I} R_i$, where I is any set and each R_i is a ring with either of the properties:

i) The two-sided annihilator of each R_i is zero.

ii) $R_i^2 = R_i$ for all $i \in I$.

Then $\widetilde{R}$ is isomorphic to $\prod_{i \in I} \widetilde{R}_i$.

PROOF. Let $\varrho \in \widetilde{R}$ and let $x \in R_i$, some $i \in I$. Then, for $j \neq i$, $R_j(x\varrho) = 0 = (x\varrho) R_j$. This shows that the j — th component of $x\varrho$ is zero if condition i) holds. If ii) holds, write $x = \Sigma u_k v_k$ where $u_k, v_k \in R_i$. Then $x\varrho = \Sigma u_k (v_k \varrho) \in R_i$.

In either case, ϱ restricted to R_i maps R_i into R_i for all $i \in I$ and is thereby an element of $\widetilde{R}_i$. It is then clear that the map $\varrho \rightarrow (\dots, \varrho_i, \dots)$, where ϱ_i is the restriction of ϱ to R_i, defines the required isomorphism.

§ 3. In [8], Nagata shows that, if H is an algebra over a field of characteristic zero and if $h^n = 0$ for all $h \in H$ and a fixed integer n, then H is nilpotent. In [6], Higman extended this result to the case of a field whose characteristic is either zero or greater than n. By looking at this later proof, one can see that the following stronger result holds.

LEMMA 2. Let R be a ring such that $x^n = 0$ for all $x \in R$ and a fixed positive integer n, and suppose that, for $x \in R$, $n!\,x = 0$ implies $x = 0$. Then R is nilpotent.

In the torsion-free case (which is discussed in the sequel) Lemma 2 can be established directly from Nagata's result by embedding R in its divisible hull, D, endowed with the unique multiplication which extends that of R. D is then an algebra over the rationals and $d^n = 0$ for all $d \in D$. By Nagata, D, and therefore R, is nilpotent.

The next lemma establishes a structure for certain m-regular rings which has considerable effect on the centroid.

LEMMA 3. Let R be an m-regular ring with the properties:

i) $R^2 = R$.

ii) For $x \in R$, $m!\,x = 0$ implies $x = 0$.

Then R coincides with the ideal generated by the m-th powers of its elements.

PROOF. By m-regularity, the ideal generated by the m-th powers is $T = \{\sum_i r_i u_i^m s_i \mid r_i, u_i, s_i \in R\}$. For $x \in R$, write $\bar{x}$ for the coset $x + T$. Then $\bar{x}^m = 0$ for all $\bar{x} \in R/T$. Now let u be an element of R. We have $(m!\,u)^m = (m!\,u)^m u' (m!\,u)^m$ for some $u' \in R$. Then $u^m = m!^m u^m u' u^m$ by condition ii). Suppose that $m!\,\bar{x} = 0$ in R/T. Then $m!\,x \in T$ and so can be written in the form $\sum_i r_i u_i^m s_i = \sum_i m!^m r_i u_i^m u_i' u_i^m s_i$ as above. This shows that $x \in T$, so that $\bar{x} = 0$. By Lemma 2, R/T is nilpotent. But $R^2 = R$ implies that $R = R^n$ for all positive integers n. Therefore R/T nilpotent implies $R = T$.

We can now give sufficient conditions for the centroid to be m-regular.

THEOREM 2. Let R be an m regular ring with the properties:

i) $R^2 = R$.

ii) For $x \in R$, $m!x = 0$ implies $x = 0$.

iii) The two-sided annihilator, A, of R, is zero.

Then $\widetilde{R}$ is m-regular.

PROOF. By Lemma 3, $R = \{\sum_i r_i u_i^m s_i \mid r_i, u_i, s_i \in R\}$. Take $\varrho \in \widetilde{R}$ and $x \in R$. We can write $x = \sum_i r_i u_i^m s_i$, so that $x\varrho^m = \sum_i r_i (u_i \varrho)^m s_i = \sum_i r_i (u_i \varrho)^m u_i' (u_i \varrho)^m s_i$ for some $u_i' \in R$. This gives $x\varrho^m = y\varrho^{2m}$ for some $y \in R$, which is condition b) of Funayama for ϱ^m. Then $R = \mathrm{Im}\,\varrho^m + \mathrm{Ker}\,\varrho^m$.

