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THE PARAMETRIX OF REGULAR HYPOELLIPTIC BOUNDARY VALUE PROBLEMS

J. BARROS - NETO (1)

Let $P(D, D_t)$ be a properly hypoelliptic partial differential operator with constant coefficients and of type μ (Section 1) and let $Q_\nu(D, D_t)$, $1 \leq \nu \leq \mu$, be partial differential operators with constant coefficients all defined in $\mathbb{R}^{n+1}$. Suppose that $(P(D, D_t); (Q_\nu(D, D_t))_{1\leq\nu\leq\mu})$ defines a regular hypoelliptic boundary problem in $\mathbb{R}^{n+1}_+$ (Section 2, definition 1). Then, to $(P; (Q_\nu)_{1\leq\nu\leq\mu})$ it corresponds kernels $K(x, t)$, $K_1(x, t), \dots, K_\mu(x, t)$ satisfying the following properties:

i) $K, (K_\nu)_{1\leq\nu\leq\mu}$ are distributions belonging to $S'(\mathbb{R}^{n+1}_+)$;

ii) they are C^∞ functions in the open half space $\mathbb{R}^{n+1}_+$ which can be extended to C^∞ functions in $\overline{\mathbb{R}^{n+1}_+} - \{0\}$;

(iii) denoting with the same notation the extensions of K and $(K_\nu)_{1\leq\nu\leq\mu}$, then $K(x, t)$ is a solution of the boundary problem

$$\begin{cases} P(D, D_t)K = \delta_x \otimes \delta_t - \beta(x) \otimes \delta_t & \text{in} \quad \overline{R^{n+1}_+} \\ Q_\nu(D, D_t)K\big|_{\mathbb{R}^n_0} = 0, & 1 \leq \nu \leq \mu, \end{cases}$$

while every $K_l(x, t)$ is a solution of the boundary problem

$$\begin{cases} P(D, D_t)K_l = 0 & \text{in} \quad \mathbb{R}^{n+1}_+ \\ Q_\nu(D, D_t)K_l\big|_{\mathbb{R}^n_0} = \delta_{\nu, l}(\delta_x - \beta(x)), & 1 \leq \nu \leq \mu, \end{cases}$$

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where $\delta_{\nu, l}$ is the Kronecker symbol and $\beta(x) \in S(\mathbb{R}^n)$. The kernels $K, (K_\nu)_{1 \leq \nu \leq \mu}$ define a parametrix of the hypoelliptic preblem under consideration.

The kernels $K, (K_\nu)_{1 \leq \nu \leq \mu}$ are obtained by the same method we employed in [2] and [3] in order to get Green's and Poisson's kernels. Our proof is based on the existence and properties of the characteristic function of a boundary problem, a notion introduced by Hörmander in his paper [5], where he characterized regular hypoelliptic boundary problems.

The existence of kernels $K, (K_\nu)_{1 \leq \nu \leq \mu}$ satisfying conditions i), ii) and iii) above is proven, in our theorem of section 4, to be a necessary and sufficient condition in order that the boundary problem $(P; (Q_\nu)_{1 \leq \nu \leq \mu})$ be a regular hypoelliptic one in $\mathbb{R}^{n+1}_+$. In that theorem we prove several equivalent conditions for regular hypoellipticity of a boundary problem, one of which is Hörmander's algebraic condition proved in [5].

In section 1, we define properly hypoelliptic polynomials and establish a few properties needed later. In section 2, the definition of regular hypoelliptic boundary value problems is given and some examples are discussed. The characteristic function, some of its properties and estimates are considered in section 3. Most results of this section are contained in Hörmander [5] and reproduced here for the sake of completeness. In section 4, we state and start the proof of our main theorem which gives several necessary and sufficient conditions for a boundary problem to be regular hypoelliptic. Section 5 is devoted to the proof of existence and properties of the kernels $K, (K_\nu)_{1 \leq \nu \leq \mu}$. Finally, in section 6, we complete the proof of the theorem of section 4 by introducting the parametrix of the boundary problem.

1. Hypoelliptic polynomials.

Let $P(\zeta)$ be a constant coefficient polynomial in n complex variables $\zeta = (\zeta_1, \dots, \zeta_n)$, $\zeta_j = \xi_j + i\eta_j$, $1 \leq j \leq n$ and let

$$N = \{\zeta \in \mathbb{C}^n : P(\zeta) = 0\}$$

be the variety of zeros of $P(\zeta)$.

We say that $P(\zeta)$ is *hypoelliptic* if and only if the following condition holds:

$$(\mathrm{H}_1) \qquad \zeta \in N, \ |\zeta| \to +\infty \ \text{implies} \ |\mathrm{Im}\, \zeta| \to +\infty.$$

From condition (H_1) it follows trivially that the set

$$\{\xi \in \mathbb{R}^n : P(\xi) = 0\} = N \cap \mathbb{R}^n$$

is a compact subset of $\mathbb{R}^n$.

A polynomial $P(\xi)$ is said to be *elliptic* if

$$P_m(\xi) \neq 0, \quad \forall\, \xi \in \mathbb{R}^n \text{ with } \xi \neq 0,$$

where $P_m(\xi)$ denotes the *principal part* of P, i.e. the homogeneous part of highest degree. As it is well known, elliptic, semielliptic (for the definition see [4], [6]) and parabolic polynomials are hypoelliptic.

Consider a constant coefficient hypoelliptic polynomial $P(\zeta, \tau)$ in $n+1$ variables and suppose that the coefficient of the highest power of τ is independent of ζ thus it can be assumed to be equal to 1. Write

$$P(\zeta, \tau) = \tau^\sigma + a_1(\zeta)\,\tau^{\sigma-1} + \dots + a_\sigma(\zeta) \tag{1}$$

where $a_1(\zeta), \dots, a_\sigma(\zeta)$ are polynomials in $\zeta \in \mathbb{C}^n$. If $\xi \in \mathbb{R}^n$ and τ is a real root of the equation

$$P(\xi, \tau) = 0, \tag{2}$$

it follows from (H_1) that (ξ, τ) belongs to a compact subset of $\mathbb{R}^{n+1}$. Therefore, we can find $r > 0$ so large that $|\xi| > r$ implies that (2) has no real root τ. Since the roots depend continuously on ξ, in each connected component of the complement in $\mathbb{R}^n$ of the closed ball $\overline{B}(o, r)$, the number of roots τ of (1) with *positive imaginary part* is constant.

We say that a polynomial $(P(\zeta, \tau)$ is *properly hypoelliptic* if it is hypoelliptic and the number of τ zeros of the equation (2), with positive imaginary part, is constant, for all $\xi \in \mathbb{R}^n$ with $|\xi|$ sufficiently large. We call *type* of P the number of such zeros. It is obvious that when $n > 1$ all hypoelliptic polynomials are properly hypoelliptic. This may not be the case when $n = 1$, what can be seen by taking $P(\xi, \tau) = \xi + i\tau$. It is well known that if $P(\xi, \tau)$ is an elliptic operator and $n > 1$ then P is of even order $2m$ and its type is m.

