ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

ALDO ANDREOTTI
C. DENSON HILL

Complex characteristic coordinates and tangential Cauchy-Riemann equations

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 26, nº 2 (1972), p. 299-324
<http://www.numdam.org/item?id=ASNSP_1972_3_26_2_299_0>

COMPLEX CHARACTERISTIC COORDINATES AND TANGENTIAL CAUCHY-RIEMANN EQUATIONS

by ALDO ANDREOTTI and C. DENSON HILL*

The present work is inspired by a paper of Hans Lewy [5]. Here we extend the easy part of Lewy's results to an arbitrary first order linear system of l equations im m real variables and one unknown function.

This generalization involves the systematic introduction of complex characteristic coordinates (§ 1). Whenever a sufficient number of functionally independent characteristic coordinates exist, these provide, under appropriate constant rank assumptions on the principal part of the system, a local diffeomorphism into a submanifold M of some $\mathbb{C}^q$. When the system is viewed from M it assumes a cannonical form, splitting into a subsystem having for principal part the tangential Cauchy-Riemann equations to M, and a second subsystem having for principal part the operator of exterior differentiation along a real fibration of M (theorem 1).

In general the existence of characteristic coordinates is an open question. In theorem 2 we prove their existence under the assumption of the analyticity of the coefficients in some of the variables when the system is put in a proper normalized form. We use a trick similar to one used by Garabedian [1].

There is a connection between these questions and the theory of $C - R$-manifolds [2]. Also the solvability conditions of Hörmander [3] and Nirenberg's complex Frobenius theorem [6] are related to the problem of the foliation of M into complex submanifolds, as studied by Sommer [9]. This is illustrated in the last section.

Pervenuto alla Redazione il 28 Gennaio 1971.

(*) Research supported by the office of scientific reasearch of the U. S. A. F. under contract AF F 44625-69-C-0106 and by a N. A. T. O. fellowship in science.

1. Complex characteristic coordinates.

a) Let Ω be an open set in $\mathbb{R}^m$ where $x = (x^1, \dots, x^m)$ are the coordinates. We consider in Ω a system of l ($l < m$) complex valued vector fields

$$(1) \qquad \begin{cases} P_k \equiv \sum_1^m {}_j\, C_k^j(x) \dfrac{\partial}{\partial x_j} \\ 1 \leq k \leq l \end{cases}$$

where the $C_k^j(x)$ are functions of class C^1 in Ω and are complex valued.

By $l(x)$ we denote the dimension over $\mathbb{C}$ of the space spanned by the vectors P_k at the point x. Clearly $l(x) = \operatorname{rank}(C_k^j(x))$. We set $C(x) = (C_k^j(x))$.

Corresponding to the vectors (1) we can consider their symbols

$$P_k(x, \xi) = \sum_1^m {}_j\, C_k^j(x)\, \xi_j$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{C}^m$ is a complex covector. A *characteristic complex direction* $\xi \in \mathbb{C}^m$ at x will be by definition a complex covector $\xi \neq 0$ such that

$$\begin{cases} P_k(x, \xi) = 0. \\ 1 \leq k \leq l. \end{cases}$$

This notion is independent of the basis chosen for the linear space spanned by the vectors (1) at x, and is invariant under C^1-changes of coordinates.

Let $\zeta = \zeta(x)$ be a C^1, complex-valued function defined in Ω with the following properties:

i) $d\zeta \neq 0$ at each point of Ω

ii) ζ is a solution of the system

$$\begin{cases} P_k\, \zeta = 0 \\ 1 \leq k \leq l \end{cases}$$

Any function of this type will be called *a complex characteristic coordinate.*

Let $\zeta^1, \zeta^2, \dots, \zeta^n$ be complex characteristic coordinates. We will say that $\zeta^1, \zeta^2, \dots, \zeta^n$ are *functionally independent* at x if $d\zeta^1, d\zeta^2, \dots, d\zeta^n$ are linearly independent at x (i. e. $(d\zeta^1 \wedge d\zeta_2 \wedge \dots \wedge d\zeta^n)_x \neq 0$).

For a complex characteristic coordinate ζ the covector

$$\text{grad}\ \zeta = \left(\frac{\partial\zeta}{\partial x^1}, \frac{\partial\zeta}{\partial x^2}, \dots, \frac{\partial\zeta}{\partial x^m}\right)$$

is a characteristic complex direction. It follows then

PROPOSITION 1. *At each point $x \in \Omega$ the maximum number of functionally independent characteristic coordinates one can find in a neighborhood of x is* $n = m - l(x) = \text{corank}\,(C_k^j(x))$.

b) We now make on the system (1) the following assumptions

(A_1) *for each point $x \in \Omega$ $l(x) = l$ i. e. the l vector fields are linearly independent at each point $x \in \Omega$*

(A_2) *there exist in Ω $n = m - l$ functionally independent characteristic coordinates $\zeta^1, \zeta^2, \dots, \zeta^n$ at each point of Ω.*

If we assume that the vector fields (1) are of class C^2 we can consider the commutators

$$[P_\mu, P_\nu] \equiv P_\mu P_\nu - P_\nu P_\mu \quad (\mu, \nu = 1, 2, \dots, l)$$

Then every solution of the system $P_k\zeta = 0$ for $1 \leq k \leq l$ must also satisfy the equations

$$[P_\mu, P_\nu]\,\zeta = 0.$$

Assumptions (A_1), (A_2), imply that at each point each commutator is a linear combination of the given vector fields i. e.

PROPOSITION 2. *Under the assumptions* (A_1), (A_2), *if the vector fields* (1) *are of class C^2 then the system $\{P_k\zeta = 0$ for $1 \leq k \leq l\}$ is involutive*

$$[P_\mu, P_\nu] = \sum_{1}^{l}{}_\sigma\, k_{\mu\nu}^\sigma P_\sigma$$

where $k_{\mu\nu}^\sigma(x)$ are functions of class C^1 on Ω.

c) We now consider the complex conjugate vector fields of (1),

$$(\overline{1}) \qquad \begin{cases} \overline{P}_k \equiv \sum_{1}^{m}{}_j\, \overline{C}_k^j(x)\, \dfrac{\partial}{\partial x_j} \\ 1 \leq k \leq l \end{cases}$$

At each point $x \in \Omega$ let $l(x) + r(x)$ be the dimension of the vector space over $\mathbb{C}$ generated by the vectors (1) and $(\bar{1})$. We have

$$0 \leq r(x) \leq \min(l(x), m - l(x))$$

$$l(x) + r(x) = \operatorname{rank} \begin{pmatrix} C(x) \\ \bar{C}(x) \end{pmatrix}.$$

We will add to (A_1) (A_2) the following assumption :

(A_3) *The vectors* (1) *and* $(\bar{1})$ *generate a space of constant dimension* $l + r$ *at each point of* Ω.

PROPOSITION 3. *Consider the map*

$$\zeta : \Omega \to \mathbb{C}^n$$

given by $\zeta(x) = (\zeta^1(x), \zeta^2(x), \dots, \zeta^n(x))$ *where the* ζ^i *form a maximal system of functionally independent complex characteristic coordinates in* Ω.

If assumptions (A_1), (A_2), (A_3), *are satisfied then the map* ζ *has constant rank* $= n + r$ *(*r *as defined in* (A_3)*) in all of* Ω.

PROOF. We want to prove that

$$\operatorname{rank} \frac{\partial(\zeta, \bar{\zeta})}{\partial(x^1, \dots, x^m)} = n + r.$$

We set $L = \begin{pmatrix} \partial\zeta^1/\partial x^1, \dots, \partial\zeta^n/\partial x^1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \partial\zeta^1/\partial x^m, \dots, \partial\zeta^n/\partial x^m \end{pmatrix}$. By the assumptions (A_1) and (A_2) the columns of L form a basis for the nullspace Γ_x of C at each point $x \in \Omega$. It follows that the columns of $(L, \bar{L})$ span the space $\Gamma_x + \bar{\Gamma}_x$ i.e. $\operatorname{rank}(L, \bar{L}) = \dim_{\mathbb{C}}(\Gamma_x + \bar{\Gamma}_x)$.

