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ASYMPTOTICS FOR $\Box u = m^2 u + G(x, t, u, u_x, u_t)$, I. GLOBAL EXISTENCE AND DECAY (*)

by JOHN M. CHADAM

Equations of the type

$$\Box u = m^2 u + G(x, t, u, u_x, \dot{u}), \quad \dot{u} = u_t \tag{1}$$

are of interest inasmuch as they are mathematical prototypes of nonlinear equations which arise in Quantum Field Theory. Classical scattering for these equations is based on the knowledge of precise estimates for the decay of $\| u(t) \|_r = \| u(\cdot, t) \|_r$ and $\| \dot{u}(t) \|_r = \| \dot{u}(\cdot, t) \|_r$ as $|t| \to \infty$. A program of this sort has been carried out for the case $G(x, t, u, u_x, \dot{u}) = G(u)$ by Segal [1] and Strauss [2] for $m \neq 0$ and $m = 0$, respectively. In this paper a generalization of the method of Segal [1] will be developed and used to obtain suitable decay for a wide class of perturbations $G(x, t, u, u_x, \dot{u})$. The technique is perturbative in nature in that it depends (in the most interesting cases) on restricting the size of the Cauchy data and/or the coupling constant. Although equation (1) is the only type treated here, the technique can, in all likelihood, be used in studying the decay of a wider class of equations (e. g. perturbations of the Dirac equation, evolution equations arising in fluid dynamics).

In section 1 an abstract version of the technique is outlined. It is used in section 2 to study the decay of particular illustrative examples of equation (1); namely, $G(x, t, u, u_x, \dot{u}) = G(\dot{u}), G(u), \sum_{\varkappa=0}^{3} G_\varkappa(u)\, \partial_\varkappa u$ and the linear

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case $G(x,t)\,u$. A sequel to this paper will be devoted to the abstract (classical) scattering theory for the equation $\square u = m^2 u + G(x, t, u, u_x u_t)$. The corresponding theory for the above examples will also be treated based on the decay results obtained in section 2.

1. Decay. A step-by-step outline of the approach for obtaining the decay of the solutions of equation (1) will be given. The main result will be stated as a final summarizing theorem. Only real-valued solutions will be treated. The extension to the complex case can be easily made.

To begin, the notation is fixed and some preliminary results are discussed. Let A^2 denote the self-adjoint realization of $m^2 I - \Delta$ on (real) $L^2(E^n)$. The solution spaces $H(A, a)$, of equation (1) which are relevant in this work are, for each $a \in \mathbb{R}$, the completions of $D(A^a) \oplus D(A^{a-1})$ with respect to the inner product

$$\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)_{A,\,a} := (A^a u_1, A^a v_1) + (A^{a-1} u_2, A^{a-1} v_2),$$

where $(\cdot,\cdot)$ is the usual inner product in $L^2(E^n)$. The norm of $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H(A, a)$ will be denoted by $\left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|_{A,\,a}$.

It is standard to treat the existence theory of equation (1) in its vector-valued form

$$\frac{d}{dt}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} - \tilde{G}_t\begin{pmatrix} u \\ \dot{u} \end{pmatrix}, \tag{2}$$

where $\tilde{G}$ is the mapping: $\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} \to \begin{pmatrix} 0 \\ G(x, t, u_1(x), u_{1x}(x), u_2(x)) \end{pmatrix}$. Now $U_0(t)$: $H(A, a) \to H(A, a)$ defined by

$$U_0(t)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos tA, & A^{-1} \sin tA \\ -A \sin tA, & \cos tA \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{3}$$

can easily be shown to be a continuous one-parameter group of orthogonal transformations on $H(A,a)$ with skew-adjoint infinitesmal generator $\begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix}$. That is, $U_0(t)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a generalized solution $\left(\text{a strict solution if } \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ is in the domain of } \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix}\right)$ of the vector form of the Klein-Gordon equation

((2) with $\widetilde{G}_t \equiv 0$), or the free propagator. Thus the integrated form of equation (2) is

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = U_0(t) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \int_{t_0}^{t} U_0(t-s)\, \widetilde{G}_s \begin{pmatrix} u(s) \\ \dot{u}(s) \end{pmatrix} ds, \tag{4}$$

where $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H(A, a)$ is the Cauchy data at time t_0. Solutions of equation (4) are generalized solutions of the Cauchy problem for equation (2).

General results of Segal [3, Theorem 1, p. 343] provide conditions on $\widetilde{G}_t$ which guarantee the existence of unique local solutions of equation (4) in $H(A, a)$. More specifically if $\widetilde{G}_t : \mathbb{R} \times H(A, a) \to H(A, a)$ is continuous and semi-Lipschitz uniformly on each finite t-interval, there exists an interval $I(t_0)$ containing t_0 and a (necessarily unique) solution, $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$, of equation (4) over $I(t_0)$ such that $t \to \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$ is continuous from $I(t_0)$ into $H(A, a)$. The interval of existence is either all of $\mathbb{R}$ or extends to the $\bar{t}$ closest to t_0 for which $\left\| \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} \right\|_{A, a} \to \infty$ as $t \to \bar{t}$. In the present situation, a criterion for global existence can be obtained in terms of $\widetilde{G}_t$ [3, Corollary 1.3, p. 347] from

$$\left\| \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} \right\|_{A, a}^2 \le \left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|_{A, a}^2 + 2 \left| \int_{t_0}^{t} \left(\widetilde{G}_s \begin{pmatrix} u(s) \\ \dot{u}(s) \end{pmatrix}, \begin{pmatrix} u(s) \\ \dot{u}(s) \end{pmatrix} \right)_{A, a} ds \right|. \tag{5}$$

For the examples to be considered in section 2 there are no a priori reasons for the right hand term of (5) to be bounded throughout $\mathbb{R}$ (as, for example, in the treatment of $G(u) = u^3$ [3, Theorem 4, p. 359]). However (5), with other inequalities, will be used to establish global existence as well as decay for the solutions of these equations by means of the technique to be outlined in the remainder of this section.

In the following it is assumed that G is sufficiently regular so that the above discussion applies to establish the existence of a unique solution, $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$, of equation (4) in some open interval, $I(t_0)$, including t_0 ($|t_0| < \infty$). A more detailed discussion of these requirements on G will be reserved for the particular examples in section 2 in order to focus here on the properties of G which lead to the decay of the solutions. Since the results are perturbative it will be necessary to distinguish the coupling constant and constants which depend on the Cauchy data (lower case) from those which cannot be made small (upper case). In many instances the same upper case letter will be used for several successive inessentially different numbers. For notational convenience $G(\cdot, t, u(t), u_x(t), \dot{u}(t))$ will be replaced by $G(t, u(t))$, and $\left\| \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} \right\|_{A, a}$ by $|u(t)|_a$, while $\|u(t)\|_r = \|u(\cdot, t)\|_r$ and $\|\dot{u}(t)\|_r = \|\dot{u}(\cdot, t)\|_r$. It will also be convenient, with no loss of generality, to suppose that $t_0 < t \in I(t_0)$.

Following Segal [1, section 3] the first component of equation (4), using the relation (3), can be written as

$$u(t) = u_0(t) - \int_{t_0}^{t} A^{-1} \sin[(t-s)A]\, G(s, u(s))\, ds, \tag{6}$$

where $u_0(t)$ is the unique global solution of the Klein-Gordon equation with the same Cauchy data at t_0. If $G(s, u(s))$ is in the domain of A^b (see assumption (D_2) to follow), then

$$u(t) = u_0(t) - \int_{t_0}^{t} A^{-(b+1)} \sin[(t-s)A]\, A^b\, G(s, u(s))\, ds. \tag{7}$$

Taking the Fourier-Plancherel transform of (7) one obtains

$$\widehat{u}(\xi, t) = \widehat{u}_0(\xi, t) - \int_{t_0}^{t} \frac{\sin[(t-s)(\xi^2 + m^2)^{1/2}}{(m^2 + m^2)^{(b+1)/2}} \widehat{A^b\, G(s, u(s))}\, ds.$$

Let $E_{t,b}$ denote the inverse Fourier-Plancherel transform of $(\xi^2 + m^2)^{-(b+1)/2} \cdot \sin t(\xi^2 + m^2)^{1/2}$ (i.e. b is taken greater than 1/2 throughout this work). Now a well known variant of Parseval's Theorem shows that the inverse Fourier transform of the integrand is $E_{t,b} * A^b\, G(s, u(s))$. But the integrand is also in $L^2(E^3)$ because of the boundedness of $(\xi^2 + m^2)^{-(b+1)/2} \sin[(t-s)(\xi^2 + m^2)^{1/2}]$. Thus the inverse Fourier and inverse Fourier-Plancherel transforms agree on the integrand. Taking the inverse Fourier-Plancherel transform through the last equation results in

$$u(t) = u_0(t) - \int_{t_0}^{t} E_{t-s,b} * A^b\, G(s, u(s))\, ds \cdot \qquad \text{(a.e.x.)}. \tag{7a}$$

Since precise estimates on space-L^p norms of $E_{t,b}$ are available [1, section 4, p. 478 and 4, section 5] and a bound on the space- L^q norm of $A^b G(s, u(s))$ will be assumed (c.f. assumption (D_2)), an estimate on $\| u(t) \|_r$ can be obtained in the form

$$(8) \qquad \| u(t) \|_r \leq \| u_0(t) \|_r + \int_{t_0}^{t} \| E_{t-s,b} \|_p \| A^b G(s, u(s)) \|_q \, ds,$$

where $1 + r^{-1} = p^{-1} + q^{-1}$. All of the conditions for the validity of the above discussion are met by making the following assumptions:

(D_1) $\quad \| u(t) \|_r , \| \dot{u}(t) \|_r \leq \text{Const.} \, | u(t) |_a ,$

(D_2) $\quad \| A^b G(t, u(t)) \|_q \leq g \ \text{Const.} \ | u(t) |_a^{2\alpha} \| u(t) \|_r^{\beta} \| \dot{u}(t) \|_r^{\gamma}$

for $1 + r^{-1} = p^{-1} + q^{-1}$ as well as $q = 2$.

REMARK. $\| A^b G(t, u(t)) \|_q$, as the previous discussion suggests, will denote throughout this paper, the space-L^q norm of the image under A^b: $L^2(E^3) \rightarrow L^2(E^3)$ of the square integrable function $G(t, u(t))$. In the particular cases to be discussed in the next section bounds for $\| A^b G(t, u(t)) \|_q$ are obtained in terms of polynomials in $| u(t) |_a$, $\| u(t) \|_r$ and $\| \dot{u}(t) \|_r$, reducing to the single term in (D_2) only by using (D_1). Both inequalities will follow from the use of Sobolev inequelities. The « Const. » arises in this manner, while g is the coupling constant. Assumption (D_1) used only in a technical fashion here to simplify (D_2), is necessary in a much more essential manner in later developments.

With (D_2), inequality (8) can be replaced by

$$(9) \quad \| u(t) \|_r \leq \| u_0(t) \|_r + g \, C \int_{t_0}^{t} \| E_{t-s,b} \|_p \, | u(s) |_a^{2\alpha} \| u(s) \|_r^{\beta} \| \dot{u}(s) \|_r^{\gamma} \, ds.$$

Similarly,

$$(10) \qquad \dot{u}(t) = \dot{u}_0(t) - \int_{t_0}^{t} \cos[(t-s)A] \, G(s, u(s)) \, ds.$$

$$= \dot{u}_0(t) - \int_{t_0}^{t} A^{-b} \cos[(t-s)A] \, A^b \, G(s, u(s)) \, ds.$$

$$(11) \qquad = \dot{u}_0(t) - \int_{t_0}^{t} F_{t-s,\, b-1} * [A^b \, G(s, u(s))] \, ds,$$

where $F_{t,b}$ is the inverse Fourier-Plancherel transform of $(m^2 + y^2)^{-(b+1)/2}$ $\cos\,[t\,(m^2 + y^2)^{1/2}]$. Thus

$$(12)\qquad \|\dot{u}(t)\|_r \leq \|\dot{u}_0(t)\|_r + \int_{t_0}^{t} \|F_{t-s,\,b-1}\|_{p'}\,\|A^b\,G(s,u))\|_{q'}\,ds,$$

where $1 + r^{-1} = p'^{-1} + q'^{-1}$. Assuming

$$(D_2')\; \|A^b\,G(s,u(s))\|_{q'} \leq g\ \text{Const.}\,|u(t)|_a^{2\alpha}\,\|{}_r^{\beta'}\,\|\dot{u}(t)\|_r^{\gamma'},$$

for $1 + r^{-1} = p'^{-1} + q'^{-1}$ and $q' = 2$.
inequality (12) can be replaced by

$$(13)\qquad \|\dot{u}(t)\|_r \leq \|\dot{u}_0(t)\|_r + gC' \int_0^t \|F_{t-s,\,b-1}\|_{p'}\,|u(s)|_a^{2\alpha}\,\|u(s)\|_r^{\beta'}\,\|\dot{u}(s)\|_r^{\gamma'}.$$