Suppose, now, that the polynomial (1) is properly hypoelliptic and of type μ. Denote by $\mathcal{A}$ the set of all $\zeta \in \mathbb{C}^n$ such that equation (2) has precisely *μ roots with positive imaginary part and none that it is real.* Since the roots are continuous functions of ζ, it follows that $\mathcal{A}$ is an open set in $\mathbb{C}^n$. Also, since P is properly hypoelliptic the set $\mathcal{A}$ contains a suitable neighborhood of infinity in $\mathbb{R}^n$. More precisely, in [5], Hörmander has proved the following results on the set $\mathcal{A}$:

1) *Let $P(\zeta, \tau)$ be properly hypoelliptic and of type μ. Then, given $A > 0$, there is $B > 0$ such that $\mathcal{A}$ contains the following set*

$$\{\zeta \in \mathbb{C}^n : |\operatorname{Im} \zeta| \leq A,\ |\operatorname{Re} \zeta| \geq B\}. \tag{3}$$

2) *Under the same assumptions above, there are constants* $\varrho \geq 1$ *and* $C > 0$ *such that* $\mathcal{A}$ *contains the set*

$$\{\zeta \in \mathbb{C}^n : |\operatorname{Re} \zeta| \geq C(1 + |\operatorname{Im} \zeta|^{\varrho}\}. \tag{4}$$

The proof uses the fact that the hypoellipticity condition (H_1) is equivalent to the following conditions ([4] and [6]):

(H_2) *given* $A > 0$ *there is* $B > 0$ *such that* $P(\zeta) \neq 0$ *whenever* $|\operatorname{Im} \zeta| \leq A$ *and* $|\operatorname{Re} \zeta| \geq B$.

(H_3) *there are constants* $C > 0$ *and* $\varrho \geq 1$ *such that* $\zeta \in \mathbb{C}^n$ *and* $|\operatorname{Re} \zeta| \geq$ $\geq C(1 + |\operatorname{Im} \zeta|^{\varrho})$ *imply* $P(\zeta) \neq 0$.

The proof of 1) (resp. 2)) above goes as follows. If $\zeta \in \mathbb{C}^n$ belongs to (3) (resp. (4)) then, for every real number τ, we have

$$|\operatorname{Im}(\zeta, \tau)| = |\operatorname{Im} \zeta| \leq A \text{ and } |\operatorname{Re}(\zeta, \tau)| \geq |\operatorname{Re} \zeta| \geq B$$

$$(\text{resp. } |\operatorname{Re}(\zeta, \tau)| \geq |\operatorname{Re} \zeta| \geq C(1 + |\operatorname{Im} \zeta|^{\varrho}) = C(1 + |\operatorname{Im}(\zeta, \tau)|^{\varrho})),$$

hence by (H_2) (resp. (H_3)) we must have $P(\zeta, \tau) \neq 0$. Therefore the equation in τ $P(\zeta, \tau) = 0$ has *no* real root when ζ belongs to (3) (resp. (4)). It then follows that on every connected component of the set (3) (resp. (4)) the number of τ roots of $P(\zeta, \tau) = 0$ with positive imaginary part is constant. Since every connected component contains real vectors $\xi \in \mathbb{R}^n$ with $|\xi|$ sufficiently large and for those the number of roots with positive imaginary part is μ, the assertion 1) (resp. 2)) follows at once, q. e. d.

2. Regular hypoelliptic boundary value problems.

Denote by $\mathbb{R}^{n+1}_+$ the set of all $(x, t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1}$ such that $t > 0$ and by $\overline{\mathbb{R}^{n+1}_+}$ the closure of $\mathbb{R}^{n+1}_+$ in $\mathbb{R}^{n+1}$. Let

$$\mathbb{R}^n_0 = \{(x, o) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}.$$

Denote by $C^\infty(\overline{\mathbb{R}^{n+1}_+})$ (resp. $C_c^\infty(\overline{\mathbb{R}^{n+1}_+})$) the set of all C^∞ functions (resp. with compact support) in $\overline{\mathbb{R}^{n+1}_+}$. If Ω is an open subset of $\mathbb{R}^{n+1}_+$ and if ω is a plane piece of the boundary of Ω contained in $\mathbb{R}^n_0$, we have a similar definition for $C^\infty(\Omega \cup \omega)$ and $C_c^\infty(\Omega \cup \omega)$. Let

$$C^\infty(\overline{\mathbb{R}_+^{n+1}}\,;\,\mathbb{R}_0^n,\mu) = C^\infty(\overline{\mathbb{R}_+^{n+1}}) \times C^\infty(\mathbb{R}_0^n) \times \dots \times C^\infty(\mathbb{R}_0^n)$$

with μ copies of $C^\infty(\mathbb{R}_0^n)$ in the last product. Analogously,

$$C_c^\infty(\overline{\mathbb{R}_+^{n+1}}\,;\,\mathbb{R}_0^n,\mu) = C_c^\infty(\overline{\mathbb{R}_+^{n+1}}) \times C_c^\infty(\mathbb{R}_0^n) \times \dots \times C_c^\infty(\mathbb{R}_0^n)$$

with μ copies of $C_c^\infty(\mathbb{R}_0^n)$. In the same way we define $C^\infty(\Omega \cup \omega\,;\,\omega,\mu)$ and $C_c^\infty(\Omega \cup \omega\,;\,\omega,\mu)$. An element $\mathcal{F} \in C^\infty(\Omega \cup \omega\,;\,\omega,\mu)$ is, then, of the form

$$\mathcal{F} = (f\,;\,g_1,\dots,g_\mu)$$

with $f \in C^\infty(\Omega \cup \omega)$ and $g_j \in C^\infty(\omega)$, $1 \le j \le \mu$.

Let $D = (D_1,\dots,D_n)$ where $D_j = \dfrac{1}{i}\dfrac{\partial}{\partial x_j}$, $1 \le j \le n$ and let $D_t = \dfrac{1}{i}\dfrac{\partial}{\partial t}$. Given a set $(P(D,D_t)\,;\,Q_1(D,D_t),\dots,Q_\mu(D,D_t))$ of partial differential operators with constant coefficients, define the operator

$$\mathcal{P} : u \in C^\infty(\Omega \cup \omega) \to (P(D,D_t)\,u\,;\,Q_1(D,D_t)\,u,\dots,Q_\mu(D,D_t)\,\mu) \in C^\infty(\Omega \cup \omega\,;\,\omega,\mu)$$

where $Q_j(D,D_t)\,u$ indicates, here, the restriction of $Q_j(D,D_t)\,u$ to the plane piece of boundary ω.

DEFINITION 1. *Let $P(\zeta,\tau)$ be a polynomial of the form* (1) *and suppose that P is properly hypoelliptic and of type μ. Let $Q_1(\zeta,\tau),\dots,Q_\mu(\zeta,\tau)$ be μ polynomials with constant coefficients. We say that the set of differential operators*

$$(P(D,D_t)\,;\,Q_1(D,D_t),\dots,Q_\mu(D,D_t))$$

defines a regular hypoelliptic boundary problem in $\Omega \cup \omega$ if, given $(f\,;\,g_1,\dots,g_\mu) \in C^\infty(\Omega \cup \omega\,;\,\omega,\mu)$, every $u \in C^k(\Omega \cup \omega)$ [1] *solution of the boundary problem*

$$(5)\qquad \begin{cases} P(D,D_t)\,u = f & \text{in } \Omega \\ \lim\limits_{t \to 0+} Q_\nu(D,D_t)\,u(x,t)\,|_\omega = g_\nu & \text{in } \omega,\ 1 \le \nu \le \mu, \end{cases}$$

belongs to $C^\infty(\Omega \cup \omega)$.

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[1] Here k denotes the maximum order of $P, Q_1,\dots,Q_\mu$.

EXAMPLES. 1) The Dirichlet and Neumann problems are regular elliptic (hence hypoelliptic) problems. More generally, every regular elliptic boundary problem (see, for instance, [1] or [4], Chap. 10) is a regular hypoelliptic problem ([2]).

