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ON THE CONVERSION OF BINARY ALGEBRAS INTO SEMI-PRIMAL ALGEBRAS

D. JAMES SAMUELSON

As a generalization of the concept of functional completeness, Foster and Pixley [4] introduced the notion of semi-primality and developed a structure theory in this extended realm which subsumed that obtained in the primal case [1; 2]. In order to enrich the scope of applicability of this extended theory, the construction of classes of semi-primal algebras of as comprehensive a nature as possible is fundamental. Recent results along these lines have been obtained for the important subclass of subprimal algebras. The following theorem was proved by Moore and Yaqub [5]: suppose $(B; \times)$ is a finite, associative binary algebra which contains a nilpotent element $\eta \neq 0$ and an invertible element $z \neq 1$; then there exists a permutation, $^{\cap}$, on B such that $(B; \times, ^{\cap})$ is a regular subprimal algebra. In the present communication we obtain a generalization of this result: the associativity requirement is dropped and the nilpotent-unit condition is replaced by a weaker power assumption on the elements of B. Moreover, it is shown that every finite binary algebra can be converted into a singular subprimal algebra by the addition of a single binary operation to the species.

1. Preliminaries.

We collect several of the concepts which will be used subsequently into

DEFINITION 1.1. Let $\mathcal{U} = (A; (f_0, f_1, \ldots, f_\alpha, \ldots)_{\alpha<\beta})$ be an algebra of species $S = (r_0, r_1, \ldots, r_\alpha, \ldots)_{\alpha<\beta}$ where each primitive operation f_α is a mapping $A^{r_\alpha} \rightarrow A$ of finitary rank $r_\alpha \geq 0$. If $r_\alpha = 0$ we set $A^{r_\alpha} = \{\emptyset\}$ and call f_α a nullary operation.

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(1⁰) An *A-function* is any set theoretic mapping $f(\xi_0, \dots, \xi_{n-1})$ of A^n into A for any integer $n \geq 0$.

(2⁰) If $(\bar{f}_0, \bar{f}_1, \dots, \bar{f}_\gamma, \dots)_{\gamma<\delta}$ is a sequence of A-functions of species $\bar{S}$, $\beta \leq \delta$, and $\bar{f}_\gamma = f_\gamma$ whenever $\gamma < \beta$, then the algebra $\bar{\mathcal{U}} = (A; (\bar{f}_0, \bar{f}_1, \dots$ $\dots, \bar{f}_\gamma, \dots)_{\gamma<\delta})$ is called a *conversion* of $\mathcal{U}$ from species S to $\bar{S}$.

(3⁰) An *S expression* is any indeterminate symbol $\xi_0, \xi_1, \dots$ or any formal composition of these indeterminate symbols via the primitive operation symbols f_α of $\mathcal{U}$.

(4⁰) We say an A-function $f(\xi_0, \dots, \xi_{n-1})$ is *S-expressible* provided there exists an *S-expression* $\Phi(\xi_0, \dots, \xi_{n-1})$ such that $\Phi(\xi_0, \dots, \xi_{n-1}) = f(\xi_0, \dots, \xi_{n-1})$ for all $\xi_0, \dots, \xi_{n-1}$ in A. In particular, an element a in A is said to be S-expressible if there exists a unary S-expression $\Lambda_a(\xi_0)$ such that $\Lambda_a(\xi_0) = a$ for all ξ_0 in A.

(5⁰) An A-function $f(\xi_0, \dots, \xi_{n-1})$ is said to be *conservative* if for each subalgebra $\mathcal{B} = (B; \Omega)$ of $\mathcal{U}$ and sequence of elements $b_0, \dots, b_{n-1}$ of B, it follows that $f(b_0, \dots, b_{n-1}) \in B$ also.

(6⁰) The algebra $\mathcal{U}$ is said to be *primal* (respectively, *semi-primal*) if A is a finite set of at least two elements and every A-function (respectively, every conservative A-function) is *S-expressible.*

(7⁰) We say that $\mathcal{U}$ is a *subprimal* algebra if it is a semiprimal algebra which possesses exactly one proper ($\neq \mathcal{U}$) subalgebra $\mathcal{U}^* = (A^*; \Omega)$ called the *core* of $\mathcal{U}$. If A^* contains at least two elements, $\mathcal{U}$ is said to be a *regular subprimal*; otherwise, it is called a *singular subprimal.*

(8⁰) If $\mathcal{U}$ is subprimal and $a \in A \setminus A^*$ then a is said to be *ex expressible* provided there exists an S-expression $\Gamma_a(\xi_0)$ such that $\Gamma_a(\xi_0) = a$ for all ξ_0 in $A \setminus A^*$.

(9⁰) The algebra $\mathcal{U}$ is said to possess a *frame* $[0, 1, \times, {}^\cap]$ if there exists elements $0, 1$ in A $(0 \neq 1)$ and A-functions $\times$ (binary) and ${}^\cap$ (unary) such that

(i) $0 \times \xi_0 = \xi_0 \times 0 = 0,\ 1 \times \xi_0 = \xi_0 \times 1 = \xi_0$ (all ξ_0 in A);

(ii) $\xi_0 \times \xi_1$ and $\xi_0^\cap$ are S-expressible;

(iii) $\xi_0^\cap$ is a permutation of A with $0^\cap = 1$;

(iv) 0 and 1 are S-expressible.

(10⁰) We say $\mathcal{U}$ possesses a *singular coupling* $[0, \times, T; 1, 1^0]$ if it possesses elements $0, 1$ with $0 \neq 1$ and two binary A-functions $\times, T$ such that (i) above holds in addition to

(v) 0 is S-expressible;

(vi) $\xi_0 \times \xi_1$ and $\xi_0\, T\, \xi_1$ are S expressible;

(vii) $0\, T\, \xi_0 = \xi_0\, T\, 0 = \xi_0$ (all ξ_0 in A);

(viii) there exists 1^0 in A such that $1\, T\, 1^0 = 1^0\, T\, 1 = 0$.

(11^0) If $\mathcal{U}$ contains distinct elements $0, 1$ satisfying (i) above, then the *characteristic function* of an element a in A, denoted by $\delta_a(\xi_0)$, is the unary A-function satisfying

$$\delta_a(\xi_0) = \begin{cases} 1 \text{ if } \xi_0 = a \\ 0 \text{ if } \xi_0 \neq a. \end{cases}$$

The following results from the literature are important in the sequel. For any set A we use $|A|$ to denote the cardinality of A.

THEOREM 1.1. (Foster and Pixley [4]). *An algebra $\mathcal{U} = (A; \Omega)$ of species S is a regular subprimal if and only if*

(*a*) $3 \leq |A| < \infty$;

(*b*) *$\mathcal{U}$ possesses a unique proper ($\neq \mathcal{U}$) subalgebra $\mathcal{U}^* = (A^*; \Omega)$ and $|A^*| \geq 2$;*

(*c*) *$\mathcal{U}$ possesses a frame;*

(*d*) *for each $a \in A$, the characteristic function $\delta_a(\xi_0)$ is S-expressible;*

(*e*) *there exists an $a \in A \setminus A^*$ which is ex-expressible.*

THEOREM 1.2. (Foster and Pixley [4]). *An algebra $\mathcal{U} = (A; \Omega)$ of species S is a singular subprimal if and only if*

(*a*) $2 \leq |A| < \infty$;

(*b*) *$\mathcal{U}$ possesses a unique proper ($\neq \mathcal{U}$) subalgebra $\mathcal{U}^* = (A^*; \Omega)$ and $|A^*| = 1$;*

(*c*) *$\mathcal{U}$ possesses a singular coupling;*

(*d*) *for each $a \notin A^*$, the characteristic function $\delta_a(\xi_0)$ is S-expressible;*

(*e*) *there exists an element $a \in A \setminus A^*$ which is ex-expressible.*

DEFINITION 1.2. A *binary algebra* is an algebra $\mathcal{B} = (B; \times)$ of species $S = (2)$ which contains elements $0, 1$ $(0 \neq 1)$ such that

$$0 \times \xi_0 = \xi_0 \times 0 = 0, 1 \times \xi_0 = \xi_0 \times 1 = \xi_0 \quad (\text{all } \xi_0 \text{ in } B).$$

The element 0 is called the *null* of $\mathcal{B}$; the element 1 is called the *identity*.