At this stage, if an explicit bound for $|u(t)|_a$ in terms of $\|u(t)\|_r$ and $\|\dot{u}(t)\|_r$ were available [1, (3.13b) p 468], the coupled inequalities (9) and (13) could be used to obtained information concerning $\|u(t)\|_r$ and $\|\dot{u}(t)\|_r$. However, in the examples of interest here, relationships of this type cannot be obtained in general. Instead, one further inequality coupling the three norms $|u(t)|_a$, $\|u(t)\|_r$ and $\|\dot{u}(t)\|_r$ is sought. Specifically, from (5), the inequalily

$$(14)\qquad |u(t)|_a^2 \leq |u_0(t)|_a^2 + \int_{t_0}^{t} \|A^{a-1}\,G(s,u(s)\|_2\,\|A^{a-1}\dot{u}(s)\|_2\,ds$$

follows. Noticing that $\|A^{a-1}u(s)\|_2 \leq |u(t)|_a$ and making the further restriction on G,

$$(D_3)\qquad \|A^{a-1}\,G(s,u(s))\|_2 \leq \text{Const.}\,|u(t)|_a^{2(\alpha''-1/2)}\,\|u(t)\|_r^{\beta''}\,\|u(t)\|_r^{\gamma''}.$$

inequality (14) reduces to

$$(15)\qquad |u(t)|_a^2 \leq |u_0(t)|_a^2 + gD\int_{t_0}^{t} |u(s)|_a^{2\alpha''}\,\|u(s)\|_r^{\beta''}\,\|\dot{u}(s)\|_r^{\gamma''}\,ds.$$

Suppose $t^{-\varepsilon}$ and $t^{-\delta}$, $\delta, \varepsilon \geq 0$, are the anticipated decays for $\|\dot{u}(t)\|_r$ and $\|u(t)\|_r$, respectively, as $|t| \to \infty$. Define, as usual, $x(t) = \sup\limits_{t_0 < s < t \in I(t_0)}$

$(1 + |s|)^{\varepsilon} \| u(s) \|_r$

$$\dot{x}(t) = \sup_{t_0 < s < t \in I(t_0)} (1 + |s|)^{\delta} \| \dot{u}(s) \|_r \text{ and } y(t) = \sup_{t_0 < s < t \in I(t_0)} |u(s)|_a^2 .$$

Because $U(t)$ is a orthogonal group on $H(A, a)$,

$\sup_{t_0 < s < t \in I(t_0)} |u_0(s)|_a^2 = |u_0(t_0)|_a^2 = y_0$, a constant which is independent of the interval $I(t_0)$ and dependent only on the Cauchy data $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Based on the extensive available knowledge of the decay of solutions of the Klein-Gordon equation [1, section 4], suppose further that ε, δ are chosen small enough and the Cauchy data $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ are chosen smooth enough so that

$$(D_4) \qquad \begin{cases} \sup_{s \in \mathbb{R}} (1 + |s|)^{\varepsilon} \| u_0(s) \|_r = x_0 , \text{ and} \\ \sup_{s \in \mathbb{R}} (1 + |s|)^{\delta} \| \dot{u}_0(s) \|_r = \dot{x}_0 . \end{cases}$$

REMARK. It can be shown that if $\varepsilon, \delta \leq 3/2$ and u_1 and u_2 are sufficiently smooth so that $A^p u_i \in L^1(R^n) \cap L^2(R^n)$ $i = 1,2$ for a suitably large p then $x_0 = x_0(t_0, u_1, u_2) \leq Q(t_0)(\| A^p u_2 \|_1)$, where $Q(t_0)$ is locally bounded. Similarly for $\dot{x}_0$. Thus $x_0, \dot{x}_0$ are independent of $I(t_0)$ and for each finite t_0 can be made arbitrarily small by choosing u_1, u_2 small in appropriate sense.

On multiplying (9) by $(1 + |t|)^{\varepsilon}$, replacing t by t' and taking supremums over $t_0 < t' < t \in I(t_0)$,

$$x(t) \leq x_0 + gC \sup_{t_0 < t' < t \in I(t_0)} (1 + |t'|)^{\varepsilon} \int_{t_0}^{t'} \| E_{t'-s, b} \|_p |u(s)|_a^{2\alpha} \| u(s) \|_r^{\beta} \| u(s) \|_r^{\gamma} ds.$$

Now

$$\sup_{t_0 < s < t' \in I(t_0)} (1 + |s|)^{\varepsilon} \| u(s) \|_r = x(t') \leq x(t).$$

Thus

$\| u(s) \|_r \leq (1 + |s|)^{-\varepsilon} x(t)$ for all $s \in (t_0, t')$. Similar bounds can be obtained $\| \dot{u}(s) \|_r$ and $|u(s)|_a$, resulting in

$$(16) \qquad x(t) \leq x_0 + gCy(t)^\alpha x(t)^\beta x(t)^\gamma \sup_{t_0 < t' < t \in I(t_0)} (1 + |t'|)^\varepsilon \int_{t_0}^{t'} \| E_{t'-s,\, b} \|_p (1 + |s|)^{-(\beta\varepsilon + \delta\gamma)}\, ds.$$

Using the known estimates [1, section 4 and 4, section 5]

$$(D_5) \qquad \begin{cases} \| E_{t,\, b} \|_p \leq \text{Const.}\ (1 + |t|)^{-\varrho}\ \forall t \in \mathbb{R},\ \varrho \geq 0 \ \text{ and} \\ \| F_{t,\, b-1} \|_{p'} \leq \text{Const.}\ (1 + |t|)^{-\sigma}\ \forall t \in \mathbb{R},\ \sigma \geq 0, \end{cases}$$

the last part of (16) can be bounded by

$$K \sup_{t_0 < t' < t \in I(t_0)} (1 + |t'|)^\varepsilon \int_{t_0}^{t'} (1 + |t' - s|)^{-\varrho} (1 + |s|)^{-(\beta\varepsilon + \delta\gamma)}\, ds$$

$$\leq K \sup_{t' < t \in I(t_0)} (1 + |t'|)^\varepsilon \int_{-\infty}^{t'} (1 + |t' - s|)^{-\varrho} (1 + |s|)^{-(\beta\varepsilon + \delta\gamma)}\, ds.$$

If $\varrho, \beta\varepsilon + \delta\gamma > 0$ and $\max(\varrho, \beta\varepsilon + \delta\gamma) > 1$, [1, Lemma 3.1, p. 467],

$$\int_{-\infty}^{t'} (1 + |t' - s|)^{-\varrho} (1 + |s|)^{-(\beta\varepsilon + \delta\gamma)}\, ds \leq \text{Const}\,(1 + |t'|)^{-\min(\varrho, \beta\varepsilon + \delta\gamma)}.$$

Thus, if $\min(\varrho, \beta\varepsilon + \gamma\delta) \geq \varepsilon$

$$(17\text{a}) \qquad x(t) \leq x_0 + gCy(t)^\alpha x(t)^\beta \dot{x}(t)^\gamma$$

where all of the new constants (which are independent of $I(t)$) are incorporated into C. In exactly the same fashion, the final set of coupled inequalities

$$(17\text{a}) \qquad x(t) \leq x_0 + gCy(t)^\alpha x(t)^\beta \dot{x}(t)^\gamma$$

$$(17\text{b}) \qquad \dot{x}(t) \leq \dot{x}_0 + gC'y(t)^{\alpha'} \dot{x}(t)^{\beta'} x(t)^{\gamma'}$$

$$(17\text{c}) \qquad y(t) \leq y_0 + gDy(t)^{\alpha''} x(t)^{\beta''} \dot{x}(t)^{\gamma''}$$

can be obtained provided

$$(D_6) \qquad \begin{cases} \max(\varrho, \beta\varepsilon + \gamma\delta) > 1, \ \min(\varrho, \beta\varepsilon + \gamma\delta) \geq \varepsilon, \\ \max(\sigma, \beta'\varepsilon + \gamma'\delta) > 1, \ \min(\sigma, \beta'\varepsilon + \gamma'\delta) \geq \delta, \\ \beta''\varepsilon + \gamma''\delta > 1. \end{cases}$$

The boundedness of $x(t)$, $\dot{x}(t)$, $y(t)$ and hence the decay of $\| u(t) \|_r$ and $| u(t) \|_r$ and boundedness of $| u(t) |_a$ will follow from the inequalities (17) using the next technical result.

LEMMA 1.1. Suppose $x_i(t), i = 1, \dots n$, are real valued functions defined on some interval I and satisfy

$$(18) \qquad 0 \leq x_i(t) \leq x_{i\delta} + h_i(x_1(t), \dots, x_n(t)) \quad \forall t \in I,$$

where x_{i0} are constants and h_i are non-negative polynomials of n variables (i. e. $h_i(x_1(t), \dots x_n(t))$

$$= \sum_j g_{ij}\, x_1(t)^{\alpha_{ij}^1} \dots x_n(t)^{\alpha_{ij}^n}, \ \tau_{ij} = \sum_{k=1}^{n} \alpha_{ij}^k \Big)$$

i) Suppose $\tau_{ij} < 1$ for all i, j, then $x_i(t)$, $i = 1 \dots n$, is bounded on I.

ii) Suppose $\tau_{ij} > 1$ for all i, j. If

a) $x_i(t)$, $i = 1, \dots, n$, is continuous on I,

b) for some $\bar{t} \in I$ $x_i(\bar{t}) \leq x_{i0}$, and

c) either x_{i0} for all $i = 1, \dots, n$ or g_{ij} for all i, j are sufficiently small, then $x_i(t)$, $i = 1, \dots, n$ is bounded on I.

iii) Suppose the τ_{ij}'s are arbitrary non-negative exponents. If

a) $x_i(t)$, $i = 1, \dots, n$ is continuous on I

b) for some $\bar{t} \in I$, $x_i(\bar{t}) \leq x_{i0}$, and

c) g_{ij} for those i, j associated with $\tau_{ij} = 1$ are sufficiently small and either

1. g_{ij} for all i, j such that $\tau_{ij} > 1$ or

2. x_{i0} for all $i = 1, \dots, n$ and g_{ij} for all i, j such that $\tau_{ij} < 1$, sufficiently small,

then $x_i(t)$, $i = 1, \dots, n$ is bounded on I.

PROOF. c. f. Arpendix I. The « sufficiently small » is made precise by the computations appearing there.

For example, in the particular case considered by Strauss and Segal [e. g. 2, Lemma 3.7, p. 437]

$$0 \leq x(t) \leq x_0 + g\, x(t)^{\gamma},\ \gamma > 1,$$

the specific restriction on the size of the constants is

$$x_0\, g^{(\gamma-1)^{-1}} < (1-\gamma^{-1})\, \gamma^{-(\gamma-1)^{-1}}$$

and this results in the bound $x(t) < x_0 (1-\gamma^{-1})^{-1}$.

THEOREM 1.2. Suppose equation (4) has a unique solution, $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$, in $H(A, a)$ for some a over some interval $I(t_0)$, such that $t \to \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$ is continuous from $I(t_0) \to H(A, a)$. If assumptions $(D_1) \dots (D_6)$ are satisfied and either

i) $\alpha + \beta + \gamma,\ \alpha' + \beta' + \gamma',\ \alpha'' + \beta'' + \gamma'' < 1$

ii) $\alpha + \beta + \gamma,\ \alpha' + \beta' + \gamma',\ \alpha'' + \beta'' + \gamma'' > 1$ and x^0, $\dot{x}_0$, y_0 small or g small,

iii) $\alpha, \beta, \dots, \beta'', \gamma''$ arbitrary with x_0, $\dot{x}_0$, y_0 and g small,

then the solution can be extended to all $t \in \mathbb{R}$ and $\| u(t) \|_r = 0(|t|^{-\varepsilon})$, $\| \dot{u}(t) \|_r = 0(|t|^{-\delta})$ and $|u(t)|_a = 0(1)$ as $|t| \to \infty$.

PROOF. Assumptions $(D_1) \dots (D_6)$ imply inequalities (17) for $t \in I(t_0)$. Using Lemma 1.1 with $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, $g_{ij} = g$, $\tau_{11} = \alpha + \beta + \gamma$, $\tau_{21} = \alpha' + \beta' + \gamma'$ $\tau_{31} = \alpha'' + \beta''\, \gamma''$, it follows that $\| u(t) \|_r \leq K_1 (1 + |t|)^{-\varepsilon}$, $\| \dot{u}(t) \|)_r \leq K_2 (1 + |t|)^{-\delta}$ and $|u(t)|_a \leq K_3$ for all $t \in I(t_0)$ where K_i, $i = 1, 2, 3$, are locally bounded functions of t_0 and independent of $I(t_0)$. Thus the solution $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$ exists globally and the same estimates obtain for all $t \in \mathbb{R}$.

REMARK. The above discussion is intended as the summary of a a method which will apply to several explicit examples rather than as an abstract result There are many minor variations of the scheme which may be more effective in certain instances but it seems inappropriate to discuss them in this general setting. They will, however, be pointed out in the next section with reference to more specific interactions. In fact the following results will indicate tnat simple generalizations of the above are possible but have been purposely omitted in favor of simplicity in the presentation of this abstract summary.