2) Let

$$P = iD_t + D_x^2 + D_y^2$$

be the *heat operator* in R^3 and consider the following boundary problem

$$(6)\qquad \begin{cases} P\,u\,(t, x, y) = f & \text{in } \mathbb{R}^3_+ \\ \lim\limits_{y \to 0+} u\,(t,x,y) = g & \text{in } \mathbb{R}^2 . \end{cases}$$

(Observe that this is not the Cauchy problem for the heat equation). The chracteristic polynomial

$$P\,(\eta, \xi, \tau) = i\eta + \xi^2 + \tau^2$$

is semi-elliptic, hence hypoelliptic. Moreover, if $(\xi^2 + i\eta)^{\frac{1}{2}}$ denotes the square root with positive real part, for all $(\xi, \eta) \neq 0$, then

$$\tau = i\,(\xi^2 + i\eta)^{\frac{1}{2}}$$

is the root of $P\,(\eta, \xi, \tau)$ with positive imaginary part, for all $(\xi, \eta) \neq 0$. Therefore, P is properly hypoelliptic of type 1. We shall see, in section 4, that (4) defines a regular hypoelliptic boundary value problem in $\mathbb{R}^3_+$.

3. The characteristic function of a hypoelliptic boundary problem.

Let $f_1, \dots, f_\mu$ be μ analytic functions of a complex variable and let

$$k\,(\tau) = \prod_1^\mu\,(\tau - \tau_j)$$

be a polynomial in τ with μ complex roots $\tau_1, \dots, \tau_\mu$ not necessarily distinct. In what follows we are going to allow $\tau_1, \dots, \tau_\mu$ to vary but always belon-

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([2]) In the elliptic case, the solutions of (5) are analytic functions up to the boundary when the data are analytic functions.

ging to some bounded set in $\mathbb{C}$. Define

$$R(k; f_1, \dots, f_\mu) = \frac{\det f_\nu(\tau_j)}{\prod_{k<j} (\tau_j - \tau_k)} \tag{7}$$

when the zeros are distinct and by continuity otherwise.

PROPOSITION. *$R(k; f_1, \dots, f_\mu)$ is an analytic function of the complex variables $(\tau_1, \dots, \tau_\mu)$.*

PROOF. Suppose for a moment that $\tau_1, \dots, \tau_\mu$ are distinct and define the divided differences

$$f(\tau_1, \tau_2) = \frac{f(\tau_1) - f(\tau_2)}{\tau_1 - \tau_2} \tag{8}$$

and

$$f(\tau_1, \dots, \tau_n) = \frac{f(\tau_1, \dots, \tau_{n-1}) - f(\tau_2, \dots, \tau_n)}{\tau_1 - \tau_n}. \tag{9}$$

Clearly $f(\tau_1, \dots, \tau_n)$ is a symmetric function of its variables. Let, now, C be a Jordan curve in the complex plane surrounding all the roots $\tau_1, \dots, \tau_\mu$ and suppose that f is analytic in some neighborhood of the bounded region defined by C. We have

$$f(\tau_1) - f(\tau_2) = \frac{1}{2\pi i} \int_C \frac{\tau_1 - \tau_2}{(z - \tau_1)(z - \tau_2)} f(z)\, dz$$

hence

$$f(\tau_1, \tau_2) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \tau_1)(z - \tau_2)}\, dz. \tag{8}$$

In general

$$f(\tau_1, \dots, \tau_n) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \tau_1) \dots (z - \tau_n)}\, dz. \tag{9'}$$

This formula shows that $f(\tau_1, \dots, \tau_n)$ is an analytic function of all its variables and also that the divided differences are well defined in the case of coinciding zeros. By using (9), it is easy to see that (7) can be written as follows

$$R(k; f_1, \dots, f_\mu) = \det \left(f_\nu(\tau_1, \dots, \tau_j)\right)_{\substack{\nu=1, \dots, \mu \\ j=1, \dots, \mu}} \tag{7'}$$

Therefore $R(k; f_1, \dots, f_\mu)$ is an analytic function of the variables $\tau_1, \dots, \tau_\mu$, q. e. d.

Suppose that $\tau_1 \neq \tau_2$ then (8) can be written as follows

$$(8'') \qquad f(\tau_1, \tau_2) = \int_0^1 f'(t_1 \tau_1 + (1 - t_1) \tau_2) \, dt_1$$

and this formula makes sense even if $\tau_1 = \tau_2$. By an induction argument one can show that

$$(9'') \qquad f(\tau_1, \tau_2, \dots, \tau_n) =$$

$$= \int_0^1 t_1^{n-2} \left\{ \int_0^1 t_2^{n-3} \dots \int_0^1 t_{n-2} \left[\int_0^1 f^{(n-1)} (t_1 (\dots (t_{n-1} \tau_1 + (1 - t_{n-1}) \tau_n) \dots) + \right.\right.$$

$$\left.\left. + (1 - t_1) \tau_{n-1}) \, dt_{n-1} \right] dt_{n-2} \dots \right\} dt_1 .$$

By observing that

$$\int_0^1 t_1^{n-2} \left\{ \int_0^1 t_2^{n-3} \dots \left(\int_0^1 t_{n-2} \, dt_{n-2} \right) \dots dt_2 \right\} dt_1 = \frac{1}{(n-1)!}$$

we get the following inequality

$$|f(\tau_1, \tau_2, \dots, \tau_n)| \leq \frac{1}{(n-1)!} \sup_{z \in K} |f^{(n-1)}(z)|$$

where K denotes the convex hull of the points $\tau_1, \dots, \tau_\mu$. In all this discussion we are supposing that f is an analytic function in some neighborhood of K. Inserting the last inequality in (7′) we get

$$(10) \qquad | R(k; f_1, \dots, f_\mu) | \leq \prod_{\nu=1}^{\mu} \left(\sum_{j=0}^{\mu-1} \sup_K \frac{|f_\nu^{(j)}(z)|}{j!} \right)$$

where K is the convex hull of $\tau_1, \dots, \tau_\mu$.

We now go back to our properly hypoelliptic polynomial (1) of type μ. For every $\zeta \in \mathscr{A}$ let us denote by $\tau_1(\zeta), \dots, \tau_\mu(\zeta)$ the roots of $P(\zeta, \tau) = 0$ with positive imaginary part and set

$$(11) \qquad k_\zeta(\tau) = \prod_{j=1}^{\mu} (\tau - \tau_j(\zeta)) .$$

Given μ polynomials $Q_1(\zeta, \tau), \dots, Q_\mu(\zeta, \tau)$ with constant coefficients, the function

$$C(\zeta) = R(k_\zeta\,;\, Q_1, \dots, Q_\mu) = \frac{\det Q_\nu(\zeta, \tau_j(\zeta))}{\prod\limits_{k<j} (\tau_j(\zeta) - \tau_k(\zeta))} \tag{12}$$

defined on $\mathscr{A}$ is said to be the *characteristic function* of the set $(P\,;\, Q_1, \dots, Q_\mu)$ or the boundary problem defined in $\mathbb{R}^{n+1}_+$ by the partial differential operators $(P(D, D_t)\,;\, Q(D, D_t), \dots, Q_\mu(D, D_t))$.