THEOREM 1.3. (Foster [3]). *Let $\mathcal{B} = (B; \times)$ be a finite binary algebra with null 0 and identity 1. Then there exists a cyclic permutation, $^\cap$, on B such that $(B; \times, ^\cap)$ is primal. Moreover, if $\mathcal{B} \not\sim \mathcal{G}_2 \times \mathcal{G}_2$ (where $\mathcal{G}_2$ is the two-element binary algebra), then $^\cap$ can further be chosen to satisfy $0^\cap = 1$.*

THEOREM 1.4. (Moore and Yaqub [5]). *Suppose $(B; \times)$ is a finite binary algebra with null 0 and identity 1 in which $|B| \geq 3$. Let $B^* = \{0, 1\}$ and* *let* $\cap$ *be a permutation on* B *of the form* $\cap = (0, 1)(b_1, \ldots, b_m)$ *where* $B = \{0, 1, b_1, \ldots, b_m\}$. *Suppose* $(B^*; \times, \cap)$ *is the unique proper subalgebra of* $(B; \times, \cap)$. *If for some* $b_i \in B \setminus B^*$, *the characteristic function* $\delta_{b_1}(\xi_0)$ *is* $(\times, \cap)$-*expressible, then* $(B; \times, \cap)$ *is a regular subprimal algebra with core* $(B^*; \times, \cap)$.

In order to simplify the work of § 2 and § 3 we introduce the convenient.

NOTATION 1.1. If $(B; \Omega)$ is an algebra, f is a unary B-function, and s is a positive integer, we define

$$f^{(s)}(\xi_0) = f(\ldots f(f(\xi_0))\ldots), \quad s \text{ iterations.}$$

In particular, if $\xi_0^{\cap}$ is a permutation on B, we define

$$\xi_0^{\cap_s} = (\ldots(\xi_0^{\cap})^{\cap}\ldots)^{\cap}, \quad s \text{ iterations};$$

$\xi_0^{\cup_s}$ is defined similarly, where $\xi_0^{\cup}$ denotes the inverse of $\xi_0^{\cap}$.

Observe that if $\xi_0^{\cap}$ is a permutation on a finite set B and $\xi_0^{\cup}$ is its inverse, then there exists an integer s such that $\xi_0^{\cap_s} = \xi_0^{\cup}$. Hence, any $(\cap, \cup)$-expression is simply a $(\cap)$-expression.

2. Regular Subprimal Conversion.

From Theorem 1.1 we see that with each regular subprimal algebra $\mathcal{B} = (B; \Omega)$ is associated a frame $[0, 1, \times, \cap]$ and hence a finite binary algebra $(B; \times)$ having 0 as null and 1 as identity. It seems interesting, therefore, to pose the following converse problem: for what type of finite binary algebra $(B; \times)$ with null 0 and identity 1 is it possible to convert $(B; \times)$ into a regular subprimal algebra $(B; \times, \cap)$ of species $S = (2, 1)$ for which $[0, 1, \times, \cap]$ is a frame? That a solution is not universally possible is a consequence of

THEOREM 2.1. *For each integer* $n \geq 3$ *there exists a binary algebra* $(B; \times)$ *of order* n *such that no permutation,* $\cap$, *on* B *renders* $(B; \times, \cap)$ *a subprimal algebra.*

PROOF. Let $B = \{0, 1, b_1, \ldots, b_m\}$ be a set of at least three elements. Define a binary operation, $\times$, on B by

$$0 \times b = b \times 0 = 0, \; 1 \times b = b \times 1 = b \text{ (each } b \text{ in } B);$$

$$b_i \times b_j = b_{\min(i,j)} \text{ whenever } 1 \leq i, j \leq m. \tag{2 1}$$

Let $\cap$ be a permutation on B. Then $\cap$ can be written as a product of disjoint cycles. From (2.1) it follows that each such cycle is a subalgebra of $\mathcal{B} = \text{def} = (B; \times, \cap)$. Hence, $\mathcal{B}$ has either zero or at least two proper $(\neq \mathcal{B})$ subalgebras according as $\cap$ is or is not cyclic on B. In either case, it follows that $\mathcal{B}$ is not subprimal.

The next theorem provides a positive result.

THEOREM 2.2. *Let $(B; \times)$ be a finite binary algebra with null 0 and identity 1 where $|B| \geq 4$. Suppose there exists a unary $(\times)$-expression, $p(\xi_0)$, and an element $\beta \in B \setminus \{0, 1\}$ such that*

(a) $p(\beta) = 0$ and

(b) $p(\xi_0)$ is not identically 0 on $B \setminus \{0, 1\}$.

Then there exists a permutation, $\cap$, on B such that $(B; \times, \cap)$ is a regular subprimal algebra with frame $[0, 1, \times, \cap]$.

REMARKS. Note that the operation $\times$ on B is assumed to be neither associative nor commutative and that $p(\xi_0)$ is just a power of ξ_0 under some fixed association. Throughout the proof, $a_1 \cdot a_2$ or juxtaposition $a_1 a_2$ will freely be used in lieu of $a_1 \times a_2$, and whenever no association of the product of three or more terms is explicitly denoted, the product will be assumed to be associated from the left, although in most, if not all, situations, any association of the terms will do, i.e. $a_1 \dots a_n$ will be understood to mean $(\dots(a_1 \times a_2) \times a_3 \dots) \times a_n$.

PROOF. Let $B_0 = B \setminus \{0, 1\}$ and define subsets $N^{[i]}$, $U^{[j]}$ of B_0 $(i, j \geq 1)$ inductively by

$$N^{[1]} = \{b \in B_0 \mid p(b) = 0\}; \quad U^{[1]} = \{b \in B_0 \mid p(b) = 1\};$$

$$N^{[i+1]} = \{b \in B_0 \mid p(b) \in N^{[i]}\} \text{ for } i \geq 1; \tag{2.2}$$

$$U^{[j+1]} = \{b \in B_0 \mid p(b) \in U^{[j]}\} \text{ for } j \geq 1.$$

Since B is finite, there exist values r, s of i, j, respectively, such that $N^{[r]} \neq \emptyset$, $U^{[s]} \neq \emptyset$, $N^{[i]} = \emptyset$ if $i > r$, and $U^{[j]} = \emptyset$ if $j > s$. Thus,

$$B = \{0, 1, \alpha_1, \dots, \alpha_c, \beta_1, \dots, \beta_d, \gamma_1, \dots, \gamma_e\}$$

where $\{\alpha_1, \dots, \alpha_c\} = \bigcup_{i=1}^{r} N^{[i]}$, $\{\beta_1, \dots, \beta_d\} = \bigcup_{j=1}^{s} U^{[j]}$, and $\{\gamma_1, \dots, \gamma_e\} = B_0 \setminus \{\alpha_1, \dots, \alpha_c, \beta_1, \dots, \beta_d\}$. Clearly, $p(\gamma_k) \in \{\gamma_1, \dots, \gamma_e\}$ for $1 \leq k \leq e$. Let (see

Notation 1.1)

$$K(\xi_0) = \text{def} = p^{(w)}(\xi_0), \quad w = \max\{r, s\}. \tag{2.3}$$

From (2.2) it follows immediately that

$$K(\xi_0) = \begin{cases} 0 \text{ if } \xi_0 = 0, \ \alpha_1, \dots, \alpha_c \\ 1 \text{ if } \xi_0 = 1, \ \beta_1, \dots, \beta_d; \end{cases}$$

$$K(\gamma_k) \in \{\gamma_1, \dots, \gamma_e\}, \quad 1 \leq k \leq e.$$

The existence of the permutation, $\cap$, will now be shown by considering several cases. In each case, the regular subprimality of $(B; \times, \cap)$ will follow either from Theorem 1.1 or Theorem 1.4. By assumption, $c \geq 1$.

