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Asymptotics for $\Box u = m^2 u + G(t, x, u, u_x, u_t)$,, II. Scattering theory

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ASYMPTOTICS FOR $\square u = m^2 u + G(x, t, u, u_x, u_t)$, II. SCATTERING THEORY(*)

by JOHN M. CHADAM

In this paper the scattering theory of equations of the form

$$\square u = m^2 u + G(t, x, u, u_x, u_t) \tag{1}$$

will be discussed based on the decay estimates obtained in a previous work [1]. The problem in its simplest analytic form is: Given a Banach space B which is physically relevant (e. g. the finite energy solution space of the unperturbed equation) and a solution of the free equation, $u_-(t)$, which for each $t \in \mathbb{R}$ is in B, to find

i) a solution of the perturbed equation (1), $u(t)$, which for each $t \in \mathbb{R}$ is in B and which is asymptotically equivalent to $u_-(t)$ at $t = -\infty$; i. e.

$$\| u(t) - u_-(t) \|_B \longrightarrow 0 \quad \text{at } t \longrightarrow -\infty, \tag{2}$$

ii) for the u of i), to find a solution of the free equation, $u_+(t)$, which for each $t \in \mathbb{R}$ is in B and which is asymptotically equivalent to $u(t)$ at $t = +\infty$; i. e.

$$\| u(t) - u_+(t) \|_B \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty. \tag{3}$$

The correspondence between u_- and u_+ is a description of the scattering or dispersion experienced by the free solution u_- when it is propogated by means of the quasi-linear equation (1). The scattering operator discussed

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in the time-dependent approach can be obtained via the isomorphism of solutions of the free $(K - G)$ equation and the Cauchy data at any finite time.

In section 1 an abstract version of the basic tecnique will be systematically outlined. The general properties required of G for the existence of a scattering theory will be explicitly verified in section 2 for the specific examples discussed in reference 1, section 2. The general technique is in large part a combination of the ideas used by Segal [2] and Strauss [3] in discussing similar questions for perturbations $G(u)$. The results to be presented here are, as are those of [1] and [2], perturbative in nature in that they depend upon restricting the size of u_- and/or the coupling constant.

1. Scattering. The notations and definitions introduced in part I of this work, [1], will be used throughout this paper. For convenience the most basic of these are listed below without motivation. A^2 will denote the self-adjoint realization of $m^2 I - \Delta$ on $L^2(E^n)$. The real solution spaces of the $K - G$ equation, $H(A, a)$, which are relevant in this work are, for each $a \in \mathbb{R}$, the completions of $D(A^a) \oplus D(A^{a-1})$ with respect to the inner product $\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)_{A,a} := (A^a u, A^a v_1) + (A^{a-1} u_2, A^{a-1} v_2)$. The norm of $\begin{pmatrix} u(t) \\ u(t) \end{pmatrix} \in H(A, a)$ will be denoted by $|u(t)|_a$ as opposed to the usual L^r-norm, $\| u(t) \|_r$, and $G(\cdot, t, u(t), u_x(t), \dot{u}(t))$ will be shortened to $G(t, u(t))$. Generalized solutions of the $K - G$ equation can be written in the form

$$U_0(t) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos tA, & A^{-1} \sin tA \\ - A \sin tA, & \cos tA \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{4}$$

in view of the fact that $U_0(t)$, so defined, is a continuous one — parameter group of orthogonal transformations on $H(A, a)$ with skew-adjoint infinitesmal generator $\begin{pmatrix} 0 & I \\ - A^2 & 0 \end{pmatrix}$. The corresponding generalized solutions of the perturbed equation (1) which will be discussed here are the $H(A, a)$ — valued solutions of the integrated form of (1),

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = U_0(t) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \int_{t_0}^{t} U_0(t - s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds, \tag{5}$$

where $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H(A, a)$ is the Cauchy data at time t_0.

Conditions on G are available which guarantee the existence of unique local solutions to such equations in more general settings [4, Theorem 1, p. 343] as well as in the specific cases to be considered here [1, section 2]. In order to focus attention on the requirements demanded on G for a scattering theory to exist, throughout this section it will be assumed that the map $(t, u_1, n_2) \to \begin{pmatrix} 0 \\ G(t, x, u_1(x), u_{1x}(x), u_2(x)) \end{pmatrix} : \mathbb{R} \times H(A, a) \to H(A, a)$ is continuous and semi Lipschitz uniformily on each finite t-interval thus guaranteeing the existence of solutions of (5) in $H(A, a)$ locally for finite t_0. In addition, to avoid unnecessary technical problems, we assume that $G(x, t, 0, 0, 0) = 0$ (which is the case in the examples to be treated in section 2).

Turning now to the scattering theory of equation (1), the Hilbert space $H(A, a)$ is a suitable canditate for the space in which problems i) and ii) should be investigated in view of the fact that $H\left(A, \frac{1}{2}\right)$ and $H(A, 1)$ are, respectively, the Lorentz-invariant and finite energy solution spaces of the $K - G$ equation. However, not all $H(A, a)$ — solutions $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$ can be used because for technical reasons it is necessary to know that the space — $L^r(E^n)$ norm of the components of these solutions, $\| u(t) \|_r$ and $\| \dot{u}(t) \|_r$, satisfy suitable decay estimates of the type developed in part I. The relevent subclass of $H(A, a)$-valued solutions can be described most conveniently as follows. In anticipation of the use of condition (D_1) of part I (i. e., r and a are such that $\| u(t) \|_r$, $\| \dot{u}(t) \|_r \leq C \mid u(t) \mid_a$), if the expected decay of each component of the solution is $\| u(t) \|_r = 0(\mid t \mid^{-\varepsilon})$ and $\| \dot{u}(t) \|_r = 0(\mid t \mid^{-\delta})$, then for $f(\cdot) = \begin{pmatrix} f_1(\cdot) \\ f_2(\cdot) \end{pmatrix} : (-\infty, T) \to H(A, a)$, define

$$(6) \qquad]f[_T := \sup_{t > T} \left(|f(t)|_a + k(1 + |t|)^{\varepsilon} \|f_1(t)\|_r + \dot{k}(1 + |t|)^{\delta} \|f_2(t)\|_r \right),$$

where k, $\dot{k}$ are 0 or 1 (depending, in the examples, on which component is relevant in the computation). The ε, δ, a and r dependence are supressed because they remain fixed throughout the argument. The type of $H(A, a)$-valued solutions which are important here are those which lie in

$$B_T := \left\{ f(\cdot) = \begin{pmatrix} f_1(\cdot) \\ f_2(\cdot) \end{pmatrix} : \begin{array}{c} t \to f(t) : (-\infty, T) \to H(A, a) \text{ is continuous} \\ \text{and }]f[_T < \infty \end{array} \right\}.$$

In view of (D_1) of part I, B_T with the norm $]\cdot[_T$ can be easily shown to be a Banach space. For ε, $\delta \leq 3/2$, the discussion in part I shows that B_∞

contains a large class of smooth solutions of the $K - G$ equation (1) (i. e. $\begin{pmatrix} u(\cdot) \\ \dot{u}(\cdot) \end{pmatrix} \in B_\infty$, but for notational convenience, in the future this will be written as $u \in B_\infty$, with the implication being that for solutions the second component is understood to be $\dot{u}(\cdot)$).

The proof of the existence of a scattering theory for equation (1) will proceed as follows. With certain conditions on G, it will be proved that equation (—),

$$(7) \qquad \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} u_-(t) \\ \dot{u}_-(t) \end{pmatrix} - \int_{-\infty}^{t} U_0(t-s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds,$$

has a solution $u \in B_T$ for sufficiently large negative T for any solution of the free equation $u_- \in B_\infty$. Next, u will be shown to be a generalized solution of equation (1) which tends to u_- in $H(A, a)$ as $t \to -\infty$ because of the representation (7). The solution u will then be extended to a global solution in B_∞ by using the techniques of part I. This will show the existence in B_∞ of solutions of equation (5) which tend in $H(A, a)$ to a prescribed free solution $u_- \in B_\infty$. The uniqueness of the solution of this problem will be proved under the prevailing assumptions on G, thus completing the discussion of problem i). Finally the techniques and computations used in the above will allow a short proof of the existence of a unique solution in B_∞ of the free equation u_+ to which u tends in $H(A, a)$ as $t \to +\infty$, thus dealing with problem ii). The scattered solntion u_+ can be explicitly obtained either from equation (+),

$$(8) \qquad \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} u_+(t) \\ \dot{u}_+(t) \end{pmatrix} + \int_{t}^{\infty} U_0(t-s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds,$$

(the analogue of equation (—) at $t = +\infty$), or in terms of the incoming free solution and u by means of

$$(9) \qquad \begin{pmatrix} u_+(t) \\ \dot{u}_+(t) \end{pmatrix} = \begin{pmatrix} u_-(t) \\ \dot{u}_-(t) \end{pmatrix} - \int_{-\infty}^{\infty} U_0(t-s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds.$$

In addition to the decay assumptions $(D_1) \dots (D_6)$ required of G and the given data, variants of several of these will be required; namely if

$u, v \in B_T$

$$(S_2)\ \| A^b [G(s, u(s)) - G(s, v(s))] \|_q \leq$$

$$\leq g(1 + |s|)^{-\alpha} \gamma(\,]u[_T,]v[_T)\,]u - v[_T, \text{[1]}.$$

for all $s \in (-\infty, T)$ and some $b \leq a$ for a pair $1 \leq q,\ q' \leq 2$ (with corresponding α', γ'), and

$$(S_3)\ \| A^{a-1} [G(s, u(s)) - G(s, v(s))] \|_2 \leq$$

$$\leq g(1 + |s|)^{-\alpha''} \gamma''(\,]u[_T,]v[_T)\,]u - v[_T,$$

for all $s \in (-\infty, T)$, where the functions γ, γ' and $\gamma'' : \mathbb{R}^2 \to \mathbb{R}^+$ are bounded on bounded sets and monotonically non-decreasing in each variable. Corresponding to (D_6) it will also be assumed that all the constants and exponents appearing above can be chosen consistent with

$$\begin{aligned} &\max(\varrho, \alpha) > 1, && \min(\varrho, \alpha) > \varepsilon, \\ (S_6) \quad &\max(\sigma, \alpha') > 1, && \min(\sigma, \alpha') > \delta, \\ &\alpha'' > 1, \end{aligned}$$

where ϱ and σ are the temporal decays of the fundamental solutions of the $K - G$ equation, $E_{t,b}$ and $F_{t,b-1}$, in the L^p and $L^{p'}$-norm respectively with $p^{-1} + q^{-1} = 1 + r^{-1}$ and $p'^{-1} + q'^{-1} = 1 + r^{-1}$ [1, equations (6)...(8) and (D_5)].