Since $Q_1, \dots, Q_\mu$ are polynomials in the variables (ζ, τ) then

$$\frac{\det Q_\nu(\zeta, \tau_j)}{\prod\limits_{k<j} (\tau_j - \tau_k)}$$

is a polynomial in the variables $(\zeta_1, \dots, \zeta_n, \tau_1, \dots, \zeta_\mu)$. Moreover, for every $\zeta = (\zeta_1, \dots, \zeta_n)$, it is a symmetric polynomial in $(\tau_1, \dots, \tau_\mu)$. By replacing τ_j by $\tau_j(\zeta)$ a root of $P(\zeta, \tau) = 0$ with positive imaginary part, it follows from a well known theorem in Algebra that the function $C(\zeta)$ is a polynomial in the coefficients of the polynomial k_ζ, for all $\zeta \in \mathscr{A}$. But these coefficients are analytic functions on $\mathscr{A}$ because they are the elementary symmetric functions of $\tau_1(\zeta), \dots, \tau_\mu(\zeta)$ which, when $\zeta \in \mathscr{A}$, are all the roots of $P(\zeta, \tau) = 0$ with positive imaginary part. Therefore, $C(\zeta)$ is an analytic function on $\mathscr{A}$.

4. Some characterizations of regular hypoelliptic boundary problems.

In this section we shall prove a theorem which gives necessary and sufficient conditions in order that a hypoelliptic boundary problem be a regular one.

THEOREM. *The following are equivalent conditions:*

1) $(P(D, D_t)\,;\, Q_1(D, D_t), \dots, Q_\mu(D, D_t))$ *defines a regular hypoelliptic boundary value problems in* $\Omega \cup \omega$;

2) *Every* $u \in C^k(\Omega \cup \omega)$ *solution of the homogeneous boundary value problem*

$$\begin{cases} P(D, D_t)\, u = 0 & \text{in } \Omega \\ Q_\nu(D, D_t)\, u\,|_\omega = 0 & \text{in } \omega, \quad 1 \le \nu \le \mu, \end{cases} \tag{13}$$

is an infinitely differentiable function in $\Omega \cup \omega$.

3) *Let* $C(\zeta)$ *be the characteristic function of the boundary problem and let* $N = \{\zeta \in \mathbb{C}^n : C(\xi) = 0\}$, *then,*

$$\zeta \in \mathcal{A} \cap N, \ |\zeta| \to +\infty \text{ implies } |\operatorname{Im} \zeta| \to +\infty.$$

4) *Given* $A > 0$ *we can find* $B > 0$ *such that* $\zeta \in \mathbb{C}^n$, $|\operatorname{Im}\zeta| \leq A$, $|\operatorname{Re}\zeta| \geq B$ *implies* $\zeta \in \mathcal{A}$ *and* $C(\zeta) \neq 0$.

5) *There are constants* $M > 0$ *and* $\gamma \geq 1$ *such that* $\zeta \in \mathbb{C}^n$, $|\operatorname{Re}\zeta| \geq$ $\geq M(1 + |\operatorname{Im}\zeta|^\gamma)$ *implies* $\zeta \in \mathcal{A}$ *and* $C(\zeta) \neq 0$.

6) *There are distributions* $K(x, t)$, $K_1(x, t), \ldots, K_\mu(x, t)$ *in* $S'(\mathbb{R}^{n+1}_+)$ *such that* $K, K_1, \ldots, K_\mu$ *are infinitely differentiable functions in* $\mathbb{R}^{n+1}_+$ *which can be extended to infinitely differentiable functions in* $\overline{\mathbb{R}^{n+1}_+} - \{0\}$. *If we keep the same notations for the extended functions, then* $K(x, t)$ *satisfies the boundary problem*

$$(14) \qquad \begin{cases} P(D, D_t)\, K = \delta_x \otimes \delta_t - \beta(x) \otimes \delta_t & \text{in } \overline{\mathbb{R}^{n+1}_+} \\ Q_\nu(D, D_t)\, K|_{\mathbb{R}^n_0} = 0, & 1 \leq \nu \leq \mu \end{cases}$$

where $\beta \in \mathcal{S}(\mathbb{R}^n)$ *and every* $K_l(x, t)$ *satisfies the boundary problem*

$$(15) \qquad \begin{cases} P(D, D_t)\, K_l = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ Q_\nu(D, D_t)\, K_l|_{\mathbb{R}^n_0} = \delta_{\nu, l}(\delta_x - \beta(x)), & 1 \leq \nu \leq \mu, \end{cases}$$

where $\delta_{\nu, l}$ *is the Kronecker symbol and* $\beta \in \mathcal{S}(\mathbb{R}^n)$.

7) *There is a continuous linear map* $\mathcal{E} : C^\infty_c(\Omega \cup \omega; \omega, \mu) \to C^\infty(\Omega \cup \omega)$ *such that*

$$\mathcal{P}\mathcal{E}\mathcal{F} = \mathcal{F} - \mathcal{L}\mathcal{F}, \qquad \forall\, \mathcal{F} \in C^\infty_c(\Omega \cup \omega; \omega, \mu),$$

where $\mathcal{L}$ *is a continuous linear map from* $C^\infty_c(\Omega \cup \omega; \omega, \mu)$ *into* $C^\infty(\omega)$. *The map* $\mathcal{E}$ *is said to be a parametrix of the boundary problem.*

PROOF. Condition 1) implies trivially condition 2). 2) $\Longrightarrow$ 3). This implication is proved in Hörmander [5] § 4. One establishes, first the estimate

$$\sum_{|\alpha| \leq k+1} \sup_{x \in \Omega'} |D^\alpha u(x)| \leq C \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u(x)|$$

for all solutions $u \in C^k(\Omega \cup \omega)$ of the homogeneous problem (13), where k denotes the maximum order of $P(D, D_t)$ and of $Q_\nu(D, D_t)$, $1 \leq \nu \leq \mu$, and Ω' is an open subset such that its closure is contained in $\Omega \cup \omega$ but not in Ω. Then, applying the estimate to exponential solutions, i. e. solutions of the form $u(x, t) = e^{i \langle x, \zeta \rangle} v(t)$ of (10) the algebraic condition 3) follows.

3) $\Longrightarrow$ 4). We already know by property 1) of section 1 that given $A > 0$ there is $B' > 0$ such that the set

$$\{\zeta \in \mathbb{C}^n : |\operatorname{Im} \zeta| \leq A, \ |\operatorname{Re} \zeta| \geq B'\}$$

is contained in $\mathcal{A}$. On the other hand, if 3) is satisfied, given $A > 0$ we can find $B'' > 0$ such that

$$\zeta \in \mathbb{C}^n, |\operatorname{Im} \zeta| \leq A \text{ and } |\operatorname{Re} \zeta| \geq B'' \text{ imply } C(\zeta) \neq 0.$$

It then suffices to take $B = \max(B', B'')$.

It is very easy to check that 4) $\Longrightarrow$ 3), hence condition 3) is equivalent to condition 4).

5) $\Longrightarrow$ 3). Indeed, suppose that $\zeta \in \mathcal{A} \cap N$ and that $|\zeta| \to +\infty$. If $|\operatorname{Im} \zeta|$ is bounded by a constant $A > 0$ then $|\operatorname{Re} \zeta|$ must tend to infinity. In this case, we can find $\zeta \in \mathcal{A} \cap N$ (hence $C(\zeta) = 0$) such that

$$|\operatorname{Re} \zeta| \geq M(1 + |\operatorname{Im} \zeta|^\gamma)$$

which contradicts 5).

3) $\Longrightarrow$ 5) Suppose that there is a real number t_0 such that for all $\zeta \in \mathcal{A}$ with $|\operatorname{Re} \zeta| > t_0$ we have $C(\zeta) \neq 0$. By property 2) of section 1, there are constants $C > 0$ and $\varrho > 1$ such that $\mathcal{A}$ contains the set

$$\{\zeta \in \mathbb{C}^n : |\operatorname{Re} \zeta| \geq C(1 + |\operatorname{Im} \zeta|^\varrho)\}.$$

If we take $M = \max(t_0, C)$ and $\gamma = \varrho$ then the set defined in condition 5) is contained in $\mathcal{A}$ and $C(\zeta) \neq 0$ in that set.