Case 1 $(c = d = 1, \ e = 0)$. In this instance, $B = \{0, 1, \alpha_1, \beta_1\}$. Since $p(\alpha_1) = 0$ and $p(\beta_1) = 1, \alpha_1^2 \notin \{1, \alpha_1\}$ and $\beta_1^2 \neq \beta_1$. If $\alpha_1^2 = 0$, let

$$\cap = (0, 1, \alpha_1)(\beta_1).$$

Then, $\delta_{\beta_1}(\xi_0) = p(\xi_0) \times p(\xi_0^{\cap})$, the element β_1 is ex-expressed by $\Gamma_{\beta_1}(\xi_0) = \xi_0$, $\{0, 1, \alpha_1\}$ is the unique proper subalgebra of $(B; \times, \cap)$, $\delta_1(\xi_0) = p(\xi_0[p([\delta_{\beta_1}(\xi_0)]^{\cap})])$, $\delta_{\alpha_1}(\xi_0) = \delta_1(\xi_0^{\cap^2})$, $\delta_0(\xi_0) = \delta_1(\xi_0^{\cap})$, and $[0, 1, \times, {}^{\cap}]$ is a frame (see Theorem 1.1).

If $\alpha_1^2 = \beta_1$, let

$$\cap = (0, 1)(\alpha_1, \beta_1).$$

Then, $\delta_{\alpha_1}(\xi_0) = p([\xi_0 p(\xi_0^{\cap})]^2)$ (see Theorem 1.4).

Case 2 $(d \geq 1, \ c + d \geq 3, \ e = 0)$. Let

$$\cap = (0, 1)(\alpha_1, \dots, \alpha_c, \beta_1, \dots, \beta_d).$$

If $d \geq 2$, then $\delta_{\beta_d}(\xi_0) = K(\xi_0) K(\xi_0^{\cup}) \dots K(\xi_0^{\cup^{d-1}})$, whereas, if $c \geq 2$, then $\delta_{\alpha_1}(\xi_0) = [K(\xi_0)]^{\cap} [K(\xi_0^{\cap})]^{\cap} \dots [K(\xi_0^{\cap^{c-1}})]^{\cap}$.

Case 3 $(c \geq 2, d \geq 1, e \geq 1)$. Let

$$\cap = (0, 1)(\alpha_1, \dots, \alpha_c, \beta_1, \dots, \beta_d, \gamma_1, \dots, \gamma_e) \tag{2.4}$$

and $I = \{1, \dots, e\}$. Among the $K(\gamma_k) = \text{def} = \gamma_{t_k}$, $k \in I$, there exists a maximum subscript $t_{k'} \geq t_k$, $k \in I$. Let s_1 be the smallest positive *even* integer such that $[K(\gamma_{k'})]^{\cap s_1} \in \{\alpha_1, \alpha_2\}$. If $g_1(\xi_0) = K([K(\xi_0)]^{\cap s_1})$ and $I_1 = \{k \in I \mid g_1(\gamma_k) \neq 0\}$, then

$$g_1(\xi_0) = \begin{cases} 1 & \text{if } \xi_0 = 1, \beta_1, \dots, \beta_d \\ 0 & \text{if } \xi_0 = 0, \alpha_1, \dots, \alpha_c, \gamma_k \end{cases} \qquad (k \in I \setminus I_1);$$

$$g_1(\gamma_k) \in \{\gamma_1, \dots, \gamma_e\}, \quad k \in I_1 .$$

Among the $g_1(\gamma_k) = \text{def} = \gamma_{v_k}$, $k \in I_1$, there exists a maximum subscript $v_{k''} \geq v_k$, $k \in I_1$. Let s_2 be the smallest positive *even* integer such that $[g_1(\gamma_{k''})]^{\cap s_2} \in \{\alpha_1, \alpha_2\}$. If $g_2(\xi_0) = K([g_1(\xi_0)]^{\cap s_2})$ and $I_2 = \{k \in I_1 \mid g_2(\gamma_k) \neq 0\}$, then

$$g_2(\xi_0) = \begin{cases} 1 & \text{if } \xi_0 = 1, \beta_1, \dots, \beta_d \\ 0 & \text{if } \xi_0 = 0, \alpha_1, \dots, \alpha_c, \gamma_k \end{cases} \qquad (k \in I \setminus I_2);$$

$$g_2(\gamma_k) \in \{\gamma_1, \dots, \gamma_e\}, k \in I_2 .$$

After a finite number of steps, the above process leads to an integer m for which $I_m = \emptyset$. It then follows that $g_m(\xi_0) = \text{def} = K([g_{m-1}(\xi_0)]^{\cap s_m})$ is a $(\times, {}^{\cap})$ expression and

$$g_m(\xi_0) = \begin{cases} 1 & \text{if } \xi_0 = 1, \beta_1, \dots, \beta_d \\ 0 & \text{if } \xi_0 = 0, \alpha_1, \dots, \alpha_c, \gamma_1, \dots, \gamma_e . \end{cases}$$

Thus, $\delta_{\alpha_c}(\xi_0) = [g_m(\xi_0)]^{\cap} [g_m(\xi_0^{\cup})]^{\cap} \dots [g_m(\xi_0^{\cup e+c-1})]^{\cap}$.

Case 4 ($d \geq 2$, $e \geq 1$). Let ${}^{\cap}$ be defined on B as in (2.4). We proceed similarly to Case 3. Among the $K(\gamma_k) = \text{def} = \gamma_{t_k}$, $k \in I = \text{def} = \{1, \dots, e\}$, there exists a minimum subscript $t_{k'} \leq t_k$, $k \in I$. Let s_1 be the smallest positive *even* integer such that $[K(\gamma_{k'})]^{\cup s_1} \in \{\beta_{d-1}, \beta_d\}$. If $g_1(\xi_0) = \text{def} = K([K(\xi_0)]^{\cup s_1})$ and $I_1 = \{k \in I \mid g_1(\gamma_k) \neq 1\}$, then

$$g_1(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0, \alpha_1, \dots, \alpha_c \\ 1 & \text{if } \xi_0 = 1, \beta_1, \dots, \beta_d, \gamma_k \end{cases} \qquad (k \in I \setminus I_1);$$

$$g_1(\gamma_k) \in \{\gamma_1, \dots, \gamma_e\}, k \in I_1 .$$

Among the subscripts of the $g_1(\gamma_k) = \text{def} = \gamma_{v_k}$, $k \in I_1$, there exists a minimum subscript $v_{k''} \leq v_k$, $k \in I_1$. Let s_2 be the smallest positive *even* integer such that $[g_1(\gamma_{k''})]^{\cup s_2} \in \{\beta_{d-1}, \beta_d\}$. If $g_2(\xi_0) = \text{def} = K([g_1(\xi_0)]^{\cup s_2})$ and $I_2 =$

$= \{k \in I_1 \mid g_2(\gamma_k) \neq 1\}$, then

$$g_2(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0, \alpha_1, \dots, \alpha_c \\ 1 & \text{if } \xi_0 = 1, \beta_1, \dots, \beta_d, \gamma_k \end{cases} \qquad (k \in I \setminus I_2);$$

$$g_2(\gamma_k) \in \{\gamma_1, \dots, \gamma_e\},\ k \in I_2 .$$

Continue this process for m steps, until $I_m = \emptyset$. Then $g_m(\xi_0) = \text{def} = K([g_{m-1}(\xi_0)]^{\cup_{s_m}})$ is a $(\times, {}^{\cap})$-expression and

$$g_m(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0, \alpha_1, \dots, \alpha_c \\ 1 & \text{if } \xi_0 = 1, \beta_1, \dots, \beta_d, \gamma_1, \dots, \gamma_e. \end{cases}$$

It follows immediately that $\delta_{\gamma_e}(\xi_0) = g_m(\xi_0)\, g_m(\xi_0^{\cup}) \dots g_m(\xi_0^{\cup_{d+e-1}})$.

