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J. F. PRICE

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MEANS OF VECTOR-VALUED FUNCTIONS AND PROJECTIONS WHICH COMMUTE WITH THE ACTION OF A GROUP

J. F. PRICE

ABSTRACT - This paper falls into two parts. In the first part we extend the concept of a mean of scalar-valued functions to that of a mean of vector-valued functions and then study some of the properties of this concept.

In the second part we use these results to provide a solution to the following problem. Suppose that V is a Hausdorff locally convex topological vector space which is continuously acted upon by a topological group G (of continuous endomorphisms of V) and suppose further that there exists a continuous idempotent endomorphism (that is a projection) of V onto a subspace U of V which is invariant under the action of G. What are conditions on V, G and U which ensure the existence of a projection Q from V onto U which commutes with each of the operators in G? Results of this type are known and used in several areas (for example, the theory of representations of finite groups and compact groups, and the theory of operators which commute with translations) but the general unified approach given here appears to include the known results.

Finally these results are applied to some cases when V is a space of functions, measures or distributions over a locally compact group G, which in turn acts on V by left translations.

§ 1. Notation and preliminaries.

The notation introduced in this section will be used unchanged throughout the sequel and will in general follow that of [6]. Whenever we deal with a set which has a topology, this topology will always be assumed to be Hausdorff.

The presentation of the results will be considerably simplified by letting the symbol $L^\infty(X)$ play a dual role. Thus whenever X is merely a nonvoid set, we let $L^\infty(X)$ denote, as an expedient abuse of notation, the set of

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bounded complex-valued functions on X. The maximum modulus of each $f \in L^\infty(X)$ will be denoted by $\|f\|_\infty$. In the second (and more useful) role, whenever $X = G$ is a nonvoid set which has the structure of a locally compact ($= LC$) group, we suppose that it is equipped with a fixed left Haar measure λ_G and then let $L^\infty(G)$ denote the set of equivalence classes of functions on G which are locally essentially bounded. This time $\|f\|_\infty$ denotes the local essential supremum of the modulus of (a representative of) f. When $1 \leq p < \infty$, $L^p(G)$ will denote the other usual spaces of equivalence classes of functions on G, with $\|\cdot\|_p$ denoting the corresponding norm.

1.1 Means of scalar-valued functions. Let X be a nonvoid set (which may, of course, have the structure of a LC group) and let $\mathcal{F}$ denote a subspace of $L^\infty(X)$. Then $\mathcal{F}$ is said to posses a *mean* M if

(i) $\mathcal{F}$ contains the constant function $\mathbf{1} : X \to \{1\}$, and

(ii) there exists a linear functional M on $\mathcal{F}$ having the property that

$$-S(-f) \leq M(f) \leq S(f)$$

whenever f is a real-valued function in $\mathcal{F}$, where $S(f)$ denotes the supremum or the local essential supremum of f depending on the role of $L^\infty(x)$.

The above definition is slightly more general than that used in Greenleaf [8]. Some authors prefer to consider only real-valued means on sets of real-valued functions (see, for example, Day [4]) but in fact there is little to choose between this « real » approach and the « complex » approach described in 1.1 above. We adopt the definition in the form of 1.1 because every real valued mean on a set $\mathcal{G}$ of real-valued functions has a unique extension to a mean on $\mathcal{F} = \mathcal{G} + i\mathcal{G}$. On the other hand, interesting sets $\mathcal{F}$ are generally of the form $\mathcal{F} = \mathcal{G} + i\mathcal{G}$, where $\mathcal{G}$ is a real-linear subspace of $L^\infty(X)$ consisting of real-valued functions. See § 2.1 below.

Let V denote a (Hausdorff) locally convex topological vector space ($= LCTVS$), and let V^* and V' denote the algebraic dual of V and the topological dual of V respectively.

1.2. Means of vector-valued functions. Let $\mathcal{F}$ be any linear set of functions from X into V. Then $\mathcal{F}$ is said to *possess a mean* if the linear envelope of $V' \circ \mathcal{F} = \{v' \circ f : v' \in V', f \in \mathcal{F}\}$ is a subset of $L^\infty(X)$ which possesses a mean in the sense of definition 1.1. If M is a mean on $V' \circ \mathcal{F}$, then the corresponding mean z of $f \in \mathcal{F}$ is defined as the unique member of V'^* satisfying

$$\langle z, v' \rangle = M(v' \circ f)$$

for all v' in V'. We then say that M is a mean on $\mathcal{F}$ and usually write $M(f)$, or even $M_{x \in G}(f(x))$, in place of z.

1.3 REMARKS (1). Note that definition 1.2 is a proper extension of definition 1.1. since in the special case when V is the $LCTVS$ of complex numbers it is easily seen that the two definitions are equivalent. To exploit this more general definition, in the sequel $\mathcal{F}$ will always denote a set of functions from G into V, but further remarks will be made from time to time on the consequences of restricting V to be the space of complex numbers.