Suppose, next, that for every positive real number t there is $\zeta \in \mathcal{A}$ with $|\operatorname{Re} \zeta| > t$ and such that $C(\zeta) = 0$. Define the following function

$$M(t) = \inf\{|\operatorname{Im} \zeta| : \zeta \in \mathcal{A}, \ |\operatorname{Re} \zeta| \geq t \text{ and } C(\zeta) = 0\}.$$

One can see that $M(t)$ is the infimum of all λ such that the following system of equations and inequalities hold:

$$|\operatorname{Re} \zeta| \geq t^2, \ |\operatorname{Im} \zeta|^2 = \lambda^2, \ \lambda > 0, \ P(\xi, \tau) = \Pi(\tau - \tau_j)$$

$$\operatorname{Im} \tau_1 > 0, \dots, \operatorname{Im} \tau_\mu > 0, \ \operatorname{Im} \tau_{\mu+1} < 0, \dots, \operatorname{Im} \tau_o < 0$$

$$k_\zeta(\tau) = \Pi(\tau - \tau_j), \ 0 = R(k_\zeta(\tau); \ Q_1(\zeta, \tau), \dots, Q_\mu(\zeta, \tau)).$$

By using Seidenberg's theorem, Hörmander has shown ([5], Pg. 259) that $M(t)$ is a piecewise algebraic function of t. If condition 3) holds, it implies that $M(t) \to \infty$ as $t \to +\infty$. By using the Puisseux expansion of $M(t)$ at infinity we get:

$$M(t) = c t^\varepsilon (1 + 0(1)) \text{ when } t \to +\infty,$$

with $\varepsilon > 0$ and $c > 0$. But this condition is equivalent to $\dfrac{M(t)}{ct^\varepsilon} \to 1$ as $t \to +\infty$. Hence we get

$$t \leq C \, |M(t)|^{\frac{1}{\varepsilon}} \text{ when } t \text{ is large}$$

Taking $t = |\operatorname{Re} \zeta|$ with $\zeta \in \mathscr{A}$ and $C(\zeta) = 0$ we get

$$|\operatorname{Re} \zeta| \leq C \, |\operatorname{Im} \zeta|^{\frac{1}{\varepsilon}} \text{ when } |\operatorname{Re} \zeta| \text{ is large.}$$

Hence, for every $\zeta \in \mathscr{A}$ with $C(\zeta) = 0$ we have

$$|\operatorname{Re} \zeta| < M(1 + |\operatorname{Im} \zeta|^{\frac{1}{\varepsilon}})$$

where M is a suitable constant. Finally, choosing $\gamma = \max\left(\varrho, \dfrac{1}{\varepsilon}\right)$ we obtain condition 5). Conditions 3), 4) and 5) are, then, all equivalent and in section 5 we shall prove that each of them imply condition 6).

5. The construction of kernels $K, K_1, \dots, K_\mu$.

Suppose that u is a smooth solution in $\overline{\mathbb{R}^{n+1}_+}$ of the boundary problem (3) with $(f; g_1, \dots, g_\mu) \in C_c^\infty(\overline{\mathbb{R}^{n+1}_+}; \mathbb{R}^n, \mu)$. Then, taking partial Fourier transform with respect to the x variable, $\widehat{u}(\xi, t)$ is a solution of the initial value problem

$$(16) \qquad \begin{cases} P(\xi, D_t)\, \widehat{u}(\xi, t) = \widehat{f}(\xi, t) \\ Q_\nu(\xi, D_t)\, \widehat{u}(\xi, 0) = \widehat{g_\nu}(\xi), \quad 1 \leq \nu \leq \mu. \end{cases}$$

The kernel $K(x, t)$ will be defined as an inverse Fourier transform of a solution of the initial value problem derived from (14) while the kernels $K_1(x, t), \dots, K_\mu(x, t)$ will be inverse Fourier transforms of solutions of the initial value problem derived from (15). First of all let

$$G_0(\xi, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{it\tau}}{P(\xi, \tau)} d\tau$$

the integral being absolutely convergent if σ, the degree of $P(\xi, \tau)$ in τ, is ≥ 2 and convergent if $\sigma = 1$. We clearly have

$$P(\xi, D_t)\, G_0(\xi, t) = \delta_t,$$

i. e., $G_0(\xi, t)$ is a fundamental solution of $P(\xi, D_t)$.

Next, for every $1 \leq \nu \leq \mu$, define

$$H_\nu(\xi, t) = \frac{R(k_\xi;\, Q_1(\xi, \tau(\xi)), \dots, e^{it\tau(\xi)}, \dots, Q_\mu(\xi, \tau(\xi)))}{C(\xi)} \tag{17}$$

where $C(\xi)$ is the characteristic function of the boundary problem and $\tau(\xi)$ indicates any of the μ roots of (2) with positive imaginary part and the exponential factor occurs in the ν th place. It follows from condition 5) that for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq M$, $C(\xi) \neq 0$, hence $H_\nu(\xi, t)$ is well defined. Moreover, $H_\nu(\xi, t)$ is the unique solution of the initial value problem

$$\begin{cases} P(\xi, D_t)\, H_\nu(\xi, t) = 0 \\ Q_l(\xi, D_t)\, H_\nu(\xi, 0) = \delta_{l,\nu}, \quad 1 \leq l \leq \mu. \end{cases} \tag{18}$$

We modify, now, the fundamental solution $G_0(\xi, t)$ as follows. For all $|\xi| \geq M$, set

$$G(\xi, t) = G_0(\xi, t) - \sum_{\nu=1}^{\mu} (Q_\nu(\xi, D_t)\, G_0)(\xi, 0)\, H_\nu(\xi, t). \tag{19}$$

The function $G(\xi, t)$ is a solution of the following initial value problem

$$\begin{cases} P(\xi, D_t)\, G(\xi, t) = \delta_t \\ Q_\nu(\xi, D_t)\, G(\xi, 0) = 0, \quad 1 \leq \nu \leq \mu, \end{cases} \tag{20}$$

for all $|\xi| > M$.

Let $\chi(\xi) \in C_c^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 1$, for all $|\xi| \leq M$ and $\chi(\xi) = 0$, for all $|\xi| \geq M + 1$. We shall see later on that for all $t \geq 0$,

$$(1 - \chi(\xi))\, G(\xi, t) \quad \text{and} \quad (1 - \chi(\xi))\, H_\nu(\xi, t), \quad 1 \leq \nu \leq \mu, \tag{21}$$

are tempered distributions in $\mathbb{R}^n$. Thus we can define

$$K(x, t) = \mathcal{F}_\xi^{-1}((1 - \chi(\xi))\, G(\xi, t)) \tag{22}$$

and

$$K_\nu(x, t) = \mathcal{F}_\xi^{-1}((1 - \chi(\xi))\, H_\nu(\xi, t)), \quad 1 \leq \nu \leq \mu. \tag{23}$$

Operating formally we have:

$$P(D, D_t)\, K = \mathcal{F}_\xi^{-1}((1 - \chi(\xi))\, P(\xi, D_t)\, G(\xi, t)) =$$

$$= \mathcal{F}_\xi^{-1}((1 - \chi(\xi)) \otimes \delta_t) = \delta_x \otimes \delta_t - \beta(x) \otimes \delta_t$$

and also

$$\lim_{t \to 0} Q_\nu(D, D_t)\, K(x, t) = \mathcal{F}_\xi^{-1}((1 - \chi(\xi))\, P(\xi, D_t)\, G(\xi, 0)) = 0, \quad 1 \leq \nu \leq \mu.$$

In the same way we get equations (15).