Remark. Although the number of conditions has been raised above that required for decay, the new ones, (S_2), (S_3) and (S_6), do not further restrict the types of perturbations which may be considered. This will be seen in the discussion of the examples to follow in that the same conditions on G required for (D_2), (D_3) and (D_6) are sufficient to deduce (S_2), (S_3) and (S_6) and by essentially the same methods. In fact it may be possible, but hardly profitable or interesting, to show that a version of the scattering conditions can be given which in general imply the corresponding decay conditions.

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(1) More explicity $u(.) = \begin{pmatrix} u_1(.) \\ u_2(.) \end{pmatrix}$, $G(s, u(s))$ denotes $G(s, x, u_1(s, x), u_x(s, x), u_2(s, x))$ and $]u[_T$ is given by (6). Similarly for v.

The proof of the above assertions will begin by establishing the local existence at $t = -\infty$ of solutions of equation (—).

THEOREM 1.1. Suppose $u_- \in B_\infty$ is a solution of the $K - G$ equation and conditions (D_1), (S_2), (S_3), (D_5) and (S_6) can be simultaneously satisfied. Then for a sufficiently large negative T, there exists a unique solution of equation (—),

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} u_-(t) \\ \dot{u}_-(t) \end{pmatrix} - \int_{-\infty}^{t} U_0(t-s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds,$$

over the interval $(-\infty, T)$ with $u \in B_T$.

PROOF. For $u(\cdot) = \begin{pmatrix} u_1(\cdot) \\ u_2(\cdot) \end{pmatrix} \in B_T$ define the operator L by

$$L \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = -\int_{-\infty}^{t} U(t-s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds \tag{10}$$

for $t < T$ (the integral is considered to be $H(A, a)$-valued). The proof consists of showing that there exists a T such that $L : B_T \to B_T$ is a contraction.

For $\begin{pmatrix} u_1(\cdot) \\ u_2(\cdot) \end{pmatrix}, \begin{pmatrix} v_1(\cdot) \\ v_2(\cdot) \end{pmatrix} \in B_T$

$$L \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} - L \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = - \tag{11}$$

$$- \int_{-\infty}^{t} \begin{pmatrix} A^{-1} \sin[(t-s)A] \cdot [G(s, u(s)) - G(s, v(s))] \\ \cos[(t-s)A] \cdot [G(s, u(s)) - G(s, v(s))] \end{pmatrix} ds.$$

Taking the $H(A, a)$-norm of equation (11), using the spectral theorem to remove the trigonometrical terms and observing that each component gives the same contribution, one obtains

$$\left\| L \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} - L \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} \right\|_{A, a} \leq \sqrt{2} \int_{-\infty}^{t} \| A^{a-1} [G(s, u(s)) - G(s, v(s))] \|_2 \, ds. \tag{12}$$

$$\leq g C \gamma'' (\,]u[_T, \,]v[_T)]u - v[_T \int_{-\infty}^{t} (1 + |s|)^{-\alpha''} ds. \tag{13}$$

Incidentally, estimate (13) with $v = 0$ shows that $L\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in H(A, a)$ for each $t \in (-\infty, T)$, and the continuity of the map follows from the existence of the defining $H(A, a)$-valued integral (10) and the continuity of $G(s, u(s))$ as assumed for the local existence theory. The assumptions (D_1), (S_2) and (D_5) along with the discussion in obtaining inequality (8) from equation (6) in part I justify the following steps in estimating the L^r-norm of the first component of $L\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} - L\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$.

$$(14) \qquad \left\| \int_{-\infty}^{t} A^{-1} \sin[(t-s)A]\,[G(s, u(s)) - G(s, v(s))]\, ds \right\|_r$$

$$\leq \int_{-\infty}^{t} \| E_{t-s,\, b} \|_p \, \| A^b [G(s, u(s)) - G(s, v(s))] \|_q \, ds$$

$$\leq g\, \gamma'(]u[_T,]v[_T)\,]u - v[_T \int_{-\infty}^{t} (1 + |t - s|)^{-\sigma} (1 + |s|)^{-\alpha}\, ds.$$

From the estimates (13), (14) and (15) one obtains from the definition

$$(16) \qquad]Lu - Lv[_T \leq g C \left\{ \int_{-\infty}^{T} (1 + |s|)^{-\alpha''}\, ds \; + \right.$$

$$+ \; k \sup_{t<T} (1 + |t|)^{\varepsilon} \int_{-\infty}^{t} (1 + |t - s|)^{-\varrho} (1 + |s|)^{-\alpha}\, ds$$

$$\left. + \; k \sup_{t<T} (1 + |t|)^{\delta} \int_{-\infty}^{t} (1 + |t - s|)^{-\sigma} (1 + |s)^{-\alpha'}\, ds \right\} \Gamma(]u[_T,]v[_T)\,]u - v[_T,$$

where $\Gamma(.,.)$ is a function like the $\gamma' s$, being the sum of such things. Finally, using a technical result of Segal [2, Lemma 3.1, p. 467] to estimate the integrals in (16) and the assumption (S_6),

$$(17) \qquad]Lu - Lv[_T \leq g C\, \Gamma(]u[_T,]v[_T)\, |T|^{-\tau}\,]u - v[_T,$$

where $\tau > 0$. The version of the contraction mapping principle given by Strauss [3, Lemma 1.5, p. 416] can now be applied (taking $\theta(]u[_T +]v[_T) = g C \Gamma(]u[_T +]v[_T,]u[+]v[_T) |T|^{-\tau})$ to obtain the desired conclusion (as in [3, Lemma 1.7, p. 417]).

REMARK. Suppose $]u_-[_\infty = \mu$, then $]u_-[_T \leq]u_-[_\infty = \mu$ for all $T \in \mathbb{R}$. The remainder of the proof of Theorem 1.1 amounts to showing that inequality (17) guarantees the existence of a $T = T(\mu, G)$ such that the right hand side of equation (—) defines a mapping from the closed ball of radius 2μ, centered at the origin in B_T into the same ball, and at the same time is a contraction. Thus the solution u of equation (—) lies in the same ball; that is

$$]u[_T \leq 2\,]u_-[_\infty, \tag{18}$$

an observation which will be useful in later developments.

The next step in the program is to show that the u given in Theorem 1.1 is a solution of equation (5) for some finite t_0 so that the results of part I can be applied to prove that it can be extended to $t = +\infty$. To this end it is convenient to have the following intuitively obvious representation result. There is no loss in generality to assume that the solution $\begin{pmatrix} u(\cdot) \\ \dot{u}(\cdot) \end{pmatrix}$ of equation (—) obtained in Theorem 1.1 is defined at $t = T$ (since the interval of existence can be taken slightly smaller than the maximum provided by Theorem 1.1).

PROPOSITION 1.2. Let u_T denote the solution of the $K - G$ equation with Cauchy data $\begin{pmatrix} u(T) \\ \dot{u}(T) \end{pmatrix}$ at $t = T$. With the hypotheses and resulting solution u of Theorem 1.1,

$$\begin{pmatrix} u_T(t) \\ \dot{u}_T(t) \end{pmatrix} = \begin{pmatrix} u_-(t) \\ \dot{u}_-(t) \end{pmatrix} - \int_{-\infty}^{T} U_0(t-s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds, \tag{19}$$

as $H(A, a)$-valued functions for all $t \in \mathbb{R}$.

PROOF. It suffices to show that $\int_{-\infty}^{T} \begin{pmatrix} -A^{-1} \sin sA\, G(s, u(s)) \\ \cos sA\, G(s, u(s)) \end{pmatrix} ds$ exists as an improper Riemann integral with values in $H(A, a)$. Then, because for each t, $U_0(t)$ is a bounded (orthogonal) transformation in $H(A, a)$,

$$U_0(t)\int_{-\infty}^{T}\begin{pmatrix}-A^{-1}\sin sA\,G(s, u(s))\\ \cos sA\,G(s, u(s))\end{pmatrix}ds = \int_{-\infty}^{T}U_0(t)\begin{pmatrix}-A^{-1}\sin sA\,G(s, u(s))\\ \cos sA\,G(s, u(s))\end{pmatrix}ds,$$

and a simple calculation under the integral as in equation (11) provides the equation

$$\begin{pmatrix}u_-(t)\\ \dot{u}_-(t)\end{pmatrix} - U(t)\int_{-\infty}^{T}\begin{pmatrix}-A^{-1}\sin sA\,G(s, u(s))\\ \cos sA\,G(s, u(s))\end{pmatrix}ds$$

$$= \begin{pmatrix}u_-(t)\\ \dot{u}_-(t)\end{pmatrix} - \int_{-\infty}^{T}U_0(t-s)\begin{pmatrix}0\\ G(s, u(s))\end{pmatrix}ds.$$

The left member of the above equation is a solution of the $K - G$ equation in $H(A, a)$, while the second part shows that at $t = T$ it is precisely $\begin{pmatrix}u(T)\\ \dot{u}(T)\end{pmatrix}$. The result now follows from the uniquess of solution to the $K - G$ equation in $H(A, a)$.

The existence of $\int_{-\infty}^{T}\begin{pmatrix}-A^{-1}\sin sA\,G(s, u(s))\\ \cos sA\,G(s, u(s))\end{pmatrix}ds = \int_{-\infty}^{T}U_0(-s)\begin{pmatrix}0\\ G(s, u(s))\end{pmatrix}ds$ will follow by Bochner's Theorem [e. g. 5, Theorem 1, p. 133] from the continuity of $\left|U_0(-s)\begin{pmatrix}0\\ G(s, u(s))\end{pmatrix}\right|_a$ and the fact that its decay in s is sufficiently rapid so that its improper Riemann integral at $-\infty$ exists. The continuity is a consequence of the combined continuity of the maps $s \to \begin{pmatrix}u(s)\\ \dot{u}(s)\end{pmatrix} : (-\infty, T) \to H(A, a)$ (Theorem (1.1)), $(s, u, \dot{u}) \to \begin{pmatrix}0\\ G(s, u(s))\end{pmatrix} : \mathbb{R} \times H(A, a) \to H(A, a)$ (implicitly assumed throughout this work) and the fact that $U_0(s)$ is a strongly continuous group on $H(A, a)$. The decay on the other hand follows from inequality (13) with $v = 0$ in view of the assumption that $G(x, s, 0, 0, 0) = 0$.