Case 5 $(c = d = 1,\ e \geq 1)$.

Case 5(*a*) (K is $1-1$ on $\{\gamma_1, \dots, \gamma_e\}$). By assumption, there exists a positive integer N such that $K^{(N)}(\xi_0)$ is the identity on $\gamma_1, \dots, \gamma_e$. Thus,

$$K^{(N)}(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0, \alpha_1 \\ 1 & \text{if } \xi_0 = 1, \beta_1 \\ \gamma_k & \text{if } \xi_0 = \gamma_k,\ 1 \leq k \leq e. \end{cases}$$

Let ${}^{\cap}$ be defined on B by

$${}^{\cap} = (0, 1)(\alpha_1, \gamma_1, \beta_1, \gamma_2, \dots, \gamma_e).$$

If e is odd, let $g_1(\xi_0) = K^{(N)}([K^{(N)}(\xi_0)]^{\cap_2})$, $g_2(\xi_0) = K^{(N)}([g_1(\xi_0)]^{\cap_2}), \dots,$ and $g_e(\xi_0) = K^{(N)}([g_{e-1}(\xi_0)]^{\cap_2})$. It is easy to check that

$$g_e(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0, \alpha_1, \gamma_1, \dots, \gamma_e \\ 1 & \text{if } \xi_0 = \beta_1, 1. \end{cases}$$

Thus, $\delta_{\gamma_1}(\xi_0) = [g_e(\xi_0)]^{\cap}[g_e(\xi_0^{\cup})]^{\cap} \dots [g_e(\xi_0^{\cup_e})]^{\cap}$.

If e is even, let $g_0(\xi_0) = K^{(N)}([K^{(N)}(\xi_0)]^{\cup})$, $g_1(\xi_0) = K^{(N)}([g_0(\xi_0)]^{\cup}), \dots$ $\dots, g_{e-2}(\xi_0) = K^{(N)}([g_{e-3}(\xi_0)]^{\cup})$. Then,

$$g_{e-2}(\xi_0) = \begin{cases} 1 & \text{if } \xi_0 = 0,\ \alpha_1, \gamma_2, \gamma_4, \dots, \gamma_e \\ 0 & \text{if } \xi_0 = 1,\ \beta_1, \gamma_1, \gamma_3, \dots, \gamma_{e-1}. \end{cases}$$

We have, $\delta_{\gamma_1}(\xi_0) = [g_{e-2}(\xi_0)]^{\cap}[g_{e-2}(\xi_0^{\cap})]^{\cap}$.

Case 5 (*b*) (K is not $1-1$ on $\{\gamma_1, \dots, \gamma_e\}$). In this case there exists a $\gamma_i \notin \text{Range}(K)$. Without loss of generality, assume that $\gamma_1 \notin \text{Range}(K)$ and $K(\gamma_1) = \gamma_2$. Let $^{\cap}$ be defined on B as in (2.4), let $g_0(\xi_0) = K([K(\xi_0)]^{\cup_2})$, let $I = \{2, \dots, e\}$, and let $I_0 = \{k \in I \mid g_0(\gamma_k) \neq 1\}$;

$$g_0(\xi_0) = \begin{cases} 0 & \text{if} \quad \xi_0 = 0, \alpha_1 \\ 1 & \text{if} \quad \xi_0 = 1, \beta_1, \gamma_1, \gamma_k \end{cases} \qquad (k \in I \setminus I_0);$$

$$g_0(\gamma_k) \in \{\gamma_2, \dots, \gamma_e\}, \quad k \in I_0.$$

Among the $g_0(\gamma_k) = \text{def} = \gamma_{t_k}$, $k \in I_0$, there exists a minimal subscript $t_{k'}$. Let s_1 be the smallest positive even integer such that $[g_0(\gamma_{k'})]^{\cup_{s_1}} \in \{\beta_1, \gamma_1\}$. If $g_1(\xi_0) = g_0([g_0(\xi_0)]^{\cup_{s_1}})$ and $I_1 = \{k \in I_0 \mid g_1(\gamma_k) \neq 1\}$, then

$$g_1(\xi_0) = \begin{cases} 0 & \text{if} \quad \xi_0 = 0, \alpha_1 \\ 1 & \text{if} \quad \xi_0 = 1, \beta_1, \gamma_1, \gamma_k \end{cases} \qquad (k \in I \setminus I_1);$$

$$g_1(\gamma_k) \in \{\gamma_2, \dots, \gamma_e\}, \qquad k \in I_1.$$

Continue this process, similarly to Case 4, until $I_m = \emptyset$, defining successively $g_2(\xi_0) = g_0([g_1(\xi_0)]^{\cup_{s_2}}), \dots, g_m(\xi_0) = g_0([g_{m-1}(\xi_0)]^{\cup_{s_m}})$. Since $I_m = \emptyset$, the $(\times, ^{\cap})$-expression g_m satisfies

$$g_m(\xi_0) = \begin{cases} 0 & \text{if} \quad \xi_0 = 0, \alpha_1 \\ 1 & \text{if} \quad \xi_0 = 1, \beta_1, \gamma_1, \dots, \gamma_e. \end{cases}$$

Thus, $\delta_{\gamma_e}(\xi_0) = g_m(\xi_0)\, g_m(\xi_0^{\cup}) \dots g_m(\xi_0^{\cup_e})$.

Case 6 ($c \geq 2$, $d = 0$, $e \geq 1$). Define $^{\cap}$ on B by

$$^{\cap} = (0, 1)(\alpha_1, \dots, \alpha_c, \gamma_1, \dots, \gamma_e). \tag{2.5}$$

Let $g_0(\xi_0) = K([K(\xi_0)]^{\cap})$, $I = \{1, \dots, e\}$, and $I_0 = \{k \in I \mid g_0(\gamma_k) \neq 0\}$. Then,

$$g_0(\xi_0) = \begin{cases} 1 & \text{if} \quad \xi_0 = 0, \alpha_1, \dots, \alpha_c \\ 0 & \text{if} \quad \xi_0 = 1, \gamma_k \end{cases} \qquad (k \in I \setminus I_0);$$

$$g_0(\gamma_k) \in \{\gamma_1, \dots, \gamma_e\}, \qquad k \in I_0.$$

Among the $g_0(\gamma_k) = \text{def} = \gamma_{t_k}$, $k \in I_0$, there exists a maximum subscript $t_{k'} \geq t_k$, $k \in I_0$. Let s_1 be the smallest positive even integer such that

$[g_0(\gamma_{k'})]^{\cap s_1} \in \{\alpha_1, \alpha_2\}$, let $g_1(\xi_0) = K([g_0(\xi_0)]^{\cap s_1})$, and let $I_1 = \{k \in I_0 \mid g_1(\gamma_k) \neq 0\}$. Then

$$g_1(\xi_0) = \begin{cases} 1 & \text{if} \quad \xi_0 = 0, \alpha_1, \dots, \alpha_c \\ 0 & \text{if} \quad \xi_0 = 1, \gamma_k \end{cases} \qquad (k \in I \setminus I_1);$$

$$g_1(\gamma_k) \in \{\gamma_1, \dots, \gamma_e\}, \qquad k \in I_1 .$$

Similarly to Case 3, continue this process for m steps until $I_m = \emptyset$, successively defining $g_2(\xi_0) = K([g_1(\xi_0)]^{\cap s_2}), \dots, g_m(\xi_0) = K([g_{m-1}(\xi_0)]^{\cap s_m})$ where

$$g_m(\xi_0) = \begin{cases} 1 & \text{if} \quad \xi_0 = 0, \alpha_1, \dots, \alpha_c \\ 0 & \text{if} \quad \xi_0 = 1, \gamma_1, \dots, \gamma_e . \end{cases}$$

Then $\delta_{\alpha_1}(\xi_0) = g_m(\xi_0)\, g_m(\xi_0^{\cap}) \dots g_m(\xi_0^{\cap c-1})$.