2. **Decay for specific examples.** In order to reduce the number of parameters, throughout this section the number of space dimensions (n) will be three, and only uniform decay ($r = \infty$) will be studied. In order that assumption (D_1) be satisfied a is taken ≥ 3 (or ≤ 2 if $\dot{u}$ does not enter the discussion) because for any $f : \mathbb{R} \to D(A^a)$, $a \geq 2$, $\| f(t) \|_\infty \leq$ Const. $\| A^2 f(t) \|_2 \leq$ Const. $m^{-(a-2)} \| A^a u(t) \|_2$. In addition for $\lambda \geq 0$, the completion of $D(A^\lambda)$ with respect to the inner product $(A^\lambda \cdot, A^\lambda \cdot)$ is precisely $D(A^\lambda)$ (i. e. no completion is reqnired). Thus $H(A, \lambda) = D(A^\lambda) \oplus \oplus D(A^{\lambda-1}) \subset L^2(E^3) \oplus L^2(E^3)$ for $\lambda \geq 1$ and $\| u(t) \|_2 \leq m^{-\lambda} \| A^\lambda u(t) \|_2 \leq \leq m^{-\lambda} | u(t) |_\lambda$ and $\| \dot{u}(t) \|_2 \leq m^{-(\lambda-1)} \| A^{\lambda-1} \dot{u}(t) \|_2 \leq m^{-(\lambda-1)} | u(t) |_\lambda$. In the case of interest in the following, $a \geq 2$, the boundedness of $| u(t) |_a$ thus guarantees the boundedness of $\| u(t) \|_2$ and $\| \dot{u}(t) \|_2$ hence the decay of $\| u(t) \|_r$ and $\| \dot{u}(t) \|_r$, $2 < r < \infty$, can be obtained by interpolation by means of

$$\| f(t) \|_r \leq \| f(t) \|_\infty^{1-2/r} \, \| f(t) \|_2^{2/r}. \tag{19}$$

In many case, (19) provides the same results as the direct approach.

(*a*) $G(x, t, u, u_x, u_t) = G(u)$. This example is considered not only because it is technically the most simple but also because the present method allows the decay estimates of Segal [1, section 4, p. 478] to be generalized to the solution where a priori boundedness of the energy is not required. First the question of local existence is treated.

THEOREM 2.1. Suppose $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H(A, 2)$ and $G(x, t, u, u_x, u_t) = G(u)$ where $G \in C^2$, is real-valued and $\left| \dfrac{d^j G}{d\lambda^j}(\lambda) \right| \leq g \, |\lambda|^{\alpha_j}$ for all $\lambda \in \mathbb{R}$ $j = 0, 1, 2$ with $0 \leq \alpha_j < \infty$. Then there is an interval $I(t_0)$ containing t_0 such that the (integrated form of the) equation

$$\Box\, u = m^2 u + G(u), \tag{20}$$

with Cauchy data $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ at t_0 has a unique solution $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$, in $H(A, 2)$ over the interval $I(t_0)$. Moreover the map $t \to \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$ is continuous from $I(t_0)$ into $H(A, 2)$.

PROOF. Appealing to the previously mentioned result of Segal [3, Theorem 1, p 343] all that requires checking is the semi-Lipschitz character of $\widetilde{G}_t$. The continuity of $\widetilde{G}_t$ follows trivially from this semi-Lipschitz property and the fact that G is not explicitly time dependent. Let $\begin{pmatrix} v \\ - \\ v \end{pmatrix}, \begin{pmatrix} w \\ - \\ w \end{pmatrix} \in$ $\in H(A, 2)$. Then $\left\| \widetilde{G}_t \begin{pmatrix} v \\ - \\ v \end{pmatrix} - \widetilde{G}_t \begin{pmatrix} w \\ - \\ w \end{pmatrix} \right\|_{A,2} = \| A \ (G\ (v) - G\ (w) \|_2 \leq$ Const. $\| G(v) - G(w) \|_{1,2}$ where $\| \varphi \|_{k,p} = \{ \sum_{0 \leq |\alpha| \leq k} \| D^\alpha \varphi \|_p^p \}^{1/p}$ is the usual norm in the Sobolev space $W^{k\ p}(E^n)$. The last inequality follows from a result of Calderòn [5, Theorem 7, p. 36] which in this context gives $C^{-1} \| \varphi \|_{k,p} \leq \| A^k \varphi \|_p \leq C \| \varphi \|_{k,p}$ for k a positive integer and $1 < p < \infty$, where $C = C_{p,k}$.

By the mean-value theorem $G(v) - G(w) = G'(av + bw)[v - w]$ where $a, b \geq 0$, $a + b = 1$. Thus $\| G(v) - G(w) \|_2 \leq \| G'(av + bw) \|_\infty \| v - w \|_2 \leq \leq \| av + bw \|_\infty^{\alpha_1} \| v - w \|_2$. But as previously mentioned for $f \in D(A^2)$, $\| f \|_\infty \leq$ Const. $\| A^2 f \|_2$ and $\| f \|_2 \leq m^{-2} \| A^2 f \|_2$ by Sobolev inequalities and the spectral theorem respectively. Therefore $\| G(v) - G(w) \|_2 \leq \leq C(\| A^2 v \|_2 + \| A^2 w \|_2)^{\alpha_1} \| A^2 (v - w) \|_2$.

To obtain a similar estimate for the remaining term (denoting the partial derivative with respect to x_i by ∂_i), write [6, Lemma 1, p. 87] $\partial_i (G(v) - G(w)) = G'(v) \partial_i v - G'(w) \partial_i w = G'(v)(\partial_i v - \partial_i w) + [G'(v) - G'(w)] \partial_i w = = G'(v) \partial_i (v - w) + G''(av + bw)(v - w) \partial_i w$. Then $\| \partial_i (G(v) - G(w)) \|_2 \leq \leq \| G'(v) \|_\infty \| \partial_i (v - w) \|_2 + \| G''(av + bw) \|_\infty \| v - w \|_\infty \| \partial_i w \|_2 \leq \{ C_1 \| A^2 v \|_2^{\alpha_1} + + C_2 (\| A^2 v \|_2 + \| A^2 w \|_2)^{\alpha_2} \| A^2 w \|_2 \} \| A^2 (v - w) \|_2$ (where in addition to the Sobolev inequality and the Spectral Theorem, the result of Calderón was used in the form $\| \partial_i \varphi \|_2 \leq C \| A \varphi \|_2 \leq C m^{-1} \| A^2 \varphi \|_2$). Since $\| A^2 \varphi \|_2 \leq \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|_{A,2}$, the above estimates can be combined to give

$$\left\| \widetilde{G}_t \begin{pmatrix} v \\ - \\ v \end{pmatrix} - \widetilde{G}_t \begin{pmatrix} w \\ - \\ w \end{pmatrix} \right\|_{A,2} \leq C \left(\left\| \begin{pmatrix} v \\ - \\ v \end{pmatrix} \right\|_{A,2}, \left\| \begin{pmatrix} w \\ - \\ w \end{pmatrix} \right\|_{A,2} \right) \left\| \begin{pmatrix} v \\ - \\ v \end{pmatrix} - \begin{pmatrix} w \\ - \\ w \end{pmatrix} \right\|_{A,2}, \tag{21}$$

where $C(\cdot,\cdot)$ is bounded on bounded sets. This is precisely the statement that $\widetilde{G}_t : H(A, 2) \to H(A, 2)$ is semi Lipschitz as required to complete the proof.

REMARK. The result could be proved just as easily with the less restrictive condition $\left| \dfrac{d^j G(\lambda)}{d\lambda^j} \right| = 0(|\lambda|^\alpha j)$ as $|\lambda| \to \infty$. However this condition is not sufficient for global existence and decay and so this more general framework is omitted.

A decay result for the particular case of $G(u)$ will now be obtained by showing through detailed calculation that the abstract result of the previous section is applicable.

THEOREM 2.2. Suppose $G \in C^2(\mathbb{R})$, is real-valued and $\left| \frac{d^j G}{d\lambda^j}(\lambda) \right| \leq g \mid \lambda \mid^{\beta - j}$ for all λ and $j = 0, \ldots, 2$, with $\beta \geq 3$. If the Cauchy data $\binom{u_1}{u_2} \in H(A, 2)$ are sufficienty smooth so that condition (D_4) is satisfied with $\varepsilon = 3/2$ then equation (20) has a unique global solution with $\| u(t) \|_\infty = 0(\mid t \mid^{-3/2})$ and $\mid u(t) \mid_2 = 0(1)$ as $\mid t \mid \to \infty$, provided that either the coupling constant g or the Cauchy data, $x_0 + y_0$, is sufficiently small.

PROOF. Inequality (8), in the present situation, is

$$\| u(t) \|_\infty \leq \| u_0(t) \|_\infty + \int_{t_0}^{t} \| E_{t-s, 2} \|_p \| A^2 G(u(s)) \|_q \, ds, \tag{22}$$

where $p = q(q-1)^{-1}$. First observe, relative to assumption (D_5) that if $1 \leq q \leq 2$ $\|E_{t,2}\|_p \leq C(1 + \mid t \mid)^{-(3/2 - 3/p)}$ by means of uniform boundedness of $\| E_{t,2} \|_\infty$ (private communication from W. Strauss) and Nelson's [4, Section 5] decay estimate $\| E_{t,2} \|_\infty = 0(\mid t \mid^{-3/2})$ as $\mid t \mid \to \infty$. The interpolating trick at the beginning of this section gives the global estimate $\| E_{t,2} \|_{q(q-1)^{-1}} \leq C(1 + \mid t \mid)^{-3/q + 3/2}$ for all t and $1 \leq q \leq 2$. The estimate corresponding to (D_2) for similar values of q is summarized in the following

LEMMA 2.3. With the hypotheses of Theorem 2.2 and $1 \leq q \leq 2$

$$\| A^2 G(u(s)) \|_q \leq g C \| u(s) \|_\infty^{\beta - 2/q} \mid u(s) \mid_2^{2/q}. \tag{23}$$

PROOF. In view of the equivalence of weak and strong derivatives [eg. 7, Theorem 1, p. 1031], $\| A^2 G(u) \|_q = \| (m^2 - \Delta) G(u) \|_q \leq C \| G(u) \|_{2,q}$ if the last term of the inequality is finite. In view of the growth condition on G, $\| G(u) \|_q \leq g \| \mid u \mid^\beta \|_q = g \| \mid u \mid^{\beta q - 2} \mid u \mid^2 \|_1^{1/q} \leq g \| u \|_\infty^{\beta - 2/q} \| u \|_2^{2/q} \leq$ $\leq g C \| u \|_\infty^{\beta - 2/q} \| A^2 u \|_2^{2/q}$ provided that $\beta q \geq 2$, which is guaranteed by the choice of β and q. Next (c. f. Remark 1 to follow for first equality) $\| \partial_i (G(u)) \|_q = \| G'(u) \partial_i u \|_q \leq \| G'(u) \|_{2q/2-q} \| \partial_i u \|_2 \leq g C \| \mid u \mid^{\beta - 1} \|_{2q/2-q} \| A^2 u \|_2 \leq$ $\leq g C \| \mid u \mid^{2q(\beta-1)(2-q)^{-1} - 2} \mid u \mid^2 \|_1^{(2-q)/2q} \| A^2 u \|_2 \leq g C \| u \|_\infty^{\beta - 2/q} \| A^2 u \|_2^{2/q}$ provided that $2q(\beta - 1)(2 - q)^{-1} \geq 2$, which again is guaranteed by the given conditions on β and the choice of q. The last term (c. f. Remark 1 to follow for first inequality) $\| \partial_{ik}^2 G(u) \|_q \leq \| G'(u) \partial_{ik}^2 u \|_q + \|G''(u) \partial_i u \, \partial_k u \|_q$. The first term can be handled in precisely the same manner as $\| \partial_i (G(u)) \|_q$ was treated by replacing $\| \partial_i u \|_2 \leq C \| A^2 u \|_2$ with $\| \partial_{ik}^2 u \|_2 \leq C \| A^2 u \|_2$. For the last term

$$\| G''(u)\, \partial_i u\, \partial_k u \|_q \leq \| G''(u) \|_{2q/2-q} \| \partial_i u \|_4 \| \partial_k u \|_4 \leq g C \| u \|_\infty^{\beta-1-2/q} \| A^4 u \|_2^{2/q} \| u \|_\infty ,$$

the last inequality following from the Sobolev inequality [8, p. 125 and Remark 2 to follow] $\| \partial_i u \|_4 \leq C \max_{|\alpha|=2} \| D^\alpha u \|_2^{1/2} \| u \|_\infty^{1/2}$ and the above trick applied to $\| G''(u) \|_{2q/2-q}$, requiring $2q(\beta - 2)(2 - q)^{-1} \geq 2$, which is again satisfied.