Now let $x \in \mathrm{Ker}\,\varrho^m \cap \mathrm{Im}\,\varrho^m$ and let y be any element of R. Write $y = a\varrho^m + b$, where $a \in R$ and $b \in \mathrm{Ker}\,\varrho^m$. Then $xy = xa\,\varrho^m + xb = (x\varrho^m)\,a + z\,(b\varrho^m) = 0$, where $x = z\varrho^m$, $z \in R$. Similarly, $yx = 0$. By condition iii), we have $x = 0$ and so $R = \mathrm{Im}\,\varrho^m \oplus \mathrm{Ker}\,\varrho^m$ which implies that ϱ^m is a regular element.

If R is an m-regular ring with the (necessary) conditions Ai and Aii, the above result can be generalized as follows. Let $p_1, \ldots, p_h$ be the prime factors of m! Using (B) write $R = R_{p_1} \oplus \dots \oplus R_{p_h} \oplus R_0$, where the decomposition is ring theoretical and R_0 has the property ii) of Theorem 2. The other two conditions, i) and iii), will therefore ensure that $\widetilde{R}_0$ be m-regular. (Note that a ring-direct summand of an m-regular ring is m-regular.) It is easy to see that, for $\varrho \in \widetilde{R}$, ϱ restricted to R_{p_i} is an element of $\widetilde{R}_{p_i}$ for $i = 1, \dots, h$ and the same is true for R_0 by condition i). As in Lemma 1, we have $\widetilde{R} = \widetilde{R}_{p_1} \oplus \dots \oplus \widetilde{R}_{p_h} \oplus \widetilde{R}_0$. This establishes the following corollary to Theorem 2.

COROLLARY 1. Let R be an m-regular ring with conditions **Ai** and **Aii**. Using the above notation, assume also that:

i) $R_0^2 = R_0$.

ii) The two-sided annihilator of R_0 is zero.

iii) For each $i = 1, \dots, h$, $\widetilde{R}_{p_i}$ is m_i-regular for some m_i. [For example, R_{p_i} Artinian would be sufficient.]

Then $\widetilde{R}$ is m_0-regular, where m_0 is the least common multiple of m, $m_1, \dots, m_h$.

§ 4. We now turn to the case where R is an algebra over a field of characteristic zero or greater than m. First we show that this is more general than the torsion-free case.

LEMMA 4. Let R be an m-regular ring with properties:

i) $R^2 = R$.

ii) R is torsion-free.

Then R is an algebra over the rationals, Q.

PROOF. By Lemma 3, $R = \{\Sigma r_i u_i^m s_i \mid r_i, u_i, s_i \in R\}$. Let $x \in R$ and choose any positive integer n. There exists an $x' \in R$ such that $n^m x^m = n^m x^m x' n^m x^m$, so that x^m is divisible by n. It follows that R is divisible and therefore an algebra over Q.

The next result sharpens the structure found in Lemma 3 in a manner which affects the annihilator.

LEMMA 5. Let R be any ring with the properties:

i) $R^2 = R$.

ii) R is an algebra over a field F, which is of characteristic zero or greater than m.

Then $R = I_m = \{\sum_i f_i u_i^m \mid f_i \in F, u_i \in R\}$.

Proof. It is proved in [6] that, under condition ii), I_m is an ideal in R. R/I_m is then an algebra over F and $\bar{x}^m = 0$ for $x \in R/I_m$, so that R/I_m is nilpotent. Condition i) then gives $R = I_m$ as before.

It might be noted that if the characteristic of F is zero, $R = I_n$ for any positive integer n. Otherwise, $R = I_n$ for $n \leq m$.

We now state two properties which imply the absence of annihilators in certain m-regular rings.

1) For all $a, x \in R$, there exist $b, y \in R$ such that $ax^m = b^m y$ for a fixed integer m.