Case 7 ($d = e = 0$). In this case, $B = \{0, 1, \alpha_1, \dots, \alpha_c\}$. Let $N^{[i]} = \{\alpha_{i1}, \dots, \alpha_{i, n(i)}\}$ for $i = 1, \dots, r$ (see (2.2)). Since $p(\xi_0)$ is not identically 0 on B_0, it follows that $r \geq 2$. Let

$$\cap = (0, 1)(\alpha_{11}, \dots, \alpha_{1, n(1)}, \dots, \alpha_{r1}, \dots, \alpha_{r, n(r)}).$$

If $n_1 + \dots + n_r = 2$, then $B = \{0, 1, \alpha_{11}, \alpha_{21}\}$ where $p(\alpha_{11}) = 0$ and $p(\alpha_{21}) = \alpha_{11}$. In this case, let $h_1(\xi_0) = [p(\xi_0^{\cap})]^{\cap}$ and $h_2(\xi_0) = \xi_0 [p^{(2)}([p(\xi_0)]^{\cap})]$. Then, $\delta_{\alpha_{11}}(\xi_0) = [p([h_1(h_2(\xi_0))]^{\cap})]^{\cap}$. Now assume that $n_1 + \dots + n_r \geq 3$. Consider the unary $(\times, \cap)$-expression $g_0(\xi_0) = \text{def} = p^{(2)}([p^{(r-1)}(\xi_0)]^{\cap})$. If $n_1 + \dots + n_{r-1} \geq 2$, then $\delta_{\alpha_{11}}(\xi_0) = g_0(\xi_0)\, g_0(\xi_0^{\cap}) \dots g_0(\xi_0^{\cap n_1 + \dots + n_{r-1} - 1})$. If $n_1 + \dots + n_{r-1} = 1$, then $n_r \geq 2$ and $\delta_{\alpha_{r, n(r)}}(\xi_0) = [g_0(\xi_0)]^{\cap} [g_0(\xi_0^{\cup})]^{\cap} \dots [g_0(\xi_0^{\cup n(r)-1})]^{\cap}$.

Case 8 ($c = e = 1$, $d = 0$). In this case, $B = \{0, 1, \alpha_1, \gamma_1\}$ where $p(\alpha_1) = 0$ and $p(\gamma_1) = \gamma_1$. Hence, $\gamma_1^2 \neq 0$ and $\alpha_1^2 \notin \{1, \alpha_1\}$. If $\gamma_1^2 \neq \alpha_1$, then let

$$\cap = (0, 1, \gamma_1)(\alpha_1).$$

It is easy to verify that $\delta_{\alpha_1}(\xi_0) = [p(\xi_0)]^{\cap} [p(\xi_0^{\cup})]^{\cap} [p(\xi_0^{\cup_2})]^{\cap}$, $\delta_1(\xi_0) = g(p(\xi_0) \cdot [p(\xi_0)]^{\cap})$ where $g(\xi_0) = (\xi_0^{\cap} \xi_0^{\cap_2})^{\cup_2}$, and that the conditions (a)-(e) of Theorem 1.1 hold.

If $\alpha_1^2 = \gamma_1$, then let $\cap$ be defined on B as in (2.5). Then $\delta_{\alpha_1}(\xi_0) = g_1(g_2(\xi_0))$ where $g_1(\xi_0) = [p([p(\xi_0)]^{\cap})]^{\cap}$ and $g_2(\xi_0) = (\xi_0 [p([p(\xi_0)]^{\cap})])^2$.

If $\gamma_1^2 = \alpha_1$ and $\alpha_1^2 = 0$, let

$$\cap = (0, 1, \alpha_1)(\gamma_1).$$

Then $\delta_1(\xi_0) = p(\xi_0^2)$ and $\delta_{\gamma_1}(\xi_0) = g(\xi_0)\, g(\xi_0^{\cap})\, g(\xi_0^{\cap^2})$ where $g(\xi_0) = ([\delta_1(\xi_0)]^{\cap})^2$. Again, conditions (a)-(e) of Theorem 1.1 are easily verified.

Case 9 ($c = 1$, $d = 0$, $e \geq 2$, and K is $1-1$ on $\gamma_1, \ldots, \gamma_e$). Since K is $1-1$ on $\gamma_1, \ldots, \gamma_e$, there exists an integer N such that $K^{(N)}(\xi_0)$ is the identity on $\gamma_1, \ldots, \gamma_e$. Thus,

$$K^{(N)}(\xi_0) = \begin{cases} 0 & \text{if} \quad \xi_0 = 0, \alpha_1 \\ 1 & \text{if} \quad \xi_0 = 1 \\ \gamma_k & \text{if} \quad \xi_0 = \gamma_k, \ 1 \leq k \leq e. \end{cases}$$

Case 9 (a) (e is even). Let $\cap$ be defined on B as in (2.5) and let $g_1(\xi_0) = K^{(N)}([K^{(N)}(\xi_0)]^{\cap})$, $g_2(\xi_0) = K^{(N)}([g_1(\xi_0)]^{\cap})$, $g_3(\xi_0) = K^{(N)}([g_2(\xi_0)]^{\cap}), \ldots, g_e(\xi_0) = K^{(N)}([g_{e-1}(\xi_0)]^{\cap})$. Then

$$[g_e(\xi_0)]^{\cap} = \begin{cases} 0 & \text{if} \quad \xi_0 = 1, \gamma_2, \gamma_4, \gamma_6, \ldots, \gamma_e \\ 1 & \text{if} \quad \xi_0 = 0, \alpha_1, \gamma_1, \gamma_3, \gamma_5, \ldots, \gamma_{e-1}. \end{cases}$$

We have, $\delta_{\alpha_1}(\xi_0) = [g_e(\xi_0)]^{\cap}[g_e(\xi_0^{\cap})]^{\cap}$.

Case 9 (b) (e is odd). Since $p(\gamma_t) \in \{\gamma_1, \ldots, \gamma_e\}$, $\gamma_t^2 \neq 0$, $1 \leq t \leq e$.

Case 9 (b) (i) (there exists a t_0 such that $\gamma_{t_0}^2 = 1$, α_1, or γ_j where $j \neq t_0$). Let $\cap$ be defined on B as in (2.5) and let $g_1(\xi_0), \ldots, g_e(\xi_0)$ be defined as in Case 9 (a). Then,

$$g_e(\xi_0) = \begin{cases} 1 & \text{if} \quad \xi_0 = 0, \alpha_1, \gamma_2, \gamma_4, \ldots, \gamma_{e-1} \\ 0 & \text{if} \quad \xi_0 = 1, \gamma_1, \gamma_3, \ldots, \gamma_e. \end{cases}$$

Now let $h_1(\xi_0) = K^{(N)}(\xi_0 \cdot g_e(\xi_0))$ and $h_i(\xi_0) = K^{(N)}([h_{i-1}(\xi_0)]^{\cup_2})$ for $i = 2, 3, \ldots, \frac{e-1}{2}$. Then

$$h(\xi_0) = \text{def} = h_{\left(\frac{e-1}{2}\right)}(\xi_0) = \begin{cases} 0 & \text{if} \quad \xi_0 \neq \gamma_{e-1} \\ \gamma_2 & \text{if} \quad \xi_0 = \gamma_{e-1}. \end{cases} \tag{2.6}$$

If $\gamma_{t_0}^2 = 1$, assume that $t_0 = 2$. Then $\delta_{\gamma_{e-1}}(\xi_0) = [h(\xi_0)]^2$. If $\gamma_{t_0}^2 = \alpha_1$, assume that $t_0 = 1$. Then $\delta_{\gamma_{e-1}}(\xi_0) = (K^{(N)}[([h(\xi_0)]^{\cup})^2])^{\cap}$. If $\gamma_{t_0}^2 = \gamma_j$, $j \neq t_0$, assume that $\gamma_3^2 = \gamma_2$. Then $\delta_{\gamma_{e-1}}(\xi_0) = (K^{(N)}[([h(\xi_0)]^{\cap}[h(\xi_0)]^{\cap})^{\cup_2}])^{\cap}$.