REMARK 1. In treating terms like $\| \partial_i G(u) \|_q$ it convenient to view $\partial_i G(u)$ in the strong sense in $L^2(E^3)$ (the usual formulation of ∂_i as a closed operator in $L^2(E^3)$). The technical result of Segal [6, Lemma 1, p. 87] can be applied (since $\| u \|_2$, $\| u \|_\infty \leq \| A^2 u \|_2$) to give the desired result $\partial_i G(u) = G'(u)\, \partial_i u$ where the derivatives on either side can be interpreted as strong $L^2(E^3)$ or equivalently weak derivatives. For higher order derivatives, $\partial^2_{ik} G(u)$, the desired results follow by a combination of the above reasoning along with the Leibniz formula for weak derivatives written as follows in a form which is most appropriate for this discussion: Suppose f_1 and $f_2 \in L^2(E^n)$ have weak derivatives $D^\alpha f_1$ and $D^\alpha f_2 \in L^2(E^n)$ for all $|\alpha| \leq j$ then the product $f_1 f_2$ has weak derivatives up to order j and they are given by the Leibniz formula in terms of the appropriate weak derivatives.

REMARK 2. The very general version of the Sobolev inequalities given by Nirenberg [8, p. 125] will be the basic tool in obtaining inequalities of the form (D_2) and (D_3) for the examples to be considered. By recalling that $D(A^a) = W^{a,2}$ and passing to the limit one obtains the specific form in which this result will be used; namely,

PROPOSITION 2.4. Let $D^\alpha f$ denote the $\alpha\, th$ weak derivative of f and and define $\| D^i f \|_p = \max_{|\alpha| = i} \| D^\alpha f \|_p$. Suppose $f \in D(A^a)$ with $a \geq 2$. Then if $2 \leq p < \infty$, $2 \leq k \leq a$, $0 \leq j < k$,

$$\| D^j f \|_p \leq \text{Const.} \| D^k f \|_2^\gamma \| f \|_\infty^{1-\gamma} ,$$

where $p^{-1} = j/3 + \gamma(1/2 - k/3)$ for all γ in the interval $j/k \leq \gamma \leq 1$.

REMARK 3. The method used in handling the term $\| G(u) \|_q$ appears to be the only one available. From a technical point of view then, this term typifies the best possible result of the form (23) that can be obtained. The remainder of the problem in obtaining estimates of the form (23) (or more generally (D_2) and (D_3)) consists of showing, by means of Sobolev inequalities, that the same product also dominates the derivative terms.

Returning to the proof of Theorem 2.2, inequalities (22), (23) and the estimate on $\| E_{t,2} \|_\infty$ result in

$$(24) \qquad \| u(t) \|_\infty \leq \| u_0(t) \|_\infty + gC \int_{t_0}^{t} (1 + | t - s |)^{-3/q+3/2} \| u(s) \|_\infty^{\beta-2/q} | u(s) |_2^{2/q} \, ds.$$

The inequality for the escalated energy norm can be obtained similarly. With $a = 2$, inequality (14) is

$$(25) \qquad | u(t) |_2^2 \leq | u_0(t) |_2^2 + 2 \int_{t_0}^{t} \| A G(u(s)) \|_2 \, | u(s) |_2 \, ds.$$

Using Lemma 2.3, $\| A G(u(s)) \|_2 \leq D \| G(u(s)) \|_{1,2} \leq gD \| u(s) \|_\infty^{\beta-1} | u(s) |_2$. Thus

$$(26) \qquad | u(t) |_2^2 \leq | u_0(t) |_2^2 + gD \int_{t_0}^{t} \| u(s) \|_\infty^{\beta-1} | u(s) |_2^2 \, ds.$$

Using the definitions and techniques of the previous section, the analogue of inequality (16) can be obtained in the form

$$(27a) \qquad x(t) \leq gC\, x(t)^{\beta-2/q} y(t)^{1/q} \sup_{t_0 < t' < t \in I(t_0)} (1 + | t' |)^{\varepsilon} \cdot \int_{t_0}^{t} (1 + | t' - s |)^{-3/q+3/2} (1 + | s |)^{-(\beta-2/q)\varepsilon} \, ds$$

$$(27b) \qquad y(t) \leq y_0 + gD\, x(t)^{\beta-1} y(t) \sup_{t_0 < t' < t \in I(t_0)} \int_{t_0}^{t'} (1 + | s |)^{-(\beta-1)\varepsilon} \, ds.$$

In order to reduce the above to abstract inequalities analogous to (17) so that Lemma 1.1 can be applied, the explicitly time-dependent factors must be removed from inequalities (27). This will be done by direct checking that condition (D_6) is satisfied. Technically the arguments is to show that it is possible to choose a q, $1 \leq q \leq 2$, so that the equalities are satisfied with $\varepsilon = 3/2$. It is quite simple in this case. If $q=1$, $\max(3/q - 3/2, (\beta - 2/q)\varepsilon) = \max(3/2, (\beta - 2)\, 3/2) \geq 3/2$ and $\min(3/2, (\beta - 2)\, 3/2) = 3/2 = \varepsilon$ while $(\beta - 1)\, 3/2 \geq 3$ because $\beta \geq 3$.

Thus choosing $q = 1$, inequalities (27) can be reduced to

(28a) $$x(t) \leq x_0 + gC\,x(t)^{\beta-2}\,y(t),$$

(28b) $$y(t) \leq y_0 + gD\,x(t)^{\beta-1}\,y(t).$$

The degree of the non linear terms are $\beta - 2 + 1 = \beta - 1 \geq 2$ and $\beta - 1 + 1 = \beta \geq 3$. Thus part ii) of Lemma 1.1 gives the desired result over $I(t_0)$ provided that either the coupling constant, g, or the Cauchy data $x_0 + y_0$ is sufficiently small. The global existence of the solutions $\begin{pmatrix} u(t) \\ \cdot \\ u(t) \end{pmatrix}$ in $H(A, 2)$, and the fact that the same estimates for $\| u(t) \|_\infty$ and $| u(t) |_2$ obtain throughout all of $\mathbb{R}$ follows trivially as in the abstract Theorem 1.2, thus concluding the proof of Theorem 2.2.

It should be mentioned that in this simple case where the perturbation G does not depend on $\dot{u}$ explicitly, the inequality associated with $\| \dot{u}(t) \|_\infty$ can be avoided if one is only interested in the decay of $\| u(t) \|_\infty$. On the other hand the above results, along with the analogue of (12),

(29) $$\| \dot{u}(t) \|_\infty \leq \| \dot{u}_0(t) \|_\infty + \int_{t_0}^{t} \| F_{t-s,1} \|_{q'(q'-1)^{-1}} \| A^2 G(u(s)) \|_{q'}\, ds,$$

can be used to obtain a decay result for $\| \dot{u}(t) \|_\infty$ as follows. For $1 < q' \leq 2$, it can be show that $\| F_{t,1} \|_{q'(q'-1)^{-1}} = 0(| t |^{(2q'-3)/q'})$ as $| t | \to 0$ (unpublished joint work of W. Strauss and the author) and $0(| t |^{-1/q'})$ as $| t | \to \infty$ for $1 < q' < 4/3$ [4, section 5]. But, for $q' > 1$, $-(2q' - 3)\,q'^{-1} < 1/q'$. Thus $\| F_{t,q} \|_{q'(q'-1)^{-1}} \leq C | t |^{-1/q'}$ for all t. The analogue of (27a) then is

(30) $$\| \dot{u}(t) \|_\infty \leq \dot{x}_0 (1 + | t |)^{-3/2} + g\dot{C} \int_{t_0}^{t} | t - s |^{-1/q'} (1 + | s |)^{-(\beta - 2/q')3/2}\, ds.$$

By choosing q' close to 1, $(\beta - 2/q')\,3/2 > 1$. Thus by a technical result of Shenk and Thoe [9, Lemma 3.1] the integral is bounded by $(1 + | t |)^{-1/q'}$.

COROLLARY 2.5. With the hypothesis of Theorem 2.2, the global solution of (20) has $\| \dot{u}(t) \|_\infty = 0(| t |^{-\delta})$ as $| t | \to \infty$ with δ arbitrary but < 1, provided that either the coupling constant g or the Cauchy data $x_0 + y_0$ is sufficiently small.

(b) $G(x, t, u, u_t, u_x) = G(u_t)$. Decay and asymptotics for perturbations of this type have not as yet been considered in the literature. However, in the context of the abstract discussion of section 1, it is only a technical variant of the preceding example. In fact, most of the basic estimates which were proved there can be applied directly in checking the details here.

THEOREM 2.6. Suppose $G \in C^3(\mathbb{R})$, is real-valued and $\left|\frac{d^j G}{d\lambda^j}(\lambda)\right| \leq g\,|\lambda|^{\beta-j}$ for all λ and $j = 0, \dots, 3$, $\beta \geq 3$. If the Cauchy data $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H(A, 3)$ are sufficiently smooth so that condition (D_4) is satisfied with $|t|^{-3/2}$ decay then (the integrated from of)

$$\square\, u = m^2 u + G(u_t) \tag{31}$$

has a unique global solution $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} \in H(A, 3)$ with $\| u(t) \|_\infty = 0(|t|^{-\varepsilon})$, where $\varepsilon = \varepsilon(\beta) = 3/2$ for $\beta > 7/2$ and $\varepsilon(\beta) < 3(\beta - 1)/5$ for $3 \leq \beta \leq 7/2$, $\| \dot{u}(t) \|_\infty = 0(|t|^{-\delta})$ where δ is arbitrary but < 1 and $|u(t)|_3 = 0(1)$ as $|t| \to \infty$, provided that either the coupling constant, g, or the Cauchy data $x_0 + y_0$ is sufficiently small.

PROOF. Again the existence of local solutions follows if $\widetilde{G}_t$ is semi-Lipschitz. To this end let $\begin{pmatrix} \bar{v} \\ v \end{pmatrix}, \begin{pmatrix} \bar{w} \\ w \end{pmatrix} \in H(A, 3)$. Then

$$\left| \widetilde{G}_t \begin{pmatrix} \bar{v} \\ v \end{pmatrix} - \widetilde{G}_t \begin{pmatrix} \bar{w} \\ w \end{pmatrix} \right|_3 = \| A^2 [G(v) - G(w)] \|_2 \tag{32}$$

$$\leq m^2 \| G(v) - G(w) \|_2 + \| \Delta [G(v) - G(w)] \|_2 .$$

As in the proof of Theorem 2.1, $\| G(v) - G(w) \|_2 \leq C(\| A^2 v \|_2 + \| A^2 w \|_2)^{\beta-1} \| A^2(v - w) \|_2$. Using the procedure outlined in Remark 1 following Lemma 2.3, $\Delta[G(v) - G(w)] = G'(v)\,\Delta v - G'(w)\,\Delta w + G''(v)(\Delta v)^2 - G''(w)(\Delta w)^2 = G'(v)(\Delta v - \Delta w) + (G'(v) - G'(w))\,\Delta w + G''(v)((\Delta v)^2 - (\Delta w)^2) + (G''(v) - G''(w))(\Delta w)^2$. A suitable estimate for the $L^2(E^3)$-norm of the first two terms can be obtained as in the last paragraph of the proof of Theorem 2.1. For the third term

$$\| G''(v)[(\Delta v)^2 - (\Delta w)^2] \|_2 \leq g \| v \|_\infty^{\beta-2} \left\| \sum_{i=1}^{3} (\partial_i v - \partial_i w)(\partial_i v + \partial_i w) \right\|_2 \leq$$

$$\leq gC \| A^2 v \|_2^{\beta-2} \| D(v + w) \|_4 \| D(v - w) \|_4$$

using the notation of Proposition 2.4. The same result gives $\| Df \|_4 \leq$ $\leq$ Const. $\| D^2 f \|_2^{1/2} \| f \|_\infty^{1/2}$ for $f \in D(A^2)$ which, along with $\|f\|_\infty \leq$ Const $\| A^2 f \|_2$ provides a suitable bound for the third term. Finally $\| G''(v) - G''(w)(\Delta w)^2 \|_2 \leq$ $\leq \| G'''(av + bw) \|_\infty \| v - w \|_\infty \| (\Delta w)^2 \|_2 \leq g\, C\, (\| A^2 v \|_2 + \| A^2 w \|_2)^{\beta - 3} \cdot$ $\cdot \| A^2 (v - w) \|_2 \| A^2 w \|_2^2$. As in Theorem 2.1, the above estimates can be assembled to show that

$$\left| \tilde{G}_t \begin{pmatrix} \bar{v} \\ v \end{pmatrix} - \tilde{G}_t \begin{pmatrix} \bar{w} \\ w \end{pmatrix} \right|_3 \leq C \left(\left| \begin{pmatrix} \bar{v} \\ v \end{pmatrix} \right|_3, \left| \begin{pmatrix} \bar{w} \\ w \end{pmatrix} \right|_3 \right) \left| \begin{pmatrix} \bar{v} \\ v \end{pmatrix} - \begin{pmatrix} \bar{w} \\ w \end{pmatrix} \right|_3 .$$

As for the decay, the analogue of inequality (24) using Lemma 1.3, is

$$\text{(33a)} \quad \| u(t) \|_\infty \leq \| u_0(t) \|_\infty + gC \int_{t_0}^{t} (1 + | t - s |)^{-3/q + 3/2} \| \dot{u}(s) \|_\infty^{\beta - 2/q} | u(s) |_3^{2/q} \, ds.$$