2) For all $a, x \in R$, there exist $b, y \in R$ such that $a^m x = by^m$ for a fixed integer m.

Theorem 3. Let R be an m-regular ring with the properties:

i) $R^2 = R$.

ii) R is an algebra over a field F of characteristic zero or greater than m.

iii) R has either of the properties 1) or 2) with the integer m.

Then the two sided annihilator A of R, is zero

Proof. Suppose that R has property 1) and take $a \in A$. By Lemma 5, there exist $f_i \in F$, $u_i \in R$, such that $a = \sum_i f_i u_i^m = \sum_i f_i u_i^m u_i' u_i^m$ for some $u_i' \in R$. This is of the form $\sum_i a_i u_i^m$, with $a_i \in R$. Let $a = \sum_{i=1}^{n} a_i u_i^m$ be a shortest possible representation of a in this way. If $n \geq 1$, $a = \sum_{i=1}^{n-1} a_i u_i^m + a_n u_n^m = \sum_{i=1}^{n-1} a_i u_i^m + v^m b$ for some $v, b \in R$, by 1). Then $0 = v^m v' a = \sum_{i=1}^{n-1} v^m v' a_i u_i^m + v^m b$, where $v^m = v^m v' r^m$. This gives $a_n u_n^m = - \sum_{i=1}^{n-1} s_i u_i^m$, where s_i are in R, so that $a = \sum_{i=1}^{n-1} (a_i - s_i) u_i^m$. This is a contradiction unless $a = 0$, which establishes the result in this case. The other case is similar.

Corollary 2. Let R be an m-regular ring with the properties:

i) $R^2 = R$.

ii) R is an algebra over a field of characteristic zero or greater than m.

iii) All idempotents of R are central.

Then $\widetilde{R}$ is m-regular.

Proof. Let $a, x \in R$. Then $ax^m = ax^m x' x^m$ for some $x' \in R$ and $x^m x'$ is an idempotent. Then $ax^m = x^m (x' ax^m)$ so that condition 1) holds with m. It follows by Theorem 3 that the annihilator is zero and Theorem 2 applies.

In the case where R is an algebra, condition iii) of Theorem 2 can be weakened. We shall make use of the following lemma.

LEMMA 6. Let R be an m-regular ring with the properties:

i) $R^2 = R$.

ii) For $x \in R$, $m!\,x = 0$ implies $x = 0$.

Then $\widetilde{R}$ is m-regular if and only if it is π-regular.

PROOF. Suppose that $\widetilde{R}$ is π-regular and let $\varrho \in \widetilde{R}$. Then ϱ^n is regular for some positive integer n. If $n < m$, it is easy to see that ϱ^m is regular, since $\widetilde{R}$ is commutative, so we assume that $n > m$. In the proof of Theorem 2, it is seen that conditions i) and ii) are enough to give condition *b*) of Funayama for any ϱ^m, so that $\mathrm{Im}\,\varrho^n = \mathrm{Im}\,\varrho^m$ and $R = \mathrm{Im}\,\varrho^m + \mathrm{Ker}\,\varrho^m$. Because of i), ϱ^n regular implies $R = \mathrm{Im}\,\varrho^m \oplus \mathrm{Ker}\,\varrho^n$, so that $\mathrm{Im}\,\varrho^m \cap \mathrm{Ker}\,\varrho^m$ is contained in $\mathrm{Im}\,\varrho^n \cap \mathrm{Ker}\,\varrho^n = 0$. We then have $R = \mathrm{Im}\,\varrho^m \oplus \mathrm{Ker}\,\varrho^m$, which means that ϱ^m is regular.

It is clear that if R is an algebra over a field, so is the two-sided annihilator A of R. We have the following result.

THEOREM 4. Let R be an m-regular ring with the properties:

i) $R^2 = R$.

ii) R is an algebra over a field F of characteristic zero or greater than m.

iii) The two-sided annihilator A of R is of finite dimension over F.

Then $\widetilde{R}$ is m-regular.