Now $\Gamma_x \cap \bar{\Gamma}_x =$ nullspace of $\begin{pmatrix} C \\ \bar{C} \end{pmatrix}$ which has dimension $m - (l + r) = n - r$ (by assumption (A_3))). Thus

$$\dim_{\mathbb{C}}(\Gamma_x + \bar{\Gamma}_x) = \dim_{\mathbb{C}} \Gamma_x + \dim_{\mathbb{C}} \bar{\Gamma}_x - \dim \Gamma_x \cap \bar{\Gamma}_x$$

$$= n + n - (n - r) = n + r$$

since $\dim_{\mathbb{C}} \Gamma_x = \dim_{\mathbb{C}} \bar{\Gamma}_x = n$.

COROLLARY. *By shrinking Ω and relabeling the variables we may assume that the first $n+r$ rows of*

$$(L, \bar{L}) = \frac{\partial (\zeta, \bar{\zeta})}{\partial (x^1, \dots, x^m)}$$

are linearly independent over $\mathbb{C}$.

Set $x' = (x^1, \dots, x^{n+r})$, $x'' = (x^{n+r+1}, \dots, x^m)$. Then the map

$$\psi : \Omega \to \mathbb{C}^n \times \mathbb{R}^{l-r}$$

defined by

$$\psi (x', x'') = (\zeta (x', x''), x'')$$

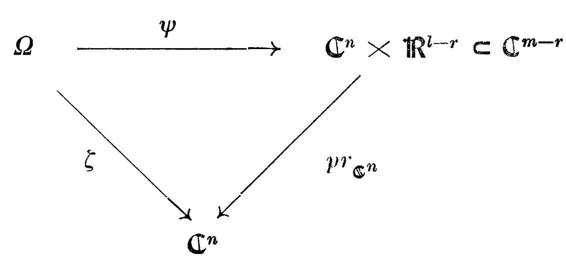
is an immersion of Ω into the target space, and if Ω is sufficiently small an imbedding onto a locally closed (1) *submanifold M of $\mathbb{C}^n \times \mathbb{R}^{l-r}$.*

PROOF. Indeed the jacobian of the map ψ is given by

$$\frac{\partial (\zeta, \bar{\zeta}, x'')}{\partial (x', x'')} = \begin{pmatrix} L' & \bar{L}' & O \\ L'' & \bar{L}'' & I \end{pmatrix}$$

where we have set $(L, \bar{L}) = \begin{pmatrix} L & \bar{L}' \\ L'' & \bar{L}'' \end{pmatrix}$ corresponding to the splitting of the variables x into x' and x''. The rank of that matrix is $n+r+(m-(n+r)) = m$.

We can consider $\mathbb{R}^{l-r}$ as the subspace of points with real coordinates in $\mathbb{C}^{l-r}$. Thus ψ can be considered as an immersion (or imbedding) of Ω into $\mathbb{C}^n \times \mathbb{C}^{l-r} = \mathbb{C}^{m-r}$. Note that the following diagram is commutative



Shrinking Ω we may assume that $\zeta(\Omega) = N$ is a locally closed submanifold of $\mathbb{C}^n$ of dimension $n+r$. Then ψ is an isomorphism of Ω onto an open subset of $N \times \mathbb{R}^{l-r}$.

---

(1) A set is locally closed if it is a closed subset of some open set.

In the sequel to avoid pedantic repetitions the vector fields (1) and characteristic coordinates will be assumed to be of class C^∞.

2. Tangential Cauchy-Riemann equations.

a) Let M be a real smooth submanifold imbedded and locally closed in $\mathbb{C}^q$ with $\dim_{\mathbb{R}} M = m$.

The manifold M can be given, in a neighborhood of a point $p \in M$, either by a system of (real) equations.

$$\begin{cases} f_\alpha(z) = 0 & \text{with } (df_1 \wedge \ldots \wedge df_{2q-m})_p \neq 0, \quad z = (z^1, \ldots, z^q) \in \mathbb{C}^q, \\ 1 \leq \alpha \leq 2q - m \end{cases}$$

or by parametric equations

$$\begin{cases} z^j = \varphi_j(t^1, \ldots, t^m) & \text{with rank } \left(\dfrac{\partial(\varphi_1, \ldots, \varphi_q \bar{\varphi}_1, \ldots, \bar{\varphi}_q)}{\partial(t^1, t^2, \ldots, t^m)}\right)_p = m. \\ 1 \leq j \leq q. \end{cases}$$

The functions f_α and φ_j will be of class C^∞.

By $\mathcal{I}(M)_p$ we denote the ideal of germs of C^∞ functions at p vanishing on M. It is generated by the functions $f_1, \ldots, f_{2q-m}$.

A complex valued tangent vector at p in $\mathbb{C}^q$ is given by an expression

$$X \equiv \sum_1^q a_j \frac{\partial}{\partial z^j} + \sum_1^q b_j \frac{\partial}{\partial \bar{z}^j}$$

where a_j and b_j are complex numbers. Such a vector is called *holomorphic* if all the $b_j' s$ are zero, *antiholomorphic* if all the $a_j' s$ are zero.

Let $p \in M$. The vector X will be called *tangent to M at p* if

$$Xf = 0 \text{ for every } f \in \mathcal{I}(M)_p.$$

It is sufficient to verify this condition on the generators $f_\alpha (1 \leq \alpha \leq 2q - m)$.

In particular the holomorphic tangent vectors at p will be defined by the conditions

$$HT_p(M) \equiv \left\{ X \equiv \Sigma\, a_j \frac{\partial}{\partial z^j} \;\middle|\; \Sigma\, a_j \frac{\partial f_\alpha}{\partial z^j}(p) = 0 \text{ for } 1 \leq \alpha \leq 2q - m \right\}.$$

Clearly $HT_p(M)$ is a complex vector space with the following properties:

(i) *if* $r(p) = \dim_{\mathbb{C}} HT_p(M)$ *we have*

$$m - q \leq r(p) \leq \left[\frac{m}{2}\right].$$

Setting for $X = \Sigma\, a_j \dfrac{\partial}{\partial z^j}$ tangent to M at p, $a_j = a_j' + i a_j''$, with a', a'' real then the tangent vector of $\mathbb{R}^{2q} = \mathbb{C}^q$

$$v(a) \equiv \Sigma\, a_j' \frac{\partial}{\partial x^j} + \Sigma\, a_j'' \frac{\partial}{\partial y^j} \qquad (z^j = x^j + i y^j)$$

has the property that $v(a)$ and $J\, v(a)$ are real tangent vectors at p to M, J being the operator corresponding to multiplication by i. And conversely; hence

(ii) $HT_p(M)$ *is isomorphic to the maximal complex subspace contained in the real tangent space to* M *at* p.

We note that since $f_\alpha(\varphi(t)) \equiv 0$ it follows that the m vectors

$$\frac{\partial}{\partial t^k} \equiv \Sigma_j \left(\frac{\partial \varphi_j}{\partial t^k}\right)_p \frac{\partial}{\partial z^j} + \Sigma_j \left(\frac{\partial \bar{\varphi}_j}{\partial t^k}\right)_p \frac{\partial}{\partial \bar{z}^j}$$

are linearly independent and span the full complexified tangent space to M at p. We set

$$\mathcal{L}_p = \begin{pmatrix} \dfrac{\partial \varphi_1}{\partial t^1}, \dots, \dfrac{\partial \varphi_1}{\partial t^m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{\partial \varphi_q}{\partial t^1}, \dots, \dfrac{\partial \varphi_q}{\partial t^m} \end{pmatrix}_p$$

(and call it the *semijacobian* of the map φ at p). Then we have

(iii) *a vector* $X \equiv \Sigma\, a_j \dfrac{\partial}{\partial z^j}$ *is a holomorphic tangent vector to* M *at* p *if and only if there exists a*

$$t = \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} \in \mathbb{C}^m \ \textit{such that}$$

$$a = \mathcal{L}_p\, t, \quad \bar{\mathcal{L}}_p\, t = 0 \qquad a = \begin{pmatrix} a_1 \\ \vdots \\ a_q \end{pmatrix}$$

thus $r(p) = m - \operatorname{rank} \mathcal{L}_p$.

b) Analogous considerations can be made for the space $\overline{HT_p(M)}$ of the antiholomorphic tangent vectors at p to M.