Case 9 (*b*) (*ii*) ($\gamma_t^2 = \gamma_t$, $1 \leq t \leq e$). In this case either (I) there exist t_0, s_0 ($t_0 \neq s_0$) such that $\gamma_{t_0}\gamma_{s_0} = \alpha_1$ or (II) $\gamma_t \gamma_s \neq \alpha_1$ for $1 \leq t, s \leq e$ ($t \neq s$). If (I) holds, let $^{\cap}$ be defined on B as in (2.5) and let $h(\xi_0)$ be as in (2.6). Assume that $\gamma_1 \gamma_3 = \alpha_1$. Then $\delta_{\gamma_{e-1}}(\xi_0) = [K^{(N)}([h(\xi_0)]^{\cup}[h(\xi_0)]^{\cap})]^{\cap}$. Suppose (II) holds. Then $\overline{B} = \text{def} = \{0, 1, \gamma_1, \dots, \gamma_e\}$ is a finite binary algebra closed under $\times$. By Theorem 1.3, there exists a cyclic permutation $\hat{} = (0, 1, \gamma_{t_1}, \dots \dots, \gamma_{t_e})$ on $\overline{B}$ such that ($\overline{B}$; $\otimes$, $\hat{}$) is primal ($\otimes$ being the restriction of $\times$ to $\overline{B}$). We can assume that $t_i = i$, $1 \leq i \leq e$, so that $\hat{} = (0, 1, \gamma_1, \dots, \gamma_e)$ on $\overline{B}$. Because of primality, the characteristic functions $\overline{\delta}_0(\xi_0)$, $\overline{\delta}_1(\xi_0)$ of 0, 1, respectively, in $\overline{B}$ are ($\otimes$, $\hat{}$)-expressible. Define $^{\cap}$ on B by

$$^{\cap} = (0, 1, \gamma_1, \dots, \gamma_e)(\alpha_1)$$

and let $g_0(\xi_0)$, $g_1(\xi_1)$ be the ($\times$, $^{\cap}$)-expressions obtained from $\overline{\delta}_0(\xi_0)$, $\overline{\delta}_1(\xi_0)$, respectively, by replacing each occurrence of $\otimes$, $\hat{}$ by $\times$, $^{\cap}$, respectively. In B, then, $^{\bullet}\delta_{\alpha_1}(\xi_0) = [K^{(N)}(\xi_0)]^{\cap}[K^{(N)}(\xi_0^{\cap})]^{\cap} \dots [K^{(N)}(\xi_0^{\cap_{e+1}})]^{\cap}$ and $\delta_0(\xi_0) = g_0(\xi_0) \cdot g_1([\delta_{\alpha_1}(\xi_0)]^{\cap})$. The remaining conditions (*a*)-(*e*) of Theorem 1.1 are easily verified.

Case 10 ($c = 1$, $d = 0$, $e \geq 2$, K is not $1-1$ on $\gamma_1, \dots, \gamma_e$).

Case 10 (*a*) (there exist k_1, k_2 with $k_1 \neq k_2$ such that $K(\gamma_{k_2}) = K(\gamma_{k_1}) = \gamma_{k_1}$). Assume that $K(\gamma_1) = K(\gamma_2) = \gamma_1$. Let $S^{[1]} = \{\gamma_k \mid K(\gamma_k) = \gamma_1,\ 1 \leq k \leq e\}$ and $S^{[t]} = \{\gamma_k \mid K(\gamma_k) \in S^{[t-1]},\ 1 \leq k \leq e\}$ for $t \geq 2$. Then there exists an integer N such that $S^{[N]} \neq \emptyset$ but $S^{[t]} = \emptyset$ if $t > N$. Assume that the γ_k are subscripted in such a fashion that $\{\gamma_1, \dots, \gamma_s\} = \bigcup_{t=1}^{N} S^{[t]}$. If $K'(\xi_0) = K^{(N)}(\xi_0)$, then

$$K'(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0, \alpha_1 \\ 1 & \text{if } \xi_0 = 1 \\ \gamma_1 & \text{if } \xi_0 = \gamma_1, \dots, \gamma_s; \end{cases}$$

$$K'(\{\gamma_{s+1}, \dots, \gamma_e\}) \subseteq \{\gamma_{s+1}, \dots, \gamma_e\}.$$

Define $^{\cap}$ on B as in (2.5).

Case 10 (*a*) (i) (K' is $1-1$ on $\gamma_{s+1}, \dots, \gamma_e$). Let P be an integer for which $K''(\xi_0) = \text{def} = K'^{(P)}(\xi_0)$ is the identity on $\{\gamma_{s+1}, \dots, \gamma_e\}$. Then, letting $g_0(\xi_0) = K''([K''(\xi_0)]^{\cup})$ and $g_r(\xi_0) = K''([g_{r-1}(\xi_0)]^{\cup})$ for $r = 1, \dots, e-s$ one has

$$g_{e-s}(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0,\ \alpha_1,\ \gamma_{s+1},\ \gamma_{s+3}, \dots \\ 1 & \text{if } \xi_0 = 1,\ \gamma_1, \dots, \gamma_s,\ \gamma_{s+2},\ \gamma_{s+4}, \dots \end{cases} \qquad \text{(if } e-s \text{ is odd)};$$

$$g_{e-s}(\xi_0) = \begin{cases} 1 & \text{if } \xi_0 = 0,\ \alpha_1,\ \gamma_{s+1},\ \gamma_{s+3}, \dots \\ 0 & \text{if } \xi_0 = 1,\ \gamma_1, \dots, \gamma_s,\ \gamma_{s+2},\ \gamma_{s+4}, \dots \end{cases} \qquad \text{(if } e-s \text{ is even)}.$$

Hence

$$\delta_{\gamma_s}(\xi_0) = \begin{cases} g_{e-s}(\xi_0)\, g_{e-s}(\xi_0^{\cup}) \dots g_{e-s}(\xi_0^{\cup_{s-1}}), & \text{if } e-s \text{ is odd} \\ [g_{e-s}(\xi_0)]^{\cap}\, [g_{e-s}(\xi_0^{\cup})]^{\cap} \dots [g_{e-s}(\xi_0^{\cup_{s-1}})]^{\cap}, & \text{if } e-s \text{ is even.} \end{cases}$$

Case 10 (*a*) (ii) (K' is not $1-1$ on $\gamma_{s+1}, \dots, \gamma_e$). Assume that $\gamma_{s+1} \notin$ $\notin$ Range (K'). Let $g_0(\xi_0) = [K'([K'(\xi_0)]^{\cup})]^{\cup}$. It is readly verified that

$$g_0(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0,\ \alpha_1 \\ 1 & \text{if } \xi_0 = 1,\ \gamma_1, \dots, \gamma_s; \end{cases}$$

$$g_0(\{\gamma_{s+1}, \dots, \gamma_e\}) \subseteq \{\gamma_{s+1}, \dots, \gamma_{e-1}\}.$$

Let $I = \{s+1, \dots, e\}$. Among the subscripts t_k arising from $g_0(\gamma_t) = \text{def} = \gamma_{t_k}$, $t \in I$, let $t_{k'}$ be minimal and let s_1 be the smallest positive even integer such that $(\gamma_{t_{k'}})^{\cup_{s_1}} \in \{\gamma_{s-1}, \gamma_s\}$. If $g_1(\xi_0) = g_0([g_0(\xi_0)]^{\cup_{s_1}})$ and $I_1 = \{t \in I \mid g_1(\gamma_t) \neq 1\}$, then

$$g_1(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0,\ \alpha_1 \\ 1 & \text{if } \xi_0 = 1,\ \gamma_1, \dots, \gamma_s,\ \gamma_k \end{cases} \qquad (k \in I \setminus I_1);$$

$$g_1(\gamma_k) \in \{\gamma_{s+1}, \dots, \gamma_{e-1}\}, \quad k \in I_1\,.$$

Continue this process, similarly to Case 4, through m steps until $I_m = \emptyset$, successively defining $g_2(\xi_0) = g_0([g_1(\xi_0)]^{\cup_{s_2}}), \dots, g_m(\xi_0) = g_0([g_{m-1}(\xi_0)]^{\cup_{s_m}})$ where

$$g_m(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0,\ \alpha_1 \\ 1 & \text{if } \xi_0 = 1,\ \gamma_1, \dots, \gamma_e. \end{cases}$$

Then $\delta_{\gamma_e}(\xi_0) = g_m(\xi_0)\, g_m(\xi_0^{\cup}) \dots g_m(\xi_0^{\cup_{e-1}})$.