In order to justify these formal calculations we shall prove that (21) are tempered distributions and that the distributions $K(x, t)$, $1 \leq \nu \leq \mu$, are C^∞ functions in $\mathbb{R}_+^{n+1}$ that can be extended to C^∞ functions in $\overline{\mathbb{R}_+^{n+1}} - \{0\}$. The proof will be based on sharp estimates on the derivatives with respect to t of $G(\xi, t)$ and of $H_\nu(\xi, t)$, $1 \leq \nu \leq \mu$, that were established by Hörmander in his paper [5], namely

Lemma 1. *Suppose that condition* 5) *holds and let* $D = \{\zeta \in \mathbb{C}^n : |\operatorname{Re} \zeta| \geq M(1 + |\operatorname{Im} \zeta|^\gamma)\}$. *Then, the functions* $G^{(j)}(\zeta, t) = D_t^j\, G(\zeta, t)$ *and* $H_\nu^{(j)}(\zeta, t) = D_t^j\, H_\nu(\zeta, t)$ *are analytic functions of* ζ *in* D *and there are constants* M', c *and* γ' *such that*

$$|G^{(j)}(\zeta, t)| \leq M' |\zeta|^{\gamma'} e^{-ct|\zeta|^{\frac{1}{\gamma}}}, \quad \zeta \in D, \quad t \geq 0 \tag{24}$$

$$|H_\nu^{(j)}(\zeta, t)| \leq M' |\zeta|^{\gamma'} e^{-ct|\zeta|^{\frac{1}{\gamma}}}, \quad \zeta \in D, \quad t \geq 0. \tag{25}$$

The proof of this lemma can be found in Hörmander's paper [5] at pg. 253. The estimates (24) and (25) show that, for every $t \geq 0$, the functions

$$(1 - \chi(\xi))\, G(\xi, t) \quad \text{and} \quad (1 - \chi(\xi))\, H_\nu(\xi, t), \qquad 1 \leq \nu \leq \mu,$$

define tempered distributions on $\mathbb{R}_+^{n+1}$, hence (22) and (23) are tempered distributions on $\mathbb{R}_+^{n+1}$.

We shall give here the proof of lemma 1 because, later on, we shall use some of the estimates appearing in the proof. First of all we quote the following result: *if condition* 5) *holds, there are constants* M_1 *and* c_1 *such that*

$$\frac{1}{|C(\zeta)|} \leq M_1 |\zeta|^{c_1}, \quad \forall \zeta \in D, \tag{26}$$

where D *is the set defined in lemma* 1. (See Hörmander [5], lemma 5.4 at pg. 259).

By property 2) of section 1, there are constants $\varrho \geq 1$ and $C > 0$ such that

$$|\operatorname{Re}(\zeta, \tau)| \geq C(1 + |\operatorname{Im}(\zeta, \tau)|^{\varrho}) \quad \text{implies} \quad P(\zeta, \tau) \neq 0.$$

If τ is a complex zero of $P(\zeta, \tau) = 0$, we have

$$|\operatorname{Re}\zeta| \leq |\operatorname{Re}(\zeta, \tau)| \leq C(1 + |\operatorname{Im}(\zeta, \tau)|^{\varrho}) \leq C_1(1 + |\operatorname{Im}\zeta|^{\varrho} + |\operatorname{Im}\tau|^{\varrho})$$

hence

$$C_1 |\operatorname{Im}\tau|^{\varrho} \geq |\operatorname{Re}\zeta| - C_1(1 + |\operatorname{Im}\zeta|^{\varrho}). \tag{27}$$

We may assume that for all $\zeta \in D$, $|\operatorname{Re}\zeta| \geq 2C_1(1 + |\operatorname{Im}\zeta|^{\varrho})$ so that by (27) and because $\varrho \leq \gamma$ we have

$$P(\zeta, \tau) = 0 \text{ and } \zeta \in D \text{ imply } |\operatorname{Im}\tau| \geq C_2 |\operatorname{Re}\zeta|^{\frac{1}{\gamma}}.$$

The last inequality can be replaced by

$$|\operatorname{Im}\tau| \geq C_3 |\zeta|^{\frac{1}{\gamma}} \tag{28}$$

because $\gamma \geq 1$ and in D we can estimate $|\operatorname{Im}\zeta|$ by $|\operatorname{Re}\zeta|$. On the other hand, it is easy to see that all roots τ of $P(\zeta, \tau) = 0$ satisfy the inequality

$$|\tau| \leq c(|\zeta|^{d} + 1) \tag{29}$$

where c and d are suitable constants. Therefore, the inequalities (28) and (29) show that the convex hull K of the zeros of $P(\zeta, \tau) = 0$ with positive imaginary part is contained in the circle $|\tau| \leq C(|\zeta|^{d} + 1)$ and in the half plane $|\operatorname{Im}\tau| \geq C_3 |\zeta|^{\frac{1}{\gamma}}$.

Now

$$H_{\nu}^{(j)}(\zeta, t) = \frac{R(k_{\zeta};\, Q_1, \ldots, (i\tau)^{j} e^{it\tau}, \ldots, Q_{\mu})}{C(\zeta)}$$

thus, by using (10), (26) and noticing that the polynomials $Q_\nu(\zeta, \tau)$ can be estimated by a power of $|\zeta|$ while $|e^{it\tau}| \leq e^{-tC_3|\zeta|^{\frac{1}{\gamma}}}$ we obtain (25).

Finally, (24) follows from (25) plus the inequality

$$|G_0^{(j)}(\zeta, t)| \leq 2^{\sigma+j}(c(|\zeta|^d + 1))^j e^{-c_4|\zeta|^{\frac{1}{\gamma}}t}, \quad \forall \zeta \in D \tag{30}$$

(see Hörmander [5], pg. 260), q. e. d.

Lemma 2. *The distributions $K(x, t)$ and $K_\nu(x, t)$, $1 \leq \nu \leq \mu$, are C^∞ functions in $\mathbb{R}_+^{n+1}$.*

Proof. Indeed they are all solutions of $P(D, D_t)u = 0$ in $\mathbb{R}_+^{n+1}$ and P is a hypoelliptic partial operator, q. e. d..

Lemma 3. *The distributions $K(x, 0) = \mathcal{F}_\xi^{-1}(1 - \chi(\xi))\, G(\xi, 0))$ and $K_\nu(x, 0) = \mathcal{F}_\xi^{-1}((1 - \chi(\xi))\, H_\nu(\xi, 0))$, $1 \leq \nu \leq \mu$, are C^∞ functions in $\mathbb{R}^n - \{0\}$.*

Proof. Denoting by $L(\zeta)$ any one of the functions $G(\zeta, 0)$ and $H_\nu(\zeta, 0)$, $1 \leq \nu \leq \mu$, then, by lemma 1, $L(\zeta)$ is an analytic function defined on D and satisfying the inequality

$$|L(\zeta)| \leq M'|\zeta|^{\gamma'}, \quad \forall \zeta \in D.$$

Using Cauchy's integral formula for analytic functions of several variables and Cauchy's inequalities it is easy to show that there is a constant C such that

$$|D_\xi^\beta L(\xi)| \leq C|\xi|^{\gamma' - \frac{|\beta|}{\gamma}}$$

when $\xi \in \mathbb{R}^n$ and $|\xi| \geq M + 1$. This will imply, recalling the definition of $\chi(\xi)$, that

$$|x|^k K(x, 0) \quad \text{and} \quad |x|^k K_\nu(x, 0), \qquad 1 \leq \nu \leq \mu,$$

are continuous functions of x, provided that k is a number sufficiently large. The same argument applied to any derivative of $K(x, 0)$ and $K_\nu(x, 0)$, $1 \leq \nu \leq \mu$, will show that $K(x, 0)$ and $K_\nu(x, 0)$, $1 \leq \nu \leq \mu$, are C^∞ functions in $\mathbb{R}^n - \{0\}$, q. e. d..