Let

$$X_k(p) \equiv \Sigma\, a_k^j(p) \frac{\partial}{\partial z^j} \qquad 1 \leq k \leq r(p)$$

be a basis for $HT_p(M)$ then

$$\overline{X}_k(p) \equiv \Sigma\, \overline{a}_k^j(p) \frac{\partial}{\partial \overline{z}^j}$$

is a basis for the antiholomorphic tangent space to M at p.

The system

$$\overline{X}_k(p)\, u = 0 \qquad 1 \leq k \leq r(p) \tag{2}$$

of $r(p)$ equations in one unknown function u is called the *system of tangential Cauchy Riemann equations to* M at p.

By the way an antiholomorphic tangent vector is defined, equations (2) are equivalent to the condition

$$(\overline{\partial} u)_p \equiv 0 \mod ((\overline{\partial} f_1)_p, \dots, (\overline{\partial} f_{2q-m})_p). \tag{3}$$

If $r(p) = r$ is constant along M then one can select r C^∞ vector fields X_k in a neighborhood U of p giving a basis for $HT_{p'}(M)$, $p' \in U$. Then (2) becomes a system of first order partial differential equations

$$\begin{cases} \overline{X}_k\, u = 0 \\ 1 \leq k \leq r \end{cases} \tag{2}$$

or

$$\overline{\partial}\, u \equiv 0 \mod \mathcal{J}^{01}(M) \tag{3'}$$

where $\mathcal{J}^{01}(M)$ is the differential ideal of forms in the space $\mathbb{C}^q$ of the following type:

$$\Sigma_\alpha f_\alpha\, \varphi_\alpha + \Sigma_\alpha\, \beta_\alpha\, \overline{\partial} f_\alpha\,,$$

with φ_α forms of type (0,1), and β_α C^∞ functions.

The system of partial differential equations thus obtained on M will be denoted by $\overline{\partial}_M$.

To find the expression of $\overline{\partial}_M$ in terms of local coordinates $t^1, \dots, t^m$ on M one can use the formulas of (ii)-(iii). However it is not necessary to pass through that expression to compute $\overline{\partial}_M u$. Given u, C^∞ on $U \subset M$, one can extended u to $\tilde{u}$, C^∞ in a neighborhood of U in $\mathbb{C}^q$, then apply the operators $\overline{X}_k$ to $\tilde{u}$ and restrict the result to U. Since $\overline{X}_k \mathcal{J}(M) \subset \mathcal{J}(M)$ one sees also directly that the result is independent of the choice of the extension $\tilde{u}$. Using the operator $\bar{\partial}$ and the fact that $\bar{\partial}\, \mathcal{J}(M) \subset \mathcal{J}^{01}(M)$, a similar remark applies to (3′).

c) The special case that shall be of interest to us is the case where $q \leq m \leq 2q$ and where $r(p) = r$ is constant and minimal :

$$r = m - q.$$

This means that $\mathcal{L}_p$ has constant rank q and $q = m - r$. *A manifold M of this nature will be called generic.*

In particular we have

PROPOSITION 4. *Under the assumptions of proposition 3, if Ω is sufficiently small $N = \zeta(\Omega)$ is a locally closed submanifold of C^n, of $\dim_{\mathbb{R}} N = n + r$ and with complex tangent space of complex dimension r (hence N is generic).*

PROOF. We have, with the notation used in proposition 3, that rank $L = n$.

For the map $\psi : \Omega \to \mathbb{C}^{m-r}$ of the corollary to proposition 3 we have

$$\mathcal{L}_p = \begin{pmatrix} & {}^tL & \\ O & & I \end{pmatrix}$$

where I is the $(l - r) \times (l - r)$ identity matrix. Since the last $l - r$ columns in ${}^tL$ are linear combinations of the first $n + r$, and since rank $L = n$ we can find among the first $n + r$ columns n that are linearly independent. Thus rank $\mathcal{L}_p = n + l - r = m - r$. We obtain therefore the following

COROLLARY. *Under the assumptions of the Corollary to proposition 3, if Ω is sufficiently small, $M = \psi(\Omega)$ is a locally closed submanifold of $\mathbb{C}^{m-r}$ with $\dim_M = m$ and with a complex tangent space of complex dimension r. (hence M is generic).*

3. Reduction to canonical form.

We consider on $\Omega \subset \mathbb{R}^m$ the system

$$\text{(4)} \qquad \begin{cases} P_k u \equiv \sum_{1}^{m}{}_j\, C_k^j(x) \dfrac{\partial u}{\partial x^j} = 0 \\ 1 \leq k \leq l \end{cases}$$

for which we make the assumptions (A_1) (A_3); i.e. at each point $x \in \Omega$,

$$\operatorname{rank} C = l, \qquad \operatorname{rank} \begin{pmatrix} C \\ \overline{C} \end{pmatrix} = l + r \qquad C = (C_k^j(x)).$$

If $b(x) = (b_{k'}^k(x))$ is an $l \times l$ matrix of C^∞ functions and maximal rank l everywhere in Ω, we can substitute (4) with the new system $P'_{k'} u = 0$ for $1 \leq k' \leq l$ where

$$P'_{k'} = \sum_{k=1}^{1} b_{k'}^k P_k \, .$$

By shrinking Ω and relabeling, if necessary, the variables, we may assume that the last l columns of C are linearly independent at all points $x \in \Omega$. Taking for b the inverse matrix of that of the last l columns of C, we reduce C to the form

$$C' = (C\, I)$$

where I is the $l \times l$ identity matrix and where

$$C = \begin{pmatrix} C_1^1 \dots C_1^n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ C_l^1 \dots C_l^n \end{pmatrix}$$

is $l \times n$.

Note that

$$\begin{pmatrix} C' \\ \overline{C'} \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & \overline{b} \end{pmatrix} \begin{pmatrix} C \\ \overline{C} \end{pmatrix}.$$

Hence, since the matrix $\begin{pmatrix} b & 0 \\ 0 & \overline{b} \end{pmatrix}$ is non singular, we get that

$$\operatorname{rank} \begin{pmatrix} C & I \\ \overline{C} & I \end{pmatrix} = l + r.$$

By shrinking Ω once more, by relabeling the operators P_k and relabeling correspondingly the last l independent variables we can assume that:

the first $l + r$ rows of $\begin{pmatrix} C & I \\ \overline{C} & I \end{pmatrix}$ are linearly independent.