The inequality for the derivative, analogous to inequality (30) is

$$\text{(33b)} \quad \| \dot{u}(t) \|_\infty \leq \| \dot{u}_0(t) \|_\infty + gC' \int_{t_0}^{t} \| F_{t-s,\,1} \|_{q'(q'-1)^{-1}} \| \dot{u}(s) \|^{\beta - 2/q'} | u(s) |_3^{2/q'} \, ds.$$

The counterpart for inequality (26) involving the escalated energy norm is

$$\text{(33c)} \quad | u(t) |_3^2 \leq | u_0(t) |_3^2 + gD \int_{t_0}^{t} \| \dot{u}(s) \|_\infty^{\beta - 1} | u(s) |_3^2 \, ds.$$

As in Theorem 2.2 only the last two inequalities are needed to obtain global existence (i. e. boundedness of $| u(t) |_3$) and a decay result for $\| \dot{u}(t) \|_\infty$. Using the definitions and techniques of the previous section

$$\text{(34b)} \quad \dot{x}(t) \leq \dot{x}_0 + gC\, \dot{x}(t)^{\beta - 2/q'}\, y(t)^{1/q'} \sup_{t_0 < t' < t \in I(t_0)} (1 + | t' |)^\delta \int_{t_0}^{t'} \| F_{t'-s,\,1} \|_{q'(q'-1)^{-1}} (1 + | s |)^{-(\beta - 2/q')\delta} \, ds,$$

$$\text{(34c)} \quad y(t) \leq y_0 + gD\, \dot{x}(t)^{\beta - 1}\, y(t) \sup_{t_0 < t' < t \in I(t_0)} \int_{t_0}^{t'} (1 + | s |)^{-(\beta - 1)\delta} \, ds.$$

In order to apply Lemma 1.1, it must be shown that q' can be chosen so that for arbitrary $\delta < 1$, the explicitly time dependent terms are bounded or equivalently checking that condition (D_6) is satisfied.

Since $\beta \geq 3$ and δ is (taken for the best result) arbitrarily close to $1, (\beta - 1)\delta > 1$ so that the integral in (34c) is bounded by

$$\int_0^\infty (1 + |s|)^{-(\beta-1)\delta} ds < \infty.$$

If q' is chosen such that $1 < q' < 4/3$, then the estimate in Corollary 2.5 can be used for $\| F_{t-s,1} \|_{q'(q'-1)^{-1}}$. If it is also possible to find a q' such that $(\beta - 2/q')\delta > 1$ then the computation in Corollary 2.5 shows that the integral in (34b) is bounded by $C(1 + |t'|)^{-1/q'}$, and thus the whole term is bounded by a constant if in addition $1/q' \geq \delta$. Thus condition (D_6) is guaranteed if q' can be chosen to simultaneausly satisfy $3/4 < 1/q'$ and $\delta < 1/q' < \frac{1}{2}(\beta - 1/\delta)$. Now $\delta < \frac{1}{2}(\beta - 1/\delta)$ if $\frac{1}{4}(\beta - \sqrt{\beta^2 - 8}) < \delta < \frac{1}{4}\beta + \sqrt{\beta^2 - 8})$. With $\beta \geq 3$, it is clearly possible them to find q' satisfying the second pair of inequalities for arbitrary $1/2 < \delta < 1$. Since large δ's are of interest, there is no problem in taking $\delta > 3/4$, in which case $1/q' > 3/4$. If $\beta < 2 + \delta^{-1}$ then $1/q' < \frac{1}{2}(\beta - 1/\delta) < 1$ automatically; otherwise this is an added restriction on the choice of q' which can clearly be realized since $\delta < 1$.

Having chosen a q' from the non-empty set satisfying $\delta \leq 1/q' < \min(1, \frac{1}{2}(\beta - 1/\delta))$, (34b) and (34c) reduce to (after incorporating the additional constants)

$$\text{(35b)} \qquad \dot{x}(t) \leq \dot{x}_0 + g\, C'\, \dot{x}(t)^{\beta - 2/q'}\, y(t)^{1/q'},$$

$$\text{(35c)} \qquad y(t) \leq y_0 + g\, D\, \dot{x}(t)^{\beta-1}\, y(t).$$

But the degree of the non-linear terms are $\beta - 1 + 1 = \beta \geq 3$ and $\beta - 2/q' + 1/q' > \delta^{-1} + \delta > 1$. Thus part ii) of Lemma 1.1 gives the boundedness of $x(t)$ and $y(t)$ over $I(t_0)$ provided either $x_0 + y_0$ or g is sufficiently small.

In order to obtain the indicated decay of $\| u(t) \|_\infty$, first notice that $3/q - 3/2 > 1$ if $1 \leq q < 6/5$ and $(\beta - 2/q)\delta \geq 4\delta/3 > 1$ for $q \geq 6/5$ and

$\beta \geq 3$. As a result the integral in

$$(34a) \qquad x(t) \leq x_0 + g\, C\, \dot{x}(t)^{\beta - 2/q}\, y(t)^{1/q} \sup_{t_0 < t' < t \in I(t_0)} (1 + |t'|)^{\varepsilon} \int_{t_0}^{t'} (1 + |t' - s|)^{-3/q + 3/2} (1 + |s|)^{(\beta - 2/q)\delta}\, ds,$$

is bounded by Const. $(1 + |t|)^{-\theta}$ where $\theta = \min(3/q - 3/2,\ (\beta - 2/q)\,\delta)$. If q is taken to be 1, then $\theta = 3/2$ for $\beta > 7/2$ giving the boundedness of $x(t)$. For $3 \leq \beta \leq 7/2$, taking $q = 5(\beta + 3/2)^{-1}$ implies that the second factor is the minimum so that $\theta = 3(\beta - 1)\,\delta/5$ giving the boundedness of $x(t)$ for this range of β and hence the indicated decay of $\| u(t) \|_\infty$. The global existence of the solusions $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$, and the fact that the same estimates for $\| u(t) \|_\infty$, $\| \dot{u}(t) \|_\infty$ and $| u(t) |_3$ obtain throughout all of $\mathbb{R}$ follow trivially es in the abstract Theorem 1.2 thus concluding the proof of Theorem 2.5.

REMARK 1. *A* $|t|^{-3/2}$ decay for $\| u(t) \|_\infty$ (and for $\| \dot{u}(t) \|_\infty$) can be obtained as a straightforward exercise by using the escalated energy norm $\| \ \|_{A,4}$ ($\| \ \|_{A,5}$, respectively) requiring a higher degree of differentiability for G and leading to the technical problem of checking whether inequalities of the form (D_2) and (D_3) are still available for the perturbations of interest. This « best-possible » decay is not required for doing scattering theory for this equation and hence is not pursued further.

REMARK 2. None of the examples considered thus far have required the use of the full set of non-linear inequalities. An example weich does is the relativistically invariant perturbation $G(\nabla u \cdot \nabla u - u_t^2)$. This also typifies the most difficult kind of perturbation to handle by these methods because of the presence of both spatial and temporal derivatives. Rather than examine this technically complicated case, an example which possesses these undesirable features but for which the details are more simple will be treated in the next section. The explicit presence of x and especially t variables in the perturbation serve only to enhance the decay. This will be shown by means of a simple example at the end of this section.

(c) $G(x, t, u, u_x, u_t) = G^{\varkappa}(u)\, \partial_{\varkappa} u$. In the above expression the summation convention is used over $\varkappa = 0, 1, 2, 3$ whith $\partial_0 u = u_t$. Homogeneous non-linear terms of this sort, with further conditions on $G^{\varkappa}$ to guarantee Lorentz covariance and positively of the energy, arise from general derivative couplings. For this example the investigation will emphasize producing suitable decay for scattering with the weakest conditions on the perturbation, rather than the best decay for arbitrary powers as in the last example.

THEOREM 2.7. Suppose that for each $\varkappa = 0, 1, 2, 3$, $G^\varkappa \in G^3(\mathbb{R})$, is real-valued, $\left|\frac{d^j G^\varkappa}{d\lambda^j}(\lambda)\right| \leq g\,|\lambda|^{\alpha^\varkappa - j}$ for all λ with $\alpha^\varkappa \geq 2$ for all λ with $\alpha^\varkappa \geq 2$ for $j = 0, 1, 2$, and $\left|\frac{d^3 G^\varkappa}{d\lambda^3}(\lambda)\right| \leq g\,|\lambda|^{\alpha_3^\varkappa}$ for all λ with $0 \leq \alpha_3^\varkappa < \infty$. If at $t = t_0$ the Cauchy data $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H(A, 3)$ are sufficiently smooth so that condition (D_4) is satisfied, then the (integrated form of the) equation

$$\square n = m^2 u + G^\varkappa(u)\, \partial_\varkappa u \tag{3}$$

has a unique global solution, $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$, with $\| u(t) \|_\infty = 0(|t|^{-1})$, $\| \dot{u}(t) \|_\infty = 0(|t|^{-5/6})$ and $|u(t)|_3 = 0(1)$ as $|t| \to \infty$, provided that either the coupling constant, g, or the Cauchy data, $x_0 + x_0 + y_0$, is sufficiently small.

PROOF. As in Theorems. 2.1, and 2.6 the existence of local solutions is established by checking the semi-Lipschitz character of $\widetilde{G}_t$. It will be convenient in the argument to use the summation convention over 1, 2, 3. This will be indicated by the use of Roman letters; i. e. $G^k(u)\,\partial_k(u) + G^0(u)\,\partial_0 u$. Let $\begin{pmatrix} v \\ \bar{v} \end{pmatrix}$ and $\begin{pmatrix} w \\ \bar{w} \end{pmatrix} \in H(A, 3)$.

$$\left|\widetilde{G}_t\begin{pmatrix} v \\ \bar{v} \end{pmatrix} - \widetilde{G}_t\begin{pmatrix} w \\ \bar{w} \end{pmatrix}\right|_3 = \| A^2 (G^k(v)\,\partial_k(v) + G^0(v)\,\bar{v} - G^k(w)\,\partial_k w - G^0(w)\,\bar{w}) \|_2 \tag{37}$$

$$\leq m^2 \| G^k(v)\,\partial_k v + G^0(v)\,\bar{v} - G^k(w)\,\partial_k w - G^0(w)\,\bar{w} \|_2 +$$

$$+ \| \Delta [G^k(v)\,\partial_k v + G^0(v)\,\bar{v}\, G^k(\omega)\,\partial_k w - G^0(w)\,\bar{w}] \|_2 .$$

A suitable estimate for the first term can be obtained by observing that it is bounded by $m^2 \{ \| G^k(v)\,\partial_k(v - w) \|_2 + \| G^k(w)\,\partial_k w \|_2 + \| G^0(v)\,(\bar{v} - \bar{w}) \|_2 + \| (G^0(v) - G^0(w))\, w \|_2 \}$. Each of these terms can be estimated as in Theorems 2.1, and 2.6. A similar estimate can be obtained for the second term is inequality (37) along the lines of the proof of Theorem 2.6. Since the details are extremely tedious but entirely staaightforward they will be omitted.

Continuing, then, to the decay result, if, $1 \leq q \leq 2$, then

$$(38) \qquad \| u(t) \|_\infty \leq \| u_0(t) \|_\infty + \text{Const.} \int_{t_0}^{t} (1 + |t - s|)^{-3/q + 3/2} \| A^2 (G^\varkappa(u(s)) \, \partial_\varkappa u(s)) \|_q \, ds$$

An estimate like (D_2) for the term under the integral is summarized in

LEMMA 2.8. With $G^\varkappa$ and $\alpha^\varkappa$ as in Theorem 2.7,

$$\text{i)} \quad \| A^2 (G^\varkappa (u(s)) \|_q \leq g C \, | u(s) |_3^{2/q} \sum_{\varkappa=0}^{3} \| u(s) \|_\infty^{\alpha^\varkappa + 1 - 2/q}.$$

if $\max (1, 6 (3\alpha^\varkappa - 1)^{-1}) \leq q \leq 6/5$, and

$$\text{ii)} \quad \| A^2 (G^\varkappa (u(s)) \, \partial_\varkappa u(s)) \|_2 \leq g D \, | u(s) |_3 \left\{ \sum_{\varkappa=0}^{3} \| u(s) \|_\infty^{\alpha^\varkappa} + \right.$$

$$\left. + \| u(s) \|_\infty^{\alpha^0 - 1} \| \dot{u}(s) \|_\infty + \| u(s) \|_\infty^{\alpha^0 - 1/2} \| \dot{u}(s) \|_\infty^{1/2} \right\}.$$

PROOF. Appendix II.

REMARK. The estimates in Lemma 2.8 are slight generalizations of those in assumption (D_2) and (D_3) in that they are linear combinations of the terms appearing there. However, the general method of section 1 applies to this situation as well as can be seen from the subsequent discussion of of this example. The appearance of $\| u(s) \|_\infty$ in the above inequality will demand that the full set of coupled inequalities be used.