(2) If $\mathcal{F}$ satisfies the requirements of definition 1.1, first of all we must have $\mathcal{F} \subseteq L^\infty(X)$. Something similar is true when $\mathcal{F}$ satisfies definition 1.2. For if $\mathcal{F}$ possesses a mean and $f \in \mathcal{F}$, then *either*

(*a*) X does not have the structure of a LC group in which case $f(X)$ is a weakly [i. e. $\sigma(V, V')$] bounded subset of V, *or*

(*b*) $X = G$ does have the structure of a LC group in which case for each $w \in V'$ there exists a locally null subset H_w of G for which $w \circ f$ is bounded on $G \setminus H_w$, and so $f(G \setminus \bigcup_{w \in V'} H_w)$ is a weakly bounded subset of V.

1.4 LEFT INVARIANT MEANS AND AMENABILITY. When $X = G$ is a group (not necessarily a LC group) and $\mathcal{F}$ possesses a mean M we say that M is a *left invariant mean* if

(i) $\mathcal{F}$ is closed under left translation [that is, if $\tau_a f \in \mathcal{F}$ whenever $f \in \mathcal{F}$ and $a \in G$, where $\tau_a f : x \longrightarrow f(a^{-1} x)$, and

(ii) $M(\tau_a f) = M(f)$ for all $f \in \mathcal{F}$ and $a \in G$.

(We will have no need of the analogous concept of a right invariant mean and only remark here that it is trivial to show that if $\tilde{f}$ denotes the function $x \longrightarrow f(x^{-1})$ and if $\tilde{f} \in \mathcal{F}$ whenever $f \in \mathcal{F}$, then $\mathcal{F}$ possesses a left invariant mean if and only if it possesses a «right invariant» mean. See Lemma 1.1.1 of [8] for this result when $\mathcal{F}$ consists of complex-valued functions.)

A group G is said to be *amenable* if $L^\infty(G)$ possesses a left invariant mean. Important examples of amenable groups are the Abelian groups (and hence the LC Abelian groups), the soluble groups and the compact groups-see Greenleaf [8] for details of these results.

1.5 A function f from a LC group G into a $LCTVS$ V is said to be *scalarwise measurable* if $x \longrightarrow \langle f(x), v' \rangle$ is a measurable function on G for each $v' \in V'$. Let $B(G, V)$ denote the set of scalarwise measurable functions f from G into V such that $f(G)$ is a bounded set. Then it is clear that $B(G, V)$ *possesses a left invariant mean if G is amenable.*

For the definition of $B(G, V)$ certainly entails that $V' \circ B(G, V) \subseteq L^\infty(G)$, and so, if G is amenable with mean M,

$$\begin{aligned}\langle M(\tau_a f), v' \rangle &= M(v' \circ \tau_a f) = M_{x \in G}(\langle \tau_a f(x), v' \rangle) \\ &= M_{x \in G}(\langle \tau_a f(x), v' \rangle) = M_{x \in G}(\langle f(x), v' \rangle) \\ &= \langle M(f), v' \rangle\end{aligned}$$

for all $f \in B(G, V)$, $a \in G$, and $v' \in V'$, with the conclusion that $M(\tau_a f) = M(f)$.

Definition 1.2 of a mean on a set of vector-valued functions was suggested by a similar definition of an integral of vector-valued functions; see [6, p. 558]. As in the case of integration, it is important to know conditions on G, V and $\mathcal{F}$ which ensure that $z = M(f)$ belongs to V'', or even V, whenever $f \in \mathcal{F}$. Some simple results in this direction are given in § 2, along with several other related basic results. In § 3 these results are applied to the problem mentioned in the abstract, namely to the problem of the existence of projections on V which commute with the action of a group. In this connection note that throughout we will take *projection* to mean a continuous linear idempotent operator and *closed complemented subspace* to mean a (necessarily closed) subspace which is the range of a projection. In § 4 we apply the general results of § 3 to the particular case when V is a space of functions, measures or distributions over a *LC* group G and G acts on V as the group of left translation operators.

Forerunners of some results in §§ 3 and 4 of this paper (with G compact and no overt reference to means) were proved by the author while he was a doctoral student at the Australian National University. The author is grateful to his supervisor, Dr. R. E. Edwards, and later to colleagues at the Universities of Sheffield and Genoa for helpful comments and discussions relating to this work.

§ 2 Membership of $M(f)$ in V'' or V'.

2.1 Continuity of means. To obtain worthwhile results in this section it appears to be necessary to assume that there exists a mean M on $\mathcal{F}$ which is scalarwise continuous, by which we mean that there exists a mean M on $\mathcal{F}$ and a real number k for which

$$|M(g)| \leq k \, \| g \|_\infty \tag{2.1}$$

for all $g \in V' \circ \mathcal{F}$. If the mean M on $\mathcal{F}$ is derived from a mean M on

$\{g, |g| : g \in V' \circ \mathcal{F}\}$, then the inequality (2.1) follows from the inequality

$$(2.2) \qquad « \, |M(g)| \leq kM(|g|)$$

for all $g \in V' \circ \mathcal{F}$» and the definition of M. In many cases of interest (2.2), with $k = 1$, follows from the definition of M by a proof analogous to that of Theorem (11.5) of [9].