For k in the range $r + 1 \leq k \leq l$ each operator $\overline{P}_k$ is a linear combination of the operators $\{P_1, \dots, P_l, \overline{P}_1, \dots, \overline{P}_r\}$. According to the special form of the matrix of the system, this means that we can also assume

$$\bar{P}_k = P_k \quad \text{for} \quad r+1 \leq k \leq l. \tag{5}$$

We now make the assumption (A_2); i.e. that there exist in $C^\infty(\Omega)$ n functionally independent solutions $\zeta^1, \dots, \zeta^n$ of (4). Because of (5) we get

$$P_k \zeta^h = 0, \qquad P_k \bar{\zeta}^h = 0 \quad \text{for} \quad r+1 \leq k \leq l \tag{6}$$

and $h = 1, 2, \dots, n$. Therefore

$$\frac{\partial \zeta^h}{\partial x^{n+k}} = -\sum_{1}^{n} {}_j C_k^j \frac{\partial \zeta^h}{\partial x^j} \qquad 1 \leq h \leq n$$

$$\frac{\partial \bar{\zeta}^h}{\partial x^{n+k}} = -\sum_{1}^{m} {}_j C_k^j \frac{\partial \bar{\zeta}^h}{\partial x^j} \qquad 1 \leq h \leq n$$

for all k in the range $r+1 \leq k \leq l$. It follows that in the matrix

$$(L, \bar{L}) = \frac{\partial(\zeta, \zeta)}{\partial(x^1, \dots, x^m)}$$

the last $l-r$ rows are linear combinations of the first $n+r$ rows. Since the matrix (L, L) has rank $n+r$ the assumption in the corollary to proposition 3 is verified.

We consider the map $\zeta : \Omega \to \mathbb{C}^n$ and its lifting $\psi : \Omega \to \mathbb{C}^n \times \mathbb{R}^{l-r} \subset \mathbb{C}^{m-r}$. We set $\zeta(\Omega) = N$, $\psi(\Omega) = M$ where M is an open set of $N \times \mathbb{R}^{l-r}$. We know that ψ is an immersion, and if we take Ω sufficiently small, an imbedding.

The semijacobian of the map ζ is given by the matrix ${}^tL$, where $L = \begin{pmatrix} \partial\zeta^1/\partial x^1, \dots, \partial\zeta^n/\partial x^1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \partial\zeta^1/\partial x^m, \dots, \partial\zeta^n/\partial x^m \end{pmatrix}$, and the semijacobian of ψ is given by the matrix $\mathcal{L} = \begin{pmatrix} {}^tL & \\ 0 & I \end{pmatrix}$, where I is the $(l-r) \times (l-r)$ identity matrix.

Since $\zeta^1, \dots, \zeta^n$ are characteristic coordinates we have that

$$(C\, I)\, L = 0.$$

By taking the conjugate transpose of this matrix we get

$${}^t\bar{L} \begin{pmatrix} {}^t\bar{C} \\ I \end{pmatrix} = 0$$

and the columns of the matrix ${}^t(\overline{C}\,I)$ span the full system of solutions of ${}^t\overline{L}X = 0$. It follows that the column vectors of the matrix

$$ {}^tL \begin{pmatrix} {}^t\overline{C} \\ I \end{pmatrix} $$

give the components (with respect to the basis $\frac{\partial}{\partial \zeta^1}, \dots, \frac{\partial}{\partial \zeta^n}$) of a basis for the holomorphic tangent vectors to N.

If we take into account the relations (6) and write

$$ (C\,I) = \begin{pmatrix} C_1 & I_1 & 0 \\ C_2 & 0 & I_2 \end{pmatrix} $$

with C_1 $r \times n, I_1$ $r \times r$ and consequently C_2 $(l-r) \times n$ and I_2 $(l-r) \times (l-r)$, we get

$$ (C_2 \quad 0 \quad I_2)\,\overline{L} = 0 \,. $$

Therefore the space $HT_p(N)$ is generated by the r vectors with components given by the columns of

$$ {}^tL \begin{pmatrix} {}^t\overline{C}_1 \\ I \\ 0 \end{pmatrix} . $$

These are the vectors

$$ X_k(p) = \sum_{h=1}^{n} (\overline{P}_k\,\zeta^h)(x) \frac{\partial}{\partial \zeta^h} \qquad \text{for} \quad 1 \leq k \leq r. $$

Thus the operator $\overline{\partial}_N$ is represented by the system

$$ \overline{X}_k(p) = \sum_{h=1}^{n} (P_k\,\overline{\zeta}^h)(x) \frac{\partial}{\partial\,\overline{\zeta}^h} \qquad \text{for} \quad 1 \leq k \leq r. \tag{7} $$

Similarly for the mapping ψ, since we have

$$ \begin{pmatrix} {}^t\overline{L} \\ 0 \end{pmatrix} \begin{pmatrix} {}^t\overline{C}_1 \\ I_1 \\ 0 \end{pmatrix} = 0, $$

we derive that the system (7) also represents $\overline{\partial}_M$ on $M \subset C^{m-r}$ (where the coordinates are $\zeta^1, \dots, \zeta^n, \zeta^{n+1}, \dots, \zeta^{m-r}$).

Functions $U(x)$ on Ω and functions $u(p)$ on M are in one-to-one correspondence if Ω is small enough so that ψ is an imbedding:

$$U(x) = U(x', x'') = u(\zeta(x), x'').$$

Suppose that u is extended to a neighborhood of M in $\mathbb{C}^{m-r}$ by a function of class C^1. On M we have

$$P_k U = \sum_{h=1}^{n} (P_k \zeta^h) \frac{\partial u}{\partial \zeta^h} + \sum_{h=1}^{n} (P_k \overline{\zeta}^h) \frac{\partial u}{\partial \overline{\zeta}^h} + \frac{\partial u}{\partial x^{n+k}}.$$

Since $P_k \zeta^h = 0$ the first sum on the right disappears. Moreover if $1 \leq k \leq r$ u is independent of x^{n+k} so that we get

$$P_k U = \overline{X}_k u \quad \text{for} \quad 1 \leq k \leq r, \tag{8}$$

if $r+1 \leq k \leq l$ we obtain from (6) the equations

$$P_k U = \frac{\partial u}{\partial x^{n+k}} \qquad \text{for} \quad r+1 \leq k \leq l. \tag{9}$$

Note that because $\overline{X}_k$ and $\dfrac{\partial}{\partial x^{n+k}}$ are tangent to M the result is independent of the choice of the extension of u to a neighborhood of M.

Considering $M \subset \mathbb{C}^n \times \mathbb{R}^{l-r} \subset \mathbb{C}^{m-r}$ we see that equations (8) represent the operator $\overline{\partial}_M$ while equations (9) represent the operator $d_{\mathbb{R}^{l-r}}$ of exterior differentiation along the factor $\{\zeta\} \times \mathbb{R}^{l-r}$. Also we remark that $\overline{\partial}_M$ coincides with the operator $\overline{\partial}_N$ on the factors $N \times \{x''\}$.

We can summarise this discussion in the following

THEOREM 1. *Consider on an open set $\Omega \subset \mathbb{R}^m$ a system of first order partial differential equations with C^∞ coefficients*

$$(\alpha) \begin{cases} P_k u \equiv \sum_{1}^{m} {}_j \, C_k^j(x) \dfrac{\partial u}{\partial x^j} = 0 \\ 1 \leq k \leq l \end{cases}$$

on which we make the assumptions (A_1), (A_2), (A_3).

Consider the map

$$\zeta : \Omega \to \mathbb{C}^n$$

given by a maximal system of functionally independent characteristic coordi-

nates, and a $\mathbb{C}^\infty$ *lifting* ψ *of* ζ

$$\psi : \Omega \rightarrow \mathbb{C}^n \times \mathbb{R}^{l-r} \subset \mathbb{C}^{m-r}$$

so that the following diagram is commutative

$$\begin{array}{ccc} & & \mathbb{C}^n \times \mathbb{R}^{l-r} \\ & \overset{\psi}{\nearrow} & \Big\downarrow pr_{\mathbb{C}^n} \\ \Omega & \xrightarrow[\zeta]{} & \mathbb{C}^n \end{array}$$

Then for each point $x \in \Omega$ *we can find an open neighborhood* ω *such that* $N = \zeta(\omega)$ *is a generic locally closed submanifold of* $\mathbb{C}^n$ *of* $\dim_{\mathbb{R}} N = n + r$, $M = \psi(\omega)$ *is a generic locally closed submanifold of* $\mathbb{C}^{m-r}$, *is diffeomorphic to* ω, *and is an open set in* $N \times \mathbb{R}^{l-r}$.