Thus if $6 (3\alpha^\varkappa - 1)^{-1} \leq q \leq 6/5$, inequality (38) can be replaced by

$$(39a) \qquad \| u(t) \|_\infty \leq \| u_0(t) \|_\infty + g C \int_{t_0}^{t} (1 + |t - s|)^{-3/q + 3/2} \, |u(s)|_3^{2/q} \sum_{\varkappa=0}^{3} \| u(s) \|_\infty^{\alpha^\varkappa + 1 - 2/q} \, ds.$$

Similarly, using the estimate of Corollary 4.5, if $1 < q' < 4/3$, and

$$6 (3\alpha^\varkappa - 1)^{-1} \leq q' \leq 6/5$$

$$(39b) \qquad \| \dot{u}(t) \|_\infty \leq \| \dot{u}_0(t) \|_\infty + g C' \int_{t_0}^{t} | t - s |^{-1/q'} \| A^2 (G^\varkappa (u(s)) \, \partial_\varkappa u(s)) \|_{q'} \, ds$$

$$\leq \| \dot{u}_0(t) \|_\infty + g C' \int_{t_0}^{t} | t - s |^{-1/q'} \, | u(s) |_3^{2/q'} \sum_{\varkappa=0}^{3} \| u(s) \|_\infty^{\alpha^\varkappa + 1 - 2/q'} \, ds.$$

And finally the appropriate inequality for the energy is

$$\text{(39c)}\qquad |u(t)|_3^2 \leq |u_0(t)|_3^2 + gD \int_{t_0}^{t} \Big\{ \sum_{\varkappa=0}^{3} \big(\| u(s) \|_\infty^{\alpha^\varkappa} \big) + \| u(s) \|_\infty^{\alpha^0 - 1} \| \dot{u}(s) \|_\infty + + \| u(s) \|_\infty^{\alpha^0 - 1/2} \| \dot{u}(s) \|_\infty^{1/2} \Big\} |u(s)|_3^2 \, ds.$$

Taking $q, q' = 6/5$ so that all $\alpha^\varkappa \geq 2$ can be treated simultaneously, and $x(t) = \sup_{t_0 < s < t \in I(t_0)} (1 + |s|) \| u(s) \|_\infty$,

$$\dot{x}(t) = \sup_{t_0 < s < t \in I(t_0)} (1 + |s|)^{5/6} \| \dot{u}(s) \|_\infty \quad y(t) = \sup_{t_0 < s < t \in I(t_0)} |u(s)|_3^2 .$$

then the inequalities (39) imply, as in (16),

$$\text{(40a)}\qquad x(t) \leq x_0 + gC \sum_{\varkappa=0}^{3} \Big\{ x(t)^{\alpha^\varkappa - 2/3} y(t)^{5/6} \sup_{t_0 < t' < t \in I(t_0)} (1 + |t'|) \cdot \cdot \int_{t_0}^{t} (1 + |t' - s|)^{-1} (1 + |s|)^{\alpha^\varkappa + 2/3} ds \Big\}$$

$$\text{(40b)}\qquad \dot{x}(t) \leq \dot{x}_0 + gC' \sum_{\varkappa=0}^{3} \Big\{ x(t)^{\alpha^\varkappa - 2/3} y(t)^{5/6} \sup_{t_0 < t' < t \in I(t_0)} (1 + |t'|)^{5/6} \int_{t_0}^{t} |t' - s|^{-5/6} (1 + |s|)^{-\alpha^\varkappa + 2/3} ds \Big\}$$

$$\text{(40c)}\qquad y(t) \leq y_0 + gD \Big[\sum_{\varkappa=0}^{3} \Big\{ x(t)^{\alpha^\varkappa} y(t) \int_{t_0}^{t} (1 + |s|)^{-\alpha^\varkappa} ds \Big\} + + x(t)^{\alpha^0 - 1} \dot{x}(t)\, y(t) \int_{t_0}^{t} (1 + |s|)^{-\alpha^0 + 1 - 5/6} ds + + x(t)^{\alpha^0 - 1/2} \dot{x}(t)^{1/2} y(t) \int_{t_0}^{t} (1 + |s|)^{-\alpha^0 + 1/2 - 5/12} ds \Big] .$$

Because $\alpha^{\varkappa} \geq 2$, it is straightforward to check that all the integrals converge and cancel the appropriate powers of $(1 + |t'|)$ to give

(41a)
$$x(t) \leq x_0 + gC \sum_{\varkappa=0}^{3} x(t)^{\alpha^{\varkappa}-2/3} y(t)^{5/6},$$

(41b)
$$\dot{x}(t) \leq \dot{x}_0 + gC \sum_{\varkappa=0}^{3} x(t)^{\alpha^{\varkappa}-2/3} y(t)^{5/6},$$

$$y(t) \leq \dot{y}_0 + g\, Dy(t) \left[\sum_{\varkappa=0}^{3} (x(t)^{\alpha^{\varkappa}}) + x(t)^{\alpha_0-1} \dot{x}(t) + x(t)^{\alpha^0-1/2} \dot{x}(t)^{1/2} \right].$$

Part ii) of Lemma 1.1 again applies to give the desired conclusion of Theorem 2.7.

(d) $G(x, t, u, u_x, \dot{u}) = gG(x, t)\, u$. This linear example is included mainly to indicate, in this simplest possible situation, how the abstract method of section 1 may be trivially modified to handle perturbations which are explicitly dependent on t. It is also interesting in several other technical respects; namely, it is an example which is most profitably treated by using (4) to obtain a bound for $|u(t)|_a$ rather than (5) and working with $y(t) = \sup_{t_0 < t' < t \in I(t_0)} |u(t)|_a$ which necessitates the use of part iii) of the basic Lemma 1.1. The decay of $G(x, t)$ used in the subsequent discussion is that suggested by taking $G(x, t) = (v(x, t))^2$ where v is a solution of the $K - G$ equation with smooth Cauchy data so that it is differentiable to high order with respect to x and t and it and its derivatives decay in the uniform norm like $|t|^{-3/2}$. This corresponds mathematically, to the first variational equation for $\Box u = m^2 u + gu^3$ or, physically to the simpliest equation proposed for the weak interaction of the field with an external meson field.

The basic estimate, which in this linear case will suffice for both the local existence theorem and the decay theorem, can be summarized in the following

LEMMA 2.9. If, for all t, $G(\cdot, t)$ is an element of the Sobolev space $W^{2,p}(E^3)$ for $p = 1$ and ∞ (or equivalently for all $1 \leq p \leq \infty$) and $w \in D(A^3)$, then for any $1 \leq q \leq 2$

(42)
$$\| A^2 (G(\cdot, t)\, w \|_q \leq C \{ (\| G(\cdot, t) \|_q + \| D^2 G(\cdot, t) \|_q) \| w \|_\infty +$$
$$\| DG(\cdot, t) \|_{6q(6-q)^{-1}} \| A^3 w \|_2^{1/3} \| w \|_\infty^{2/3} + \| G(\cdot, t) \|_{3q(3-q)^{-1}} \| A^3 w \|_2^{2/3} \| w \|_\infty^{1/3} \},$$

(using the notation of Proposition 2.4).

PROOF. $\| A^2 G(\cdot, t) w \|_q \leq m^2 \| G(\cdot, t) w \|_q + \| \Delta (G(\cdot, t) w) \|_q$. Because $G(\cdot, t)$, w and all their weak derivatives are in $L^2(E^3)$, the Leibniz formula may be applied to the last term to obtain $\Delta(G(\cdot, t) w) = w \Delta G(\cdot, t) + 2\nabla w \cdot \nabla G(\cdot, t) + G(\cdot, t) \Delta w$. Thus $\| A^2 G(\cdot, t) w \|_q \leq \text{Const.} \{ \| G(\cdot, t) \|_q \| w \|_\infty + \| \Delta G(\cdot, t) \|_q \| w \|_\infty + \| DG(\cdot, t) \|_{6q(6-q)^{-1}} \| Dw \|_6 + \| G(\cdot, t)_{3q(3-q)^{-1}} \| D^2 w \|_3 \}$. The result then follows from $\| Dw \|_6 \leq \text{Const.} \| A^3 w \|_2^{1/3} \| w \|_\infty^{2/3}$ and $\| D^2 w \|_3 \leq \text{Const.} \| A^3 w \|_2^{2/3} \| w \|_\infty^{1/3}$ which are particular cases of Proposition 2.4.

THEOREM 2.10. Suppose that for all t, $G(\cdot, t) \in W^{2,p}(E^3)$ for $1 \leq p \leq \infty$ and $\| G(\cdot, t) \|_{2,p}$ is a continuous function of $t \in \mathbb{R}$ with $\| G(\cdot, t) \|_{2,1}$ uniformly bounded in t and $\| G(\cdot, t) \|_{2,\infty} = 0(|t|^{-3})$ as $|t| \to \infty$. If the Cauchy data $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H(A, 3)$ are sufficiently smooth so that condition (D_4) is satisfied with $\varepsilon = 3/2$, then (the integrated form of)

$$\square u = m^2 u + gG(x, t) u \tag{43}$$

has a unique global solution $\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} \in H(A, 3)$ with

$$\| u(t) \|_\infty = 0(|t|^{-3/2}) \quad \| \dot{u}(t) \|_\infty = 0(|t|^{-\delta}),$$

δ arbitrary but < 1, and $|u(t)|_3 = 0(1)$ as $|t| \to \infty$ provided that the coupling constant is sufficiently small.

PROOF. For this linear equation, a simplier version of Segal's Theorem [3, Corollary 1.2, p. 346] can be used to get the local (in fact global) existence of solutions. All that is required is that $\widetilde{G}_t \colon H(A, 3) \to H(A, 3)$ has locally bounded operator bound. This follows directly from Lemma 2.9 and the hypotheses because

$$\left| \widetilde{G}_t \begin{pmatrix} w \\ - \\ w \end{pmatrix} \right|_3 = \| A^2 (gG(\cdot, t) w) \|_2 \leq gC \{ \| G(\cdot, t) \|_{2,2} + \| G(\cdot, t) \|_{1,3} + \| G(\cdot, t) \|_6 \} \| A^3 w \|_2 .$$

As for the decay result, only the inequalities involving $\| u(t) \|_\infty$ and $|u(t)|_3$ are need. As usual, with $q = 1$,

$$\| u(t) \|_\infty \leq \| u_0(t) \|_\infty + gC \int_{t_0}^{t} (1 + |t - s|)^{-3/2} \} \| G(\cdot, s) \|_{2,1} \| u(s) \|_\infty + \| G(\cdot, s) \|_{1, 6/5} \| u(s) \|_\infty^{2/3} |u(s)|_3^{1/3} + \| G(\cdot, s) \|_{3/2} \| u(s) \|_\infty^{1/3} |u(s)|_3^{2/3} \, ds. \tag{44a}$$

However the usual approach for the $|u(t)|_3^2$ inequality is avoided in order to get a better result. For if the standard approach was used then (using the notation of section 1) the degree of the terms on the right hand side of the inequalities for $x(t)$ and $y(t)$ would be $\alpha+\beta$ and $\alpha''+\beta''$ respectively where $\alpha+2\beta=1$ and $\alpha''+2\beta''=2$. This implies that $\alpha+\beta= =1/2<1$ since $\alpha<1$ is some terms while $\alpha''+\beta''=1+\alpha''/2\geq 1$ necessitating the ultimate use of part iii), c), 2) of Lemma 1.1. This in term requires that both the coupling constant g and the Cauchy data $x_0+x_0+y_0$ are sufficiently small. The approach here will instead be directed to getting the exponents $\alpha+\beta=\alpha'+\beta'=\alpha''+\beta''=1$ thus by part iii), c) requiring only that g be small. To this end consider equation (4). A straightforward calculation gives

$$(44c)\qquad |u(t)|_3\leq 2\left\{|u_0(t)|_3+\int_{t_0}^{t}\|A^2(g\,G(\cdot,s)\,u(s))\|\,ds\right\}$$

$$\leq 2\,|u_0(t)|_3+g\,D\int_{t_0}^{t}\{\|G(\cdot,s)\|_{2,2}\,\|u(s)\|_\infty+$$

$$+\|G(\cdot,s)\|_{1,3}\,\|(s)\|_\infty^{2/3}\,|u(s)|_3^{1/3}+G(\cdot,s)\|_6\,\|u(s)\|_\infty^{1/3}\,|u(s)|_3^{2/3}\}ds.$$

Now for $f\in W^{2,1}(E^3)\cap W^{2,\infty}(E^3)$, $1\leq p\leq\infty$, $k=0$, 1, or 2, $\|f\|_{k,p}< <\|f\|_{2,p}\leq\|f\|_{2,\infty}^{1-1/p}\,\|f\|_{2,1}^{1/p}$. A bound for all of the terms involving $G(\cdot,t)$ in (44a) and (44c) can now be obtained by using the continuity (hence local boundedness) of $\|G(\cdot,t)\|_{2,p}$ and the $|t|^{-3}$ decay rate as $|t|\to\infty$ of $\|G(\cdot,t)\|_{2,\infty}$. Specifically $\|G(\cdot,s)\|_{2,1}\leq C$, $\|G(\cdot,s)\|_{1,6/5}\leq C(1+|s|)^{-1/2}$, $\|G(\cdot,s)\|_{3/2}\leq C(1+|s|)^{-1}$, $\|G(\cdot,s)\|_{1,3}\leq C(1+|s|)^{-2}$ and $\|G(\cdot,s)\|_6\leq \leq C(1+|s|)^{-5/2}$, $\|G(\cdot,s)\|_{2,2}\leq C(1+|s|)^{-3/2}$.