PROOF. Suppose, by way of contradiction, that $\widetilde{R}$ is not π-regular. Then there exists $\varrho \in \widetilde{R}$ such that ϱ^n is not regular for any positive integer n. Take $n > m$. As in the proof of Theorem 2, $R = \mathrm{Im}\,\varrho^m + \mathrm{Ker}\,\varrho^m$ and $\mathrm{Ker}\,\varrho^{n+1} \cap \mathrm{Im}\,\varrho^m \subseteq A$. By assumption, there exists an $x \in R$ such that $0 \neq x \in \mathrm{Ker}\,\varrho^{n+1} \setminus \mathrm{Ker}\,\varrho^n$. Write $x = y_1 + y_2$, where $y_1 \in \mathrm{Im}\,\varrho^m$ and $y_2 \in \mathrm{Ker}\,\varrho^m$. Clearly, $y_1 = x - y_2 \in \mathrm{Ker}\,\varrho^{n+1} \cap \mathrm{Im}\,\varrho^m$, but $y_1 \notin \mathrm{Ker}\,\varrho^n$ since $x \notin \mathrm{Ker}\,\varrho^n$. But condition i) implies that all kernels and images are subalgebras over F and the above shows that the dimension of $\mathrm{Ker}\,\varrho^n \cap \mathrm{Im}\,\varrho^n$ is less than that of $\mathrm{Ker}\,\varrho^{n+1} \cap \mathrm{Im}\,\varrho^m$ for all integers $n > m$ and both are contained in A. This means that A is of infinite dimension, which contradicts iii). We conclude that $\widetilde{R}$ is π-regular and therefore m-regular by Lemma 6.

The question remains as to whether condition iii) can be removed from Theorem 2 or Theorem 4. At least we have the following result.

THEOREM 5. Let R be an m-regular ring with the properties:

i) $R^2 = R$.

ii) For $x \in R$, $m!\,x = 0$ implies $x = 0$.

Then $\widetilde{R}$ has a unique maximal m-regular ideal.

PROOF. We have $R = \operatorname{Im} \varrho^m + \operatorname{Ker} \varrho^m$ and $\operatorname{Im} \varrho^m = \operatorname{Im} \varrho^n$ for any $n \geq m$ and any $\varrho \in \widetilde{R}$. It is easy to see that, for any positive integer n, $\operatorname{Ker} \varrho^n \cdot R \subseteq \operatorname{Ker} \varrho^m$. Then $\operatorname{Ker} \varrho^n = R$ implies that $R = R^2 = \operatorname{Ker} \varrho^n \cdot R \subseteq \subseteq \operatorname{Ker} \varrho^m$, so that ϱ is nilpotent of index $\leq m$. This means that $\widetilde{R}$ has bounded index for its nilpotent elements and it follows, by Lemma 4 of [9], that $\widetilde{R}$ has a unique largest π-regular. The result follows by the proof of Lemma 6.

Finally, in this section, we remark that everything in § 3 and § 4 holds if R is left or ringht m-regular, but not, in general, if R is only π-regular. If R has the additional property of having bounded index for its nilpotent elements, then Theorem 2 of [9] and Corollary 6 of [2] show that (left or right) π-regularity implies (strong) m-regularity so that everything holds. In all cases considered, π-regularity implies strong $\overline{m}$-regularity for $\widetilde{R}$.

§ 5. We now look for a set of sufficient conditions with-out the hypothesis $R^2 = R$. The next lemma shows that, at least, the powers of R cannot form an infinite chain.

LEMMA 7. Let R be an m-regular ring such that, for $x \in R$, $m!\,x = 0$ implies $x = 0$. Then $R^n = R^{n+1}$ for some positive integer n.

PROOF. Let $T = \{\sum_i r_i u_i^m s_i \mid r_i, u_i, s_i \in R\}$. It has been seen in the proof of Lemma 3, that R/T has the properties of Lemma 2 and is therefore nilpotent. Therefore $R^n \subseteq T$ for some positive integer n. But $T \subseteq R^k$ for any positive integer k by m-regularity. Therefore $R^n = T = T^2$ and the result follows.