Thus for $t \in (-\infty, T)$

$$(20)\qquad \begin{pmatrix}u(t)\\ \dot{u}(t)\end{pmatrix} = \begin{pmatrix}u_-(t)\\ \dot{u}_-(t)\end{pmatrix} - \int_{-\infty}^{t}U_0(t-s)\begin{pmatrix}0\\ G(s, u(s))\end{pmatrix}ds$$

$$= \begin{pmatrix}u_T(t)\\ \dot{u}_T(t)\end{pmatrix} - \int_{T}^{t}U_0(t-s)\begin{pmatrix}0\\ G(s, u(s))\end{pmatrix}ds.$$

That is the solution, $\begin{pmatrix} u(t) \\ \cdot \\ u(t) \end{pmatrix}$, of equation (—) given in Theorem 1.1 is a solution of equation (5) (the integrated for of $\square u = m^2 u + G(s, u(s))$) with Cauchy data given at time T. The implicit assumptions on G guarantee the existence, locally, of solutions of equation (5) for $t > T$ and globally, by part I, if the inequalities $(D_1) \ldots (D_6)$ can be simultaneously satisfied with some choice of the constants and exponents and if the coupling constant g and/or the Cauchy data $\begin{pmatrix} u_T(T) \\ \cdot \\ u_T(T) \end{pmatrix}$ is appropriately small [1, Theorem 1.2]. All except (D_4) will be assumed and the decay estimate for $\begin{pmatrix} u_T(t) \\ \cdot \\ u_T(t) \end{pmatrix}$ corresponding to (D_4) will be obtained in terms of $]u_-[_\infty$ and independent of T. Thus restricting the size of u_T can be accomplished by choosing $]u_-[_\infty$ small in a sense made specific in the next result.

PROPOSITION 1.3. With the hypotheses and notations of Theorem 1.1 and Proposition 1.2, $u_T \in B_\infty$ and

$$]u_T[_\infty \leq C\,]u_-[_\infty [1 + g\,\Gamma(2]u_-[_\infty, 0)], \tag{21}$$

where $\Gamma(.,.)$ is the function appearing in inequality (16).

PROOF. From equation (19) and the estimates in Theorem 1.1

$$]u_T - u_-[_\infty \leq g\tilde{C}\,\Gamma(]u[_T, 0)]u[_T \sup_{t<T} \Big\{ \int_{-\infty}^{T} (1+|s|)^{-\alpha''} ds +$$

$$k(1+|t|)^{\varepsilon} \int_{-\infty}^{T} (1+|t-s|)^{-\varrho} (1+|s|)^{-\alpha} ds +$$

$$+ \dot{k}(1+|t|)^{-\delta} \int_{-\infty}^{T} (1+|t-s|)^{-\sigma} (1+|s|)^{-\alpha'} ds \Big\}$$

$$\leq g\,\tilde{C}\,\Gamma(]u[_T, 0)]u[_T \sup_{t \in \mathbb{R}} \Big\{ \int_{-\infty}^{\infty} (1+|s|)^{-\alpha''} ds +$$

$$k(1+|t|)^{\varepsilon} \int_{-\infty}^{\infty} (1+|t-s|)^{-\varrho} (1+|s|)^{-\alpha} ds +$$

$$+ \dot{k}(1+|t|)^{-\delta} \int_{-\infty}^{\infty} (1+|t-s|)^{-\sigma} (1+|s|)^{-\alpha} ds \Big\}.$$

Because the term in brackets is bounded by const. $(1 + |t|)^{-\tau}$ which attains its maximum at $t = 0$,

$$]\, u_T - u_- [_\infty \leq g\tilde{\tilde{C}}\, \Gamma(\,]\, u\, [_T, 0)\,]\, u\, [_T\,.$$

Finally, by inequality (18),

$$]\, u_T [_\infty \leq\,]\, u_- [_\infty + 2\, g\tilde{\tilde{C}}\, \Gamma(2\,]\, u_- [_\infty, 0)\,]\, u_- [_\infty\,,$$

because of the monotonicity of $\Gamma(.\,,.)$ in each variable.

The above results may be summarized as part of the solution of problem i) in the following form.

THEOREM 1.4. Suppose that $u_- \in B_\infty$ is a given solution of the $K - G$ equation and the inequalities (D_1), (D_2), (D_3), (D_5) and (D_6) of part I and the stricter forms (S_2), (S_3) and (S_6) can be simultaneously satisfied with some choice of the constants and exponents. If, in addition, the size of the coupling constant g and/or $]\, u_- [_\infty$ are appropriately restricted (as in Theorem 1.2 of part I), then there exists a unique global solution, $u \in B_\infty$, of (the integrated form of)

$$\square u = m^2 u + G(s, u(s))$$

such that $|u(t) - u_-(t)|_\alpha \to 0$ as $t \to -\infty$.

PROOF. The solution, u, of equation (5) constructed above will suffice to establish the question of existence provided that it can be shown to behave like u_- near $t = -\infty$ in the prescribed sense. This will follow from the fact that for large negative times u satisfies equation (—). As a result, for $t < T$,

$$|u(t) - u_-(t)|_\alpha \leq\,]\, u - u_- [_t \leq g C\, \Gamma(2\,[\, u_- [_\infty, 0)\,]\, u_- [_\infty (1 + |t|)^{-\tau}\,, \tag{22}$$

using the techniques of Proposition 1.3.

All that remains to be shown then is the uniqueness of the solution of problem i) as formulated in the statement of this theorem. The essential technical part of the argument consists of proving a converse to the remarks surrounding Proposition 1.2; i.e. any solution of the problem stated in the Theorem necessarily satisfies equation (—). To this end suppose that $u \in B_\infty$ is a solution of the problem. Then $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$ satisfies equation (5),

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} u_\tau(t) \\ \dot{u}_\tau(t) \end{pmatrix} - \int_\tau^t U_0(t - s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds, \tag{23}$$

for any $|\tau| < \infty$ and all $t \in \mathbb{R}$, where as before u_τ is the solution of the $K - G$ equation which agrees with u at $t = \tau$. The above can be written in the more convenient form

$$(24) \qquad \begin{pmatrix} u_\tau(t) \\ \dot{u}_\tau(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} - \int_t^\tau U_0(t-s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds.$$

Then, just as in Proposition 1.2, for all $t \in \mathbb{R}$

$$\int_t^\tau U_0(t-s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds = U_0(t) \int_t^\tau \begin{pmatrix} -A^{-1} \sin s A\, G(s, u(s)) \\ \cos s A\, G(s, u(s)) \end{pmatrix} ds$$

as elements of $H(A, a)$. Similarily,

$$\int_t^\tau \begin{pmatrix} -A^{-1} \sin s A\, G(s, u(s)) \\ \cos s A\, G(s, u(s)) \end{pmatrix} ds \to \int_{-\infty}^\tau \begin{pmatrix} -A^{-1} \sin s A\, G(s, u(s)) \\ \cos s A\, G(s, u(s)) \end{pmatrix} ds$$

in $H(A, a)$ as $t \to -\infty$. Thus

$$U_0(t) \int_t^\tau \begin{pmatrix} -A^{-1} \sin s A\, G(s, u(s)) \\ \cos s A\, G(s, u(s)) \end{pmatrix} ds \to U_0(t) \int_{-\infty}^\tau \begin{pmatrix} -A^{-1} \sin s A\, G(s, u(s)) \\ \cos s A\, G(s, u(s)) \end{pmatrix} ds$$

in $H(A, a)$ as $t \to -\infty$ because, for each $t \in \mathbb{R}$, $U_0(t)$ is orthogonal in $H(A, a)$. On the other hand, directly from the hypotheses,

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} \to \begin{pmatrix} u_-(t) \\ \dot{u}_-(t) \end{pmatrix} \text{ as } t \to -\infty$$

in $H(A, a)$. The above may then be assembled to show that

$$\begin{pmatrix} u_\tau(t) \\ \dot{u}_\tau(t) \end{pmatrix} \to \begin{pmatrix} u_-(t) \\ \dot{u}_-(t) \end{pmatrix} - U_0(t) \int_{-\infty}^\tau \begin{pmatrix} -A^{-1} \sin s A\, G(s, u(s)) \\ \cos s A\, G(s, u(s)) \end{pmatrix} ds$$

in $H(A, a)$ as $t \to -\infty$. But if two solutions of the $K - G$ equation agree asymptotically in $H(A, a)$, they necessarily agree in all of $\mathbb{R}$. So corresponding to equation (19), one has

$$(25) \qquad \begin{pmatrix} u_\tau(t) \\ \dot{u}_\tau(t) \end{pmatrix} = \begin{pmatrix} u_-(t) \\ \dot{u}_-(t) \end{pmatrix} - \int_{-\infty}^\tau U_0(t-s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds.$$

Thus from equations (23) and (25) any solution of problem i) as specified in the theorem is a solution of equation (—).

Suppose now that there are two solutions u and v of the problem. Then

$$\begin{pmatrix} u(t) \\ \dot u(t) \end{pmatrix} - \begin{pmatrix} v(t) \\ \dot v(t) \end{pmatrix} = -\int_{-\infty}^{t} U_0(t-s) \left[\begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} - \begin{pmatrix} 0 \\ G(s, v(s)) \end{pmatrix} \right] ds$$

$$= L \begin{pmatrix} u(t) \\ \dot u(t) \end{pmatrix} - L \begin{pmatrix} v(t) \\ \dot v(t) \end{pmatrix},$$

where L is defined in equation (10). Thus, as in equality (17)

$$]\, u - v\, [_t \leq g C \Gamma(\,]\, u\, [_\infty\,,\,]\, v\, [_\infty)\, (1 + |t|)^{-\tau}\,]\, u - v\, [_t$$

for all $t \in R$, using the facts that $\Gamma(\cdot,\cdot)$ is monotonic and $]f[_t \leq]f[_\infty$. Thus if t is taken large enough in the negative direction so that the coefficient of $]\, u - v\, [_t$ on the right side of inequality (26) is less than unity then $]\, u - v\, [_t$ and hence $|\, u(s) - v(s)\, |_a = 0$ for all $s < t$. The uniqueness part of Segal's result for such equations [4, Theorem 1, p. 343] implies they must agree as elements of $H(A, a)$ throughout their entire interval of existence.