Thus using $\varepsilon=3/2$ and $y(t)=\sup_{t_0<t'<t\,\in\,I(t_0)}|u(t')|$ inequalities (44a) and (44c) reduce to

$$(45a)\qquad x(t)\leq x_0+g\,C\,\{x(t)+x(t)^{2/3}\,y(t)^{1/3}+x(t)^{1/3}\,y(t)^{2/3}\}$$

$$\sup_{t_0<t'<t\,\in\,I(t_0)}(1+|t'|)^{3/2}\int_{t_0}^{t'}(1+|t'-s|)^{-3/2}\,(1+|s|)^{-3/2}\,ds,$$

$$(45c)\qquad y(t)\leq 2\,y_0+g\,D\,\{x(t)+x(t)^{2/3}\,y(t)^{1/3}+x(t)^{1/3}\,y(t)^{2/3}\}$$

$$\sup_{t_0<t'<t\,\in\,I(t_0)}\int_{t_0}^{t'}(1+|s|)^{-3}\,ds.$$

Clearly the terms on the far right of expressions (45a) and (45c) are bounded and hence part iii), c) of Lemma 1.1 gives the global existence and desired decay for $\| u(t) \|_\infty$ and $| u(t) |_3$ provided the coupling constant, g, is sufficiently small.

The cited decay for $\| \dot{u}(t) \|_\infty$ can be obtained as a consequence as folfows. If $1 < q' < 4/3$,

$$\| \dot{u}(t) \|_\infty \leq \| \dot{u}_0(t) \|_\infty + g\, C' \int_{t_0}^{t} \| F_{t-s,\,1} \|_{q'} \, \widehat{(q' - 1)^{-1}} \| A^2 (G(\cdot, s)\, u(s) \|_{q'}\, ds$$

$$\leq \| \dot{u}_0(t) \|_\infty + g\, C' \int_{t_0}^{t} | t - s |^{-1/q'} \{ \| G(\cdot, s) \|_{2, q'} \| u(s) \|_\infty +$$

$$\| G(\cdot, s) \|_{1,\, 6q'(6-q')^{-1}} \| u(s) \|_\infty^{2/3} + \| G(\cdot, s) \|_{3q'(3-q')^{-1}} \| u(s) \|_\infty^{1/3} \}\, ds$$

$$(46) \qquad \leq x_0 (1 + | t |)^{-3/2} + g\, C' \int_{t_0}^{t} | t - s |^{-1/q'} \{ (1 + | s |)^{-[3(1-1/q')+3/2]} +$$

$$+ (1 + | s |)^{-[3(7q'-6)/6q'+1]} + (1 + | s |)^{-[3(4q'-3)/3q'+1/2]} \}\, ds.$$

Since all the exponents of $1 + | s |$ in (46) are < -1, the technical result of Shenk and Thoe [9, Lemma 3.1] can be used to give the cited decay of $\| \dot{u}(t) \|_\infty$.

REMARK 1. This example is interesting from a technical viewpoint in that it shows how the abstract theory of section 1 can be generalized in several directions; specifically, i) the inclusion of a time-dependent factor in (D_2) and (D_3), ii) a linear combination of similar terms in (D_2) and (D_3) and, iii) an alternative for (5) as an inequality for the escalated energy.

REMARK 2. As was pointed out previously, Theorem 2.10 is suitable for treating the equation proposed for the weak interaction of the u field with en external meson field (i.e. $G(x, t) = (v(x, t))^2$ where v satisfies the $K - G$ equation). Coupled interations of this sort can be treated by a variant of this method. However perturbations of type $\sum_{k=0}^{3} G^{\varkappa}(x, t)\, \partial_{\varkappa} u + G^4(x, t)\, u$, which are mathematical generalizations of the equation governing the interaction of a meson field u with an external electromagnetic field, require more severe damping in the $G^{\varkappa\prime}s$ than is suggested by the physical problem.

It seems then that in these linear cases the damping of the time-dependent required to make this method work is fairly stringent. On the other hand when $G(x, t, u, u_x, u_t) = g\, G(x)\, u$ other methods [10] (which can presumably be generalized to treat $g\left(\sum_{\varkappa=0}^{3} G^{\varkappa}(x)\, \partial_{\varkappa} u + G^4(x)\, u\right)$) are available for treating the asymptotics of the perturbed equation.

The scattering theory for equations of this type, in the abstract as well as for the particular cases discussed in section 2, will be presented in a sequel to this paper.

APPENDIX I. Proof of Lemma 1.1.

Consider the individual terms in the polynomials h_i.

$$g_{ij}\, x_1(t)^{\alpha_{ij}^1} \dots x_n(t)^{\alpha_{ij}^n} = g_{ij}\,[x_1(t)^{\tau_{ij}}]^{\alpha_{ij}^1/\tau_{ij}} \dots [x_n(t)^{\tau_{ij}}]^{\alpha_{ij}^n/\tau_{ij}}$$

$$\leq g_{ij}\left[\frac{\alpha_{ij}^1}{\tau_{ij}}\, x_1(t)^{\tau_{ij}} + \dots + \frac{\alpha_{ij}^n}{\tau_{ij}}\, x_n(t)^{\tau_{ij}}\right] \leq g_{ij}\, z(t)^{\tau_{ij}}$$

where $z(t) = \sum_{i=1}^{n} x_i(t)$. Thus

$$0 \leq x_i(t) \leq x_{i0} + \sum_j g_{ij}\, z(t)^{\tau_{ij}},$$

and hence.

$$0 \leq z(t) \leq z_0 + \sum_{i=1}^{n} \sum_j g_{ij}\, z(t)^{\tau_{ij}},$$

where $z_0 = \sum_{i=1}^{n} x_{i0}$. Suppose, with no loss of generality, that each polynomial h_i has no more than N terms (i. e. $j \leq N$).

i) If $\tau_{ij} < 1$, then

$$g_{ij}\, z(t)^{\tau_{ij}} = (nN)^{\tau_{ij}}\, g_{ij}\left(\frac{z(t)}{nN}\right)^{\tau_{ij}} = [\{(nN)^{\tau_{ij}}\, g_{ij}\}^{(1-\tau_{ij})^{-1}}]^{1-\tau_{ij}}\left[\frac{z(t)}{nN}\right]^{\tau_{ij}}$$

$$\leq (1-\tau_{ij})\,\{(nN)^{\tau_{ij}}\, g_{ij}\}^{(1-\tau_{ij})^{-1}} + \frac{\tau_{ij}}{nN}\, z(t)$$

Thus

$$0 \leq z(t) \leq z_0 + \sum_{i=1}^{n} \sum_j (1-\tau_{ij})\,\{g_{ij}\,(nN)^{\tau_{ij}}\}^{(1-\tau_{ij})^{-1}} + \left[\sum_{i=1}^{n} \sum_{j=1} \tau_{ij}\right] \frac{z(t)}{nN}\,.$$

But, $\tau_{ij} < 1$ implies that $\sum_{i=1}^{n} \sum_j \tau_{ij} < nN$ and hence

$$0 \leq z(t) \leq \left[1 - (nN)^{-1} \sum_{i=1}^{n} \sum_i \tau_{ij}\right]^{-1} \left[z_0 + \sum_{i=1}^{n} \sum_j (1-\tau_{ij})\,\}g_{ij}\,(nN)^{\tau_{ij}}\{^{(1-\tau_{ij})}\right].$$

Since $z(t) = \sum_{i=1}^{n} x_i(t)$, $x_i(t) \geq 0$, the same bound as above obtains for each $x_i(t)$.

ii) Choose $\tau > \max \{\tau_{ij}\}$. Let $\theta \geq 0$ be arbitrary.

$$g_{ij}\, z(t)^{\tau_{ij}} = \left[(\theta^{\tau_{ij}} g_{ij})^{(1-\tau_{ij}/\tau)^{-1}}\right]^{1-\tau_{ij}/\tau} \left[\left(\frac{z(t)}{\theta}\right)^{\tau}\right]^{\tau_{ij}/\tau}$$

$$\leq \left(1 - \frac{\tau_{ij}}{\tau}\right)(\theta^{\tau_{ij}} g_{ij})^{(1-\tau_{ij}/\tau)^{-1}} + \frac{\tau_{ij}}{\tau}\theta^{-\tau} z(t)^{\tau}.$$

Tus

$$0 \leq z(t) \leq \left[z_0 + \sum_{i=1}^{n}\sum_{j}\left(1 - \frac{\tau_{ij}}{\tau}\right)(\theta^{\tau_{ij}} g_{ij})^{(1-\tau_{ij}/\tau)^{-1}}\right] + \theta^{-\tau}\left[\sum_{i=1}^{n}\sum_{j}\frac{\tau_{ij}}{\tau}\right] z(t)^{\tau}.$$

Appealing to a result of Segal and Strauss [e. g. 2, Lemma 3.7, p 437] since $z(t) = \sum_{i=1}^{n} x_i(t)$ is continuous, $\tau > 1$, $z(\bar{t}) = \sum_{i=1}^{n} x_i(\bar{t}) \leq \sum_{i=1}^{n} x_{i0} = z_0 \leq z_0 + \sum_{i=1}^{n}\sum_{j}\left(1 - \frac{\tau_{ij}}{\tau}\right)(\theta^{\tau_{ij}} g_{ij})^{(1-\tau_{ij}/\tau)^{-1}}$ for all $\theta \geq 0$, $z(t)$ is bounded on I if

$$\left[z_0 + \sum_{i=j}^{n}\sum_{j}\left(1 - \frac{\tau_{ij}}{\tau}\right)(g_{ij}\,\theta^{\tau_{ij}})^{(1-\tau_{ij}/\tau)^{-1}}\right]\left[\sum_{i=1}^{n}\sum_{j}\frac{\tau_{ij}}{\tau}\right]^{(\tau-1)^{-1}}\theta^{-\tau(\tau-1)^{-1}} < (1-\tau)^{-1}\tau^{-(\tau-1)^{-1}}.$$

The last inequality can be achieved by using the remaining hypothesis, ii), c). First, holding the g_{ij} fixed, since z_0 can be made arbitrarily small independent of θ, the left side of the inequality is $0\,(\theta^{\tau_{ij}\,\tau(\tau-\tau_{ij})^{-1} - \tau(\tau-1)^{-1}})$ as $\theta \to 0$. Thus the left side can be made sufficiently small by choosing z_0 and θ small provided that the exponent $\tau_{ij}(\tau - \tau_{ii})^{-1} - \tau(\tau - 1)^{-1} > 0$ for all i, j. This condition is identical to

$$(\tau - \tau_{ij})^{-1}(\tau - 1)^{-1}[\tau_{ij}\,\tau(\tau - 1) - \tau(\tau - \tau_{ij})] = (\tau - \tau_{ij})^{-1}(\tau - 1)^{-1}\tau^2(\tau_{ij} - 1) > 0,$$

which is clearly the case. If z_0 is to remain fixed then the left side can be made small by first choosing θ large so that $z_0\,\theta^{-\tau(\tau-1)^{-1}}$ is small and then taking g_{ij}'s so that $g_{ij}\,\theta^{\tau_{ij}}$ is small.

iii) By repeating the calculations in i),

$$0 \leq z(t) \leq \left[z_0 + \sum_{i,j,\tau_{ij}<1}(1 - \tau_{ij})\{g_{ij}(4nN)^{\tau_{ij}}\}^{(1-\tau_{ij})^{-1}}\right] +$$

$$+ \left[(4nN)^{-1}\sum_{i,j;\tau_{ij}<1}\tau_{ij} + \sum_{i,j;\tau_{ij}=1} g_{ij}\}\right] z(t) + \sum_{i\,j;\tau_{ij}>1} g_{ij}\, z(t)^{\tau_{ij}}.$$

If the g_{ij} for those i associated with $\tau_{ij} = 1$ are chosen so that $\sum_{i,j;\tau_{ij}=1} g_{ij} < 1/4$, then the above implies

$$0 \leq z(t) \leq 2z_0'(g_{ij}) + \sum_{i,j;\tau_{ij}>1} 2\, g_{ij}\, z(t)^{\tau_{ij}},$$

where

$$z_0'(g_{ij}) = z_0 + \sum_{i,j;\tau_{ij}<1} (1 - \tau_{ij}) \{g_{ij}(4nN)^{\tau_{ij}}\}^{(1-\tau_{ij})^{-1}}.$$

Repeating the argument of ii), the boundedness of $z(t)$ on I is established if

$$\left[z_0 + \sum_{i,j;\tau_{ij}<1} (1 - \tau_{ij}) \{g_{ij}(4nN)^{\tau_{ij}}\}^{(1-\tau_{ij})^{-1}} + \sum_{i,j;\tau_{ij}>1} \left(1 - \frac{\tau_{ij}}{\tau}\right) (g_{ij}\,\theta^{\tau_{ij}})^{(1-\tau_{ij}/\tau)^{-1}} \right] \cdot \left[\sum_{i,j;\tau_{ij}>1} \frac{\tau_{ij}}{\tau} \right] \theta^{-\tau(\tau-1)^{-1}} < (1-\tau)^{-1}\, \tau^{-(\tau-1)^{-1}},$$

which is possible by taking $z_0'(g_{ij})$ and θ small or the g_{ij}'s associated with $\tau_{ij} > 1$ small and θ large.