The given system of differential equations is equivalent in ω *to the system*

$$(\beta) \begin{cases} \bar{\partial}_N u = 0 & (r \text{ equations}) \\ d_{\mathbb{R}^{l-r}} u = 0 & (l - r \text{ equations}) \end{cases}$$

Only the fact that the form (β) given to the system is independent of the choice of the lifting ψ of the characteristic map ζ needs to be proved.

Let η be any other lifting of ζ. Since ψ is a diffeomorphism we may factor η trough ψ so that we can write $\eta = \theta \circ \psi$ with

$$\theta(\zeta, x'') = (\zeta, g''(\zeta, x''))$$

and we must have rank $\dfrac{\partial (g''(\zeta, x''))}{\partial (x'')} = l - r$. Then we get:

$$\frac{\partial}{\partial x_r''} = \sum_{s=1}^{l-r} \frac{\partial g_s''}{\partial x_r} \frac{\partial}{\partial g_s''}$$

$$\Sigma\, a_i \frac{\partial}{\partial \bar{\zeta}^i} = \Sigma_i\, a_i \left(\frac{\partial}{\partial \bar{\zeta}^i} + \sum_{s=1}^{l-r} \frac{\partial g_s''}{\partial \bar{\zeta}^i} \frac{\partial}{\partial g_s''} \right)$$

where we have set $x'' = (x_1'', \ldots, x_{l-r}'')$ $g'' = (g_1'', \ldots, g_{l-r}'')$. Now we let the $\Sigma\, a_i \dfrac{\partial}{\partial \bar{\zeta}^i}$ describe a basis for holomorphic tangent vectors to N, from these equations we get the conclusion since the matrix $(\partial g_s'' / \partial x_r'')$ is invertible.

REMARKS 1. If one considers a general system of first order

$$\begin{cases} L_k u \equiv P_k u + d_k(x) u = f_k(x) \\ 1 \leq k \leq 1 \end{cases}$$

and if the principal parts (α) satisfy the assumptions of the previous theorem, then we can conclude that on M the system is equivalent to one in which the principal parts are given by (β).

2. If $r = 0$ in theorem 1 then the system (α) is equivalent to one in which all the operators P_k are real. Then the fact that the system is involutive is enough to ensure existence of characteristic coordinates by virtue of the real Frobenius theorem. Also if the P_k's are real we can take a system of real characteristic coordinates. Then M is an open set of $\mathbb{R}^n \times \mathbb{R}^l$ where the given system takes the form $d_{\mathbb{R}^l} u = 0$.

3. In any case $0 \leq r \leq \min(l, m - l)$. We can thus distinguish two cases

(a) $1 \leq l \leq \left[\frac{m}{2}\right]$ then $0 \leq r \leq l$ and we have $l + 1$ possibilities. For $r = l$ the vector fields (1) and ($\overline{1}$) span two complex vector spaces $\mathcal{H}_x$ and $\overline{\mathcal{H}}_x$ at each point $x \in \Omega$, each of dimension l, and such that at each point $\mathcal{H}_x \cap \overline{\mathcal{H}}_x = 0$.

(b) $\left[\frac{m}{2}\right] < l \leq m$ then $0 \leq r \leq m - l$ and we have $m - l + 1$ possibilities. In this case $l - r$ can take the values $2l - m$, $2l - m + 1, \dots, l$ and the part $d_{\mathbb{R}^{l-r}}$ is therefore always present.

4. Existence of characteristic coordinates.

Consider on an open set $\Omega \subset \mathbb{R}^m$ a system of first order partial differential equations with C^∞ coefficients

$$(\alpha) \begin{cases} P_k u \equiv \sum_1^m{}_j \, C_k^j(x) \dfrac{\partial u}{\partial x^j} = 0 \\ 1 \leq k \leq l. \end{cases}$$

Our aim is to prove the local existence of characteristic coordinates postulated in theorem 1 by the assumption (A_2).

In wiew of proposition 2 it is natural to make the following assumptions :

The vector fields P_k are linearly independent over $\mathbb{C}$ at each point of Ω and the system (α) *is involutive i. e.*.

$$[P_\mu, P_\nu] = \sum_\sigma k^\sigma_{\mu\nu}(x)\, P_\sigma \qquad (\mu, \nu = 1, 2, \dots, l)$$

where $k^\sigma_{\mu\nu}(x)$ are C^∞ functions on Ω.

We shall need to assume analiticity of the coefficients $C_k^j(x)$ with respect to some, but not all, of the independent variables.

Without loss of generality we can assume that locally the system (α) has a matrix of coefficients in the normalized form $(C\, I)$, where I is the $l \times l$ identity matrix. If we write the coordinates $x \in \mathbb{R}^m$ as $(x, t) \in \mathbb{R}^n \times \mathbb{R}^l$, the system (α) takes the form

$$(\alpha') \begin{cases} P_k\, u \equiv \sum_1^n{}_j\, C_k^j(x, t) \dfrac{\partial u}{\partial x^j} + \dfrac{\partial u}{\partial t^k} = 0 \\ 1 \leq k \leq l. \end{cases}$$

We can now state the existence theorem

THEOREM 2. *We assume that the system* (α) *is involutive and in $\Omega_0 \subset \Omega$ has the normalized form* (α').

We assume that the real and imaginary parts of the coefficients $C_k^j(x, t)$ are C^∞ functions in all the variables and are real analytic functions of x $(j = 1, \dots, n;\ k = 1, \dots, l)$.

Then any point of Ω_0 has an open neighborhood ω in which one can find n characteristic coordinates, functionally independent at each point of ω.

PROOF. We may assume that the point in question is at the origin in $\mathbb{R}^m$.

Suppose we can find local C^∞ solutions $u^h = u^h(x, t)$ to the Cauchy problem for the system (α') with initial data

$$(\beta') \quad u\,|_{t=0} = x^h \qquad 1 \leq h \leq n.$$

Then by taking $\zeta^h = u^h$ for $1 \leq h \leq n$ we get a system of characteristic coordinates with the property that $d\zeta^1 \wedge \dots \wedge d\zeta^n \neq 0$ in a neighborhood of $\Omega_0 \cap \{t = 0\}$. In fact the n-form induced by $d\zeta^1 \wedge \dots \wedge d\zeta^n$ on $t = 0$ is $dx^1 \wedge \dots \wedge dx^n$ which is $\neq 0$.

Thus it is sufficient to prove the existence of a smooth solution u to the problem $(\alpha') - (\beta')$. To this purpose we introduce complex variables $z^j = x^j + iy^j\, (1 \leq j \leq n)$ and write $z = x + iy \in \mathbb{C}^n$ using an obvious notation. Then the coefficients $C_k^j(x, t)$ can be extended to functions

$$C_k^j(z, t) = C_k^j(x, y, t)$$

which are holomorphic in z in some neighborhood of the origin in $\mathbb{C}^n \times \mathbb{R}^l$.

In what follows we use summation convention: the indices j, γ, λ run from 1 to n; the indices k, μ, ν run from 1 to l.