REMARK. Because only a restricted class of $H(A, a)$ solutions are being used in this discussion, the convergence of u to u_- as $t \to -\infty$ is in fact better than that proved in the above theorem. In particular, just as in inequality (22), we have that for all $t < T$

$$(27) \qquad (1 + |t|)^{\varepsilon}\, \|\, u(t) - u_-(t)\, \|_r \leq g C \Gamma(2\,]\, u_-\, [_\infty\,, 0)\,]\, u_-\, [_\infty\, (1 + |t|)^{-\tau},$$

and

$$(28) \qquad (1 + |t|)^{\delta}\, \|\, \dot u(t) - \dot u_-(t)\, \|_r \leq g C \Gamma(2\,]\, u_-\, [_\infty\,, 0)\,]\, u_-\, [_\infty\, (1 + |t|)^{-\tau}.$$

Problem ii) can be treated using many of the above techniques and estimates. In summary we give

THEOREM 1.5. Assuming the hypotheses, notation and conclusions of Theorem 1.4, then there exists a unique solution, $u_+ \in B_\infty$, of the $K-G$ equation such that $|\, u(t) - u_+(t)\, |_a \to 0$ as $t \to +\infty$.

PROOF. To begin, u satisfies equation (—),

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} u_-(t) \\ \dot{u}_-(t) \end{pmatrix} - \int_{-\infty}^{t} U_0(t-s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds.$$

Then, just as in Theorem 1.4,

$$\int_{-\infty}^{t} U_0(t-s) \begin{pmatrix} 0 \\ G(s, u, (s)) \end{pmatrix} ds \to U_0(t) \int_{-\infty}^{\infty} \begin{pmatrix} -A^{-1} \sin s A \; G(s, u(s)) \\ \cos s A \; G(s, u(s)) \end{pmatrix} ds$$

in $H(A, a)$ as $t \to +\infty$. Thus

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} \to \begin{pmatrix} u_-(t) \\ \dot{u}_-(t) \end{pmatrix} - U_0(t) \int_{-\infty}^{\infty} \begin{pmatrix} -A^{-1} \sin s A \; G(s, u(s)) \\ \cos s A \; G(s, u(s)) \end{pmatrix} ds$$

in $H(A, a)$ as $t \to +\infty$, the limit being a solution of the $K - G$ equation The uniqueness follows from the observation that solutions of the $K - G$ equation which agree asymptotically agree throughout $\mathbb{R}$. Thus

$$(29) \qquad \begin{pmatrix} u_+(t) \\ \dot{u}_+(t) \end{pmatrix} = \begin{pmatrix} u_-(t) \\ \dot{u}_-(t) \end{pmatrix} - U_0(t) \int_{-\infty}^{\infty} \begin{pmatrix} -A^{-1} \sin s A \; G(s, u(s)) \\ \cos s A \; G(s, u(s)) \end{pmatrix} ds$$

$$(30) \qquad = \begin{pmatrix} u_-(t) \\ \dot{u}_-(t) \end{pmatrix} - \int_{-\infty}^{\infty} U_0(t-s) \begin{pmatrix} 0 \\ G(s, u(s)) \end{pmatrix} ds.$$

$$(31) \qquad = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} - \int_{t}^{\infty} U_0(t-s) \begin{pmatrix} 0 \\ G \cdot (s, u(s)) \end{pmatrix} ds,$$

the first equality following from Bochner's Theorem for such integrals, and the second from equation (—).

REMARK. As above the convergence of u to u_+ is actually better than just in $H(A, a)$; namely $(1 + |t|)^{\varepsilon} \| u(t) - u_+(t) \|_r$ and $(1 + |t|)^{\delta} \| \dot{u}(t) - \dot{u}_+(t) \|_r \to 0$ as $t \to +\infty$.

2. EXAMPLES. Throughout this section, as in the corresponding section of part I, $n = 3$, $r = \infty$ and $a \geq 3$ (or ≥ 2 if $\dot{u}$ does not enter the discussion). We shall show here that the scattering theory for the examples introduced in part I can be treated by means of the general technique discribed above. In view of the fact that part I shows that inequalities (D_1), (D_2), (D_3), (D_5) and (D_6) can be simultaneously satisfied along with the conditions required for existence of local solutions, all that requires checking in that (S_2), (S_3) and (S_6) can likewise be satisfied with ε and δ*'s* compatible with those appearing in the decay conditions. For these examples, as previously mentioned, the technique of verification will be quite similar to that of the correspondingly numbered decay conditions. The arguments will be directed towards showing that some realistic values of ε and δ (i.e. $\leq 3/2$) can be found so that the existence of scattering can be verified for each perturbation without attempting to estimate the minimal decay that u_- must have for the procedure to be applicable. In other words the existence of scattering will only be checked for a class of smooth solutions which decay sufficiently rapidly. A large class which falls within the scope of the following discussion is that containing solutions whose Cauchy data at some time have Fourier transforms in $C_c^\infty(R^3)$ (for which $\varepsilon = \delta = 3/2$). On the other hand lower bounds on ε and δ which presumably are not the best do appear as a result of the need to satisfy the technical parts of the procedure.

(*a*) $G(x, t, u, u_x, u_t) = G(u)$. The decay results for this example (c. f. also [2, section 4, p. 478]) were obtained, in Part I, for perturbations which, for all essential matters, looked like $G(\lambda) \approx g\,|\lambda|^\beta$ with $\beta \geq 3$ However a better result is possible; namely that if the Cauchy data were prescribed at some time in $H(A, 2)$ for which the solution of the $K - G$ equation decayed uniformly in space like $|t|^{-\varepsilon}$ with $3(3\beta - 5)^{-1} < \varepsilon \leq 2^{-1}(3\beta - 6)$ for $8/3 < \beta \leq 3$ and $\max(1, 3(2\beta - 4)^{-1} < \varepsilon \leq 3/2$ for $\beta > 3$, then the solution of the perturbed equation exists globally and possesses the same decay property provided either g or the Cauchy data was suitably small. (Actually the result was proved only for $\varepsilon = 3/2$ in Part I but the condition (D_6) for this case, appearing between inequalities (27*b*) and (28*b*), can readily be checked for the cited range of ε using the algebraic computations appearing below). For this example it will turn out that such ε's are also suitable for showing the existence of scattering and discussion of $\dot{u}$ can be avoided (i. e. $k = 1$ and $\dot{k} = 0$ in defining equation (7)). More precisely, the result we shall prove is

THEOREM 2.1. Suppose $G(x, t, u, u_x, u_t) = G(u)$ where $G \in C^3(\mathbb{R})$, is real-valued and $\left| \dfrac{d^j G}{d\lambda^j}(\lambda) \right| \leq g\,|\lambda|^{\beta - j}$ for all λ and $j = 0, \dots, 3$ with $\beta > 8/3$.

If the given solution u_- of the $K - G$ equation is in B_∞, with $a = 2$, $3(3\beta - 5)^{-1} < \varepsilon < 2^{-1}(3\beta - 6)$ for $8/3 < \beta \leq 3$ and $\max(1, 3(2\beta - 4)^{-1}) < < \varepsilon < 3/2$ for $\beta > 3$, and g or $]u_-[_\infty$ is appropriately small, then there exists a unique global solution, $u \in B_\infty$, of the perturbed equation and a unique solution, $u_+ \in B_\infty$, of the $K - G$ equation such that

$$| u(t) - u_-(t) |_2 \to 0 \quad \text{as} \quad t \to -\infty,$$

and

$$| u(t) - u_+(t) |_2 \to 0 \quad \text{as} \quad t \to +\infty.$$

In addition $(1 + |t|)^\varepsilon \| u(t) - u_-(t) \|_\infty$ and $(1 + |t|)^\varepsilon \| u(t) - u_+(t) \|_\infty \to 0$ as $t \to -\infty$ and $+\infty$ respectively.

PROOF. With $a = 2$ and taking $b = 2$ an inequality of the form (S_2) and (S_3) which suffices for the present situation as well as for the next example is that summarized in

LEMMA 2.2. If G satisfies the hypotheses of Theorem 2.1 and $1 \leq q \leq 2$; then for arbitrary $u, v \in B_T$ and $s < T$

$$\| A^2 [G(u(s)) - G(v(s))] \|_q \leq g (1 + |s|)^{-(\beta - 2/q)\varepsilon} \gamma(]u[_T,]v[_T)]u - v[_T, \tag{32}$$

where $\gamma : \mathbb{R}^2 \to \mathbb{R}^+$ is bounded on bounded sets and monotonically non-decreasing in each variable.

PROOF. Appendix 1.

From inequality (32) and the well-known decay estimate $\| E_{t,2} \|_{q(q-1)^{-1}} \leq \leq C(1 + |t|)^{-3/q + 3/2}$ for $1 \leq q \leq 2$ (c.f. [1, Theorem 2.2]), the exponents appearing in (S_6) for this specific example are $\alpha = (\beta - 2/q)\varepsilon$, $\varrho = 3/q - 3/2$ and $\alpha'' = (\beta - 1)\varepsilon$ (noting that $\| Af \|_2 \leq \| A^2 f \|_2$ by the spectral theorem). Now in view of the hypotheses $\alpha'' = (\beta - 1)\varepsilon > 2\varepsilon > 2$ for $\beta > 3$ and $> (3\beta - 3)(3\beta - 5)^{-1} > 1$ for $8/3 < \beta \leq 3$. All that remains then is to check that it is possible to choose $1 \leq q \leq 2$ such that $\max(3/q - 3/2, (\beta - 2/q)\varepsilon) > 1$ and $\min(3/q - 3/2, (\beta - 2/q)\varepsilon) > \varepsilon$. This will be done very crudely by splitting the problem into two parts. First with $\beta > \varepsilon$ and $\max(1, 3(2\beta - 4)^{-1}) < \varepsilon < 3/2$ choose $1 < q < 6(2\varepsilon + 3)^{-1}$ ($< 6/5$ since $\varepsilon > 1$). Such q's exist because if $\varepsilon < 3/2$ then $1 < 6(2\varepsilon + 3)^{-1}$. The conclusion will clearly follow by showing that this choice implies that $(\beta - 2/q)\varepsilon > 3/q - 3/2 > \varepsilon$ and $(\beta - 2/q)\varepsilon > 1$. The first part of the first inequality is equivalent to $q > (4\varepsilon + 6)(2\beta\varepsilon + 3)^{-1}$. But $\varepsilon > 3(2\beta - 4)^{-1}$ implies that $(4\varepsilon + 6)(2\beta\varepsilon + 3)^{-1} < 1$. Our choice of $q > 1 > (4\varepsilon + 6)(2\beta\varepsilon + 3)^{-1}$ thus verifies the first part. The second part is equivalent to $q < 6(2\varepsilon + 3)^{-1}$

The last inequality follows from the choice $q > 1$, for then $(\beta - 2/q)\,\varepsilon > (\beta - 2)\,\varepsilon > \varepsilon > 1$.