REMARK. Notice in the proof of iii) that if all $\tau_{ij} \leq 1$, the boundedness of $z(t)$ depends only on the smallness of g_{ij} for those i, j for which $\tau_{ij} = 1$. (Compare with part i))

APPENDIX II. Proof of Lemma 2.8.

First write (suppressing the s-dependence)

$$\begin{aligned} A^2(G_\varkappa(u)\,\partial_\varkappa u) = {} & m^2(G^k(u)\,\partial_k u + G^0(u)\,\dot{u}) + G^k(u)\,\partial_k(\Delta u) + 2\,G^{k\prime}(u)\,\nabla u \cdot \nabla(\partial_k u) \\ & + \Delta u\, G^{k\prime}(u)\,\partial_k(u) + (\nabla u)^2\, G^{k\prime\prime}(u)\,\partial_k u + G^0(u)\,\nabla \dot{u} + 2G^{0\prime}(u)\,\nabla u \cdot \nabla \dot{u} \\ & + \Delta u\, G^{0\prime}(u)\,\dot{u} + (\nabla u)^2\, G^{0\prime\prime}(u)\,\dot{u}, \end{aligned}$$

where $G^{k\prime}$ and $G^{k\prime\prime}$ denote the first and second derivatives of G^k. Each term is now examined in order. The same constant, C, is used for inessentially different numbers and the sum over k is understood in each term.

For Lemma 2.8 the appropriate estimates are as follows.

$$\begin{aligned} \| G^k(u)\,\partial_k u \|_q & \leq \| G^k(u) \|_{6q(6-q)^{-1}} \| \partial_k u \|_6 \\ & \leq Cg \| u \|_\infty^{\alpha^k - 2(6-q)/6q} | u |_3^{2(6-q)/6q} \| u \|_\infty^{2/3} | u |_3^{1/3} \end{aligned}$$

by the estimates of leading to (23) and $\| \partial_k u \|_6 \leq$ Const. $\| u \|_\infty^{2/3} \|A^3 u\|_2^{1/3}$ from Proposition 2.4 provided $\alpha^k 6q(6-q)^{-1} \geq 2$.

$$\| G^0(u) \dot{u} \|_q \leq \| \dot{u} \|_2 \| G^0(u) \|_{2q(2-q)^{-1}}$$
$$\leq gC \| A^2 \dot{u} \|_2 \| u \|_\infty^{\alpha^0 - 2(2-q)/2q} | u |_3^{2(2-q)/2q}$$

as in (23) provided $\alpha^0 2q(2-q)^{-1} \geq 2$.

$$\| G^k(u) \partial_k (\Delta u) \|_q \leq \| G^k(u) \|_{2q(2-q)^{-1}} \| \partial_k (\Delta u) \|_2$$
$$\leq gC \| u \|_\infty^{\alpha^k + 1 - 2/q} | u |_3^{2/q}$$

as in (23) provided $\alpha^k 2q(2-q)^{-1} \geq 2$, using $\| \partial_k (\Delta u) \|_2 \leq$ Const. $\| A^3 u \|_2$.

$$\| G^{k'}(u) \nabla u \cdot \nabla (\partial_k u) \|_q \leq \sum_{j=1}^{3} \| G^{k'}(u) \|_{2q(q-2)^{-1}} \| \partial_j u \|_6 \| \partial_j \partial_k u \|_3$$
$$\leq g C \| u \|_\infty^{\alpha^k - 1 - 2(2-q)/2q} | u |_3^{2(2-q)/2q} \| u \|_\infty | u |_3$$

as in (23) provided $(\alpha^k - 1) 2q(2-q)^{-1} \geq 2$, using $\| \partial_j u \|_6 \leq$ Const. $\| A^3 u \|_2^{1/3} \| u \|_\infty^{2/3}$ and $\| \partial_j \partial_k u \|_3 \leq$ Const. $\| A^3 u \|_2^{2/3} \| u \|_\infty^{1/3}$.

$$\| \nabla u \, G^{k'}(u) \partial_k u \|_q \leq \| \nabla u \|_3 \| G^{k'}(u) \|_{2q(2-q)^{-1}} \| \partial_k u \|_6$$
$$\leq gC \| u \|_\infty^{\alpha^k - 1 - 2(2-q)/2q} | u |_3^{2(2-q)/2q} \| u \|_\infty | u |_3$$

as above provided $(\alpha^k - 1) 2q(2-q)^{-1} \geq 2$.

$$\| (\nabla u)^2 G^{k''}(u) \partial_k u \|_q \leq \| (\nabla u)^2 \|_3 \| G^{k''}(u) \|_{6q(6-5q)^{-1}} \| \partial_k u \|_2$$
$$\leq C g \left\| \sum_{j=1}^{3} | \partial_j u | \right\|_6^2 \| u \|_\infty^{\alpha^k - 2 - 2(6-5q)/6q} | u |_3^{2(6-5q)/6q} \| A^3 u \|_2$$
$$\leq Cg \| u \|_\infty^{4/3} \| A^3 u \|_2^{2/3} \| u \|_\infty^{\alpha^k - 1/3 - 2/q} | u |_3^{2/q - 5/3} \| A^3 u \|_2$$

as above provided $(\alpha^k - 2) 6q(6-5q)^{-1} \geq 2$.

$$\| G^0(u) \Delta \dot{u} \|_q \leq \| G^0(u) \|_{2q(2-q)^{-1}} \| \Delta \dot{u} \|_2$$
$$\leq Cg \| u \|_\infty^{\alpha^0 - 2(2-q)/2q} | u |_3^{2(2-q)/2q} | u |_3$$

provided $\alpha^0 \, 2q\,(2-q)^{-1} \geq 2$.

$$\| G^{0\prime}(u) \nabla u \cdot \nabla \dot{u} \|_q \leq \| G^{0\prime}(u) \|_{3q(3-2q)^{-1}} \| \partial_j u \|_6 \| \partial_j \dot{u} \|_2$$

$$\leq gC \| u \|_\infty^{\alpha_0 - 1 - 2(3-2q)/3q} | u |_3^{2(3-2q)/3q} \| u \|_\infty^{2/3} \| A^3 u \|_2^{1/3} \| A^2 \dot{u} \|_2$$

provided $(\alpha^0 - 1)\,?q\,(3-2q)^{-1} \geq 2$.

$$\| \Delta u \, G^{0\prime}(u)\, u \|_q \leq \| \Delta u \|_3 \| G^{0\prime}(u) \|_{6q(6-5q)^{-1}} \| \dot{u} \|_2$$

$$\leq gC \, | u |_3^{2/3} \| u \|_\infty^{1/3} \| u \|_\infty^{\alpha^0 - 1 - 2(6-9q)/6q} | u |_3^{2(6-5q)/6q} | u |_3$$

provided $(\alpha^0 - 1)\, 6q\,(6-5q)^{-1} \geq 2$.

$$\| (\Delta u)^2 G^{0\prime\prime}(u)\, u \|_q \leq \| (\Delta u)^2 \|_3 \| G^{0\prime\prime}(u) \|_{6q(6-5q)^{-1}} \| \dot{u} \|_2$$

$$\leq gC \| u \|_\infty^{4/3} \| A^3 u \|_2^{2/3} \| u \|_\infty^{\alpha^0 - 1/3 - 2/q} | u |_3^{2/q - 5/3} \| A^2 \dot{u} \|_2$$

provided $(\alpha_0 - 2)\, 6q\,(6-5q)^{-1} \geq 2$.

All that remains then is to check that if $6\,(3\alpha^k - 1)^{-1} q \leq 6/5$ then all the above inequalities involving q and α^k are satisfied. The fifth and last are the worst and as such are the ones which defermines the condition in the hypothesis of Lemma 2.8. Clearly the condition is not satisfied unless $q \leq 6/5$, and if this so, $6q\,(6-5q)^{-1} \geq 2\,(\alpha^k - 2)$ is equivalent to $6q\,(\alpha^k - 2) \geq 2\,(6-5q)$ or $q \geq 6\,(3\alpha^k - 1)^{-1}$. The other inequalities follow directly from from the condition $q \geq 6\,(3\alpha^k - 1)^{-1}$: $6q(6-q) \geq 6\,(3\alpha^k - 2)^{-1} > 2/\alpha^k$; $2q\,(2-q)^{-1} \geq 6\,(3\alpha - 4)^{-1} > 2(\alpha^k - 1)^{-1} > 2/\alpha^k$; $3q(3-2q)^{-1} \geq 6\,(3\alpha^k - 5)^{-1} > 2\,(\alpha^k - 1)^{-1}$.

For part ii) all but the sixth, eighth, ninth and tenth are valid for $\alpha^k \geq 2$ and $q = 2$, and contribute to the first term in the inequality. For those listed above the estimates can be made in a sightly different manner to obtain the desired result

$$\| (\Delta u)^2 G^{k\prime\prime}(u)\, \partial_k u \|_2 \leq \| (\Delta u)^2 \|_3 \| G^{k\prime\prime}(u) \|_\infty \| \partial_k u \|_6$$

$$\leq gD \| u \|_\infty^{4/3} | u |_3^{2/3} \| u \|_\infty^{\alpha^k - 1} \| u \|_\infty^{2/3} | u |_\infty^{1/3} | u |_3^{1/3}.$$

$$\| G^{0\prime}(u)\, \Delta u \cdot \Delta \dot{u} \|_2 = D \| G^{0\prime}(u) \|_\infty \| \partial_j u \|_4 \| \partial_j \dot{u} \|_4$$

$$\leq gD \| u \|_\infty^{\alpha^0 - 1} \| u \|_\infty^{1/2} | u |_3^{1/2} \| \dot{u} \|_\infty^{1/2} | u |_3^{1/2}.$$

$$\| \Delta u \, G^{0\prime}(u) \, \dot{u} \|_2 \leq \| \Delta u \|_3 \, \| G^{0\prime}(u) \|_6 \, \| \dot{u} \|_\infty$$

$$\leq gD \, \| u \|_\infty^{1/3} \, | u |_3^{2/3} \, \| u \|_\infty^{\alpha^0 - 1 - 2/6} \, | u |_3^{2/6} \, \| \dot{u} \|_\infty$$

since $6(\alpha^0 - 1) \geq 2$. Finally

$$\| (\Delta u)^2 \, G^{0\prime\prime}(u) \, \dot{u} \|_\infty \leq \| (\Delta u)^2 \|_2 \, \| G^{0\prime\prime}(u) \|_\infty \, \| \dot{u} \|_\infty$$

$$\leq gD \, \| u \|_\infty \, | u |_3 \, \| u \|_\infty^{\alpha^0 - 2} \, \| \dot{u} \|_\infty .$$

REFERENCES

[1] I. E. Segal, *Dispersion for Non-linear Relativistic Equations,* II, Ann. Scient. Ec. Norm. Sup., ser 4, **1**, 459-497, (1968).

[2] W. A. Strauss, *Decay and Asymptotics for* $\square u = F(u)$, J. Functl. Anal., **2**, 409-457, (1968).

[3] I. E. Segal, *Non-linear Semi-groups,* Ann. Math., **78**, 339-364, (1963).

[4] S. Nelson, *On Some Solutions to the Klein-Gordon Equation Related to an Integral of Sonine, to appear.*

[5] A. P. Calderón, *Lebesgue Spaces of Differentiable Functions and Distributions,* 33-49, Proc. Symp. Pure Math. IV, Amer. Math. Soc. Providence, 1961.

[6] I. E. Segal, *Quantization and Dispersion for Non-linear Relativistic Equations,* 79-108, Proc. Conf. on Math. Theory of Elem. Patticles, M. I. T., Cambridge, 1966.

[7] R. A. Goldstein, *Equality of Minimal and Maximal Extension of Partial Differential Operators iu* $L_p(R^n)$, Proc. Amer. Math. Soc., 17, 1031-1033, (1966).

[8] L. Nirenberg, *On Elliptic Partial Differential Equations,* Ann. Scoula Norm. Sup. Pisa, **13**, 115-162, (1959).

[9] N. Shenk and D. Thoe, *Outgoing Solutions of* $(-\Delta + q - k^2) u = f$ *in an Exterior Domain,* J. Math. Anal. Appl., **31**, 81-116, (1970).

[10] J. M. Chadam, *The Asymptotic Behavior of the Klein-Gordon Equation with External Potential,* I, J. Math. Anal. Appl., **31**, 334-348, (1970) and II, Pac. J. Math., 31, 19-31, (1969).