A torsion-free, m-regular ring is not necessarily divisible, but Theorem 1 shows that divisibility is a necessary condition, in this case, for the centroid to be π-regular. Then, with n as in Lemma 7, R^n and R/R^n are vector spaces over Q. Also, for $\varrho \in \widetilde{R}$, $\operatorname{Im} \varrho$ and $\operatorname{Ker} \varrho$ are divisible and are therefore subspace of R. We have the following result.

THEOREM 6. Let R be an m-regular ring with the properties:

i) R is torsion-free and divisible.

ii) With n as in Lemma 7, the dimension over Q of the two-sided annihilator of R^n is finite.

iii) With n as in Lemma 7, R/R^n is of finite dimension k over Q. Then $\widetilde{R}$ is $(k+m)$-regular.

PROOF. Let $T = \{\sum_i r_i u_i^m s_i \mid r_i, u_i, s_i \in R\}$; then $R^n = T$. Let $\varrho \in \widetilde{R}$. Since $T\varrho \subseteq T$, ϱ induces $\bar{\varrho} \in (\widetilde{R/T})$. Now $\operatorname{Im} \bar{\varrho}^t$ and $\operatorname{Ker} \bar{\varrho}^t$ are divisible subgroups of R/T for all positive integers t, and these subgroups satisfy the ascending and descending chain conditions, with no chain of length greater than k. Therefore $\operatorname{Im} \bar{\varrho}^k = \operatorname{Im} \bar{\varrho}^{k+1}$ and $\operatorname{Ker} \bar{\varrho}^k = \operatorname{Ker} \bar{\varrho}^{k+1}$.

Given $x \in R$, $(x+T)\bar{\varrho}^k = (y+T)\bar{\varrho}^{2k+m}$ for some $y \in R$, so that $x\varrho^k = y\varrho^{2k+m} + t$, where $t \in T$. Then $x\varrho^{k+m} = y\varrho^{2(k+m)} + t\varrho^m = y\varrho^{2(k+m)} + t'\varrho^{2(k+m)}$, where the last equality follows from the fact that T has the Funayama conditions, as already seen. We have $x\varrho^{k+m} = (y+t')\varrho^{2(k+m)}$ which is condition $b)$ of Funayana for ϱ^{k+m}. Also, since $\operatorname{Ker} \bar{\varrho}^{k+m} \cap \operatorname{Im} \bar{\varrho}^{k+m} = 0$, $x \in \operatorname{Ker} \varrho^{k+m} \wedge \operatorname{Im} \varrho^{k+m}$ implies $x \in T$. It follows that $x = 0$ since $\widetilde{T}$ is m-regular. Then ϱ^{k+m} has the Funayama conditions and therefore is regular.

It might be noted that the condition « R/R^n is of finite dimension » is equivalent to the more ring-theoretic (and apparently weaker) statement that R/R^n has minimum condition on ideal subspaces. An example below shows that this condition has some relevance.

It should also be observed that Theorem 6 cannot be generalized to the case where R is an algebra as before, because, in the absence of $R^2 = R$, there is nothing to guarantee that the images and kernels are subspaces.

LEMMA 8. Let R be a ring. Then ϱ is a regular element of $\widetilde{R}$ if and only if $\operatorname{Im} \varrho$ and $\operatorname{Ker} \varrho$ are ring-theoretic direct summands of R.

PROOF. This is an easy modification of Lemma 4 in [4].

EXAMPLE 1. Let R_1 and R_2 be torsion-free, divisible abelian groups of countable dimension as vector spaces over Q, with bases $\{x_i\}_0^\infty$ and $\{y_i\}_0^\infty$, respectively. Write $R = R_1 \oplus R_2$ with multiplication defined by $x_i x_j = y_{i+j}$ and any product involving a y_i is zero. Then R is a 3-regular ring with $R \supset R^2 \supset R^3 = 0$ and satisfies all the conditions of Theorem 6 except iii). Let ϱ be an additive endomorphism of R defined by $x_i \varrho = x_{i+1}$, $y_j \varrho = y_{j+1}$. It is seen that $\varrho \in \widetilde{R}$ and $\operatorname{Im} \varrho^n$ is a group direct summand for every positive integer n. But any group complement of $\operatorname{Im} \varrho^n$ contains an element z, with a non-zero component in R_1. Then zx_n is a non-zero element of $\operatorname{Im} \varrho^n$, so that the complement is not an ideal. By Lemma 8, ϱ^n is not regular for any n.