LEMMA. Suppose that $\mathcal{F}$ is a subspace of $L^\infty(G)$ such that $\mathrm{Re}\, f$, $\mathrm{Im}\, f$ and $|f|$ belong to $\mathcal{F}$ whenever f does. Then (2.2), and hence (2.1), is valid with $k = 1$ if M is a mean on $\mathcal{F}$.

For the results in §§ 3, 4 it is important to note that the hypotheses of this lemma are satisfied when $\mathcal{F} = L^\infty(G)$ so that, by 1.5, (2.1) and (2.2) are valid when $\mathcal{F} = B(G, V)$ and G is amenable.

The following four results parallel 8.14.2, 8.14.9, 8.14.5 and 8.14.6 of Edwards [6]. Even though these results in [6] are concerned with vector-valued integration, simplified versions of their proofs are immediately applicable to their counterparts involving means, the simplification being that only finite measures need be considered.

Throughout we suppose that $f \in \mathcal{F}$, and that Γ denotes the weakly closed convex balanced envelope in V'^* of $f(G)$, while Γ_0 denotes the closed convex balanced envelope in V of $f(G)$.

2.2 LEMMA. Let $\mathcal{F}$ be a linear set of functions from G into V which possesses a mean M satisfying (2.1). For each $f \in \mathcal{F}$, $M(f)$ belongs to $k\Gamma$.

PROOF. For each $v' \in V'$

$$(2.3) \qquad |\langle M(f), v' \rangle| = |M(v' \circ f)| \leq k \, \| v' \circ f \|_\infty$$

by the definition of M and (2.1). If $M(f)$ does not belong to $k\Gamma$, 2.2.4 (2) of [6] implies the existence of v' in V' such that

$$(2.4) \qquad |\langle M(f), v' \rangle| > k \cdot \sup \{ |\langle f(x), v' \rangle| : x \in G \}.$$

This contradicts (2.3) since the right side of (2.4) is not less than $k \, \| v' \circ f \|_\infty$ and so completes the proof.

2.3 LEMMA. Suppose that $\mathcal{F}$ satisfies the hypotheses of Lemma 2.2 with the additional condition than Γ_0 is weakly compact in V. (This additional condition is always satisfied if, for example, V is semireflexive and $f(G)$ bounded in V.) Then $M(f)$ belongs to $k\Gamma_0$.

PROOF. Since Γ_0 is also weakly closed in V, the weak compactness of Γ_0 shows that Γ_0 is weakly closed in V'^*. However Γ_0 is weakly dense in Γ so that $\Gamma_0 = \Gamma$. Apply Lemma 2.2.

2.4 LEMMA. Suppose that V and W are *LCTVSs*, u is a continuous linear map from V into W, and f is a function from G into W, and f is a function from G into V. Suppose further that $V' \circ f$ possesses a mean, M say (so that $M(f)$ exists). Whenever it is known that $M(f)$ belongs to V we have that $M(u \circ f)$ exists and belongs to W and that moreover

$$u(M(f)) = M(u \circ f).$$

PROOF. Let $u' : W' \to V'$ be the adjoint of u and let w' by any element of W'; to show that $M(u \circ f)$ exists we must show that $M(w' \circ u \circ f)$ exists.

But this must be the case since

$$\begin{aligned} \langle u(M(f)), w' \rangle &= \langle M(f), u'(w') \rangle \\ &= M_{x \in G} \langle f(x), u'(w') \rangle \\ &= M_{x \in G} \langle u \circ f(x), w' \rangle, \end{aligned}$$

whence we have $M_{x \in G} \langle u \circ f(x), w' \rangle = \langle M(u \circ f), w' \rangle$, thus completing the proof.

2.5 LEMMA. Suppose that p is any continuous seminorm on V and that f is a function from G into V. Suppose further that the linear envelope of $\{g, |g| : g \in V' \circ \mathcal{F}\} \cup \{p \circ f\}$ possesses a mean, M say. Then if $M(f)$ belongs to V and if (2.2) holds, we have

$$p(M(f)) \leq k \cdot M(p \circ f).$$

PROOF. A simple proof exists by imitating the proof of 8.14.6 of [6] and using the fact for each $v \in V$,

$$p(v) = \sup \{ |\langle v, v' \rangle| : v' \in U \},$$

where $U = \{v' \in V' : |\langle v, v' \rangle| \leq 1 \text{ for all } v \text{ with } p(v) \leq 1\}$.

Some results similar to the above two are also true when it is known only that $M(f) \in V'^*$. We will have no need of these generalizations here, but the interested reader could easily deduce their forms and proofs from 8.14.5 and 8.14.6, respectively, of [6]. As a final remark, it should be stressed that the results in this section are rather simple and that more refined results are certainly possible. These could by obtained, for example, by using the results and proof in §§ 8.14 and 8.16 of [6] as guidelines.

§ 3. Projections commuting with the action of a group.