Lemma 4. *For every integer j, $H_\nu^{(j)}(\xi, t) \to H_\nu^{(j)}(\xi, 0)$, $1 \leq \nu \leq \mu$, uniformly on every compact subset of $\mathbb{R}^n$, when $t \to 0^+$.*

PROOF. By using the inequality (11), the fact that $Q_l(\xi, \tau)$ are polynomials and that the roots $\tau(\xi)$ can be estimated by (29) one can easily see that $| H_\nu^{(j)}(\xi, t) - H_\nu^{(j)}(\xi, 0) |$ can be estimated by a constant times a finite sum of terms of the form

$$|\xi|^A \sup_{\tau \in K} | e^{it\tau(\xi)} - 1 | \tag{31}$$

and

$$|\xi|^A \, |(it)^j| \sup e^{-t \operatorname{Im} \tau(\xi)} \leq |\xi|^A \, |(it)^j| \, e^{-t |\xi|^{\frac{1}{\gamma}}} \tag{32}$$

It is clear that the right side of (32) converges to zero uniformly when ξ belongs to a compact subset of $\mathbb{R}^n$. On the other hand,

$$e^{it\tau(\xi)} - 1 = \int_0^t i\tau(\xi)\, e^{is\tau(\xi)}\, ds,$$

whence

$$| e^{it\tau(\xi)} - 1 | \leq \int_0^t | \tau(\xi) | \, e^{-s \operatorname{Im} \tau(\xi)}\, ds \leq t\, | \tau(\xi) | \leq ct\, (|\xi|^d + 1)$$

by (29). Therefore, (31) converges to zero uniformly when ξ belongs to a compact subset of $\mathbb{R}^n$, q. e. d..

LEMMA 5. *For every n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, for every integer j, $D_x^\alpha D_t^j K_\nu(x, t) \to D_x^\alpha D_t^j K_\nu(x, 0)$, uniformly on every compact subset of $\mathbb{R}^n - \{0\}$, when $t \to 0+$.*

PROOF. Let L be a compact subset of $\mathbb{R}^n - \{0\}$ and let $\omega \in C^\infty(\mathbb{R}^n - \{0\})$ be such that $\omega = 1$ on L. By lemmas 2 and 3,

$$[D_x^\alpha D_t^j K_\nu(x, t) - D_x^\alpha D_t^j K_\nu(x, 0)]\, \omega(x) \in S(\mathbb{R}^n), \quad \forall\, t \geq 0.$$

Taking Fourier transform in x, we get:

$$F_t(\xi) = \int_{\mathbb{R}^n} (1 - \chi(\eta))\, (\eta^\alpha H_\nu^{(j)}(\eta, t) - \eta^\alpha H_\nu^{(j)}(\eta, 0))\, \widehat{\omega}(\xi - \eta)\, d\eta$$

and $F_t(\xi) \in S(\mathbb{R}^n)$ for all $t \geq 0$. To complete the proof of the lemma, it suffices to show that $F_t(\xi) \to 0$ in $S(\mathbb{R}^n)$ as $t \to 0+$. If r is any non-negative integer and $\beta = (\beta_1, \dots, \beta_n)$ any n-tuple of non-negative integers let

$$I = (1 + |\xi|^2)^r D_\xi^\beta F_t(\xi) =$$

$$= \int_{|\eta| \geq M} \eta^\alpha (H_\nu^{(j)}(\eta, t) - H_\nu^{(j)}(\eta, 0)) (1 + |\xi|^2)^r D^\beta \widehat{\omega}(\xi - \eta)\, d\eta$$

whence

$$(33) \quad |I| \leq \int_{M \leq |\eta| \leq A} |\eta^\alpha|\, |H_\nu^{(j)}(\eta, t) - H_\nu^{(j)}(\eta, 0)|\, (1 + |\xi|^2)^r\, |D^\beta \widehat{\omega}(\xi - \eta)|\, d\eta +$$

$$+ \int_{|\eta| > A} |\eta^\alpha|\, |H_\nu^{(j)}(\eta, t)|\, (1 + |\xi|^2)^r\, |D^\beta \widehat{\omega}(\xi - \eta)|\, d\eta$$

$$+ \int_{|\eta| > A} |\eta^\alpha|\, |H_\nu^{(j)}(\eta, 0)|\, (1 + |\xi|^2)^r\, |D^\beta \widehat{\omega}(\xi - \eta)|\, d\eta$$

where A is a number to be chosen later. From lemma 4 it follows, once we choose A, that the first integral can be made $< \frac{\varepsilon}{3}$, provided that $t > 0$ be small enough. By using Peetre's inequality [3] and (25), we can estimate the second integral by

$$(34) \qquad C \cdot \int_{|\eta| > A} |\eta|^s (1 + |\eta|^2)^r (1 + |\xi - \eta|^2)^r\, |D^\beta \widehat{\omega}(\xi - \eta)|\, d\eta.$$

Next, by choosing k to be a positive integer so large that the integral

$$\int_{\mathbb{R}^n} |\eta|^{2s} (1 + |\eta|^2)^{2(r-k)}\, d\eta$$

converges, we can estimate (34) by

$$(35) \quad C \cdot \left(\int_{|\eta| > |A|} |\eta|^{2s} (1 + |\eta|^2)^{2(r-k)}\, d\eta \right)^{\frac{1}{2}} \cdot$$

$$\cdot \left(\int_{\mathbb{R}^n} (1 + |\eta|^2)^{2k} (1 + |\xi - \eta|^2)^{2r}\, |D^\beta \widehat{\omega}(\xi - \eta)|^2\, d\eta \right)^{\frac{1}{2}}$$

---

(3) $(1 + |\xi|^2)^t \leq C (1 + |\eta|^2)^t (1 + |\xi - \eta|^2)^{|t|}$ for every real t.

where the last integral converges since it is the convolution product of the tempered distribution $(1+|\eta|^2)^{2k}$ with the function $(1+|\xi|^2)^{2r}|D^\beta\widehat{\omega}(\xi)|^2$ which belongs to S. Therefore, by choosing A sufficiently large we can make (35) and, consequently, the second integral appearing in (33), smaller than $\frac{\varepsilon}{3}$. Similarly, one can show that the third integral appearing in (33) is $<\frac{\varepsilon}{3}$, provided that A be sufficiently large, which completes the proof of the lemma, q. e. d..