Suppose now that $8/3 < \beta \leq 3$, then $0 < (\beta - 8/3)(\beta - 1)$, or equivalently $3\,(3\beta - 5)^{-1} < 2^{-1}(3\beta - 6)$. Thus it is indeed possible to find ε'*s* such that $3\,(3\beta - 5)^{-1} < \varepsilon < 2^{-1}(3\beta - 6)$. Now $\varepsilon < 2^{-1}(3\beta - 6)$ is equivalent to $2\,(\beta - 1)^{-1} < (6 + 4\varepsilon)(2\beta\varepsilon + 3)^{-1}$ so that it is possible to find q'*s* such that $1 \leq 2\,(\beta - 1)^{-1} < q < (6 + 4\varepsilon)(2\beta\varepsilon)^{-1} < 6/5$, the first inequality following from $\beta \leq 3$, while the last is equivalent to $3\,(3\beta - 5)^{-1} < \varepsilon$. Finally $q < (6 + 4\varepsilon)(2\beta\varepsilon + 3)^{-1}$ implies that $3/q - 3/2 > (\beta - 2/q)\,\varepsilon$, and $2\,(\beta - 1)^{-1} < q$ implies that $(\beta - 2/q)\,\varepsilon > \varepsilon$, while $q < 6/5$ is equivalent to $3/q - 3/2 > 1$ thus concluding the proof.

REMARK. A more careful analysis in Theorem 2.1 would presumably give better lower bounds for ε than $\max(1, 3\,(2\beta - 4)^{-1})$ for $\beta > 3$ and $3\,(3\beta - 5)^{-1}$ for $8/3 < \beta \leq 3$.

(*b*) $G(x, t, u, u_x, u_t) = G(u_t)$. For this example the decay results in part I were also obtained for G'*s* of the form $G(\lambda) \approx g\,|\,\lambda\,|^{\beta}$ with $\beta \geq 3$. However in this case the relevant solution space is $H(A, 3)$ with $b = 2$. In addition only the decay of $\dot{u}$ (shown in part I to be $|\,t\,|^{-\delta}$ $1/2 < \delta < 1$ if the same is true for the free solution) needs to be considered here to obtain scattering in $H(A, 3)$. Hence, for this example, $k = 0$ and $\dot{k} = 1$.

THEOREM 2.3. *Suppose* $G(x, t, u, u_x, u_t) = G(u_t)$ *where* $G \in C^3(\mathbb{R})$, *is real-valued and* $\left|\dfrac{d^j G}{d\lambda^j}(\lambda)\right| \leq g\,|\,\lambda\,|^{\beta - j}$ *for all* λ *and* $j = 0, \dots, 3$ *with* $\beta \geq 3$. *If the given solution* u_- *of the* K-G *equation is in* B_∞ *with* $a = 3$ *and* $1/2 < \delta < 1$, *and* g *or* $]\,u_-\,[_\infty$ *is appropriately small, then there exists a unique global solution,* $u \in B_\infty$ *of the perturbed equation and a unique solution,* $u_+ \in B_\infty$, *of the* K-G *equation such that*

$$|\,u(t) - u_-(t)\,|_3 \longrightarrow 0 \quad \text{as} \quad t \longrightarrow -\infty,$$

and

$$|\,u(t) - u_+(t)\,|_3 \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty.$$

In addition $(1 + |\,t\,|)^{\delta}\,\|\,\dot{u}(t) - \dot{u}_-(t)\,\|_\infty$ *and* $(1 + |\,t\,|)^{\delta}\,\|\,\dot{u}(t) - \dot{u}_+(t)\,\|_\infty \longrightarrow 0$ *as* $t \longrightarrow -\infty$ *and* $+\infty$ *respectively.*

PROOF. From inequality (32) which is again applicable and the estimate $\|\,F_{t,1}\,\|_{q'(q'-1)^{-1}} \leq C\,|\,t\,|^{-1/q'}$ for $1 < q' < 4/3$ (c. f. [1, Corollary 2.5]) the exponents in (S_6) are $\alpha' = (\beta - 2/q')\,\delta$, $\sigma = 1/q'$ and $\alpha'' = (\beta - 1)\,\delta$ (where in the analog of inequality (16) the ensuing integrals are estimated by using a similar technical result due to Shenk and Thoe [6, Lemma 3.1]). Clearly, $\alpha'' = (\beta - 1)\,\delta \geq 2\delta > 1$, so that it is sufficient to show that it is possible to choose $1 < q' < 4/3$ such that $(\beta - 2/q')\,\delta > 1/q' > \delta$ and $(\beta - 2/q')\,\delta > 1$. But this is precisely the result which was established in the discussion of decay for this example [1, Theorem 2.6].

(c) $G(x, t, u, u_x, u_t) = \sum_{\varkappa=0}^{3} G^{\varkappa}(u)\,\partial_{\varkappa} u$. Here $\partial_0 u = u_t$. In this example the results in part I were obtained for $G^{\varkappa}(\lambda) \approx g\,|\lambda|^{\alpha_{\varkappa}}$ with $\alpha_{\varkappa} \geq 2$. Again the relevant solution space is $H(A, 3)$ and $b = 2$. By making minimal modifications in the proof of Theorem 2.7 of part I, it can be shown that that if the free solution and its time derivative decay like $|t|^{-\varepsilon}$, $3/4 < \varepsilon \leq 1$, and $|t|^{-\delta}$, $1/4 < \delta \leq 5/6$, respectively, then the same is true for the perturbed solution if the size of g or the Cauchy data is restricted. In this case it will be necessary to have some decay for both u_- and $\dot{u}_-$ in order to prove that scattering can take place in $H(A, 3)$. (i. e. $k = \dot{k} = 1$).

THEOREM 2.4. Suppose $G(x, t, u, u_x, u_t) = \sum_{\varkappa=0}^{3} G^{\varkappa}(u)\,\partial_{\varkappa} u$ where fore ach $\varkappa = 0, \dots, 3$, $G^{\varkappa} \in C^3(\mathbb{R})$, is real-valued and for all λ $\left|\frac{d^j G^{\varkappa}}{d\lambda^j}(\lambda)\right| \leq g\,|\lambda|^{\alpha^{\varkappa}-j}$ with $\alpha^{\varkappa} \geq 2$ and $j = 0, 1, 2$ and $\left|\frac{d^3 G^{\varkappa}}{d\lambda^3}(\lambda)\right| \leq g\,|\lambda|^{\alpha_3^{k}}$ with $0 \leq \alpha_3^{\varkappa} < \infty$. If the given solution u_- of the K-G equation is in B_∞ with $a = 3$, $3/4 < \varepsilon < 1$, $1/4 < \delta < 5/6$ and g or $]\,u_-[_\infty$ is appropriately small, then there exists a unique global solution, $u \in B_\infty$, of the perturbed equation and a unique solution, $u_+ \in B_\infty$, of the $K-G$ equation such that

$$|\,u(t) - u_-(t)\,|_3 \longrightarrow 0 \quad \text{as} \quad t \longrightarrow -\infty,$$

and

$$|\,u(t) - u_+(t)\,|_3 \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty.$$

In addition $(1 + |t|)^{\varepsilon}\,\|\,u(t) - u_{\mp}(t)\,\|_\infty$ and $(1 + |t|)^{\delta}\,\|\,\dot{u}(t) - \dot{u}_{\mp}(t)\,\|_\infty \longrightarrow 0$ as $t \longrightarrow \mp\infty$.

PROOF. The inequalities of the form (S_2) and (S_3) which are valid in the present situation are summarized in

LEMMA 2.5. With hypotheses of Theorem 2.4 and $u, v \in B_T$

$$(33)\quad \text{(i)} \quad \left\| A^2 \left[\sum_{\varkappa=0}^{3} G^{\varkappa}(u(s))\,\partial_{\varkappa} u(s) - \sum_{\varkappa=0}^{3} G^{\varkappa}(v(s))\,\partial_{\varkappa} v(s) \right] \right\|_q \leq$$

$$g\,(1 + |s|)^{-a}\,\gamma\,(]\,u\,[_T)\;]\,u - v\,[_T$$

for all $s < T$ and $\max(1, 6(3\alpha^\varkappa - 1)^{-1}) \leq q \leq 6/5$ with $\alpha = \min_\varkappa (\alpha^\varkappa + 1 - 2/q)\,\varepsilon$, and

$$(34)\quad \text{(ii)}\quad \left\| A^2 \left[\sum_{\varkappa=0}^{3} G^\varkappa(u(s))\, \partial_\varkappa u(s) - \sum_{\varkappa=0}^{3} G^\varkappa(v(s))\, \partial_\varkappa v(s) \right] \right\|_1 \leq$$

$$g\,(1 + |s|)^{-\alpha''}\,\gamma''(\,]n[_T,\,]v[_T)\,]u - v[_T$$

for all $s < T$ with $\alpha'' = \min_\varkappa (\alpha^\varkappa \varepsilon, (\alpha^0 - 1)\,\varepsilon + \delta, (\alpha^0 - 1/2)\,\varepsilon + \delta/2)$. The functions $\gamma, \gamma'' : \mathbb{R}^2 \to \mathbb{R}^+$ are bounded on bounded sets and monotonically non-decreasing in each variable.

PROOF. Appendix II.

Returning to the proof of the theorem, in order to treat $\alpha^\varkappa = 2$ we take $q = q' = 6/5$. Thus $\varrho = 1$, $\sigma = 5/6$ and $\alpha = \alpha' = 4/3\varepsilon$. It is trivial to to check that (S_6) is satisfied in the form $\alpha > \varrho > \varepsilon$ with $\alpha > 1$, $\alpha' > \sigma > \delta$ with $\alpha' > 1$ and $\alpha'' > 1$.

(*d*) $G(x, t, u, u_x, n_t) = g G(x, t)\, u$. For this linear example only the decay of $\| u(t) \|_\infty$ needs to be considered in order to obtain the existence of scattering in $H(A, 3)$. The decay result obtains for $0 \leq \varepsilon \leq 3/2$ by means of a minor modification of Theorem 2.21 of part I and the computations to follow. In these calculations b is taken to be 2.