3.1 G-SPACES. Throughout this section we suppose that G is a LC group with identity e, that V is a $LCTVS$ and that they are related by the fact that V is a « G-space », by which we mean that there exists an operator Φ from $G \times V$ into W which satisfied the following four conditions:

(1) $\Phi(e, v)$ for all $v \in V$;

(2) $\Phi(g_1 g_2, v) = (g_1, \Phi(g_2, v))$ for all $g_1, g_2 \in G$ and $v \in V$;

(3) Φ is linear in the second variable; and

(4) for each $v \in V$ and $v' \in V'$ in the first map, and each $g \in G$ in the second map, the maps

$$g \longrightarrow \langle v', \Phi(g, v) \rangle, \quad v \longrightarrow \Phi(g, v)$$

are continuous from G into $\mathbf{C}$ and V into V respectively.

In other words, $g \longrightarrow \Phi(g,.)$ is a weakly continuous representation of G by continuous linear operators over V. In the sequel we will usually write gv for $\Phi(g, v)$.

The following theorem, the main result of this section, may appear to be stated in tedious generality, but in fact all the hypotheses are used precisely as stated in fairly natural examples examined in the next section.

3.2 THEOREM. Let G be a LC group, let V be a $LCTVS$ which is a G-space, and let P be a projection (a continuous linear idempotent operator) from V onto a subspace, U say, of V which is G-invariant. Further suppose that

(i) the set of operators $\{gPg^{-1} : g \in G\}$ is equicontinuous,

(ii) G is amenable,

(iii) V is semireflexive OR G is compact V is quasicomplete ([6], p. 480) and, for each $v \in V$, the map $g \longrightarrow gPg^{-1}v$ is continuous from G into V equipped with its $\sigma(V, V')$ topology.

Then we may construct a continuous projection Q from V onto U which commutes with the action of G, that is, which satisfies $Qg = gQ$ for all $g \in G$.

3.3 REMARKS. (1) A special case of the above theorem is often used to prove Maschke's theorem on the decomposition of representations of finite groups (see [3], p. 41) and in 3.7 below we give a generalization of this theorem for amenable groups. Other special cases are also known; it was proved for G compact and V a Banach space by Rudin [14, Theorem 1], and for G amenable, $V = L^p(G)$, $1 < p < \infty$ and G acting on V as the group of left translation operators by Rosenthal [13, Lemma 3.1].

Our method of proving Theorem 3.2 is similar to a technique used in [14] and avoids the fixed point theorem used in [13].

(2) Clearly condition (i) of 3.2 is satisfied if P already commutes with the action of G.

(3) With regard to the first clause of hypothesis (iii) of 3.2, the example on p. 20 of Rosenthal [13] shows that some extra condition is needed on V whenever G is non-compact.

PROOF OF 3.2. Let G, V, P and U satisfy the hypotheses of the theorem, and let M denote a left invariant mean on $L^\infty(G)$; as remarked in 1.5, M may then be extended to a left invariant mean on $B(G, V)$. To apply the methods of the previous section we first need to show that, for each $v \in V$, the function

$$\psi : g \longrightarrow gPg^{-1} v$$

from G into V belong to $B(G, V)$.

Since V is locally convex and $\{gPg^{-1} : g \in G\}$ is equicontinuous, to each continuous seminorm p on V there exists a continuous seminorm q on V such that

$$p(gPg^{-1} v) \leq q(v) \tag{3.1}$$

for all $g \in G$ and $v \in V$. But this implies that the range of ψ is bounded. Let $v' \in V'$; the complex valued function

$$\theta : (g, h) \longrightarrow \langle gPh^{-1} v, v' \rangle$$

on $G \times H$ is separately continuous by 3.1 (4). We must show that its diagonal, $\chi : g \longrightarrow \theta(g, g)$, is measurable. From [6, Proposition 4.14.9] we learn that G is the disjoint union of a locally negligible set N and a disjoint locally countable family $\{K_i\}$ of compact sets. Now according to Theorems 2 and 4 of Moran [10], on each compact $K_i \times K_i$ there exists a sequence $(\theta_n^i)_{n=1}^\infty$ of jointly continuous functions which tend pointwise to θ. Thus the functions $\chi_n^i : g \longrightarrow \theta_n^i(g, g)$ are each continuous and tend pointwise to χ on each K_i, whence we must have the required measurability of χ.

We now define an operator Q from V into V'^* by

$$Q : v \longrightarrow M_{g \in G}(gPg^{-1} v),$$

and show that it has the properties to satisfy the theorem.

If the first alternative of condition (iii) is satisfied, Lemma 2.3 is immediately applicable (with $f = \psi : g \longrightarrow gPg^{-1} v$). If the second alternative is satisfied we first note that $\psi(G)$ must be weakly compact so that Γ_0 must also be weakly compact because V is quasicomplete. [By using an extension of a result due to Krein (see [6], Theorem 8.13.1) the condition that V is quasi complete may by slightly relaxed.] Thus in either case, Lemma 2.3 is applicable and shows that $Qv \in V$ for all $v \in V$. In fact, since the range of P is G-invariant, Lemma 2.3 shows that

$$\text{ran } Q \subseteq \text{ran } P. \tag{3.2}$$

The linearity of Q follows from the linearity of M and of gPg^{-1}, while the continuity of Q follows from the remarks in 2.1, Lemma 2.5, and (3.1). For suppose that p is a continuous seminorm on V; by 2.1 and 2.5

$$p(Qv) \leq M_{g \in G}(p(gPg^{-1} v))$$

for all $v \in V$, and so, from (3.1) and definition 1.1 of a mean,

$$p(Qv) \leq h(v)$$

for all $v \in V$.