We can apply the previous results to prove that for all α and for all j, $D_x^\alpha D_t^j K_\nu(x,t)$ can be extended continuously to $\overline{\mathbb{R}_+^{n+1}}-\{0\}$ and that for $t=0$ the extension coincides with $D_x^\alpha D_t^j K_\nu(x,0)$ in $\mathbb{R}^n-\{0\}$. Indeed, let $(x_0, 0)$ be an arbitrary point in $\mathbb{R}^n-\{0\}$ and write

$$(36)\qquad D_x^\alpha D_t^j K_\nu(x,t) - D_x^\alpha D_t^j K_\nu(x_0,0) =$$

$$[D_x^\alpha D_t^j K_\nu(x,t) - D_x^\alpha D_t^j K_\nu(x,0)] + [D_x^\alpha D_t^j K_\nu(x,0) - D_x^\alpha D_t^j K_\nu(x_0,0)].$$

By lemma 5, the first term between brackets converges to zero, as $t\to 0^+$, uniformly when x belongs to a compact subset in $\mathbb{R}^n-\{0\}$, while the second term converges to zero, as $x\to x_0$, because $D_x^\alpha D_t^j K_\nu(x,0)$ is continuous in $\mathbb{R}^n-\{0\}$ (lemma 3). This shows that the left hand side of (36) converges to zero as $(x,t)\to(x_0,0)$. Therefore, the distributions $K_1(x,t),\dots$ $\dots, K_\mu(x,t)$ satisfy the condition 6) of our theorem. Finally, let us mention that similar versions of lemmas 4 and 5 hold true for $G(\xi,t)$ and $K(x,t)$ thus $K(x,t)$ also satisfy the condition 6) of the theorem.

6. The parametrix of a regular hypoelliptic boundary value problem.

Let $\mathcal{F}=(f; g_1, \dots, g_\mu)\in C_c^\infty(\Omega\cup\omega; \omega, \mu)$ and define the following operator

$$\mathcal{E}\mathcal{F} = K * f + \sum_{\nu=1}^{\mu} K_\nu *' g_\nu$$

where $*$ indicates the convolution in $\mathbb{R}_+^{n+1}$ while $*'$ indicates the convolution with respect to the variable x only. It is easy to see that $\mathcal{E}\mathcal{F}\in C^\infty(\Omega\cup\omega)$. Furthermore, we have the following

Lemma 6. *Let $f(x,t)$ (resp. $g_\nu(x)$) be a continuous function with compact support in $\overline{\mathbb{R}^{n+1}_+}$ (resp. $\mathbb{R}^n$). Then*

$$K * f \qquad (\text{resp. } K_\nu *' g_\nu)$$

is a C^∞ function in the complement in $\overline{\mathbb{R}^{n+1}_+}$ of the singular support of f (resp. g_ν).

Proof. 1) Let (x,t) with $t \neq 0$ be a point in the complement in $\overline{\mathbb{R}^{n+1}_+}$ of sing supp f and let ω be a relatively compact open neighborhood of (x,t) such that $\overline{\omega} \cap \text{sing supp} f = \emptyset$. Let $\alpha \in C_c^\infty(\mathbb{R}^{n+1}_+)$ be such that $\alpha = 1$ on $\overline{\omega}$ and $\text{supp}\, \alpha \cap \text{sing supp} f = \emptyset$. We can write

$$K * f = K * (\alpha f) + K * (1-\alpha) f.$$

Since αf is a C^∞ function with compact support, then $K*(\alpha f)$ is a C^∞ function everywhere. Since (x,t) does not belong to the support L of $(1-\alpha)f$, the origin does not belong to the set $M = \{(x-y, t-s) : (y,s) \in L\}$ and on $M \cap \overline{\mathbb{R}^{n+1}_+}$, K is a C^∞ function. Therefore

$$(K * (1-\alpha f))(x,t) = \iint_K K(x-y, t-s)(1-\alpha(y,s)) f(y,s)\, dy\, ds$$

is C^∞ at (x,t).

2) The same proof applies to a point $(x,0)$ with $x \neq 0$.

3) Finally, using lemmas 2, 3 and 5 one can see, immediately, that for all α and all j, $(D_x^\alpha D_t^j K * f)(x,t)$ converges to $(D_x^\alpha D_t^j K * f)(x_0, 0)$ as $(x,t) \to (x_0, 0)$, $x_0 \neq 0$. Therefore $K*f$ is a C^∞ function in the complement in $\overline{\mathbb{R}^{n+1}_+}$ of sing supp f. A similar proof applies to g_ν and $K_\nu *' g_\nu$, q. e. d..

By using equations (14) and (15) one can easily see that the operator $\mathcal{E}$ above defined satisfies the relation

$$\mathcal{P}\mathcal{E}\mathcal{F} = \mathcal{F} - \mathcal{L}\mathcal{F}$$

where

$$\mathcal{L}\mathcal{F} = \beta(x) *' \left(f(x,0) - \sum_{\nu=1}^{\mu} g_\nu(x) \right).$$

The operator $\mathcal{E} : C_c^\infty(\Omega \cup \omega; \omega, \mu) \to C^\infty(\Omega \cup \omega)$ is said to be a parametrix of the regular hypoelliptic boundary value problem.

We are now in a position of completing the proof of our theorem. Let $u \in C^k(\Omega \cup \omega)$ be a solution of (5), i. e. $\mathcal{P}u = \mathcal{F}$ with $\mathcal{F} = (f; g_1, \ldots, g_\mu) \in$

$\in C^\infty(\Omega \cup \omega; \omega, \mu)$. We want to prove that $u \in C^\infty(\Omega \cup \omega)$. Let Ω' be an open subset of $\mathbb{R}^{n+1}_+$, let $\omega' \subset \mathbb{R}^n_0$ be the plane piece of its boundary and suppose that the closure of $\Omega' \cup \omega'$ is compact and contained in $\Omega \cup \omega$. Let $\alpha \in C_c^\infty(\overline{\mathbb{R}^{n+1}_+})$ with compact support contained in $\Omega \cup \omega$ and such that α is equal to one on the closure of $\Omega' \cup \omega'$. From our assumptions it follows that

$$\mathcal{P}(\alpha u) = \mathcal{G} = (g\,;\, h_1, \dots, h_\mu)$$

with $\operatorname{supp} g \subset \Omega \cup \omega$, $\operatorname{supp} h_\nu \subset \omega$ and $\mathcal{G} \in C^\infty(\Omega' \cup \omega'; \omega', \mu)$. By lemma 6, it follows that $\mathcal{E}\mathcal{P}(\alpha u) \in C^\infty(\Omega' \cup \omega')$.

On the other hand,

$$\widehat{\mathcal{E}\mathcal{P}(\alpha u)} = (1 - \chi(\xi))\, G(\xi, t)\, \hat{g}(\xi, t) + \sum_{\nu=1}^{\mu} (1 - \chi(\xi))\, H_\nu(\xi, t)\, \hat{h}_\nu(\xi)$$
$$= (1 - \chi(\xi))\, \widehat{\alpha u}(\xi, t)$$

because $\widehat{\alpha u}$ is a solution of the initial value problem (16) and we have uniqueness when $|\xi|$ is sufficiently large. The last relation implies that

$$\mathcal{E}\mathcal{P}(\alpha u) = (\alpha u) - \beta *' (\alpha u).$$

Since $\mathcal{E}\mathcal{P}(\alpha u) \in C^\infty(\Omega' \cup \omega')$ and $\beta *' (\alpha u)$ is C^∞ with respect to the tangential variable and is C^k with respect to the transversal variable t, the last equation implies that on $\Omega' \cup \omega'$, the function $u(x, t)$ is infinitely differentiable in x and k times differentiable in t. But

$$P(D, D_t)\, u = D_t^\sigma u + a_1(D)\, D_t^{\sigma-1} u + \dots = f(x, t)$$

with $f \in C^\infty(\Omega \cup \omega)$. By taking derivatives in both sides we are able to conclude that indeed $u \in C^\infty(\Omega' \cup \omega')$, q. e. d..

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