THEOREM 2.6. Suppose $G(x, t, u, u_x u_t) = g\, G(x, t)\, u$ where, for all $t \in \mathbb{R}$, $G(\cdot, t) \in W^{2,p}(\mathbb{R}^3)$ (1) for $1 \leq p \leq \infty$ and $\| G(\cdot, t) \|_{2,p}$ is a continuous function of $t \in \mathbb{R}$ with $\| G(\cdot, t) \|_{2,1}$ unirormly bounded in t and $\| G(\cdot, t) \|_{2,\infty} = 0(|t|^{-3})$ as $|t| \to \infty$. If the given solution u_- of the $K - G$ equation is in B_∞ with $a = 3$, $0 \leq \varepsilon < 3/2$ and g is sufficiently small, then there exists a unique global solution, $u \in B_\infty$, of the perturbed equation and a unique solution, $u_+ \in B_\infty$, of the $K - G$ equation such that

$$|u(t) - u_-(t)|_3 \to 0 \quad \text{as} \quad t \to -\infty,$$

and

$$|u(t) - u_+(t)|_3 \to 0 \quad \text{as} \quad t \to +\infty.$$

In addition $(1 + |t|)^\varepsilon \| u(t) - u_-(t) \|_\infty$ and $(1 + |t|)^\varepsilon \| u(t) - u_+(t) \|_\infty \to 0$ as $t \to -\infty$ and $+\infty$ respectively.

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(1) $W^{2,p}(E^3)$ as usual denotes the Sobolev space with norm

$$\| f \|_{2,p} = \{ \sum_{0 \leq |\alpha| \leq 2} \| D^\alpha f \|_p^p \}^{1/p}.$$

Proof. The appropriate forms of $(S_2)\,(S_3)$ are quite similar to the result in Lemma 2.9 of part I, being most conveniently summarized in

Lemma 2.7. With the hypotheses of Theorem 2.6, $1 \le q \le 2$ and $u, v \in B_T$,

$$(35) \quad \| A^2 [g G(\cdot, s) \} u(s) - v(s) \{] \|_q \le g C (1 + |s|)^{-\alpha} \gamma ( ] u [_T,] v [_T,)] u - v [_T,$$

for all $s < T$, and $\alpha = \min\left(3 - 3/q + \varepsilon, 7/2 - 3/q + \dfrac{2\varepsilon}{3}, 4 - 3/q + \varepsilon/3\right)$ and $\gamma : \mathbb{R}^2 \to \mathbb{R}^+$ is bounded on bounded sets and monotonically nondecreasing in each variable.

Proof. Appendix III.

The remainder of the proof of Theorem 2.6 consists of showing that (S_6) can be satisfied with $\varrho = 3/q - 3/2$, $\alpha = \min\left(3 - 3/q + \varepsilon, 7/2 - 3/q + \dfrac{2\varepsilon}{3}, 4 - 3/q + \varepsilon/3\right)$ and $\alpha'' = \min\left(3/2 + \varepsilon, 2 + \dfrac{2\varepsilon}{3}, 5/2 + \varepsilon/3\right)$ by a suitable choice of $1 \le q \le 2$. Clearly $\alpha''' > 1$ for all $1 \le q \le 2$. For any $0 \le \varepsilon < 3/2$, choose $1 < q < 36(33 + 2\varepsilon)^{-1}$. Such $q's$ exist because $\varepsilon < 3/2$ implies that $1 < 36(33 + 2\varepsilon)^{-1}$. To complete the argument it is sufficient to show that $3/q - 3/2 > 4 - 3/q + \varepsilon/3 > 7/2 - 3/q + 2\varepsilon/3 > 3 - 3/q + \varepsilon > \varepsilon$ and that $3/q - 3/2 > 1$. Taking each inequality in turn, the first is equivalent to $q < 36(33 + 2\varepsilon)^{-1}$; the second and third follow from $\varepsilon < 3/2$; the fourth from $q > 1$. For the last inequality, since $0 \le \varepsilon < 3/2$, $36(33 + 2\varepsilon)^{-1} < 6/5$ and hence $3/q - 3/5 > 1$.

The technical part of the above discussion suggests several interesting exercises. As previously mentioned, better estimates on the lower bounds of ε (and δ if it appears as in (*c*)) are presumably possible. It would also be more satisfying to know that the upper bounds are actually attained. Finally it should be strightforward to check the rate of convergence of $\dot{u}$ to $\dot{u}_\pm$ in examples (*a*), (*b*) and (*d*). From another point of view it might be profitable to examine whether the abstract method, which in part generalizes that of Segal [2] and Strauss [3], can be applied to other quasi-linear situations. Current work which will be reported elsewhere iudicates that the coupled equations arising from generalized weak and electromagnetic interactions can be treated similarly.

Appendices. In this section the computational details of the basic inequalities in section 2 will be given. The technique is precisely that used to obtain similar inequalities in part I by means Sobolev type inequalities. The most basic is

Proposition. (Nirenberg, cf. Proposition 2.4 of part I). Let $D^\alpha f$ denote the α^{th} weak derivative of f and define $\| D^i f \|_p = \max_{|\alpha| = i} \| D^\alpha f \|_p$. Suppose $f \in D(A^2)$ with $a \geq 2$. Then if $2 \leq p < \infty$, $2 \leq k \leq a$, $0 \leq j < k$,

$$\| D^j f \|_p \leq \text{Const.} \| D^k f \|_2^\gamma \| f \|_\infty^{1-\gamma}. \tag{36}$$

where $p^{-1} = j/3 + \gamma (1/2 - k/3)$ for all $j/k \leq \gamma \leq 1$.

For the other definitions and concepts which are implicitly assumed (such as the chain rule and Leibniz formula for strong derivatives) see part I, especially the introductory remarks surrounding Proposition 2.4.

I. **Proof of Lemma 2.2** A straightforward computation gives (deleting the s-dependence for the moment)

$$\begin{aligned} A^2 [G(u) - G(v)] &= m^2 [G(u) - G(v)] - \Delta [G(u) - G(v)] \\ &= m^2 G'(a_1 u + b_1 v)(u - v) + G'(u)(\Delta u - \Delta u) \\ &+ G''(a_2 u + b_2 v) \Delta v (u - v) + G''(u)((\Delta u)^2 - (\Delta v)^2) \\ &+ G'''(a_3 u + b_3 v)(\Delta u)^2 (u - v), \end{aligned} \tag{37}$$

where G', G'', G''' refer to the usual derivatives of G and the derivatives of u and v are the weak, or equivalently strong — $L^2(E^3)$ derivatives, and $a_i, b_i \geq 0$, $a_i + b_i = 1$ are constants arising from the mean value theorem. The desired inequality can now we obtained by estimating the L^q-norm of each term. For convenience the subscripts of a_i, b_i will be deleted.

$$\begin{aligned} \| G'(au + bv)(u - v) \|_q &\leq g \| \, |au + bv|^{\beta - 1} \|_q \| u - v \|_\infty \\ &\leq g \| au + bv \|_\infty^{\beta - 1 - 2/q} \| au + bv \|_2^{2/q} \| u - v \|_\infty \end{aligned}$$

provided that $(\beta - 1) q \geq 2$, which is guaranteed by the restrictions on β and q. The next part of the argument, being similar for all other terms, will be represeuted at this stage in detail for the one and only time. Displaying the s-dependence

$$\begin{aligned} \| G'(au(s) + bv(s))(u(s)) - v(s)) \|_q &\leq g C (a \| u(s) \|_\infty + b \| v(s) \|_\infty)^{\beta - 1 - 2/q} \cdot \\ &(a \| A^2 u(s) \|_2 + b \| A^2 v(s) \|_2)^{2/q} \| u(s) - v(s) \|_\infty . \end{aligned}$$

But for $s < T$, $(1 + |s|)^\varepsilon \| u(s) \|_\infty$ and $\| A^2 u(s) \|_2 \leq | u(s) |_a \leq] u [_T$.

This and similar results for $v(s)$ and $u(s) - v(s)$ can be used to obtain

$$\| G'(au(s) + bv(s))(u(s) - v(s)) \|_q \leq gC(1 + |s|)^{-(\beta-1-2/q)\varepsilon}$$

$$(a\,]u[_T + b]\,v\,[_T)^{\beta-1-2/q}$$

$$\cdot (a\,]u[_T + b]\,v\,[_T)^{2/q}(1 + |s|)^{-\varepsilon}]\,u - v[_T,$$

$$\leq gC(1 + |s|)^{-(\beta - 2/q)}\varepsilon\,(a\,]u[_T + b]\,v\,[_T)^{\beta-2/q}]\,u - v\,[_T.$$

In other words the required estimate has been obtained for the first term of expression (35) with $\gamma(]u[_T, v[_T) = (a]u[_T)^{\beta-2/q}$ which is of the stated form.

The most essential details for the remainder of the terms proceed as follows.

$$\| G'(u)(\Delta u - \Delta v) \|_q \leq g \| |u|^{\beta-1} \|_{2q(2-q)^{-1}} \| \Delta(u - v) \|_2$$

$$\leq g \| |u|^{2q(\beta-1)(2-q)^{-1}-2} |u|^2 \|_1^{(2-q)/2q} \| A^2(u - v) \|_2$$

$$\leq g \| u \|_\infty^{\beta-2/q} \| u \|_2^{2/q-1} \| A^2(u - v) \|_2,$$

where the requirement $2q(\beta - 1)(2 - q)^{-1} \geq 2$ is again fulfilled.

$$\| G''(au + bv)\,\Delta v\,(u - v) \|_q \leq g \| |au + bv|^{\beta-2} \|_{2q(2-q)^{-1}} \| \Delta v \|_2 \| u - v \|_\infty$$

$$\leq gC(a\|u\|_\infty + b\|v\|_\infty)^{\beta-1-2/q}(a\|A^2 u\|_2 + b\|A^2 v\|_2)^{2/q-1}$$

$$\cdot \| A^2 v \|_2 \| u - v \|_\infty,$$

the requirement $2q(\beta - 2)(2 - q)^{-1} \geq 2$ again being fulfilled.