Case 10 (*b*) (there does not exist k_1, k_2 with $k_1 \neq k_2$ such that $K(\gamma_{k_2}) = K(\gamma_{k_1}) = \gamma_{k_1}$). Since K is not $1-1$ on $\gamma_1, \ldots, \gamma_e$ there exist pairwise non-equal integers k_1, k_2, k_3 such that $K(\gamma_{k_3}) = K(\gamma_{k_2}) = \gamma_{k_1}$. Moreover, by hypothesis, $K(\gamma_{k_1}) \neq \gamma_{k_1}$.

Case 10 (*b*) (i) ($\gamma_{k_2} \notin \mathrm{Range}(K)$ or $\gamma_{k_3} \notin \mathrm{Range}(K)$). Assume that $k_2 = 1$, $k_3 = 2$, and $k_1 = e$. Thus, $K(\gamma_1) = K(\gamma_2) = \gamma_e$, $K(\gamma_e) \neq \gamma_e$, and either $\gamma_1 \notin \mathrm{Range}(K)$ or $\gamma_2 \notin \mathrm{Range}(K)$. Assume that $\gamma_1 \notin \mathrm{Range}(K)$. Also, assume that the γ_k are subscripted so that $\{\gamma_1, \ldots, \gamma_s\} = \{\gamma_k \mid K(\gamma_k) = \gamma_e,\ 1 \leq k \leq e\}$. Define $^{\cap}$ on B as in (2.5). Let $I = \{s+1, \ldots, e\}$. If $g_0(\xi_0) = [K([K(\xi_0)]^{\cap})]^{\cup}$, then

$$g_0(\xi_0) = \begin{cases} 0 & \text{if} \quad \xi_0 = 0,\ \alpha_1 \\ 1 & \text{if} \quad \xi_0 = 1,\ \gamma_1, \ldots, \gamma_s; \end{cases}$$

$$g_0(\gamma_k) \in \{\gamma_1, \ldots, \gamma_{e-1}\} \quad \text{if} \quad k \in I.$$

Now let $T^{[1]} = \{\gamma_k \mid g_0(\gamma_k) \in \{\gamma_1, \ldots, \gamma_s\},\ k \in I\}$ and $T^{[r]} = \{\gamma_k \mid g_0(\gamma_k) \in T^{[r-1]},\ k \in I\}$ if $r \geq 2$. Then there exists an integer R such that $T^{[R]} \neq \emptyset$ but $T^{[r]} = \emptyset$ if $r > R$. Let $I_0 = \{k \in I \mid g_0^{(R+1)}(\gamma_k) \neq 1\}$. Then

$$g_0'(\xi_0) = \mathrm{def} = g_0^{(R+1)}(\xi_0) = \begin{cases} 0 & \text{if} \quad \xi_0 = 0,\ \alpha_1 \\ 1 & \text{if} \quad \xi_0 = 1,\ \gamma_1, \ldots, \gamma_s,\ \gamma_k \end{cases} \qquad (k \in I \setminus I_0);$$

$$g_0'(\gamma_k) \in \{\gamma_{s+1}, \ldots, \gamma_e\}, \quad k \in I_0.$$

Among the subscripts of the $\gamma_{t_k} = \mathrm{def} = g_0'(\gamma_k)$, $k \in I_0$, let $t_{k'}$ be minimal and let s_1 be the smallest positive even integer such that $[g_0'(\gamma_{k'})]^{\cup s_1} \in \{\gamma_{s-1}, \gamma_s\}$. Then, if $g_1(\xi_0) = g_0'([g_0'(\xi_0)]^{\cup s_1})$ and $I_1 = \{k \in I_0 \mid g_1(\gamma_k) \neq 1\}$, it follows that

$$g_1(\xi_0) = \begin{cases} 0 & \text{if} \quad \xi_0 = 0,\ \alpha_1 \\ 1 & \text{if} \quad \xi_0 = 1,\ \gamma_1, \ldots, \gamma_s,\ \gamma_k \end{cases} \qquad (k \in I \setminus I_1);$$

$$g_1(\gamma_k) \in \{\gamma_{s+1}, \ldots, \gamma_e\}, \quad k \in I_1.$$

Continue this process, similarly to Case 4, for m steps until $I_m = \emptyset$, successively defining $g_2(\xi_0) = g_0'([g_1(\xi_0)]^{\cup s_2}), \ldots, g_m(\xi_0) = g_0'([g_{m-1}(\xi_0)]^{\cup s_m})$. Then $\delta_{\gamma_e}(\xi_0) = g_m(\xi_0)\, g_m(\xi_0^{\cup}) \ldots g_m(\xi_0^{\cup e-1})$.

Case 10 (b) (ii) ($\gamma_{k_2}, \gamma_{k_3} \in \text{Range}(K)$). Assume that $k_1 = 1$, $k_2 = 2$, and $k_3 = 3$, so that $K(\gamma_2) = K(\gamma_3) = \gamma_1$ and $K(\gamma_1) \neq \gamma_1$. Since K is not $1 - 1$ on $\gamma_1, \dots, \gamma_e$ there exists a $\gamma_k \notin \text{Range}(K)$; assume that it is γ_e. If $K(\gamma_e) = \gamma_1$, then Case 10 (b) (i) applies. So assume $K(\gamma_e) \neq \gamma_1$. Also, we may assume that $\{\gamma_2, \dots, \gamma_s\} = \{\gamma_k \mid K(\gamma_k) = \gamma_1,\ 1 \leq k \leq e\}$. Define $^\cap$ on B as in (2.5). If $I = \{1, s+1, s+2, \dots, e\}$ and $g_0(\xi_0) = [K([K(\xi_0)]^\cup)]^\cap$ then

$$g_0(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0, \alpha_1 \\ 1 & \text{if } \xi_0 = 1, \gamma_2, \dots, \gamma_s; \end{cases}$$

$$g_0(\gamma_k) \in \{\gamma_2, \gamma_3, \dots, \gamma_e\}, \quad k \in I.$$

Since $\gamma_1 \notin \text{Range}(g_0)$, similarly to Case 10 ($b$) (i), there exists an integer $R > 0$ such that

$$f_0(\xi_0) = \text{def} = g_0^{(R+1)}(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0, \alpha_1 \\ 1 & \text{if } \xi_0 = 1, \gamma_2, \dots, \gamma_s, \gamma_k \end{cases} \quad (k \in I \setminus I_0);$$

$$f_0(\gamma_k) \in \{\gamma_{s+1}, \dots, \gamma_e\}, \quad k \in I_0,$$

where $I_0 = \{k \in I \mid g_0^{(R+1)}(\gamma_k) \neq 1\}$. Among the subscripts of the $\gamma_{t_k} = \text{def} = f_0(\gamma_k)$, $k \in I_0$, let $t_{k'}$ be minimal and choose s_1 to be the smallest positive even integer such that $[f_0(\gamma_{k'})]^{\cup_{s_1}} \in \{\gamma_{s-1}, \gamma_s\}$. Let $f_1(\xi_0) = f_0([f_0(\xi_0)]^{\cup_{s_1}})$. Continue this process for m steps until $I_m = \emptyset$, successively defining $f_2(\xi_0) = f_0([f_1(\xi_0)]^{\cup_{s_2}}), \dots, f_m(\xi_0) = f_0([f_{m-1}(\xi_0)]^{\cup_{s_m}})$. Then, since

$$I_m = \{k \in I \mid f_m(\gamma_k) \neq 1\} = \emptyset,$$

$$f_m(\xi_0) = \begin{cases} 0 & \text{if } \xi_0 = 0, \alpha_1 \\ 1 & \text{if } \xi_0 = 1, \gamma_1, \dots, \gamma_e \end{cases}$$

and $\delta_{\gamma_e}(\xi_0) = f_m(\xi_0) f_m(\xi_0^\cup) \dots f_m(\xi_0^{\cup_{e-1}})$.