The demonstration that Q is idempotent with range U is immediate from (3·2) and the identity

$$Qv = M_{g \in G}(gPg^{-1} v) = M_{g \in G}(v) = v,$$

valid for all $v \in U$. The following manipulation completes the proof by showing that Q commutes with each member of G. Let $h \in G$ and $v \in V$, then

$$Qhv = M_{g \in G}(gPg^{-1} hv) = M_{g \in G}(hg Pg^{-1} v)$$
$$= h(M_{g \in G}(gPg^{-1} v)) = hQv,$$

by the definition of Q, the left invariance of M and Lemma 2.4.

3.4 COROLLARY. By examination of the proof of 3.2 it may be seen that the statement of Theorem 3.2 remains valid if condition (i) is replaced by

(i) (*a*) the set $\{gP\,g^{-1}\,v : g \in G\}$ is bounded in V for each $v \in V$ and (first noting that $M_{g \in G}(gP\,g^{-1}\,v)$ exists and belongs to V for each $v \in V$) the operator $v \longrightarrow M_{g \in G}(gP\,g^{-1}\,v)$ is a continuous endomorphism of V.

When V is infrabarelled, Theorem 7.3.1 (2) of [6] shows that condition (i) of 3.2 is valid if (and only if) the first clause of (i) (a) above is valid.

3.5. EQUICONTINUITY OF $\{gP\,g^{-1} : y \in G\}$. For the set of operators $\{gP\,g^{-1} : g \in G\}$ to be equicontinuous (that is, for hypothesis (i) of 3.2 to be satisfied) it is clearly sufficient that the set of operators G be equicontinuous. A simple existension of a technique used in Rudin [14] will show that G is equicontinuous if G is compact and V is metrizable.

PROOF. Suppose that the topology of V may be defined by an increasing sequence $\{p_n\}_{n=1}^{\infty}$ of seminorms on V, and define the closed sets $E_{k,m} = \{g : p_n(gv) \leq kp_m(v)$ for all $v \in V\}$ where $k, m = 1, 2, 3, \ldots$ — see the proof of Theorem 1 in [14]. Assume that G is compact; the Baire category theorem implies the existence of a nonvoid open set F in G and positive integers k_0, m_0 such that $g \in F$ implies

$$p_n(gv) \leq k_0\,p_{m_0}(v)$$

for all $v \in V$. However, since G is compact it may be covered by a finite number of right translations of F, whence it follows that positive integers k_1, m_1 exist such that $p_n(gv) \leq k_1\,p_{m_1}(v)$ for all $v \in V$ and $g \in G$.

Thus G equicontinuous.

3.6. Suppose that V is a continuous G-space instead of merely a weakly continuous G-space, that is, suppose that in 3.1 (14) we require the map $g \longrightarrow gv$ to be *continuous* from G into V for each $v \in V$.

(In many cases a representation is continuous if and only if it is weakly continuous - see [9], (22.20).) Then a routine argument shows that the operator $g \longrightarrow gPg^{-1}\,v$ is continuous from G into V for each $v \in V$ [cf. condition (iii) of 3.2] whenever G is equicontinuous.

3.7. APPLICATION TO REPRESENTATION THEORY. A basic result in the theory of group representations states that every weakly continuous representation of a compact group (in particular, of a finite group) by operators over a finite-dimensional space may be written as the direct sum of irreducible representations of the same type. (See Theorem (10.8) of [3] and Theorem 3.20 of [1]; in the latter reference the « commuting with the action of G» part of the argument appears in slightly disguised forms in Propositions 3.16 and 3.18.)

This reduction result is immediate from the following proposition by using 3.5 above aud finite induction.

PROPOSITION. Let H be a Hilbert space which a G-space for an amenable group G and suppose that set G of operators is equicontinuous. Then corresponding to each closed G-invariant subspace U of H there exists a closed G-invariant subspace U' of H such that

$$U \oplus U' = H.$$

PROOF. Since U is a closed subspace of a Hilbert space, it is the range of a projection, P say. Apply 3.2 and the opening remarks in 3.5 to construct a projection Q from P in the manner of 3.2 and then define $U' = Q^{-1}\{0\}$.

REMARK. Example 3.21 of [1] shows that the above proposition need not hold when G is non compact and the set of operators G in is not equicontinuous.

§ 4. Projections commuting with translations.

4.0. When f is a continuous function on G, for each $a \in G$ the *left translation operator* τ_a is defined by

$$\tau_a f : x \longrightarrow f(a^{-1} x) \tag{4.1}$$

and it may then be extended to measures or distributions. In this section we will only by concerned with the case when V is a space of functions, measures or distributions over a LC group G and G acts on V as the group of left translations (and hence V must be invariant under left translations). Given such a space V, we endeavour to use the methods of the previous section to see if it has the following property:

4.1. Whenever there exists a projection from V onto a G-invariant subspace, U say, then there exists a projection onto U which commutes with left translations.