Because of the special form of (α') the involutiveness condition is equivalent to $[P_\mu, P_\nu] = 0$, which in turn is equivalent to

$$C_\mu^j \frac{\partial C_\nu^\lambda}{\partial x^j} + \frac{\partial C_\nu^\lambda}{\partial t^\mu} = C_\nu^j \frac{\partial C_\mu^\lambda}{\partial x^j} + \frac{\partial C_\mu^\lambda}{\partial t^\nu}.$$

These relations must also hold for the functions $C_k^j(z, t)$ in the complex domain; we have

$$C_\mu^j \frac{\partial C_\nu^\lambda}{\partial z^j} + \frac{\partial C_\nu^\lambda}{\partial t^\mu} = C_\nu^j \frac{\partial C_\mu^\lambda}{\partial z^j} + \frac{\partial C_\mu^\lambda}{\partial t^\nu}$$

in a neighborhood of the origin in $\mathbb{C}^n \times \mathbb{R}^l$. Also because the $C_k^j(z, t)$ are holomorphic in z we have

$$\overline{C}_\mu^j \frac{\partial C_\nu^\lambda}{\partial \overline{z}^j} = 0$$

$$\overline{C}_\nu^j \frac{\partial C_\mu^\lambda}{\partial \overline{z}^j} = 0.$$

Adding these quantities to the left and right sides of the previous equality we obtain

$$(*) \qquad C_\mu^j \frac{\partial C_\nu^\lambda}{\partial z^j} + \overline{C}_\mu^j \frac{\partial C_\nu^\lambda}{\partial \overline{z}^j} + \frac{\partial C_\nu^\lambda}{\partial t^\mu} = C_\nu^j \frac{\partial C_\mu^\lambda}{\partial z^j} + \overline{C}_\nu^j \frac{\partial C_\mu^\lambda}{\partial \overline{z}^j} + \frac{\partial C_\mu^\lambda}{\partial t^\nu}.$$

Denote the real and imaginary part of C_k^j by

$$C_k^j = a_k^j + i\, b_k^j.$$

Then $(*)$ is equivalent to

$$(**) \qquad a_\mu^j \frac{\partial C_\nu^\lambda}{\partial x^j} + b_\mu^j \frac{\partial C_\nu^\lambda}{\partial y^j} + \frac{\partial C_\nu^\lambda}{\partial t^\mu} = a_\nu^j \frac{\partial C_\mu^\lambda}{\partial x^j} + b_\nu^i \frac{\partial C_\mu^\lambda}{\partial y^j} + \frac{\partial C_\mu^\lambda}{\partial t^\nu},$$

valid in a neighborhood of the origin in $\mathbb{R}^{2n} \times \mathbb{R}^l \equiv \mathbb{C}^n \times \mathbb{R}^l$.

Our aim is to find a solution $z = z(\zeta, t)$ of the system of equations:

(a) $$\left\{ \frac{\partial z^j}{\partial t^k} = C_k^j(z, t) \right.$$

with the initial conditions

(b) $$z\,|_{t=0} = \zeta.$$

Here $\zeta = \xi + i\eta$ with components $\zeta^j = \xi^j + i\eta^j$. Separating the real and the imaginary parts, the above system gives the system of real equations:

(a′) $$\begin{cases} \dfrac{\partial x^j}{\partial t^k} = a_k^j(x, y, t), & x^j\,|_{t=0} = \xi^j \\ \dfrac{\partial y^j}{\partial t^k} = b_k^j(x, y, t), & x^j\,|_{t=0} = \eta^j. \end{cases}$$

According to the real Frobenius theorem the integrability conditions are

$$a_\mu^j \frac{\partial a_\nu^\lambda}{\partial x^j} + b_\mu^j \frac{\partial a_\nu^\lambda}{\partial y^j} + \frac{\partial a_\nu^\lambda}{\partial t^\mu} = a_\nu^j \frac{\partial a_\mu^\lambda}{\partial x^j} + b_\nu^j \frac{\partial a_\mu^\lambda}{\partial y^j} + \frac{\partial a_\mu^\lambda}{\partial t^\nu}$$

$$a_\mu^j \frac{\partial b_\nu^\lambda}{\partial x^j} + b_\mu^j \frac{\partial b_\nu^\lambda}{\partial y^j} + \frac{\partial b_\nu^\lambda}{\partial t^\mu} = a_\nu^j \frac{\partial b_\mu^\lambda}{\partial x^j} + b_\nu^j \frac{\partial b_\mu^\lambda}{\partial y^j} + \frac{\partial b_\mu^\lambda}{\partial t^\nu}.$$

These are necessary and sufficient to ensure that there exists a C^∞ solution $x = x(\xi, \eta, t)$ $y = y(\xi, \eta, t)$ of our last system in some neighborhood of the origin in $\mathbb{R}^{2n} \times \mathbb{R}^l$. But these integrability conditions are just the real and imaginary parts of the conditions (∗∗). Hence there exists a solution $z = z(\zeta, t)$ of the system (a) in a neighborhood of the origin in $\mathbb{C}^n \times \mathbb{R}^l$ that satisfies the initial conditions (b).

We now need the following

LEMMA. *The solutiou $z = z(\zeta, t)$ of the system* (a) *with initial conditions* (b) *is holomorphic in ζ.*

Proof of the lemma. We set $w_\gamma^\lambda = \dfrac{\partial z^\lambda}{\partial \bar{\zeta}^\gamma}$. We must show that $w_\gamma^\lambda \equiv 0$. Because the unital conditions (b) are holomorphic we have

(c) $$w_\gamma^\lambda\,|_{t=0} = 0.$$

Moreover for the functions w_γ^λ we have a system of differential equations

$$\text{(d)} \qquad \left\{ \frac{\partial w_\gamma^\lambda}{\partial t^k}(t) = A_{kj}^\lambda(t)\, w_\gamma^j(t) \right.$$

where

$$A_{kj}^\lambda(t) = \frac{\partial C_k^\lambda}{\partial z^j}\,(z(\zeta, t), t).$$

The system (d) is obtained by differentiating (a) with respect to the $\bar{\zeta}$'s and using the fact that the C_k^λ are complex analytic with respect to z.

In (c) and (d) we regard ζ as being fixed and the w_γ^λ as a function of t.

First we consider the equations (d) corresponding to $k = 1$. They form a system of n^2 linear ordinary differential equations with independent variable t^1 in the n^2 functions w_γ^λ. Because of the zero initial data, we deduce from the uniqueness theorem for such systems that the condition $w_\gamma^\lambda = 0$ extends from $\{\zeta\} \times \{0\} \in \mathbb{C}^n \times \mathbb{R}^l$ to $\{\zeta\} \times \mathbb{R}^1 \times \{0\} \subset \mathbb{C}^n \times \mathbb{R}^1 \times \mathbb{R}^{l-1}$. Using this extension as new initial data for the system of equations (d) with $k = 2$ (t^1 playing the role of a parameter), we similarly extend $w_\gamma^\lambda = 0$ to $\{\zeta\} \times \times \mathbb{R}^2 \times \{0\} \subset \mathbb{C}^n \times \mathbb{R}^2 \times \mathbb{R}^{l-2}$ by applying the uniqueness theorem to that system of ordinary differential equations. Continuing in this way we finally obtain $w_\gamma^\lambda = 0$ on $\{\zeta\} \times \mathbb{R}^l$. But ζ was arbitrary, hence $w_\gamma^\lambda \equiv 0$ in the domain where it is defined, and this completes the proof of the lemma.

For each fixed t in a neighborhood of the origin in $\mathbb{R}^l$, the solution $z(\zeta, t)$ defines a holomorphic map

$$T_t : \zeta \longrightarrow z \quad \text{where} \quad z = T_t \zeta = z(\zeta, t)$$

of a neighborhood of the origin in $\mathbb{C}^n$ into $\mathbb{C}^n$. For $t = 0$

$$T_t\,|_{t=0} = \text{identity}.$$

Hence if t is in a sufficiently small neighborhood of the origin in $\mathbb{R}^l$, T_t has a holomorphic inverse

$$T_t^{-1} : z \longrightarrow \zeta \quad \text{where} \quad \zeta = T_t^{-1} z = \zeta(z, t),$$

and the function $\zeta = \zeta(z, t)$ is C^∞ in all its variables.

We claim that the n components of the function $\zeta(z, t) = (\zeta^1(z, t), \ldots, \zeta^n(z, t))$ give a system of C^∞ caracteristic coordinates when restricted to the real domain. Moreover these characteristic coordinates are functionally independent in a neighborhood of the origin in $\mathbb{R}^m$.