Example 1 also gives an m-regular ring with conditions Ai and Aii whose centroid is not π-regular.

The next example shows that condition iii) of Theorem 6 is not, however, a necessary one.

EXAMPLE 2. Let R_1 be a ring with trivial multiplication which is torsion-free and divisible of any dimension. Then $\widetilde{R}_1$ is just the set of all linear transformations and is regular. Let T be a torsion-free, m-regular ring with $T^2 = T$ and no non-zero annihilators. Let $R = T \oplus R_1$. Then R has all the conditions of Theorem 6 except iii and $\widetilde{R} = \widetilde{T} \oplus \widetilde{R}_1$ (by the proof of Lemma 1) and is m-regular.

§ **6.** We now prove a result on the centroids of semigroup rings. Throughout the section, S is a semigroup with zero.

For a subset T of S, denote by $I(T)$ the two-sided ideal generated by T. The following definitions will help the statement of the theorem.

DEFINITION 1. For $s, t \in T \subseteq S$, say that s and t are *connected in* T if there exists a finite sequence $u_0, u_1, \dots, u_n$ in T, with $u_0 = s$, $u_n = t$ such that $I(u_i) \cap I(u_{i+1}) \neq 0$ for $i = 0, \dots, n-1$.

DEFINITION 2. A *connectivity component* of S is an equivalence class under the relation: $s \backsim t$ if and only if s and t are connected in S.

The semigroup S is then the disjoint union of its connectivity components and zero. If zero is adjoined to a connectivity component H_k, the component becomes a semigroup with zero, which we also denote by H_k. It is clear that if $\{H_k\}_{k \in K}$ is the set of connectivity components of S, we have $H_{k_1} H_{k_2} = 0$ for all distinct $k_1, k_2 \in K$.

If R is a ring, denote by $R[S]$ the semigroup ring of S over R, this consisting of all formal expressions $\Sigma r_i s_i$ (finite sums) where $r_i \in R$, $0 \neq s_i \in S$, with the usual definitions of equality, addition and multiplication. The zero of S is identified with that of R. It is clear that $R[S]^2 = R[S]$ if and only if $R^2 = R$ and $S^2 = S$ and the above remarks show that, if $\{H_k\}_{k \in K}$ is the set of connectivity components of S, we have $R[S] = \bigoplus_{k \in K} R[H_k]$ (ring decomposition).

The following definition concerns the main hypothesis of the theorem.

DEFINITION 3. An element $s \in S$ will be called *unmovable* if, for any finite subset F of S, there exist $s', s'' \in S$ such that:

i) $ss' = s$,

ii) $s'' s = s$,

iii) $Fs' \cap s'' F \subseteq \{ s, 0 \}$.

THEOREM 7. Let R be a ring with $R^2 = R$ and S a semigroup with zero. Let $\{H_k\}_{k \in K}$ be the set of connectivity components of S, all with zero adjoined, and assume that, for every $k \in K$, there exists a subset $T^k \subseteq H_k$ such that:

i) Any two elements of T_k are connected in T_k.

ii) $H_k = I(T_k)$.

iii) Every element of T_k is unmovable.

Then $\widetilde{R[S]}$ is isomorphic to $\prod_{k \in K} \widetilde{R}_k$, where R_k is isomorphic to R for all $k \in K$.

PROOF. Take a connectivity component H, and a subset T of H, as in the statement. Let $\bar{\varrho} \in \widetilde{R[H]}$ and describe the action of $\bar{\varrho}$ by $(xs)\bar{\varrho} = \sum_i f^s_{s_i}(x)\, s_i \in R[H]$, where $s, s_i \in H$ and $x, f^s_{s_i}(x) \in R$. It is clear that, for every pair $s, t \in H$, the function defined by $x \to f^s_t(x)$ for $x \in R$ is an additive endomorphism of R.