4.2 The following is a short list of spaces V which are G-spaces when G acts in the manner described in 4.0 and for which conditions (i), (ii) and

(iii) of Theorem 3.2 or conditions (i) (a), (ii) and (iii) of Corollary 3.4 are satisfied for every continuous linear endomorphism of V. Thus each of these space will have property 4.1.

(1) $L^p(G)$, where either G is compact and $1 \leq p < \infty$ or G is amenable and $1 < p < \infty$. (Theorem 1 of Rudin [14] shows that $L^p(G)$ has property 4.1 whenever G is compact and $1 \leq p < \infty$, while Lemma 3.1 of Rosenthal shows it when G is amenable and $1 < p < \infty$.)

PROOF. In both cases G is clearly equicontinuous so that 3.5 and 3.6 may be used to show the validity of conditions 3.2 (i) and 3.2 (iii). This argument also applies to the following case.

(2) The generalized Sobolev spaces W_p^μ defined and discussed in [16].

(3) $L^\infty(G)$ equipped with its weak (ie. $\sigma(L^\infty, L^1)$) topology where G is amenable. (When G is compact Abelian, Rosenthal [13, p. 19] shows that $L^\infty(G)$ with its weak topology has property 4.1. See also Gilbert [7].)

PROOF. Let $\mathcal{L}^\infty(G)$ denote $L^\infty(G)$ equipped with its weak topology and let P be a continuous liner endomorphism of $\mathcal{L}^\infty(G)$. To apply Corollary 3.4 we need the following two conditions.

(i) (a)′ the set $\{\tau_a P \tau_{a^{-1}} f : a \in G\}$ is (weakly) bounded in $\mathcal{L}^\infty(G)$ for each $f \in \mathcal{L}^\infty(G)$ and $f \longrightarrow M_{a \in G}(\tau_a P \tau_{a^{-1}} f)$ is continuous from $\mathcal{L}^\infty(G)$ into $\mathcal{L}^\infty(G)$; and

(iii)′ $\mathcal{L}^\infty(G)$ is reflexive.

An appeal to Theorems 1.11.4 (2), 7.1.1 (1) (b) and 8.4.2 of [6] proves (iii)′ since $L^\infty(G)$ is the topological dual of a barrelled space, $L^1(G)$. Turning to (i) (a)′, let P' denote the adjoint of P defined by

$$\int_G P' h \cdot f \, d\lambda_G = \int_G h \cdot Pf \, d\lambda_G$$

for all $h \in L^1(G)$ and $f \in L^\infty(G)$. Since by assumption P is continuous from $\mathcal{L}^\infty(G)$ into $\mathcal{L}^\infty(G)$, it is continuous from $L^\infty(G)$ into $L^\infty(G)$ and so P' is continuous from $L^1(G)$ into $L^1(G)$. Then there exists a positive number c such that

$$\| Pf \|_\infty \leq c \, \| f \|_\infty \text{ and } \| P' h \|_1 \leq c \, \| h \|_1 \tag{4.2}$$

for all $f \in L^\infty(G)$ and $h \in L^1(G)$. Using (4.2) and the left invariance of λ_G, it then follows that for $a \in G$, $h \in L^1(G)$ and $f \in L^\infty(G)$,

$$\left| \int \tau_a P \tau_{a^{-1}} f \cdot h \, d\lambda_G \right| = \left| \int f \, \tau_a P' \tau_{a^{-1}} h \, d\lambda_G \right| \leq c \, \| f \|_\infty \| h \|_1 ,$$

proving the first clause of (i) (a)′.

To prove the second clause of (i) $(a)'$ assume for the moment that $M_{a \in G}(\tau_a P' \tau_{a^{-1}} h)$ exists for each $h \in L^1(G)$ and belongs to $L^1(G)$. Then two applications of the definition of M shows that

$$\begin{aligned}\langle M_{a \in G}(\tau_a P\tau_{a^{-1}} f), h \rangle &= \int_G M_{a \in G}(\tau_a P\tau_{a^{-1}} f)\cdot h \, d\lambda_G \\ &= M_{a \in G}\left(\int_G \tau_a P\tau_{a^{-1}} f \cdot h \, d\lambda_G\right) \\ &= M_{a \in G}\left(\int_G f\cdot \tau_a P' \tau_{a^{-1}} h \, d\lambda_G\right) \\ &= \langle f, M_{a \in G}(\tau_a P' \tau_{a^{-1}} h)\rangle\end{aligned}$$

for $f \in L^\infty(G)$ and $h \in L^1(G)$, and hence that $Q : f \longrightarrow M_{a \in G}(\tau_a P\tau_{a^{-1}} f)$ is a continuous endomorphism of $\mathfrak{L}^\infty(G)$.