By applying now either Theorem 1.1 or Theorem 1.4 the permutation $^\cap$ constructed in each case renders $(B; \times, ^\cap)$ a regular subprimal algebra with frame $[0, 1, \times, ^\cap]$. This completes the proof.

Remark. If in (a) and (b) of Theorem 2.2, 0 and 1 are consistently interchanged, the resulting proposition is valid. Its proof is similar to the work above, and involves interchanging the roles of c and d, making minor modifications case by case. This fact, then, in combination with Theorem 2.2, establishes the following

THEOREM 2.3. (*Principal Theorem on Regular Subprimal Conversion*). *Let* $(B; \times)$ *be a finite binary algebra of order* $n \geq 4$ *with null* 0 *and identity* 1. *Suppose there exists a unary* $(\times)$*-expression,* $p(\xi_0)$, *and an element* $\alpha \in B \setminus \{0, 1\}$ *for which*

(*a*) $p(\alpha) = 0$ *or* $p(\alpha) = 1$;

(*b*) *p is not constant on* $B \setminus \{0, 1\}$.

Then, there exists a permutation, $\cap$, *on B such that* $(B; \times, \cap)$ *is a regular subprimal algebra with frame* $[0, 1, \times, \cap]$.

We now show that (*b*) cannot be deleted from Theorem 2.3.

EXAMPLE. Let $(B_5; \times)$ be the binary algebra with the following multiplication table.

$\times$	0	1	a_1	a_2	a_3
0	0	0	0	0	0
1	0	1	a_1	a_2	a_3
a_1	0	a_1	0	a_3	a_2
a_2	0	a_2	a_3	0	a_1
a_3	0	a_3	a_2	a_1	0

Since $a_i^2 = 0$, $1 \leq i \leq 3$, any $(\times)$-expression, $p(\xi_0)$, is identically 0 on $B_5 \setminus \{0, 1\}$. If there is a permutation, $\cap$, on B_5 for which $(B_5; \times, \cap)$ is a regular subprimal with frame $[0, 1, \times, \cap]$, then the core cannot contain more than a single a_i. The possible candidates for $\cap$ are, therefore, of the form

$$\cap = (0, 1)(a_i, a_j, a_k);$$

$$\cap = (0, 1)(a_i)(a_j)(a_k);$$

$$\cap = (0, 1)(a_i)(a_j, a_k);$$

$$\cap = (0, 1, a_i)(a_j, a_k);$$

$$\cap = (0, 1, a_i)(a_j)(a_k).$$

But, for any permutation of the first two types,

$$0 \to 0, \quad 1 \to 1, \quad a_i \to a_j, \quad a_j \to a_k, \quad a_k \to a_i$$

is a non-identical automorphism of $(B_5; \times, ^{\cap})$ while for permutations of the last three types,

$$0 \to 0, \quad 1 \to 1, \quad a_i \to a_i, \quad a_j \to a_k, \quad a_k \to a_j$$

is a non-identical automorphism. Since a semi-primal has no non-identical automorphisms [4 ; Theorem 3.3] no permutation, $^{\cap}$, on B_5 renders $(B_5; \times, ^{\cap})$ a regular subprimal with frame $[0, 1, \times, ^{\cap}]$. Similar remarks can be made if we define $a_i^2 = 1$, $1 \leq i \leq 3$, and do not change the remaining entries of the table.

3. Singular Subprimal Conversion.

From Theorem 1.2 it follows that with each singular subprimal algebra $\mathfrak{B} = (B; \Omega)$ is associated a singular coupling $[0, \times, T; 1, 1^0]$ and hence a finite binary algebra $(B; \times)$ having 0 as null and 1 as identity. It seems interesting then to inquire whether or not each finite binary algebra $(B; \times)$ with null 0 and identity 1 can be converted into a singular subprimal algebra $(B; \times, T)$ for which $[0, \times, T; 1, 1^0]$ is a singular coupling, 1^0 being some member of B. That this conversion can always be effected is a result of

THEOREM 3.1. *Let $(B; \times)$ be a finite binary algebra with null 0 and identity 1. Then there exists an element 1^0 in B and a binary operation $\xi_0 T \xi_1$ definable on B such that $(B; \times, T)$ is a singular subprimal algebra with singular coupling $[0, \times, T; 1, 1^0]$.*

PROOF. For the 2-element binary algebra $(\{0, 1\}; \times)$ it is easily verified that conditions *(a)-(e)* of Theorem 1.2 hold if $\xi_0 T \xi_1$ is defined by $0 T \xi_0 = \xi_0 T 0 = \xi_0$ and $1 T 1 = 0$. Let, then, $B = \{0, 1, b_1, \dots, b_m\}$ be the base set of a binary algebra of order $m + 2$, where $m \geq 1$. Consider the cases *(I)* $m \geq 2$ and *(II)* $m = 1$. For *(I)*, define T on B such that the following hold:

(i) $0 T \xi_0 = \xi_0 T 0 = \xi_0$, for each ξ_0 in B;

(ii) $1 T 1 = b_1$, $b_1 T b_1 = b_2, \dots, b_m T b_m = 1$;

(iii) $1 T b_1 = 1$, $b_1 T b_2 = b_2 T b_3 = \dots = b_{m-1} T b_m = b_m T 1 = 1 T b_m = 0$;

(iv) $\xi_0 T \xi_1$ is defined arbitrarily for other ξ_0, ξ_1 in B.

In case (II), define T on B by:

$$0 T \xi_0 = \xi_0 T 0 = \xi_0 \quad \text{for each } \xi_0 \text{ in } B;$$

$$1 T 1 = b_1, \quad b_1 T b_1 = 1;$$

$$1 T b_1 = b_1 T 1 = 0.$$

In either case (I) or (II), let $\xi_0^{\cap} = \xi_0 T \xi_0$. If the characteristic function $\delta_1(\xi_0)$ is $(\times, T)$-expressible then $\delta_{b_1}(\xi_0), \ldots, \delta_{b_m}(\xi_0)$, $\Gamma_1(\xi_0)$, and 0 are $(\times, T)$-expressible since

$$\delta_{b_m}(\xi_0) = \delta_1(\xi_0^{\cap}),\ \delta_{b_{m-1}}(\xi_0) = \delta_1(\xi_0^{\cap_2}), \ldots, \delta_{b_1}(\xi_0) = \delta_1(\xi_0^{\cap_m});$$

$$\Gamma_1(\xi_0) = \delta_1(\xi_0)\, T\, \delta_{b_1}(\xi_0)\, T \ldots T\, \delta_{b_m}(\xi_0)$$

$$0 = \delta_1(\xi_0) \times \delta_{b_1}(\xi_0).$$

In case (I), $\delta_1(\xi_0) = \xi_0 T \xi_0^{\cap}$, while in case (II), $\delta_1(\xi_0) = \xi_0^2$, $(\xi_0 T \xi_0^2)^2$, or $\xi_0^{\cap} T (\xi_0 \xi_0^{\cap})$, according as $b_1^2 = 0, 1$, or b_1, respectively. In each case, it is clear that $\{0\}$ is the unique subalgehra of $(B; \times, T)$ and that $[0, \times, T; 1, b_m]$ is a singular coupling. The conditions (a)-(e) of Theorem 1.2 are verified and $(B; \times, T)$ is a singular subprimal algebra.

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