Let $t \in T$ and $x, y \in R$. Let $F = \{s_1, \dots, s_n\}$ be the finite set of all $s_i \in H$ such that at least one of the values $f^t_{s_i}(x)$, $f^t_{s_i}(y)$, $f^t_{s_i}(xy)$ is non-zero. Now choose t', t'' corresponding to t and F by iii). We have:

$$(xy \cdot t)\bar{\varrho} = \sum_{i=1}^{n} f^t_{s_i}(xy)\, s_i \quad \dots \dots \quad (1).$$

$$(xy \cdot t)\bar{\varrho} = [(xt)(yt')]\bar{\varrho} = [(xt)\bar{\varrho}](yt') = \sum_{i=1}^{n} f^t_{s_i}(x)\, y \cdot s_i t' \dots \quad (2).$$

$$(xy \cdot t)\bar{\varrho} = [(xt'')(yt)]\bar{\varrho} = (xt'')[(yt)\bar{\varrho}] = \sum_{i=1}^{n} x f^t_{s_i}(y) \cdot t'' s_i \dots \quad (3).$$

Suppose that $f^t_{s_i}(xy) \neq 0$ for some i, $1 \leq i \leq n$. Then there is at least one non-zero term in s_i in (2) and in (3). Therefore $s_i = s_j t' = t'' s_k$ for some $1 \leq j, k, \leq n$, so that $s_i \in Ft' \cap t'' F$. It follows that $s_i = t$ and that $f^t_s(xy) = 0$ for all $x, y \in R$ and all $s \neq t$, $s \in H$. Since $R^2 = R$ and the functions are additive, $f^t_s = 0$ for all $s \neq t$, $s \in H$. Write $f^s_s = f^s$ for any $s \in H$. Then (1), (2) and (3) show that $f^t(xy) = f^t(x)y = xf^t(y)$ for all $x, y \in R$, so that $f^t \in \widetilde{R}$.

Now let $0 \neq g \in H$, $t \in T$. For any $x, y \in R$, we have $(xy \cdot gt)\bar{\varrho} = xg\,[(yt)\bar{\varrho}] = xg \cdot f^t(y)\, t = f^t(xy) \cdot gt$. This means that $f^t = f^{gt}$ and, similarly, $f^t = f^{tg}$, so that, for any $s \in I(t)$ and any $x \in R$, $(xs)\bar{\varrho} = f^s(x)\, s$ and we have $f^s = f^t$.

Let $t_1, t_2 \in T$. We can find $u_0, u_1, \ldots, u_n$ in T, where $u_0 = t_1$ and $u_n = t_2$ such that $I(u_i) \cap I(u_{i+1}) \neq 0$ for $i = 0, \ldots, n-1$. The preceding remark shows that $f^{t_1} = f^{t_2}$. We now have that, for any non-zero elements s_1, s_2 in H, $f^{s_1} = f^{s_2} = \varrho \in \widetilde{R}$ and $(xs)\bar{\varrho} = x\varrho \cdot s$ for all $x \in R$, $s \in H$. It is clear that the map $\varrho \to \bar{\varrho}$ is an isomorphism from $\widetilde{R}$ onto $\widetilde{R[H]}$.

As previously mentioned, $R[S] = \bigoplus_{k \in K} R[H_k]$ and it is clear that properties i), ii) and iii) imply that $H_k^2 = H_k$ so that $R[H_k]^2 = R[H_k]$ for each $k \in K$. By Lemma 1, this gives $\widetilde{R[S]} \cong \prod_{k \in K} \widetilde{R[H_k]} \cong \prod_{k \in K} \widetilde{R}$.

The following corollary is immediate.

COROLLARY 3. If R is a ring with $R^2 = R$ such that $\widetilde{R}$ is m regular and if S is a semigroup as in Theorem 7, then $R[S]$ is m regular.