Returning to $M_{a \in G}(\tau_a P' \tau_{a-1} h)$, we shall see that our assumption of its membership of $L^1(G)$ is valid by invoking Lemma 2.3; to do this we only need to show that $\{\tau_a P' \tau_{a^{-1}} h : a \in G\}$ is relatively weakly compact in $L^1(G)$. However, the Dunford-Pettis Theorem (Theorem 4.21.2 of [6]) shows that this is indeed the case since

$$\int_A |\tau_a P' \tau_{a^{-1}} h| \, d\lambda_G \leq c \int_A |h| \, d\lambda_G \tag{4.3}$$

for every measurable subset A of G. (To see (4.3), let $f \in L^\infty(G)$ and have support A; then

$$\begin{aligned}\left|\int_A \tau_a P' \tau_{a^{-1}} h \cdot f \, d\lambda_G\right| &= \left|\int_A h\cdot \tau_a P\tau_{a^{-1}} f \, d\lambda_G\right| \\ &\leq \int_A |h| \, d\lambda_G \cdot \|\tau_a P\tau_{a^{-1}} f\| \\ &\leq c\,\|f\|_\infty \int_A |h| \, d\lambda_G\end{aligned}$$

by (4.2).)

(4) A similar argument to that used in (3) shows that the space of bounded measures equipped whith its weak topology satisfies (i) (a) and (iii) for every continuous linear endomorphism.

(5) Let $\mathcal{E}$ denote the space of C^∞ functions as defined in Schwartz [15] and let $\mathcal{E}'$ denote its topological dual equipped with its strong topology. Then $\mathcal{E}'$ is the space of distributions with compact supports. It is easily shown that both $\mathcal{E}$ and $\mathcal{E}'$ satisfy the conditions of Theorem 3.2. (Firstly $\mathcal{E}$ and $\mathcal{E}'$ are reflexive and secondly the sets of translation operators on $\mathcal{E}$ and $\mathcal{E}'$ are equicontinuous.)

When V is any one of the Schwartz distribution spaces $\mathcal{D}, \mathcal{D}', \mathcal{S}$ or $\mathcal{S}'$, projections may be constructed on V which do not satisfy 3.2 (i). (However the ranges of these projections are decidely not translation invariant.)

4.3. Let N denote the set of ponts $\{(n, 0, \ldots, 0) \in R^n : n = 0, \pm 1, \pm 2, \ldots\}$ and let $\psi \in \mathcal{D}$ such that $\psi(0) = 1$ and $\psi(x) = 0$ for non-zero $x \in N$. Define an endomosphism P on $\mathcal{D}$ by

$$Pf = \psi \Sigma_{x \in N} f(x).$$

Since N is locally finite and each $f \in \mathcal{D}$ has compact support, P is well defined and is in fact a projection on $\mathcal{D}$. Let $a \in N$; then

$$\tau_a P \tau_{a^{-1}} \psi = \tau_a \psi \Sigma_{x \in N} \psi(x + a) = \tau_a \psi.$$

Now $\{\tau_a \psi : a \in N\}$ is unbounded in $\mathcal{D}$ so that $\{\tau_a P \tau_{a^{-1}} : a \in R^n\}$ cannot be equicontinuous and so P does not satisfy 3.2 (i).

A similar argument takes care of the case when $V = \mathcal{S}$ and suitable counterexample may be found for $\mathcal{D}'$ and $\mathcal{S}'$ by considering adjoints of the projections.

4.4. COMPLEMENTED SUBSPACES. When V is any one of the spaces considered in examples (1)-(5) above, characterization of the continuous linear endomorphism of V which commute with left translations are, for the most part, known. (See, for example, [2], [10, §§ 35, 36] and [11].)

Thus it follows that a description of the left translation-invariant closed complemented subspaces of the above examples of V may be obtained, since if U is a closed complemented subspace of V by definition it is the range of a projection and if U is also left translation invariant, by 4.1 it is the range of a projection which commutes with left translations. We will first illustrate this for examples (1) and (3) above when G is compact, thus obtaining a slight generalization of Theorem 2 of Rudin [14], and then we will illustrate it for example (5).

For the remainder of the paper we suppose that G is a compact group and that Γ is the set of equivalence classes of continuous irreducible unitary representations of G. If $f \in L^1(G)$ then f is uniquely represented by a Fourier series

$$f \backsim \Sigma_{\gamma \in \Gamma} d(\gamma) \, Tr\,[\widehat{f}(D_\gamma) D_\gamma(\cdot)],$$

where: D_γ is a representative (which we assume to be fixed throughout the sequel) of the class $\gamma \in \Gamma$; $d(\gamma)$ is the (finite) dimension of γ; Tr denotes the usual trace; and $\widehat{f}$ is the Fourier transform of f with respect to $\{D_\gamma : \gamma \in \Gamma\}$, that is

$$\widehat{f}(D_\gamma) = \int_G f(x)\, D_\gamma(x)^* \, d\lambda_G(x)$$

for each $\gamma \in \Gamma$, $D_\gamma(x)^*$ denoting the Hilbert adjoint of $D_\gamma(x)$.