Letting ∂_i denote the weak derivative with respect to the i^{th} coordinate the fourth term may be estimated as follows.

$$\| G''(au + bv)(\Delta u)^2 - (\Delta v)^2) \|_q \leq g \sum_{i=1}^{3} \| |au + bv|^{\beta-2} \|_{2q(2-q)^{-1}}$$

$$\| \partial_i(u + v) \|_4 \| \partial_i(u - v) \|_4$$

$$\leq gC(a\|u\|_\infty)^{\beta-1-2/q}(a\|A^2 u\|_2 + b\|A^2 v\|_2)^{2/q-1}$$

$$\| D^2(u + v) \|_2^{1/2} \| u + v \|_\infty^{1/2} \| D^2(u - v) \|_2^{1/2} \| u - v \|_\infty^{1/2}$$

The requirement $2q(\beta - 2)(2 - q)^{-1} \geq 2$ is satisfied and the result is obtained by observing that $\| D^2 f \| \leq$ Const. $\| A^2 f \|_2$. Finally,

$$\| G'''(au + bv)(\Delta v)^2 (u - v) \|_q \leq g \| au + bv \|_\infty^{\beta-3} \| (\Delta v)^2 \|_2 \| u - v \|_{2q(2-q)^{-1}}$$

$$\leq gC (a \| u \|_\infty + b \| v \|_\infty)^{\beta-3} \| Dv \|_4^2 \| u - v \|_\infty^{2-2/q} \| u - v \|_2^{2/q-1}$$

$$\leq gC (a \| u \|_\infty + b \| v \|_\infty)^{\beta-3} \| v \|_\infty \| A^2 v \|_2 \| u - v \|_\infty^{2-2/q}$$

$$\| A^2 (u - v) \|_2^{2/q-1}.$$

The conclnsion now follows because each term is of the required form and the form is preserved under sums.

II. Proof of Lemma 2.5. The summatiou convention will be used with sums on $\varkappa$ going from 0 to 3 and on k from 1 to 3, with $\partial_0 u$ denoting $\dot{u}$ and $\partial_k u$ representing $\frac{\partial u}{\partial x_k}$ for $k = 1, 2, 3$. A straightforward calculation as in equation (37) gives

$$(38) \quad A^2 G^k(u) \partial_k u = m^2 G^k(u) \partial_k u + m^2 G^0(u) \dot{u} + G^k(u) \partial_k (\Delta u)$$

$$+ 2G^{k\prime}(u) \nabla u \cdot \nabla (\partial_k u) + \Delta u G^{k\prime}(u) \partial_k u + (\nabla u)^2 G^{k\prime\prime}(u) \partial_k u$$

$$+ G^0(u) \Delta \dot{u} + 2G^{0\prime}(u) \nabla u \cdot \nabla \dot{u} + \Delta u G^{0\prime}(u) u + (\Delta u)^2 G^{0\prime\prime}(u) \dot{u}.$$

What must be estimated, however, is the difference of the above with the same expression with u replaced by v. As in Appendix I, only the essential details will be given.

$$(39) \quad \| G^k(u) \partial_k u - G^k(v) \partial_k v \|_q \leq \| [G^k(u) - G^k(v)] \partial_k u \|_q + \| G^k(v) \partial_k (u - v) \|_q$$

$$\leq \| G^{k\prime}(au + bv)(u - v) \partial_k u \|_q + \| G^k(v) \partial_k (u - v) \|_q,$$

where $a, b \geq 0$, $a + b = 1$ arise through the mean value theorem. Now

$$\| G^k(v) \partial_k (u - v) \|_q \leq gC \| v \|_\infty^{\alpha^k - 2/q + 1/3} \| A^3 v \|_2^{2/q - 1/3} \| u - v \|_\infty^{2/3} \| A^3 (u - v) \|_2^{1/3}$$

from the estimates for the first term in Lemma 2.8 of part I since $\alpha^k 6q (6 - q)^{-1} \geq 2$. For the first term in inequality

$$\| G^{k\prime}(au + bv)(u - v) \partial_k u \|_q \leq g \| | au + bv |^{\alpha^k - 1} \|_{2q(2-q)^{-1}} \| u - v \|_\infty \| \partial_k u \|_2$$

$$\leq gC (a \| u \|_\infty + b \| v \|_\infty)^{\alpha^k - 2/q} (a \| A^3 u \|_2 + b \| A^3 v \|_2)^{2/q-1} \| u - v \|_\infty \| A^3 u \|_2$$

since $(\alpha^k - 1)\, 2q\, (2 - q)^{-1} \geq 2.$

(40) $$\| G^0(u)\, \dot{u} - G^0(v)\, \dot{v} \|_q \leq \| G^{0\prime}(au + bv)(u - v)\, \dot{u} \|_q + \| G^0(v)(\dot{u} - \dot{v}) \|_q$$

Again the second term can be handled as in part I to give

$$\| G^0(v)(\dot{u} - \dot{v}) \|_q \leq gC \| v \|_\infty^{\alpha^0 + 1 - 2/q} \| A^3 v \|_2^{2/q - 1} A^2 (\dot{u} - \dot{v})_2$$

since $\alpha^0\, 2q\, (2 - q)^{-1} \geq 2$, while the first can be treated in the same manner as the first term of inequality (39).

Next

(41) $$\| G^k(u)\, \partial_k(\Delta u) - G^k(v)\, \partial_k(\Delta v) \|_q \leq \| G^{k\prime}(au + bv)(u - v)\, \partial_k(\Delta u) \|_q + \| G^k(v)\, \partial_k \Delta(u - v) \|_q$$

From part I, the second term satisfies

$$\| G^k(v)\, \partial_k \Delta(u - v) \|_q \leq gC \| v \|_\infty^{\alpha^k + 1 - 2/q} \| A^3 v \|_2^{2/q - 1} \| A^3(u - v) \|_2$$

since $\alpha^k\, 2q\, (2 - q)^{-1} \geq 2$. The first term follows along the lines of the first terms of inequalities (39) and (40). The seventh parallels the third as the second does the first.

(42) $$\| G^{k\prime}(u)\, \nabla u \cdot \nabla(\partial_k u) - G^{k\prime}(v)\, \nabla v \cdot \nabla(\partial_k v) \|_q \leq \| G^{k\prime}(u)\, \nabla(u - v) \cdot \nabla(\partial_k u) \|_q + \| G^{k\prime}(u)\, \nabla v \cdot \nabla(\partial_k [u - v]) \|_q + \| G^{k\prime}(au + bv)(u - v)\, \nabla v \cdot \nabla(\partial_k v) \|_q .$$

As in fourth term of Lemma 2.8 of part I, the first term satisfies

$$\| G^{k\prime}(u)\, \nabla(u - v) \cdot \nabla(\partial_k u) \|_q \leq gC \| u \|_\infty^{\alpha^k - 2/q} \| A^3 u \|_2^{2/q - 1} \| u - v \|_\infty^{2/3} \| A^3(u - v) \|_2^{1/3} \cdot \| u \|_\infty^{1/3} \| A^3 u \|_2^{2/3},$$

since $(\alpha^k - 1)\, 2q\, (2 - q)^{-1} \geq 2$. The second term of inequality (42) can be handled in exactly the same way to obtain

$$\| G^{k\prime}(u)\, \nabla v \cdot \nabla(\partial_k [u - v]) \|_q \leq gC \| u \|_\infty^{\alpha^k - 2/q} \| A^3 u \|_2^{2/q - 1} \| v \|_\infty^{2/3} \| A^3 v \|_2^{1/3} \cdot \| u - v \|_\infty^{1/3} \| A^3(u - v) \|_2^{2/3}.$$

For the last term in inequality (42)

$$\| G^{k''}(au + bv)(u - v) \nabla v \cdot \nabla(\partial_k v) \|_q \leq gC \| |au + bv|^{\alpha^k - 2} \|_{6q(6-5q)^{-1}} \| u - v \|_\infty \cdot$$

$$\cdot \| \partial_i v \|_2 \| \partial_i \partial_k v \|_3$$

$$\leq gC (a \| u \|_\infty + b \| v \|_\infty)^{\alpha^k - 2/q - 1/3} (a \| A^3 u \|_2 + b \| A^3 v \|_2)^{2/q - 5/3} \cdot$$

$$\cdot \| A^3 v \|_2 \| v \|_\infty^{1/3} \| A^3 v \|_2^{2/3},$$

since $(\alpha^k - 2)\, 6q(6 - 5q)^{-1} \geq 2$.

The discussion for the corresponding term with $\varkappa = 0$, that is the eighth, requires only minor modifications.

(43) $$G^{0\prime}(u) \nabla u \cdot \nabla \dot{u} - G^{0\prime}(v) \nabla v \cdot \nabla \dot{v} \|_q \leq \| G^{0\prime}(u) \nabla (u - v) \cdot \nabla \dot{v} \|_q$$

$$+ \| G^{0\prime}(u) \nabla v \cdot \nabla (\dot{u} - \dot{v}) \|_q + \| G^{0\prime\prime}(au + bv)(u - v) \nabla v \cdot \nabla \dot{v} \|_q .$$

The first two parts are handled like the eighth term in part I.

$$\| G^{0\prime}(u) \nabla (u - v) \cdot \nabla \dot{v} \|_q \leq$$

$$\leq gC \| u \|_\infty^{\alpha^0 - 2/q + 1/3} \| A^3 u \|_2^{2/q - 4/3} \| u - v \|_\infty^{2/3} \cdot \| A^3 (u - v) \|_2^{1/3} \cdot \| A^2 \dot{v} \|_2$$

and

$$\| G^{0\prime}(u) \nabla v \cdot \nabla (\dot{u} - \dot{v}) \|_q \leq$$

$$\leq gC \| u \|_\infty^{\alpha^0 - 2/q + 1/3} \| A^3 u \|_2^{2/q - 4/3} \| v \|_\infty^{2/3} \| A^3 v \|_2^{1/3} \cdot \| A^2 (\dot{u} - \dot{v}) \|_2$$

since $(\alpha^0 - 1)\, 3q(3 - 2q)^{-1} \geq 2$. As for the last term in inequality (43)

$$\| G^{0\prime\prime}(au + bv)(u - v) \nabla v \cdot \nabla \dot{v} \|_q \leq$$

$$\leq g \sum_{i=1}^{3} \| |au + bv|^{\alpha^0 - 2} \|_{6q(6-5q)^{-1}} \cdot \| u - v \|_6 \cdot \| \partial_i v \|_6 \| \partial_i \dot{v} \|_2$$

$$\leq gC (a \| u \|_\infty + b \| v \|_\infty)^{\alpha^0 - 2/q - 1/3} (a \| A^3 u \|_2 + b \| A^3 v \|_2)^{2/q - 5/3} \cdot$$

$$\cdot \| u - v \|_\infty^{2/3} \| A^3 (u - v) \|_2^{1/3} \| v \|_\infty^{2/3} \| A^3 v \|_2^{1/3} \| A^2 \dot{v} \|_2 ,$$

since $(\alpha^0 - 2)\, 6q(6 - 5q)^{-1} \geq 2$.