Let $u(z, t)$ be the $h-th$ component of $T_t^{-1} z$. We have

$$(\beta'') \qquad u\,|_{t=0} = z^h.$$

By the very definition $u(z(\zeta, t), t) = \zeta^h$; i. e. u is constant along the solutions of the system (a'); hence

$$a_k^j \frac{\partial u}{\partial x^j} + b_k^j \frac{\partial u}{\partial y^j} + \frac{\partial u}{\partial t^k} = 0.$$

This can be written also as

$$C_k^j \frac{\partial u}{\partial z^j} + \overline{C}_k^j \frac{\partial u}{\partial \overline{z}^j} + \frac{\partial u}{\partial t^k} = 0.$$

But u is holomorphic in z so that the preceding equation collapses to

$$(\alpha'') \qquad C_k^j \frac{\partial u}{\partial z^j} + \frac{\partial u}{\partial t^k} = 0.$$

Restricting z to the real domain we obtain (β') from (β'') and (α') from (α''). This, according to the remark made at the beginning, gives the proof of the theorem.

REMARK. If in the system (α) the coefficients C_k^j are real analytic functions (complex valued) of all the variables, then the characteristic coordinates given by the theorem are also complex valued real analytic functions in ω.

5. Imbedding of $C-R$-manifolds and complex analytic foliations.

a) Let M be a differentiable manifold of $\dim_{\mathbb{R}} M = m$, let $T_{\mathbb{C}}(M)$ denote the complexified tangent bundle of M and let $\mathcal{T}_{\mathbb{C}}(M)$ denote the sheaf of germs of C^∞ sections of $T_{\mathbb{C}}(M)$.

Let $H(M)$ be a complex C^∞ subbundle of $T_{\mathbb{C}}(M)$ of rank l and let $\mathcal{H}(M)$ denote the sheaf of germs of C^∞ sections of $H(M)$.

We will say that the data $(M, H(M))$ define a *$C-R$ manifold of real dimension m and $C-R$-dimension l* if the following conditions are satisfied:

$$\text{i)} \qquad [\mathcal{H}(M)_x, \mathcal{H}(M)_x] \subset \mathcal{H}(M)_x$$

$$\text{ii)} \qquad \mathcal{H}(M)_x \cap \overline{\mathcal{H}(M)_x} = 0.$$

For instance if M is a smooth real manifold imbedded and locally closed in $\mathbb{C}^q$ with $\dim_{\mathbb{R}} M = m$, and if at each point $p \in M$ the dimension $r(p)$ of the holomorphic tangent space to M at p is constant $= l$, then M

has a natural structure of a $C-R$-manifold. In particular a generic locally closed submanifold M of $\mathbb{C}^q$ of $\dim_{\mathbb{R}} M = m$ is a $C-R$ manifold of $C-R$-dimension $l = m - q$.

If M is a real analytic manifold, if $H(M)$ has a real analytic structure, and if the injection map $H(M) \longrightarrow T_{\mathbb{C}}(M)$ is real analytic then the $C-R$-manifold $(M, H(M))$ will be called a *real analytic* $C-R$ *manifold.*

If $(M, H(M))$, $(N, H(N))$ are two $C-R$ manifolds (real analytic) and if $f: M \longrightarrow N$ is a differentiable (real analytic) map from M to N, we will call f a $C-R$*-map* if it induces a bundle map $f_*: H(M) \longrightarrow H(N)$.

The notion of isomorphism of $C-R$ manifolds is then defined.

In particular locally a $C-R$-structure of real dimension m and $C-R$-dimension l is equivalent to the data of an open set $\Omega \subset \mathbb{R}^m$ and on it of a system of l C^∞ vector fields

$$(1) \qquad \begin{cases} P_k \equiv \sum_1^m C_j^k(x) \dfrac{\partial}{\partial x^j} \\ 1 \leq k \leq l \end{cases}$$

which are complex valued and verify the following conditions

i) the system (1) is involutive

ii) at each point of Ω the $2l$ vectors $P_1, \dots, P_l, \bar{P}_1, \dots, \bar{P}_l$ are linearly independent.

This local $C-R$-structure will be real analytic if (up to a linear transformation with C^∞ coefficients) the vector fields (1) are real analytic.

If follows that the local study of $C-R$-manifolds of dimension m and $C-R$-dimension l is equivalent to the study of involutive systems (1) verifying assumption (A_2) and assumption (A_3) with $r = l$.

A straightforward application of theorem 2, the remarks to theorem 2 and the remark 3 *a*) to theorem 1, gives the following consequence:

THEOREM 3. *Let M be a real analytic $C-R$-manifold of real dimension m and $C-R$-dimension l. Any point $x \in M$ has a neighborhood ω which is real analytically $C-R$ isomorphic to a locally closed and real analytic generic submanifold of $\mathbb{C}^{m-l}$.*

In particular any real analytic locally closed submanifold M with $\dim_{\mathbb{R}} M = m$ of some $\mathbb{C}^q$, on which the holomorphic tangent space $HT_p(M)$ has a constant dimension $r(p) = l$, is $C-R$-isomorphic locally to a real analytic generic locally closed submanifold of $\mathbb{C}^{m-l}$.

This fact however can be proved directly without the assumption of analyticity by the use of a « generic holomorphic projection » (a holomorphic map defined in a neighborhood of M and of maximal rank on M is in particular a $C-R$ map).

Note that $m - l$ is the minimal complex dimension in which M can be $C - R$ imbedded locally.

It is an open question if theorem 3 is valid without the assumption of real analyticity except in the case where m is even and $l = \frac{m}{2}$. In that case M is locally isomorphic to $\mathbb{C}^l$ by virtue of the Newlander-Nirenberg theorem [4,6].

Altough we were unable to find a proof of theorem 3 in the literature, that theorem seems to have been known for a long time (cfr. [8]).

b) Let us go back to the general situation where we have a system of first order equations

$$(\gamma) \qquad \begin{cases} L_k u \equiv P_k u + d_k u = f_k \\ 1 \leq k \leq l \end{cases}$$

with C^∞ coefficients defined on an open set $\Omega \subset \mathbb{R}^m$. On the system of principal parts

$$(\alpha) \qquad \begin{cases} P_k u = 0 \\ 1 \leq k \leq l \end{cases}$$

we make the assumptions (A_1), (A_2), (A_3). Let $\zeta = (\zeta^1, \dots, \zeta^n)$, $\eta = (\eta^1, \dots, \eta^n)$ be two maximal systems of functionally independent characteristic coordinates. Given a point $x \in \Omega$ we can find a small neighborhood ω of x such that $N_1 = \zeta(\omega)$ and $N_2 = \eta(\omega)$ are two locally closed generic submanifolds of $\mathbb{C}^n$ of dimension $n + r$.

From theorem 1 we deduce first that each η^i is a C^∞ function in the arguments $\zeta^1, \dots, \zeta^n$ and conversely each ζ^j is a C^∞ function in the arguments $\eta^1, \dots, \eta^n$. We thus have a natural diffeomorphism $\tau : N_1 \xrightarrow{\sim} N_2$. But the fact that both ζ and η are characteristic coordinates gives (by direct verification) that τ maps holomorphic tangent vectors into holomorphic tangent vectors. We conclude with the following

SUPPLEMENT TO THEOREM 1. *The manifold $N = \zeta(\omega)$ obtained by the characteristic map $\zeta : \Omega \to \mathbb{C}^n$ is uniquely determined up to $C - R$-isomorphisms.*

c) Let us examine in particular the case of a single equation ($l = 1$, $n = m - 1$)

$$(1) \qquad Lu = f.$$

If $r = 0$ the principal part P is real (up to a non zero factor), so the situation is trivial and reduces to ordinary differential equations (cf. remark 2 to theorem 1).