Let H_γ denote the Hilbert space of dimension $d(\gamma)$ corresponding to the representation D_γ, and let $\mathfrak{E}$ denote the set consisting of all functions μ on Γ such that $\mu(\gamma)$ is an endomorphism of H_γ for each γ. Let $\mathfrak{L}^p(G)$ denote $L^p(G)$ with its usual norm topology if $1 \leq p < \infty$ and $L^\infty(G)$ with its $\sigma(L^\infty, L^1)$-topology if $p = \infty$. We need the following two facts:

(1) If T is a continuous endomorphism of $\mathfrak{L}^p(G)$ then it commutes with left translations if and only if there exists a unique $\mu \in \mathfrak{E}$ such that

$$(Tf)^\wedge = \mu \cdot \widehat{f}$$

for each $f \in \mathfrak{L}^p(G)$. (See, for example, Theorem (35.8) of Hewitt and Ross [10]; a slightly different approach is mentioned in § 2 of Price [12].) Denote the subset of $\mathfrak{E}$ corresponding to the continuous endomorphisms of $\mathfrak{L}^p(G)$ which commute with left translations by M_p.

(2) Analogously to (4.1), define the right translation operator by $\varrho_a f(x) = f(xa^{-1})$. Then all closed subspaces of $\mathfrak{L}^p(G)$ which are closed under the family of operators τ_a, $a \in G$ [resp. ϱ_a, $a \in G$] are of the form

$$\mathfrak{L}^p_{U,l} = \{f \in \mathfrak{L}^p(G) : \text{ran } \widehat{f}(D_\gamma) \subseteq U_\gamma \text{ for all } \gamma \in \Gamma\}$$

$$[\text{resp.}\, \mathfrak{L}^p_{U,r} = \{f \in \mathfrak{L}^p(G) : \ker \widehat{f}(D_\gamma) \supseteq U_\gamma \text{ for all } \gamma \in \Gamma\}]$$

where $U = (U_\gamma)_{\gamma \in \Gamma}$ and each U_γ is a subspace of H_γ. (See, for example, Theorem (38.13) of [10].)

4.5 PROPOSITION. The following four conditions on $U = (U_\gamma)_{\gamma \in \Gamma}$ are equivalent.

(*a*) $\mathcal{L}^p_{U,l}$ is complemented in $\mathcal{L}^p(G)$.

(*b*) $\mathcal{L}^p_{U,r}$ is complemented in $\mathcal{L}^p(G)$.

(*c*) For each $\gamma \in \Gamma$ there exists a projection $\pi(\gamma)$ from H_γ onto U_γ such that $\pi : \gamma \longrightarrow \pi(\gamma)$ belongs to M_p.

(*d*) For each $\gamma \in \Gamma$ there exists a projection $\pi(\gamma)'$ on H_γ with ker $\pi(\gamma)' = U_\gamma$ such that $\pi' : \gamma \longrightarrow \pi(\gamma)'$ belongs to M_p.

PROOF. Assume that $\pi = (\pi(\gamma))$ satisfies condition (*c*) and define $\pi' : \gamma \longrightarrow \pi(\gamma)'$ where $\pi(\gamma)' = I_\gamma - \pi(\gamma)$: then $\pi' \in M_p$ showing that (*c*) implies (*d*). Similarly (*d*) implies (*c*).

The equivalence of (*a*) and (*b*) may be deduced from the fact that

$$(\mathcal{L}^p_{U,l})^\circ \overset{\text{def}}{=} \left\{ g \in \mathcal{L}^{p'}(G) : \int_G f(x)\, g(x^{-1})\, d\lambda_G(x) = 0 \text{ for all } f \in \mathcal{L}^p_{U,l} \right\} = \mathcal{L}^{p'}_{U,r}.$$

All that remains to complete the cycle of implications is a staightforward application of 4.2 (1) and 4.2 (3) to show that (*a*) and (*c*) are equivalent.

4.6 REMARK. When G is infinite compact Abelian, Theorem 1.1 of Edwards [5] shows that the idempotent elements of M_p with $p \neq 2$ form a proper subset of the idempotent elements of M_2. This fact combines with the above proposition to prove the existence of non-complemented closed translation invariant subspaces of $\mathcal{L}^p(G)$ when $p \neq 2$ and G is infinite compact Abelian. Rosenthal [13, Corollary 2.3] proves this when G is the circle group.

4.7 CLOSED COMPLEMENTED SUBSPACES OF $\mathcal{E}$ and $\mathcal{E}'$. A continuous endomorphism T of $\mathcal{E}$ [resp. $\mathcal{E}'$] commutes with translations if and only if there exists $X \in \mathcal{E}'$ such that

$$Tf = X * f \tag{4.4}$$

for all $f \in \mathcal{E}$ [resp. $\mathcal{E}'$]. Suppose that U is a closed complemented translation-invariant subspace of $\mathcal{E}$ [resp. $\mathcal{E}'$]. Combining 3.2, 4.2 (5) and (4.4) shows that there exists an idempotent element X in $\mathcal{E}'$ such that

$$U = X * \mathcal{E} \text{ [resp. } U = X * \mathcal{E}'\text{]}. \tag{4.5}$$

But the Fourier transform of each element in $\mathcal{E}'$ is continuous so that from « $\widehat{X} \cdot \widehat{X} = \widehat{X}$ on R^n » and (4.5) we must conclude that there are *no non trivial closed complemented traslation-invariant subspaces of* $\mathcal{E}$ *or* $\mathcal{E}'$.

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Institute of Advanced Studies,
Australian National University,
Canberra, Australia.