The fifth procedes exactly like the fourth. The ninth which is related to the fifth may be estimated as follows.

$$\| \varDelta u G^{0\prime}(u)\dot{u} - \varDelta v G^{0\prime}(v)\dot{v} \|_q \le \| G^{0\prime}(u) \varDelta (u-v)\dot{u} \|_q + \| G^{0\prime}(u) \varDelta v (\dot{u}-\dot{v}) \|_q + \| G^{0\prime\prime}(au+bv)(u-v)\varDelta v \dot{v} \|_q . \tag{44}$$

By part I, the first two terms satisfy

$$\| G^{0\prime}(u) \varDelta (u-v)\dot{u} \|_q \le$$

$$\le gC \| u \|_\infty^{\alpha^0-2/q+2/3} \| A^3 u \|_2^{2/q-5/3} \| u-v \|_\infty^{1/3} \| A^3(u-v) \|_2^{2/3} \| A^2 \dot{u} \|_2 ,$$

and

$$\| G^{0\prime}(u) \varDelta v (\dot{u}-\dot{v}) \|_q \le$$

$$\le gC \| u \|_\infty^{\alpha^0-2/q+2/3} \| A^3 u \|_2^{2/q-5/3} \| v \|_\infty^{1/3} \| A^3 v \|_2^{2/3} \| A^2(\dot{u}-\dot{v}) \|_2 ,$$

since $(\alpha^0-1)\,6q(6-5q)^{-1} \ge 2$. The last term is similar in part.

$$\| G^{0\prime\prime}(au+bv)(u-v)\varDelta v \dot{v} \|_q \le g \| \, | au+bv |^{\alpha^0-2} \|_{6q(6-5q)^{-1}} \| u-v \|_\infty \| \varDelta v \|_3 \| \dot{v} \|_2$$

$$\le gC (a\| u \|_\infty + b \| v \|_\infty)^{\alpha^0-2/q-1/3} (a \| A^3 u \|_2 + b \| A^3 v \|_2)^{2/q-5/3} \cdot$$

$$\cdot \| v \|_\infty^{1/3} \| A^3 v \|_2^{2/3} \| A^2 \dot{v} \|_2 ,$$

since $(\alpha^0-2)\,6q(6-5q)^{-1} \ge 2$.

The sixth term is estimated by

$$\| (\nabla u)^2 G^{k\prime\prime}(u)\partial_k u - (\nabla v)^2 G^{k\prime\prime}(v)\partial_k v \|_q \le \| [(\nabla v)^2 - (\nabla v)^2] G^{k\prime\prime}(u)\partial_k u \|_q + \| (\nabla v)^2 G^{k\prime\prime}(u)\partial_k(u-v) \|_q + \| (\nabla v)^2 G^{k\prime\prime\prime}(au+bv)(u-v)\partial_k v \|_q . \tag{45}$$

The estimates for the sixth term in part I can be effectively used on the first two terms of inequality (45) to give

$$\| [(\nabla u)^2 - (\nabla v)^2] G^{k\prime\prime}(u)\partial_k u \|_q \le$$

$$\le gC \| u+v \|_\infty^{2/3} \| A^3(u+v) \|_2^{1/3} \| u-v \|_\infty^{2/3} \| A^3(u-v) \|_2^{1/3} \cdot$$

$$\cdot \| u \|_\infty^{\alpha^k-2/q-1/3} \| A^3 u \|_2^{2/q-5/3} \| A^3 u \|_2 ,$$

and

$$\| (\nabla v)^2 G^{k\prime\prime}(u)\partial_k(u-v) \|_q \le$$

$$\leq gC \| v \|_\infty^{4/3} \| A^3 v \|_2^{2/3} \| u \|_\infty^{\alpha^k - 2/q - 1/3} \| A^3 u \|_2^{2/9 - 5/3} \| A^3 (u - v) \|_2 ,$$

since $(\alpha^k - 2)\, 6q\, (6 - 5q)^{-1} \geq 2$. The last part of inequality (45) may be done as follows.

$$\| (\nabla v)^2 G^{k'''} (au + bv)(u - v)\, \partial_k v \|_q \leq$$

$$\leq g \| (\nabla v)^2 \|_3 \| \, | au + bv |^{\alpha_3^k} \|_\infty \| u - v \|_{2q(2-q)-1} \| \partial_k v \|_6 \leq$$

$$\leq gC \| v \|_\infty^{4/3} \| A^3 v \|_2^{2/3} (a \| u \|_\infty + b \| v \|_\infty)^{\alpha_3^k} \| u - v \|_\infty^{2 - 2/q} \| A^3 (u - v) \|_2^{2/q - 1} \cdot$$

$$\cdot \| v \|_\infty^{2/3} \| A^3 v \|_2^{1/3} ,$$

since $2q(2 - q)^{-1} \geq 2$. The last term follows as above except for the term corresponding to the last in inequality (46). But

$$\| (\nabla v)^2 G^{0'''} (au + bv)(u - v)\, \dot{v} \|_q \leq$$

$$\leq g \| (\nabla v)^2 \|_3 \| \, | au + bv |^{\alpha_3^0} \|_\infty \| u - v \|_{6q(6-5q)^{-1}} \| \dot{v} \|_2$$

$$\leq gC \| v \|_\infty^{4/3} \| A^3 v \|_2^{2/3} (a \| u \|_\infty + b \| v \|_\infty)^{\alpha_3^0} \| u - v \|_\infty^{8/3 - 2/q} \cdot$$

$$\cdot \| A^3 (u - v) \|_2^{2/q - 5/3} \| A^2 \dot{v} \|_2 ,$$

since $6q(6 - 5q)^{-1} \geq 2$.

The remainder of the proof of part i) follows from the above inequalities as in Appendix I. For part ii) all but the sixth, eighth, ninth and tenth are valid for $\alpha^\varkappa \geq 2$ and $q = 2$ and contribute to a decay term $(1 + | s |)^{-\alpha^\varkappa \varepsilon}$. For the sixth term, inequality (45) may be used with $q = 2$ and the estimate for the last term is valid as it stands for $q = 2$, while the first two may be treated as in Lemma 2.8 of part I to give

$$\| [(\nabla u)^2 - (\nabla v)^2]\, G^{k''}(u)\, \partial_k u \|_2 \leq$$

$$\leq gC \| u + v \|_\infty^{2/3} \| A^3 (u + v) \|_2^{1/3} \| u - v \|_\infty^{2/3} \| A^3 (u - v) \|_2^{1/3} \cdot$$

$$\cdot \| u \|_\infty^{\alpha^k - 2} \| u \|_\infty^{2/3} \| A^3 u \|_2^{1/3} ,$$

and

$$\| (\nabla v)^2 G^{k''}(u)\, \partial_k (u - v) \|_2 \leq$$

$$gC \| v \|_\infty^{4/3} \| A^3 v \|_2^{2/3} \| u \|_\infty^{\alpha^k - 2} \| u - v \|_\infty^{2/3} \| A^3 (u - v) \|_2^{1/3} ,$$

both of which contribute to inequality (34) a term with decay factor $(1 + |s|)^{-\alpha^{k_\varepsilon}}$.

For the eighth term, the analysis of part I can be directly applied, as above, to the first two terms of inequality (43), and the last term follows from the same trick after noticing that

$$\| G^{0\prime\prime}(au + bv)(u - v) \|_\infty \leq g(a \| u \|_\infty + b \| v \|_\infty)^{\alpha^0 - 2} \| u - v \|_\infty .$$

Thus all terms hava a decay factor $(1 + |s|)^{-[(\alpha^0 - 1/2)\varepsilon + \delta/2]}$.

The ninth and tenth follow as above from the corresponding estimates in the proof of part ii), Lemma 2.8 in part I, both contributing terms with decay factor $(1 + |s|)^{-[(\alpha^0 - 1)\varepsilon + \delta]}$.

III. PROOF OF LEMMA 2.7. The proof follows quite easily from Lemma 2.9 of part I. In particular, because of the linearity, inequality (42) of part I applies to give

$$\begin{aligned} (46) \qquad & \| A^2 [G(\cdot, s) u - G(\cdot, s) v] \|_q \leq \| A^2 [G(\cdot, s)(u - v)] \|_q \\ & \leq C [(\| G(\cdot, s) \|_q + \| D^2 G(\cdot, s) \|_q) \| u - v \|_\infty \\ & + \| DG(\cdot, s) \|_{6q(6-q)^{-1}} \| u - v \|^{2/3} \| A^3 (u - v) \|_2^{1/3} \\ & + \| G(\cdot, s) \|_{3q(3-q)^{-1}} \| u - v \|_\infty^{1/3} \| A^3 (u - v) \|_2^{2/3}], \end{aligned}$$

for all $1 \leq q \leq 2$, with $\| D^j f \|_p = \max_{|\alpha| = j} \| D^\alpha f \|_p$. Now, using the decay properties of G,

$$(47a) \qquad \| G(\cdot, s) \|_q \leq \| G(\cdot, s) \|_\infty^{1-1/q} \| G(\cdot, s) \|_1 \leq C(1 + |s|)^{-(3-3/q)} .$$

Similarly,

$$(47b) \qquad \| D^2 G(\cdot, s) \|_q \leq C(1 + |s|)^{-(3-3/q)},$$

$$(47c) \qquad \| DG(\cdot, s) \|_{6q(6-q)^{-1}} \leq C(1 + |s|)^{-(7/2-3q)},$$

and

$$(47d) \qquad \| G(\cdot, s) \|_{3q(3-q)^{-1}} \leq C(1 + |s|)^{-(4-3/q)} .$$

The result now follows by repeating the argument outlined in Appendix I.

REFERENCES

[1] J. M. CHADAM, *Asymtotics for* $\square u = m^2 u + G(x, t, u, u_t, u_x)$, I. *Global Existence and Decay,* Ann. Sc. Norm. summary in Bull. Amer. Math. Soc., **76**, 1032-1035, (1970).

[2] I. E. SEGAL, *Dispersion for Non-linear Relativistic Equations,* II, Ann. Scient. Ec. Norm. Sup., ser. **4**, **1**, 459-497, (1968).

[3] W. A. STRAUSS, *Decay and Asympotics for* $\square u = F(u)$, J. Functional Anal., **2**, 409-457, (1968).

[4] I. E. SEGAL, *Non-linear Semi-groups,* Ann. Math., **78** 339-364, (1963).

[5] K. YOSIDA, *Funclional Analysis,* Springer, Berlin-Göttingen-Heidelberg, 1965.

[6] N. SHENK and D. THOE, *Outgoing Sotutions of* $(-\Delta + q - k^2)\, u = f$ *in an Exterior Domain,* J. Math Anal. Applic , **31**, 81-116, (1970).