If $r = 1$ the principal parts P and $\bar{P}$ are linearly independent in ω and the manifold M of theorem 1 reduces to the image $M = \zeta(\omega)$ of the characteristic map, it is a real $(n+1)$ dimensional locally closed generic submanifold $M \subset \mathbb{C}^n$.

At each point the holomorphic tangent space is 1-dimensional.

It may happen that M is foliated into a real $(n-1)$-parameter family of 1-dimensional complex submanifolds, so that M is $C - R$ isomorphic to an open set $M \subset \mathbb{C} \times \mathbb{R}^{n-1}$. Let z denote the complex variable on $\mathbb{C}$ and $y = (y^1, \ldots, y^{n-1})$ be the coordinates in $\mathbb{R}^{n-1}$. On M the given equation takes the form

$$\frac{\partial u}{\partial \bar{z}} + du = f$$

in which y plays the role of a parameter. This equation has local solutions for smooth d and f.

A theorem of Sommer [9] gives a necessary and sufficient condition for the existence on M of such a foliation:

If X is a holomorphic vector field on M the following integrability condition must hold:

$$[\bar{X}, X] = \beta \bar{X} + \gamma X$$

with smooth β and γ.

Writing this condition in terms of the original coordinates in ω we get

$$[P, \bar{P}] = \beta P + \gamma \bar{P}. \tag{2}$$

This is recognized as Hörmander's necessary condition for the local solvability of (1) [3]. It follows that if the characteristic image $M = \zeta(\omega)$ is not foliated, then (1) is not, for general smooth f, locally solvable. In particular the inhomogeneous Cauchy-Riemann equations

$$\bar{\partial}_M u = f \quad \text{on} \quad M$$

are always solvable if and only if M is foliated.

We should remark that the work of Nirenberg and Treves [7] deals with the more delicate case where there is a change of rank and therefore is not included in our rather superficial treatment.

d) In general let us consider the system (γ) under the usual assumptions.

First of all we recall that the system (α) of the principal part is involutive; i. e.

$$[P_\mu, P_\nu] = \sum_{\sigma=1}^{l} k^\sigma_{\mu\nu} P_\sigma \qquad (\mu, \nu = 1, 2, \ldots, l) \tag{3}$$

with smooth functions $k^{\sigma}_{\mu\nu}$ on ω. Condition (3) is implied by the existence of a maximal set of functionally independent characteristic coordinates.

Let $N = \zeta(\omega)$ be the characteristic image of ω. It is a generic locally closed submanifold of dimension $n + r$ in $\mathbb{C}^n$. Let $\{X_1, \dots, X_r\}$ be a basis for the holomorphic vector fields along N. We may ask when is N foliated into a real $(n - r)$-parameter family of r dimensional complex manifolds — so that M is $C - R$ isomorphic to an open set of the product $\mathbb{C}^r \times \mathbb{R}^{n-r}$.

The answer is supplied by Sommer's theorem [9] which gives as necessary and sufficient conditions for the existence of such a foliation, the following integrability conditions:

$$
(4) \quad
\begin{aligned}
&[X_\mu, X_\nu] = \sum_{k=1}^{r} \alpha^k_{\mu\nu} X_k && (\mu, \nu = 1, 2, \dots, r) \\
&[\overline{X}_\mu, X_\nu] = \sum_{k=1}^{r} (\beta^k_{\mu\nu} \overline{X}_k + \gamma^k_{\mu\nu} X_k) && (\mu, \nu = 1, 2, \dots, r).
\end{aligned}
$$

Let $z = (z^1, \dots, z^r)$ be the coordinates in $\mathbb{C}^r$.

On N the image of the given system has the principal part

$$\frac{\partial}{\partial \bar{z}^a} \quad (a = 1, 2, \dots, r).$$

Considering now a lifting ψ of the characteristic map, we get a diffeomorphism of ω onto $M = \psi(\omega)$, which is an open set of $N \times \mathbb{R}^{l-r}$. If $x = (x^1, \dots, x^{l-r})$ denote the coordinates in $\mathbb{R}^{l-r}$, the given system is equivalent to one with principal part

$$\frac{\partial}{\partial \bar{z}^a} \quad (a = 1, 2, \dots, r)$$

$$\frac{\partial}{\partial x^b} \quad (b = 1, 2, \dots, l - r).$$

This reduction is the content of the complex Frobenius theorem of Nirenberg.

Solvability conditions for the given system can be easily formulated by using the Poincaré lemma for the operators d and $\bar{\partial}$ (see also Hörmander [4]).

Without any loss of generality we may assume that, locally, the system (α) is in the normalized form prescribed in the proof of theorem 1. Consider the conditions for N (and thus for M) to be foliated: the first set of conditions in (4) is implied by the involutiveness because $[P_\mu, P_\nu] = 0$, due to the normalized form. For the second set of conditions we need

$$[P_\mu, \overline{P}_\nu] = \sum_{k=1}^{r} (\beta^k_{\mu\nu} P_k + \gamma^k_{\mu\nu} \overline{P}_k) \quad (\mu, \nu = 1, 2, \dots, r).$$

But $\overline{P}_k = P_k$ for $r + 1 \leq k \leq l$ because of the normalized form; hence the above conditions can be restated as

$$(5) \qquad [P_\mu, \overline{P}_\nu] = \sum_{k=1}^{l} (\beta^k_{\mu\nu} P_k + \gamma^k_{\mu\nu}, \overline{P}_k) \quad (\mu, \nu = 1, 2, \dots, l).$$

Conditions (3) and (5) are exactly the integrability conditions of Nirenberg [6]. (Note that in (5) the $\beta^k_{\mu\nu}$, $\gamma^k_{\mu\nu}$ are required only to be smooth functions on ω).

In conclusion

(i) *for a single equation* $Lu = f$ *the solvability condition of Hörmander*

$$[P, \overline{P}] = \beta P + \gamma \overline{P}$$

is equivalent to the foliation of the characteristic image.

(ii) *for a system* $\begin{cases} L_k u = f_k \\ 1 \leq k \leq l \end{cases}$ *with involutive principal part*

$$[P_\mu, P_\nu] = \sum_{\sigma=1}^{l} k^\sigma_{\mu\nu} P_\sigma \qquad (\mu, \nu = 1, 2, \dots, l),$$

the integrability conditions of Nirenberg's complex Frobenius theorem,

$$[P_\mu, \overline{P}_\nu] = \sum_{k=1}^{l} (\beta^k_{\mu\nu} P_k + \gamma^k_{\mu\nu} \overline{P}_k) \quad (\mu, \nu = 1, 2, \dots, l)$$

are equivalent to the foliation of the characteristic image.

REFERENCES

[1] P. GARABEDIAN, *Stability of Cauchy's problem in space for analytic systems of arbitrary type,* J. Math. Mech. 9 (1960), 905-914.

[2] S. J. GREENFIELD, *Chauchy-Riemann equations in several variables,* Ann. Sc. Normale Sup. 22 (1968), 275-314.

[3] L. HÖRMANDER, *Differential operators of principal type,* Math. Ann. 140 (1960), 124-146.

[4] L. HÖRMANDER, *The Frobenius-Nirenberg theorem,* Arkiv för Math. 5 (1965), 425-432.

[5] H. LEWY, *On the local character of the solution of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables,* Ann. Math. 64 (1956), 514-522.

[6] L. NIRENBERG, *A complex Frobenius theorem,* Seminar on Analytic Functions, Institute for Advanced Study (1951), 172-189.

[7] L. NIRENBERG and F. TREVES, *Solvability of a first order linear partial differential equation,* Comm. Pure Appl. Math. 26 (1963), 331-351.

[8] H. ROSSI, *Differentiable manifolds in complex euclidean space,* Proceeding of the international congress of Math. (1966), 512-516.

[9] F. SOMMER, *Komplex-analytische Blätterung reeller Mannigfaltigkeiten im C^n,* Math. Ann. 136 (1958), 111-133.

Università di Pisa
Stanford University