COROLLARY 4. If R is a ring with $R^2 = R$ and n is a positive integer, the centroid of the matrix ring R_n is isomorphic to the centroid of R.

PROOF. We can write $R_n = R[S]$ where S is the semigroup consisting of zero and the matrix units. Since S is connected and equal to the ideal generated by any non-zero element and since every element is unmovable, Theorem 7 applies.

In the case where S is a Rees matrix semigroup, we can interpret the conditions of Theorem 7 in a precise way. For complete definitions, we refer to [1], page 88.

Let I, Λ be any sets and let G° be any group G, with zero adjoined. Let P be any $\Lambda \times I$ matrix with elements from G°. If $a \in G^\circ$, $i \in I$, $\lambda \in \Lambda$, then $(a)_{i\lambda}$ denotes the *Rees $I \times \Lambda$ matrix over G°* having a in the $i\lambda$-th position and zeroes elsewhere. The set of all such elements, with multiplication defined by $(a)_{i\lambda} \circ (b)_{j\mu} = (ap_{\lambda j} b)_{i\mu}$ is called the *Rees $I \times \Lambda$ semigroup over G° with sandwich matrix P* and denoted by $M^\circ(G; I, \Lambda; P)$. We have the result:

THEOREM 8. $M^\circ(G; I, \Lambda; P)$ has the properties of Theorem 7 if and only if G is the group of one element.

PROOF. Write $G^\circ = \{0, 1\}$ and assume that $P = (p_{\lambda i})$ is not the zero matrix. Choose $i \in I$ and $\lambda \in \Lambda$ such that $p_{\lambda i} = 1$. Then $(1)_{i\lambda} \circ (1)_{i\lambda} = (1)_{i\lambda}$, so that $(1)_{i\lambda}$ has left and right identities. Now suppose that $(1)_{i\lambda} \circ (a)_{kl} = (b)_{mn} \circ (1)_{i\lambda}$ for some $(a)_{kl}, (b)_{mn} \in M^\circ(G; I, \Lambda; P)$. Clearly, l must be equal to λ, so that each product is either $(1)_{i\lambda}$ or zero and $(1)_{i\lambda}$ is unmovable.

Let $(1)_{j\mu}$ be any non-zero element of $M^\circ(G; I, \Lambda; P)$. Then $(1)_{j\mu} = (1)_{j\lambda} \circ (1)_{i\lambda} \circ (1)_{i\mu}$ shows that the ideal generated by $(1)_{i\lambda}$ is the whole semigroup. This shows sufficiency.

Suppose that $G^\circ \neq \{0, 1\}$. Let $(a)_{i\lambda}$ be any non-zero element of $M^\circ(G; I, \Lambda; P)$. If $S \circ (a)_{i\lambda} = 0$, or $(a)_{i\lambda} \circ S = 0$, then $s = (a)_{i\lambda}$ is not unmovable. Otherwise, candidates for s', s'' as in Definition 3, must be of the form $(p_{\lambda j}^{-1})_{j\lambda}$, $(p_{\mu i}^{-1})_{i\mu}$, respectively. Now take $x \in G^\circ$ such that $0 \neq x \neq a$. Then $(p_{\mu i}^{-1})_{i\mu} \circ (x)_{i\lambda} = (x)_{i\lambda} \circ (p_{\lambda j}^{-1})_{j\lambda} = (x)_{i\lambda} \neq (a)_{i\lambda}$ or zero, so that condition iii) of the definition is not satisfied and $(a)_{i\lambda}$ is not unmovable.

Finally, we remark that Theorems 7 and 8 apply to an example, by Hans Storrer, which shows that condition iii) is indispensable in Theorem 3. Let X be the ring of 3×3 matrices with real entries and zeroes in the first row and last column. Then X is a 3-regular algebra over the reals and $X^2 = X$. But the annihilator of X is not zero. Since this annihilator is of finite dimension over the reals, Theorem 4 predicts that $\widetilde{X}$ is 3-regular. In fact, X satisfies the conditions of Theorem 7 (and is a Rees matrix as in Theorem 8) so that $\widetilde{X}$ is just the reals and is therefore regular.

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